Research Article

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A Liouville theorem for the Hénon-Lane-Emden system in four and five dimensions

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Abstract: In the present article, we investigate the following Hénon-Lane-Emden elliptic system:

$$\begin{cases} -\Delta u = |x|^a v^p, & x \in \mathbb{R}^N, \\ -\Delta v = |x|^b u^q, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 2$, $p, q > 0$, $a, b \in \mathbb{R}$. We partially prove the Hénon-Lane-Emden conjecture in the case of four and five dimensions. More specifically, we show that there is no nonnegative nontrivial classical solution for the Hénon-Lane-Emden elliptic system when $a, b > -2$ and the parameter pair (p, q) meets

$$pq > 1, \quad \frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2,$$

and additionally $p, q < 4/3$ if $N = 4$ or $p, q < 10/9$ if $N = 5$.

Keywords: Liouville-type theorems, Hénon-Lane-Emden elliptic system, nonnegative nontrivial classical solution

MSC 2020: 35J60, 35B33, 35B53

1 Introduction

We consider the following Hénon-Lane-Emden elliptic system:

$$\begin{cases} -\Delta u = |x|^a v^p, & x \in \mathbb{R}^N, \\ -\Delta v = |x|^b u^q, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 2$, $p, q > 0$, $a, b \in \mathbb{R}$. The Hénon-Lane-Emden system describes the variation of pressure and density with radius in self-gravitating spheres of plasma such as stars, which plays an important role in the study of the polytropic-convective equilibrium configuration structure of self-gravitating and spherically symmetric fluids in astrophysics [10,22]. In this article, we study the nonexistence of nonnegative nontrivial classical solutions, which is the so-called Liouville-type theorem, in a certain parameter range. The Hénon-Lane-Emden conjecture is classified as a Liouville-type theorem for the Hénon-Lane-Emden system.

The Liouville-type theorem for this elliptic system with general nonlinear term has been obtained by Chen and Lu [12], who gave the conditions for the existence and nonexistence of its positive radial solutions. For the case of a single equation, the existence and nonexistence of positive solutions have been widely researched [8,9,17]. In this case, system collapses to the equation:

$$-\Delta u = |x|^a u^p, \quad x \in \mathbb{R}^N, \tag{2}$$

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where $N \geq 2$, $p > 1$. For the case $N = 1$, we refer to [31]. Equation (2) is generally called of *Lane-Emden type* if $\alpha = 0$ and it is of *Hénon* (*Hardy*) *type* if $\alpha > 0$ ($\alpha < 0$).

(i) *Lane-Emden type*. Relying on the Bochner identity

$$\frac{1}{2}\Delta\,|\nabla v|^2 = \nabla(\Delta v)\cdot\nabla v + |D^2 v|^2,$$

and text-function method, the nonexistence of positive classical solution of (2) can be established, provided that $p < p_s = \frac{N+2}{N-2}$. Furthermore, by applying the asymptotic symmetry technique, Caffarelli et al. proved that any positive classical solution of (2) is radially symmetric with respect to some point when $p = p_s$ [8].

(ii) *Hénon and Hardy type*. If $\alpha \leq -2$, then (2) admits no positive solutions in $\mathbb{R}^N$ [2]. For $\alpha > -2$, we denote the Hardy-Sobolev exponent by

$$p_\alpha = \begin{cases} \infty, & \text{if } N = 2,\\ \dfrac{N+2+2\alpha}{N-2}, & \text{if } N \geq 3, \end{cases}$$

which is the dividing point between the existence and nonexistence of positive radial solutions of (2). Existence for $p \geq p_\alpha$ and nonexistence for $p < p_\alpha$ were proved by Phan and Souplet [31] by utilizing a combination of Pohozaev identity, Sobolev inequality on S_1 and a measure argument. On the other hand, apart from the radial case, some interesting results have been obtained by Bidaut-Véron and Giacomini [2], Mitidieri and Pokhozhaev [28]. Specifically, the nonexistence of positive solutions was established for $N \geq 2$, provided that $p < \min\{p_s, p_\alpha\}$ or $p \leq \frac{N+\alpha}{N-2}$.

Let us obtain back to system (1). Before introducing the Hénon-Lane-Emden conjecture, we define the critical Sobolev hyperbola first:

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} = N-2,$$

and we call the pair (p, q) subcritical if (p, q) lies below the critical Sobolev hyperbola, i.e.,

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2.$$

Hénon-Lane-Emden conjecture *If the pair* (p, q) *is subcritical, then there is no nonnegative nontrivial classical solution of system* (1).

Although the Hénon-Lane-Emden conjecture and its related problems have been studied by many authors in recent years [1,6,11–13,15,16,18,20,23–25,29–32,36], the conjecture has not yet been completely proved. Fazly and Ghoussoub [16], Li and Zhang [23] separately gave partial and complete proofs of the conjecture in the case of dimension $N = 3$ in different ways; however, little progress has been made in the case of dimension $N \geq 4$. Therefore, it is meaningful to consider the Hénon-Lane-Emden conjecture in higher dimensions.

Next, we review some results for the case of $a = b = 0$ and the case of $a \neq 0$ or $b \neq 0$, respectively.

Case 1. $a = b = 0$.

When $a = b = 0$, system (1) degenerates into the Lane-Emden elliptic system

$$\begin{cases} -\Delta u = v^p, & x \in \mathbb{R}^N,\\ -\Delta v = u^q, & x \in \mathbb{R}^N, \end{cases} \tag{3}$$

which is widely used in physical research and has been investigated in [3,5,9,12,16,17,19,21,26]. In this case, the **Hénon-Lane-Emden Conjecture** reduces to the following conjecture.

Lane-Emden conjecture *If the pair* (p, q) *is subcritical, then there is no nonnegative nontrivial classical solution of system* (3).

For the Lane-Emden conjecture, there have been extensive studies in [5,12,27,32–35]. Mitidieri [27] proved that if the parameter pair (p, q) is subcritical and $p, q > 1$, then system (3) admits no positive radial solutions. The condition $p, q > 1$ is waived in [32] by Serrin and Zou. Souto [35] and Mitidieri [27] confirmed the Lane-Emden conjecture in dimensions $N = 1$ and $N = 2$, respectively. As for $N = 3$, Serrin and Zou [33] proved that if the parameter pair (p, q) is subcritical, then system (3) does not possess any nonnegative nontrivial classical solutions with algebraic growth at infinity. In addition, they showed that system (3) has no nonnegative nontrivial classical solutions if the parameter satisfies

$$pq \leq 1 \quad \text{or} \quad pq > 1, \quad \max\{\alpha, \beta\} \geq N - 2,$$

where $\alpha = \frac{2(p+1)}{pq-1}$, $\beta = \frac{2(q+1)}{pq-1}$. Further evidence supporting the conjecture can be found in [5], where it is shown that system (3) does not possess any nonnegative nontrivial classical solutions if the parameter pair (p, q) is subcritical and satisfies

$$\min\{\alpha, \beta\} \geq \frac{N-2}{2}, \quad (\alpha, \beta) \neq \left(\frac{N-2}{2}, \frac{N-2}{2}\right).$$

Souplet [34] confirmed this conjecture when the dimension $N = 3$ or $N = 4$ and gave a new parameter region in which the conjecture is true in the dimension $N \geq 5$: if the parameter pair (p, q) is subcritical satisfying

$$p, q > 0 \quad pq > 1, \quad \max\{\alpha, \beta\} > N - 3.$$

Case 2. $a \neq 0$ or $b \neq 0$.

In this case, Bidaut-Véron and Giacomini [2] proved that system (1) admits a positive radial solution only if the parameter pair (p, q) is above or on the Sobolev hyperbola. This means that if the parameter pair (p, q) is subcritical, then (1) has no positive radial solutions. Fazly and Ghoussoub [16] gave that if the parameter pair (p, q) is subcritical in dimension $N = 3$, then (1) has no bounded positive classical solutions. Carioli and Musina [7] showed that (1) has a positively distributed solution only in a certain parameter range. By applying variational methods, one obtains the existence of nontrivial and radially symmetric solutions of (1) when the parameters lie on the critical Sobolev hyperbola [29]. Fazly [15] confirmed that if the pair (p, q) satisfies

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2m, \tag{4}$$

in dimension $N = 2m + 1$, then the extended system of (1) – the polyharmonic Hénon-Lane-Emden system

$$\begin{cases} (-\Delta u)^m = |x|^a v^p, & x \in \mathbb{R}^N, \\ (-\Delta v)^m = |x|^b u^q, & x \in \mathbb{R}^N, \end{cases} \quad m, p, q \geq 1, \tag{5}$$

where $pq \neq 1$, a, $b \geq 0$, admits only one bounded nonnegative solution which is $(u, v) = (0, 0)$; however, the condition “bounded” was removed by Phan [30]. Furthermore, Arthur and Yan [1] showed that system (5) has no positive solutions with slow decay rates in some parameter range based on the Rellich-Pohozaev identity combined with an adapted idea of measure and feedback argument. Recently, in 2019, Li and Zhang [23] demonstrated the following result.

Theorem A. [23, Theorem 1.1]. *For $N = 3$, if $\min\{a, b\} \leq -2$ or $\min\{a, b\} > -2$ and the pair (p, q) is subcritical, then system (1) admits no nonnegative nontrivial classical solutions.*

Theorem A implies that the Hénon-Lane-Emden conjecture holds in dimension $N = 3$. The following is established when $N > 3$.

Theorem B. [23, Theorem 3.1] *For $N \geq 3$, denote $\alpha = \frac{2(p+1)+a+pb}{pq-1}$, $\beta = \frac{2(q+1)+b+qa}{pq-1}$. If the parameters satisfy*

$$\min\{a, b\} \leq -2 \quad \text{or} \quad pq \leq 1 \quad \text{or} \quad \max\{\alpha, \beta\} \geq N - 2,$$

then system (1) *does not possess positive upper solutions. Here we say that* (u, v) *is a positive upper solution of system* (1), *if* $u, v \in C^2(\mathbb{R}^N\backslash\{0\}) \cap C(\mathbb{R}^N)$, $u, v > 0$ *in* $\mathbb{R}^N\backslash\{0\}$, *and* (u, v) *satisfies*

$$\begin{cases} -\Delta u \geq |x|^a v^p, & x \in \mathbb{R}^N\backslash\{0\}, \\ -\Delta v \geq |x|^b u^q, & x \in \mathbb{R}^N\backslash\{0\}. \end{cases}$$

The main result of this article is as follows.

Theorem 1. *If* $a, b > -2$, $pq > 1$,

$$\begin{cases} p, q < 4/3, & N = 4, \\ p, q < 10/9, & N = 5, \end{cases}$$

and the pair (p, q) *is subcritical, i.e.*,

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2,$$

then system (1) *admits no nonnegative nontrivial classical solutions.*

Compared with the results in [23], we have extended the range of parameters p and q in dimensions $N = 4$ and $N = 5$. Combining Theorems 1 and B, we are led to the following.

Corollary 1. *If*

$$\begin{cases} p, q < 4/3, & N = 4, \\ p, q < 10/9, & N = 5, \end{cases}$$

and the pair (p, q) *is subcritical, i.e.*,

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2,$$

then system (1) *admits no nonnegative nontrivial classical solutions.*

The rest of the article is organized as follows. In Section 2, we give an overview of the proof. In Section 3, we present some preliminary lemmas. Section 4 is devoted to the proof of the main results. Finally, some conclusions are drawn in Section 5. Throughout the article, $\|\cdot\|_s$ represents the standard L^p-norm in space L^s, C is a general positive constant independent of u and v, which differs from line to line. B_R and S_R represent the ball and sphere of radius R in $\mathbb{R}^N$, respectively. We denote $|S_R|$ by the surface area of S_R.

2 Outline of proofs

The proof of Theorem 1 and Corollary 1 can be outlined as follows by a standard contradiction argument:

Step 1. Suppose (u, v) is a nonnegative nontrivial classical solution of system (1), then there is a constant $C = C\ (a, b, p, q, N)$ that is independent of solution (u, v), such that

$$\int_{B_1} |x|^b u^{q+1} \mathrm{d}x + \int_{B_1} |x|^a v^{p+1} \mathrm{d}x \leq C. \tag{6}$$

Step 2. It is well known if (u, v) is the solution of system (1), then for any $l \geq 1$, $(2^{l\alpha}u(2^l x), 2^{l\beta}v(2^l x))$ is also the solution of system (1).

Step 3. Noting C is independent of (u, v), substituting $(2^{l\alpha}u(2^l x), 2^{l\beta}v(2^l x))$ into (6), we obtain

$$\int_{B_{2^l}} |x|^b u^{q+1} dx + \int_{B_{2^l}} |x|^a v^{p+1} dx \le C 2^{l(N-2-\alpha-\beta)}.$$

Let $l \to \infty$, noting $\alpha + \beta > N - 2$, we yield

$$\int_{\mathbb{R}^N} |x|^b u^{q+1} dx + \int_{\mathbb{R}^N} |x|^a v^{p+1} dx = 0.$$

Recalling u and v are continuous functions, this immediately gives

$$u \equiv 0 \quad \text{and} \quad v \equiv 0, \quad x \in \mathbb{R}^N,$$

which contradicts the assumption in Step 1, and the proof is completed.

Remark 1. Since a and b can be negative in system (1), the classical solution of system (1) will be considered in the following set:

$$C^2(\mathbb{R}^N \backslash \{0\}) \cap C(\mathbb{R}^N),$$

and the classical solution satisfies system (1) in $\mathbb{R}^N \backslash \{0\}$.

3 Preliminaries

In this section, we present some preliminary results. The following Sobolev embedding inequalities on a unit sphere will be used frequently in our proofs. See, e.g., [33].

Lemma 1. (Sobolev embedding theorem on smooth compact manifolds S_1). *Let $N \ge 2$ and $j \ge 1$ be any two integers, $k \in (1, \infty)$ and $\omega = \omega(\theta) \in W^{j,k}(S_1)$.*

(*i*) *If $kj < \dim(S_1)$, then $\|\omega\|_\lambda \le C(\|D_\theta^j \omega\|_k + \|\omega\|_1)$, where $C = C(j, k, N) > 0$ and λ satisfies*

$$\frac{1}{k} - \frac{1}{\lambda} = \frac{j}{N-1}.$$

(*ii*) *If $kj = \dim(S_1)$, then for any $\lambda \in [1, \infty)$, $\|\omega\|_\lambda \le C(\|D_\theta^j \omega\|_k + \|\omega\|_1)$, where $C = C(\lambda, k, N) > 0$.*

To prepare for the proof in Section 4, we need to show that the local integral of $|D\omega|^2 \omega^{\gamma-2}$ can be estimated by the local integral of ω^γ. This idea originates from Serrin and Zou [33].

Lemma 2. *If $\omega(x) \ge 0$ is nontrivial and satisfies*

$$\Delta\omega \le 0, \quad x \in \mathbb{R}^N,$$

then for $\eta \in C_0^\infty(\mathbb{R}^N)$ and $\gamma < 1$, there is a constant $c = \frac{8}{(1-\gamma)^2} > 0$ such that

$$\int_{\mathbb{R}^N} \eta^2 |D\omega|^2 \omega^{\gamma-2} dx \le c \int_{\mathbb{R}^N} |D\eta|^2 \omega^\gamma dx.$$

The following lemma can be obtained by integrating $u(r, \theta)$ and $v(r, \theta)$ on the manifold S_1 and using Jensen's inequality. For a similar proof of this lemma, we refer to [33].

Lemma 3. *If $a, b > -2$, $pq > 1$, (u, v) is a positive classical solution of system* (1), *then there exists a positive constant $C = C(a, b, p, q, N)$ such that*

$$\int_{S_1} u(r, \theta) \le C r^{-\alpha}, \quad \int_{S_1} v(r, \theta) \le C r^{-\beta}, \quad \forall r > 0,$$

where $\alpha = \frac{2(p+1)+a+pb}{pq-1}$, $\beta = \frac{2(q+1)+b+qa}{pq-1}$.

Corollary 2. *Under the assumptions of* Lemma 3, *when* $\gamma \in (0,1)$, *we have*

$$\int_{S_1} u^\gamma(r,\theta) \le Cr^{-\alpha} + |S_1|, \quad \int_{S_1} v^\gamma(r,\theta) \le Cr^{-\beta} + |S_1|, \quad \forall r > 0.$$

Proof. For any $r > 0$, denote $S_1^{1,r} := \left\{\frac{x}{r} \middle| u(x) \ge 1, x \in S_r\right\}$ and $S_1^{2,r} := \left\{\frac{x}{r} \middle| u(x) \le 1, x \in S_r\right\}$. Direct calculation gives

$$\int_{S_1} u^\gamma(r,\theta) = \int_{S_1^{1,r}} u^\gamma(r,\theta) + \int_{S_1^{2,r}} u^\gamma(r,\theta) \le \int_{S_1^{1,r}} u(r,\theta) + |S_1| \le Cr^{-\alpha} + |S_1|.$$

Similarly, one obtains $\int_{S_1} v^\gamma(r,\theta) \le Cr^{-\beta} + |S_1|$, and the proof is completed. □

Lemma 4. [23] (Rellich-Pohozaev inequality) *If* (u, v) *is a positive classical solution of system* (1), *then for any* $R > 0$, *the following inequality holds*:

$$c_1 \int_{B_R} |x|^b u^{q+1} + c_2 \int_{B_R} |x|^a v^{p+1} \le \frac{R^{1+b}}{q+1} \int_{S_R} u^{q+1} + \frac{R^{1+a}}{p+1} \int_{S_R} v^{p+1},$$

where $c_1 = \frac{N+b}{q+1} - \frac{N-2}{2}$, $c_2 = \frac{N+a}{p+1} - \frac{N-2}{2}$. *Moreover, for any* $c \in \mathbb{R}$, *we have*

$$(c_1 - c)\int_{B_R} |x|^b u^{q+1} + (c_2 + c)\int_{B_R} |x|^a v^{p+1} \le \frac{R^{1+b}}{q+1} \int_{S_R} u^{q+1} + \frac{R^{1+a}}{p+1} \int_{S_R} u^{p+1} + c\int_{S_R} \left(\frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v\right).$$

For the detailed proof of Lemma 4, we refer to [23]. This lemma plays a key role in obtaining the integral inequality (30) later.

To estimate the integrals of Du, Dv, D^2u, and D^2v on an annulus, we introduce the following two functional inequalities. See, e.g., [4,33,34].

Lemma 5. (Interpolation inequality on an annulus.) *For* $z \in W^{2,1}(B_{2R})$, *we have*

$$\int_{B_R \setminus B_{R/2}} |Dz| \le C\left(R \int_{B_{2R} \setminus B_{R/4}} |\Delta z| + R^{-1} \int_{B_{2R} \setminus B_{R/4}} |z|\right),$$

where $C = C(N) > 0$.

Corollary 3. *For* $z \in C^2(\mathbb{R}^N \backslash \{0\}) \cap C(\mathbb{R}^N)$, *it holds*

$$\int_{B_R \setminus B_{R/2}} |Dz| \le C\left(R \int_{B_{2R} \setminus B_{R/4}} |\Delta z| + R^{-1} \int_{B_{2R} \setminus B_{R/4}} |z|\right),$$

where $C = C(N) > 0$.

Proof. Construct the following function $\zeta_R \in C^\infty(\mathbb{R}^N)$ such that

$$\zeta_R(x) = \begin{cases} 0, & |x| < R/8, \\ \in [0,1], & R/8 \le |x| \le R/4, \\ 1, & |x| > R/4. \end{cases}$$

Replacing z with $z\zeta_R \in W^{2,1}(B_{2R})$ in Lemma 5, one can easily obtain Corollary 3. □

Lemma 6. (Elliptic L^p-estimate) *For* $1 < k < \infty$, $z \in W^{2,k}(B_{2R})$, *we have*

$$\int_{B_R \setminus B_{R/2}} |D^2 z|^k \leq C\left(\int_{B_{2R} \setminus B_{R/4}} |\Delta z|^k + R^{-2k} \int_{B_{2R} \setminus B_{R/4}} |z|^k\right),$$

where $C = C(k, N) > 0$ [14].

The above lemma follows from a standard elliptic L^p-estimate for $R = 1$ and an obvious dilation argument. See, e.g., [34].

Corollary 4. *For* $1 < k < \infty$, $z \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$, *it holds that*

$$\int_{B_R \setminus B_{R/2}} |D^2 z|^k \leq C\left(\int_{B_{2R} \setminus B_{R/4}} |\Delta z|^k + R^{-2k} \int_{B_{2R} \setminus B_{R/4}} |z|^k\right),$$

where $C = C(k, N) > 0$.

Proof. By replacing z with $z\zeta_R \in W^{2,k}(B_{2R})$ in Lemma 6, one can directly derive this result. □

We find that the integrals of u^k and v^k are difficult to estimate, and the following lemma gives clever estimates for these integrals. We refer to [16, Lemma 2.6].

Lemma 7. (L^1-regularity estimate on B_R) *Let* $N > 2$ *and* $1 \leq k < \frac{N}{N-2}$. *For* z *that is sufficiently smooth in* B_{2R}, *we have*

$$\|z\|_{L^k(B_R)} \leq C\left(R^{2+N(\frac{1}{k}-1)} \|\Delta z\|_{L^1(B_{2R})} + R^{N(\frac{1}{k}-1)} \|z\|_{L^1(B_{2R})}\right),$$

where $C = C(k, N) > 0$.

Since the integrals of the higher order derivatives of u and v are difficult to control by the integrals of their lower order derivatives, we must first estimate the integrals of the higher order derivatives Δu and Δv on $B_{R/2}$.

Lemma 8. *If* $N > 3$, $a, b > -2$, $pq > 1$, (u, v) *is a positive classical solution of system* (1), *then there exists a positive constant* $C = C\ (a, b, p, q, N, R)$ *such that*

$$\int_{B_{R/2}} |\Delta u| \leq C \quad \text{and} \quad \int_{B_{R/2}} |\Delta v| \leq C.$$

Proof. Since $N > 3$, $a, b > -2$, then

$$\int_{B_R} |\Delta u| = \int_{B_R} |x|^a v^p \leq \|v^p\|_{L^\infty(B_R)} \int_0^R \int_{\partial B(0,r)} \frac{1}{|x|^{-a}} \mathrm{d}S \mathrm{d}r < +\infty, \quad \forall R > 0.$$

Note that $u \in C^2(B(0, R) \setminus \{0\}) \cap C(B(0, R))$, one can check u has second-order weak derivatives in $B(0, R)$. Choosing $\eta \in C_0^\infty(B_R)$ with $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_{R/2}$, by the definition of weak derivatives and Lemma 3, we are led to

$$\begin{aligned}\int\limits_{B_{R/2}} |\Delta u| &\le \int\limits_{B_R} |\Delta u|\eta \\ &= \int\limits_{B_R} (-\Delta u)\eta \\ &= \int\limits_{B_R} (-\Delta \eta)u \le \|\Delta\eta\|_{L^\infty(B_R)} \int\limits_{B_R\setminus B_{R/2}} u \\ &= \|\Delta\eta\|_{L^\infty(B_R)} \int\limits_{R/2}^{R} r^{N-1} \int\limits_{S_1} u(r,\theta)\mathrm{d}S\mathrm{d}r \\ &\le \|\Delta\eta\|_{L^\infty(B_R)} C(a,b,p,q,N) \int\limits_{R/2}^{R} r^{N-1-\alpha}\mathrm{d}r \\ &\le \|\Delta\eta\|_{L^\infty(B_R)} C(a,b,p,q,R,N) \\ &\le C(a,b,p,q,R,N).\end{aligned}$$

The other estimate $\int_{B_{R/2}} |\Delta v| \le C$ can be similarly yielded. □

For $\omega \in C(\mathbb{R}^N)$, we perform a spherical coordinate transform on the variables of function $\omega(x)$ and denote the spherical average of $\omega(x)$ by

$$\overline{\omega}(r) = \frac{1}{|S_r|}\int\limits_{S_r} \omega(r,\theta)\mathrm{d}S, \quad r > 0.$$

One can define $\overline{\omega}(0) = \omega(0)$, then $\overline{\omega}(r)$ is continuous in $[0,+\infty)$. Moreover, we have $\overline{\omega}(r) \in C^2((0,+\infty))$ if $\omega(x) \in C^2(\mathbb{R}^N\setminus\{0\})$.

Lemma 9. *If $\omega \in C^2(\mathbb{R}^N\setminus\{0\}) \cap C(\mathbb{R}^N)$ satisfies $-\Delta\omega \ge 0$ and $\omega \ge 0$ in $\mathbb{R}^N\setminus\{0\}$, then it holds either $\omega(x) = 0$ or $\omega(x) > 0$ in $\mathbb{R}^N$.*

Proof. If $\omega(x) \equiv 0$ in $\mathbb{R}^N\setminus\{0\}$, then the continuity of ω gives $\omega(x) \equiv 0$ in $\mathbb{R}^N$ directly.

If $\omega \not\equiv 0$, then we consider the following two cases:

Case 1. $\omega(x) > 0$ in $\mathbb{R}^N\setminus\{0\}$, it suffices to show $\omega(0) \ne 0$. By contradiction, we suppose $\omega(0) = 0$ and note that $-\Delta\omega \ge 0$ in $\mathbb{R}^N\setminus\{0\}$. By a direct calculation, we can deduce that

$$(\overline{\omega})_{rr} + \frac{N-1}{r}(\overline{\omega})_r \le 0, \quad r > 0.$$

This implies that $(r^{N-1}(\overline{\omega})_r)_r \le 0$; thus, $r^{N-1}(\overline{\omega})_r$ is nonincreasing. However, one can easily check that $\lim_{r\to 0^+} r^{N-1}(\overline{\omega})_r = 0$, which guarantees $(\overline{\omega})_r \le 0$ for all $r > 0$. Suppose for contradiction that there exists a $r_1 > 0$ such that $(\overline{\omega}(r_1))_r > 0$. By $\overline{\omega}(0) = 0$ and the continuity of $\overline{\omega}(r)$ at $r = 0$, we choose $r_2 < r_1$ such that $\overline{\omega}(r_2) < \overline{\omega}(r_1)$. Lagrange mean value theorem yields that there exists $r_3 \in (r_2, r_1)$ such that $(\overline{\omega}(r_3))_r > 0$, which yields a contradiction. Thus, $\overline{\omega}(r) \le 0$ in $(0,+\infty)$. However, $\overline{\omega}(r) \ge 0$ in $(0,+\infty)$, then $\overline{\omega}(r) = 0$ in $(0,+\infty)$. Similarly, the continuity of $\overline{\omega}(r)$ gives $\overline{\omega}(0) = \omega(0) = 0$, a contradiction.

Case 2. There exist $x_1, x_2 \in \mathbb{R}^N\setminus\{0\}$ such that $\omega(x_1) = 0$, $\omega(x_2) \ne 0$. Choosing an open set with a smooth boundary such that $x_1 \in U$, $x_2 \in \partial U$, then u attains its minimum value at the internal point x_1. Hence, by strong maximum principle, we deduce a contradiction.

Therefore, $\omega(x) > 0$ in $\mathbb{R}^N$ if $\omega(x)$ is nontrivial in $\mathbb{R}^N$. □

Corollary 5. *If $(u,v) \in (C^2(\mathbb{R}^N\setminus\{0\}) \cap C(\mathbb{R}^N))^2$ is a nonnegative classical solution of system* (1), *then $u, v > 0$ in $\mathbb{R}^N$ or $u, v \equiv 0$ in $\mathbb{R}^N$.*

Proof. Since $(u, v) \in (C^2(\mathbb{R}^N\backslash\{0\}) \cap C(\mathbb{R}^N))^2$ is a nonnegative classical solution of system (1), then we have $u, v \geq 0$ and $\Delta u, \Delta v \leq 0$ in $\mathbb{R}^N\backslash\{0\}$. Clearly, one can deduce that if $u \not\equiv 0$ and $v \not\equiv 0$ in $\mathbb{R}^N$, then $u, v > 0$ in $\mathbb{R}^N$ by Lemma 9. Therefore, it holds either $u, v > 0$ or $u, v \equiv 0$ in $\mathbb{R}^N$. □

4 Proof of Theorem 1 and Corollary 1

4.1 Proof of Theorem 1

Proof. Suppose that (u, v) is a nonnegative nontrivial classical solution of system (1), by Corollary 5, we have $u, v > 0$ in $\mathbb{R}^N$. For $\gamma \in (0, 1)$, $R > 1$, we construct a function $\eta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B_R\backslash B_{R/2}$. Lemmas 1 and 2 guarantee

$$\int_{S_1} u^\gamma(R, \theta)\mathrm{d}S \leq C_2 r^{-\alpha} + |S_1|, \quad \int_{S_1} v^\gamma(R, \theta)\mathrm{d}S \leq C_2 r^{-\beta} + |S_1|, \tag{7}$$

$$\int_{B_R\backslash B_{R/2}} u^{\gamma-2}|Du|^2\mathrm{d}x \leq \int_{\mathbb{R}^N} \eta^2 u^{\gamma-2}|Du|^2\mathrm{d}x \leq C\int_{\mathbb{R}^N} u^\gamma|D\eta|^2\mathrm{d}x, \tag{8}$$

$$\int_{B_R\backslash B_{R/2}} v^{\gamma-2}|Dv|^2\mathrm{d}x \leq \int_{\mathbb{R}^N} \eta^2 v^{\gamma-2}|Dv|^2\mathrm{d}x \leq C\int_{\mathbb{R}^N} v^\gamma|D\eta|^2\,\mathrm{d}x, \tag{9}$$

where $\alpha = \frac{2(p+1)+a+pb}{pq-1}$, $\beta = \frac{2(q+1)+b+qa}{pq-1}$. This implies

$$\int_{B_R\backslash B_{R/2}} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}x \leq C\int_{\mathbb{R}^N} u^\gamma\,|D\eta|^2\mathrm{d}x \leq C, \tag{10}$$

$$\int_{B_R\backslash B_{R/2}} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}x \leq C\int_{\mathbb{R}^N} u^\gamma|D\eta|^2\mathrm{d}x \leq C, \tag{11}$$

where $C = C(a, b, p, q, N, R, \gamma)$.

We estimate the integrals of $|Du|$ and $|Dv|$ in the annulus $B_R\backslash B_{R/2}$ by using Corollaries 3 and 4:

$$\begin{aligned}\int_{B_R\backslash B_{R/2}} |Du|\mathrm{d}x &\leq C\left(R\int_{B_{2R}\backslash B_{R/4}} |\Delta u|\mathrm{d}x + R^{-1}\int_{B_{2R}\backslash B_{R/4}} |u|\mathrm{d}x\right)\\ &\leq C\left(R\int_{B_{2R}\backslash B_{R/4}} |x|^a v^p\mathrm{d}x + R^{-1}\int_{B_{2R}\backslash B_{R/4}} |u|\mathrm{d}x\right)\\ &\leq C,\end{aligned} \tag{12}$$

$$\begin{aligned}\int_{B_R\backslash B_{R/2}} |Dv|\mathrm{d}x &\leq C\left(R\int_{B_{2R}\backslash B_{R/4}} |\Delta u|\mathrm{d}x + R^{-1}\int_{B_{2R}\backslash B_{R/4}} |u|\mathrm{d}x\right)\\ &\leq C\left(R\int_{B_{2R}\backslash B_{R/4}} |x|^a v^p\mathrm{d}x + R^{-1}\int_{B_{2R}\backslash B_{R/4}} |u|\mathrm{d}x\right)\\ &\leq C,\end{aligned} \tag{13}$$

where $C = C(a, b, p, q, N, R)$. We fix a function $\xi_R \in C^\infty(\mathbb{R}^N)$ that satisfies $0 \leq \xi_R \leq 1$ and

$$\xi_R(x) = \begin{cases} 0, & |x| < R/4, \\ \in [0,1], & R/4 \le |x| \le R/2, \\ 1, & |x| > R/2. \end{cases}$$

For any $k \in \left[1, \frac{N}{N-2}\right)$, by (12), (13), Lemmas 7, 8, 3, the integral of $|u|^k$ can be estimated as follows:

$$\begin{aligned}
\left(\int_{B_R \setminus B_{R/2}} |u|^k \mathrm{d}x \right)^{\frac{1}{k}} &\le \left(\int_{B_R} |u\xi_R|^k \mathrm{d}x \right)^{\frac{1}{k}} \\
&\le C(N,k)\left(R^{2+N\left(\frac{1}{k}-1\right)} \int_{B_{2R}} |\Delta(u\xi_R)| \mathrm{d}x + R^{N\left(\frac{1}{k}-1\right)} \int_{B_{2R}} |u\xi_R| \mathrm{d}x \right) \\
&\le C(N,k)\left(R^{2+N\left(\frac{1}{k}-1\right)} \int_{B_{2R}} |2D\xi_R \cdot Du + \Delta u \cdot \xi_R + \Delta\xi_R u| \mathrm{d}x + R^{N\left(\frac{1}{k}-1\right)} \int_{2R} |u\xi_R| \mathrm{d}x \right) \\
&\le C(N,k)\left(R^{2+N\left(\frac{1}{k}-1\right)} \left(\int_{B_{2R}} |2D\xi_R| \cdot |Du| \mathrm{d}x + \int_{B_{2R}} |\Delta u| \cdot |\xi_R| \mathrm{d}x + \int_{B_{2R}} |\Delta \xi_R| \cdot |u| \mathrm{d}x \right) \right. \\
&\quad \left. + R^{N\left(\frac{1}{k}-1\right)} \int_{2R} |u\xi_R| \mathrm{d}x \right) \\
&\le C(N,k)\left(R^{2+N\left(\frac{1}{k}-1\right)} \left(C \int_{B_{R/2}\setminus B_{R/4}} |Du| \mathrm{d}x + C \int_{B_{2R}\setminus B_{R/4}} |\Delta u| \mathrm{d}x + C \int_{B_{R/2}\setminus B_{R/4}} |u| \mathrm{d}x \right) \right. \\
&\quad \left. + R^{N\left(\frac{1}{k}-1\right)} \left(\int_{B_{2R}\setminus B_R} |u| \mathrm{d}x + \int_{B_R\setminus B_{R/2}} |u| \mathrm{d}x + \int_{B_{R/2}\setminus B_{R/4}} |u| \mathrm{d}x \right) \right) \\
&\le C(N,k)\left(R^{2+N\left(\frac{1}{k}-1\right)} \left(C(a,b,p,q,N,R) + C(R) \int_{R/4}^{R/2} \int_{S_r} u(r,\theta) \mathrm{d}S\mathrm{d}r \right) \right. \\
&\quad \left. + 3R^{N\left(\frac{1}{k}-1\right)} \int_{R/4}^{R/2} \int_{S_r} u(r,\theta) \mathrm{d}S\mathrm{d}r \right) \\
&\le C(N,k)\left(C(a,b,p,q,N,R) R^{2+N\left(\frac{1}{k}-1\right)} + 3R^{N\left(\frac{1}{k}-1\right)} C(\alpha,N,R) \right) \\
&\le C.
\end{aligned} \tag{14}$$

Similarly, one obtains

$$\left(\int_{B_R \setminus B_{R/2}} |v|^k \mathrm{d}x \right)^{\frac{1}{k}} \le C, \tag{15}$$

where $C = C(a,b,p,q,N,R,k)$.

Now, we establish the results for $N = 4$ and $N = 5$ separately.

Case 1: $N = 4$. For $p, q < \frac{4}{3}$, set $k = \frac{3}{2}, \frac{3}{2}p$, and $\frac{3}{2}q$ in (14) and (15), respectively, we derive

$$\left(\int_{B_R\setminus B_{R/2}} |u|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}} \le C, \quad \left(\int_{B_R\setminus B_{R/2}} |v|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}} \le C, \tag{16}$$

$$\left(\int_{B_R\setminus B_{R/2}} |u|^{\frac{3}{2}p}\mathrm{d}x\right)^{\frac{2}{3p}} \le C, \quad \left(\int_{B_R\setminus B_{R/2}} |v|^{\frac{3}{2}p}\mathrm{d}x\right)^{\frac{2}{3p}} \le C, \tag{17}$$

$$\left(\int_{B_R\setminus B_{R/2}} |u|^{\frac{3}{2}q}\mathrm{d}x\right)^{\frac{2}{3q}} \le C, \quad \left(\int_{B_R\setminus B_{R/2}} |v|^{\frac{3}{2}q}\mathrm{d}x\right)^{\frac{2}{3q}} \le C, \tag{18}$$

where $C = C(a, b, p, q, R)$.

Then by (16), (17), (18), and Corollary 4, we estimate the integral of $|D^2u|^{\frac{3}{2}}$ in the domain $B_R\setminus B_{R/2}$,

$$\begin{aligned}
\int_{B_R\setminus B_{R/2}} |D^2u|^{\frac{3}{2}}\mathrm{d}x &\le C\left(\int_{B_{2R}\setminus B_{R/4}} |\Delta u|^{\frac{3}{2}}\mathrm{d}x + R^{-3}\int_{B_{2R}\setminus B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x\right)\\
&= C\left(\int_{B_{2R}\setminus B_{R/4}} ||x|^a v^p|^{\frac{3}{2}}\mathrm{d}x + R^{-3}\int_{B_{2R}\setminus B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x\right)\\
&\le C\left((2R)^{\frac{3}{2}a}\int_{B_{2R}\setminus B_{R/4}} v^{\frac{3}{2}p}\mathrm{d}x + R^{-3}\int_{B_{2R}\setminus B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x\right)\\
&\le C\left((2R)^{\frac{3}{2}a}\left(\int_{B_{2R}\setminus B_R} v^{\frac{3}{2}p}\mathrm{d}x + \int_{B_R\setminus B_{R/2}} v^{\frac{3}{2}p}\mathrm{d}x + \int_{B_{R/2}\setminus B_{R/4}} v^{\frac{3}{2}p}\mathrm{d}x\right) + R^{-3}\int_{B_{R/2}\setminus B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x\right)\\
&\le C.
\end{aligned} \tag{19}$$

Similarly, one obtains

$$\int_{B_R\setminus B_{R/2}} |D^2v|^{\frac{3}{2}}\mathrm{d}x \le C. \tag{20}$$

Now, we claim that

$$\left| E_1 := \left\{ r\in[R/2,R] : \int_{S_r} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S > \frac{64}{R}C \right\} \right| < \frac{R}{64}, \tag{21}$$

$$\left| E_2 := \left\{ r\in[R/2,R] : \int_{S_r} \left|Dv^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S > \frac{64}{R}C \right\} \right| < \frac{R}{64}. \tag{22}$$

Otherwise, one can derive

$$\int_{S_r} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S > \frac{R}{64}\cdot\frac{64}{R}C = C, \quad \int_{S_r} \left|Dv^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S > \frac{R}{64}\cdot\frac{64}{R}C = C, \tag{23}$$

which contradicts with (10) and (11). By (19) and (20), a similar discussion yields

$$\left|E_3 := \left\{r \in [R/2, R] : \int_{S_r} |D^2u|^{\frac{3}{2}} \mathrm{d}S > \frac{64}{R}C\right\}\right| < \frac{R}{64},$$

$$\left|E_4 := \left\{r \in [R/2, R] : \int_{S_r} |D^2v|^{\frac{3}{2}} \mathrm{d}S > \frac{64}{R}C\right\}\right| < \frac{R}{64},$$

where $C = C(a, b, p, q, R, \gamma)$, $|\cdot|$ is the Lebesgue measure. Therefore, there exists a measurable set $E_0 := [R/2, R]\backslash\bigcup_{i=1}^{4}E_i$ satisfying $\frac{7}{16}R < |E_0| \le R/2$ such that for all $r \in E_0$

$$\int_{S_r} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S \le \frac{64}{R}C, \quad \int_{S_r} \left|Dv^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S \le \frac{64}{R}C,$$

$$\int_{S_r} |D^2u|^{\frac{3}{2}} \mathrm{d}S \le \frac{64}{R}C, \quad \int_{S_r} |D^2v|^{\frac{3}{2}} \mathrm{d}S \le \frac{64}{R}C.$$

By fixing a number $r_0(R) \in E_0$, we obtain

$$\int_{S_{r_0(R)}} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S \le \frac{64}{R}C, \quad \int_{S_{r_0(R)}} \left|Dv^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S \le \frac{64}{R}C, \tag{24}$$

$$\int_{S_{r_0(R)}} |D^2u|^{\frac{3}{2}} \mathrm{d}S \le \frac{64}{R}C, \quad \int_{S_{r_0(R)}} |D^2v|^{\frac{3}{2}} \mathrm{d}S \le \frac{64}{R}C. \tag{25}$$

We set $k = \frac{3}{2}$, $j = 2$. For any $\lambda \in [1, \infty)$, by (ii) of Lemmas 1, 3, and (25), it holds

$$\begin{aligned}
\|u(r_0(R), \theta)\|_\lambda &\le C\Big(\|D_\theta^2 u(r_0(R), \theta)\|_{\frac{3}{2}} + \|u(r_0(R), \theta)\|_1\Big)\\
&\le C\Big(r_0^2(R)\|D^2u(r_0(R), \theta)\|_{\frac{3}{2}} + \|u(r_0(R), \theta)\|_1\Big)\\
&\le C\Big(r_0^2(R)\|D^2u(r_0(R), \theta)\|_{\frac{3}{2}} + Cr_0^{-\alpha}(R)\Big)\\
&\le C\left(r_0^2(R)\left(\frac{64}{R}C\right)^{\frac{2}{3}} + Cr_0^{-\alpha}(R)\right) \le C.
\end{aligned} \tag{26}$$

Similarly, one obtains

$$\|v(r_0(R), \theta)\|_\lambda \le C, \tag{27}$$

where $C = C(a, b, p, q, R, \lambda)$.

Fix a $\gamma \in (0, 1)$, utilizing the Hölder inequality, (24) and (25), we deduce

$$\begin{aligned}
\int_{S_{r_0(R)}} |Du|v\mathrm{d}S &= \int_{S_{r_0(R)}} (|Du|^2u^{\gamma-2}u^{2-\gamma}v^2)^{\frac{1}{2}}\mathrm{d}S\\
&\le \left(\int_{S_{r_0(R)}} |Du|^2u^{\gamma-2}\mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{S_{r_0(R)}} u^{2-\gamma}v^2\mathrm{d}S\right)^{\frac{1}{2}}\\
&\le \left(\int_{S_{r_0(R)}} |Du|^2u^{\gamma-2}\mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{S_{r_0(R)}} u^{2(2-\gamma)}\mathrm{d}S\right)^{\frac{1}{4}}\left(\int_{S_{r_0(R)}} v^4\mathrm{d}S\right)^{\frac{1}{4}}\\
&= \left(\int_{S_{r_0(R)}} \left|Du^{\frac{\gamma}{2}}\right|^2 \mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{S_{r_0(R)}} u^{2(2-\gamma)}\mathrm{d}S\right)^{\frac{1}{4}}\left(\int_{S_{r_0(R)}} v^4\mathrm{d}S\right)^{\frac{1}{4}}\\
&\le C.
\end{aligned} \tag{28}$$

Similarly, one obtains

$$\int_{S_{r_0(R)}} \left|\frac{\partial v}{\partial \nu}\right| u \mathrm{d}S \le C, \tag{29}$$

where $C = C(a, b, p, q, R)$. Noting (p, q) is subcritical, which guarantees $c_1 + c_2 > 0$ in Lemma 4. By choosing $c = \frac{c_1+c_2}{2}$, $R = 4 > 2$, we are led to

$$\begin{aligned}
\int_{B_1} |x|^b u^{q+1}\mathrm{d}x + \int_{B_1} |x|^a v^{p+1}\mathrm{d}x &\le \int_{B_{r_0}} |x|^b u^{q+1}\mathrm{d}x + \int_{B_{r_0}} |x|^a v^{p+1}\mathrm{d}x \\
&\le C \int_{S_{r_0}} \left(r_0^{1+b} u^{q+1} + r_0^{1+a} v^{p+1} + \left|\frac{\partial u}{\partial \nu}\right| v + \left|\frac{\partial v}{\partial \nu}\right| u \right)\mathrm{d}S \\
&\le C \int_{S_{r_0}} (r_0^{1+b} u^{q+1} + r_0^{1+a} v^{p+1} + |Du|v + |Dv|u)\mathrm{d}S \\
&\le C\left(C + \int_{S_{r_0}} (|Du|v + |Dv|u)\mathrm{d}S \right) \\
&\le C,
\end{aligned} \tag{30}$$

where $r_0 \in [2, 4]$ denotes $r_0(R = 4)$ and $C = C(a, b, p, q)$ is a positive constant independent of the solutions u and v. It is easy to check that $(2^{l\alpha}u(2^l x), 2^{l\beta}v(2^l x))$ is also a positive solution of system (1) for each l. Replacing (u, v) by $(2^{l\alpha}u(2^l x), 2^{l\beta}v(2^l x))$ in (30), we arrive at

$$\int_{B_{2^l}} |x|^b u^{q+1}\mathrm{d}x + \int_{B_{2^l}} |x|^a v^{p+1}\mathrm{d}x \le C 2^{l(N-2-\alpha-\beta)}. \tag{31}$$

Recalling (p, q) is subcritical, which implies $\alpha + \beta > N - 2$, let $l \to \infty$ in (31), we obtain

$$\int_{\mathbb{R}^N} |x|^b u^{q+1}\mathrm{d}x + \int_{\mathbb{R}^N} |x|^a v^{p+1}\mathrm{d}x = 0,$$

which directly gives $u = 0$ and $v = 0$ a.e. $x \in \mathbb{R}^N$. Recalling that both u and v are continuous functions, hence we obtain

$$u \equiv 0 \quad \text{and} \quad v \equiv 0 \quad x \in \mathbb{R}^N,$$

which yields a contradiction. Therefore, the case of $N = 4$ is established.

Case 2: $N = 5$. For $p, q < \frac{10}{9}$, set $k = \frac{5}{3}, \frac{3}{2}p$, and $\frac{3}{2}q$ in (14) and (15) separately, we obtain similar estimates as above. Applying the estimates, Corollary 4, we bound the integrals of $|D^2u|^{\frac{3}{2}}$ in the domain $B_R \backslash B_{R/2}$ as follows:

$$\begin{aligned}
\int_{B_R \backslash B_{R/2}} |D^2u|^{\frac{3}{2}}\mathrm{d}x &\le C\left(\int_{B_{2R}\backslash B_{R/4}} |\Delta u|^{\frac{3}{2}}\mathrm{d}x + R^{-3} \int_{B_{2R}\backslash B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x \right) \\
&= C\left(\int_{B_{2R}\backslash B_{R/4}} ||x|^a v^p|^{\frac{3}{2}}\mathrm{d}x + R^{-3} \int_{B_{2R}\backslash B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x \right) \\
&\le C\left((2R)^{\frac{3}{2}a} \int_{B_{2R}\backslash B_{R/4}} v^{\frac{3}{2}p}\mathrm{d}x + R^{-3} \int_{B_{2R}\backslash B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x \right) \\
&\le C\left((2R)^{\frac{3}{2}a}\left(\int_{B_{2R}\backslash B_R} v^{\frac{3}{2}p}\mathrm{d}x + \int_{B_R\backslash B_{R/2}} v^{\frac{3}{2}p}\mathrm{d}x + \int_{B_{R/2}\backslash B_{R/4}} v^{\frac{3}{2}p}\mathrm{d}x \right) + R^{-3} \int_{B_{R/2}\backslash B_{R/4}} |u|^{\frac{3}{2}}\mathrm{d}x \right) \\
&\le C.
\end{aligned} \tag{32}$$

Similarly, one obtains

$$\int\limits_{B_R \setminus B_{R/2}} |D^2 v|^{\frac{3}{2}} \mathrm{d}x \leq C. \tag{33}$$

Therefore, by (10), (11), (32), (33), an argument similar to (21) and (23) guarantees

$$\left| F_1 := \left\{ r \in [R/2, R] : \int\limits_{S_r} \left| Du^{\frac{\gamma}{2}} \right|^2 \mathrm{d}S > \frac{64}{R} C \right\} \right| < \frac{R}{64},$$
$$\left| F_2 := \left\{ r \in [R/2, R] : \int\limits_{S_r} \left| Dv^{\frac{\gamma}{2}} \right|^2 \mathrm{d}S > \frac{64}{R} C \right\} \right| < \frac{R}{64},$$

and

$$\left| F_3 := \left\{ r \in [R/2, R] : \int\limits_{S_r} |D^2 u|^{\frac{3}{2}} \mathrm{d}S > \frac{64}{R} C \right\} \right| < \frac{R}{64},$$
$$\left| F_4 := \left\{ r \in [R/2, R] : \int\limits_{S_r} |D^2 v|^{\frac{3}{2}} \mathrm{d}S > \frac{64}{R} C \right\} \right| < \frac{R}{64},$$

where $C = C(a, b, p, q, R, \gamma)$. Then, there exists a measurable set $F_0 := [R/2, R] \setminus \bigcup_{i=1}^4 F_i$ satisfying $\frac{7}{16} R < |F_0| \leq R/2$ such that for all $r \in F_0$

$$\int\limits_{S_r} \left| Du^{\frac{\gamma}{2}} \right|^2 \mathrm{d}S \leq \frac{64}{R} C, \quad \int\limits_{S_r} \left| Dv^{\frac{\gamma}{2}} \right|^2 \mathrm{d}S \leq \frac{64}{R} C, \tag{34}$$

$$\int\limits_{S_r} |D^2 u|^{\frac{3}{2}} \mathrm{d}S \leq \frac{64}{R} C, \quad \int\limits_{S_r} |D^2 v|^{\frac{3}{2}} \mathrm{d}S \leq \frac{64}{R} C. \tag{35}$$

Now fix $r_0(R) \in F_0$ such that (34) and (35) hold true. We set $k = \frac{3}{2}$, $j = 2$, then for $\lambda = 6$, applying (i) of Lemmas 1, 3, and (35), we derive

$$\begin{aligned}
\|u(r_0(R), \theta)\|_6 &\leq C\Big(\|D_\theta^2 u(r_0(R), \theta)\|_{\frac{3}{2}} + \|u(r_0(R), \theta)\|_1 \Big) \\
&\leq C\Big(r_0^2(R) \|D^2 u(r_0(R), \theta)\|_{\frac{3}{2}} + \|u(r_0(R), \theta)\|_1 \Big) \\
&\leq C\Big(r_0^2(R) \|D^2 u(r_0(R), \theta)\|_{\frac{3}{2}} + C r_0^{-\alpha}(R) \Big) \\
&\leq C\left(r_0^2(R) \left(\frac{64}{R} C \right)^{\frac{2}{3}} + C r_0^{-\alpha}(R) \right) \\
&\leq C.
\end{aligned} \tag{36}$$

Similarly, one obtains

$$\|v(r_0(R), \theta)\|_6 \leq C, \tag{37}$$

where $C = C(a, b, p, q, R)$.

Fix $\gamma \in (0, 1)$, then $2(2 - \gamma) \in (2, 4)$. Set $S_1^{1, r_0(R)} := \left\{ \frac{x}{r_0(R)} \middle| u(x) \geq 1, x \in S_{r_0(R)} \right\}$ and $S_1^{2, r_0(R)} := \left\{ \frac{x}{r_0(R)} \middle| u(x) \leq 1, x \in S_{r_0(R)} \right\}$. We first estimate

$$\begin{aligned}\int\limits_{S_1} u^{2(2-\gamma)}(r_0(R),\theta)\mathrm{d}S &= \int\limits_{S_1^{1,r_0(R)}} u^{2(2-\gamma)}(r_0(R),\theta)\mathrm{d}S + \int\limits_{S_1^{2,r_0(R)}} u^{2(2-\gamma)}(r_0(R),\theta)\mathrm{d}S\\ &\le \int\limits_{S_1^{1,r_0(R)}} u^{6}(r_0(R),\theta)\mathrm{d}S + |S_1|\\ &\le \int\limits_{S_1} u^{6}(r_0(R),\theta)\mathrm{d}S + |S_1|\\ &\le C + |S_1| \le C.\end{aligned}$$

Similarly, one obtains $\int_{S_1} v^{2(2-\gamma)}(r_0(R),\theta)\mathrm{d}S \le C$. Then we are led to

$$\int\limits_{S_{r_0(R)}} u^{2(2-\gamma)}\mathrm{d}S \le C, \qquad \int\limits_{S_{r_0(R)}} v^{2(2-\gamma)}\mathrm{d}S \le C,$$

where $C = C(a, b, p, q, R)$.

Next, we estimate

$$\begin{aligned}\int\limits_{S_1} u^{4}(r_0(R),\theta)\mathrm{d}S &= \int\limits_{S_1^{1,r_0(R)}} u^{4}(r_0(R),\theta)\mathrm{d}S + \int\limits_{S_1^{2,r_0(R)}} u^{4}(r_0(R),\theta)\mathrm{d}S\\ &\le \int\limits_{S_1^{1,r_0(R)}} u^{6}(r_0(R),\theta)\mathrm{d}S + |S_1|\\ &\le \int\limits_{S_1} u^{6}(r_0(R),\theta)\mathrm{d}S + |S_1|\\ &\le C + |S_1| \le C.\end{aligned}$$

Similarly, one obtains $\int_{S_1} v^{4}(r_0(R),\theta)\mathrm{d}S \le C$. Thus, it holds

$$\int\limits_{S_{r_0(R)}} u^{4}\mathrm{d}S \le C, \qquad \int\limits_{S_{r_0(R)}} v^{4}\mathrm{d}S \le C,$$

where $C = C(a, b, p, q, R)$.

Applying the Hölder's inequality together with (34) and (35), we derive

$$\begin{aligned}\int\limits_{S_{r_0(R)}} |Du|v\mathrm{d}S &= \int\limits_{S_{r_0(R)}} (|Du|^2u^{\gamma-2}u^{2-\gamma}v^2)^{\frac{1}{2}}\mathrm{d}S\\ &\le \Bigg(\int\limits_{S_{r_0(R)}} |Du|^2u^{\gamma-2}\mathrm{d}S\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{S_{r_0(R)}} u^{2-\gamma}v^2\mathrm{d}S\Bigg)^{\frac{1}{2}}\\ &\le \Bigg(\int\limits_{S_{r_0(R)}} \left|Du^{\frac{\gamma}{2}}\right|^2\mathrm{d}S\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{S_{r_0(R)}} u^{2(2-\gamma)}\mathrm{d}S\Bigg)^{\frac{1}{4}}\Bigg(\int\limits_{S_{r_0(R)}} v^{4}\mathrm{d}S\Bigg)^{\frac{1}{4}}\\ &\le C.\end{aligned} \tag{38}$$

Similarly, one obtains

$$\int\limits_{S_{r_0(R)}} \left|\frac{\partial v}{\partial \nu}\right| u\mathrm{d}S \le C, \tag{39}$$

where $C = C(a, b, p, q, R)$. Note that (p, q) is subcritical, then we have $c_1 + c_2 > N - 2$ in Lemma 4. Using the same argument in 30, let $R = 4 > 2$ and $r_0 = r_0(R = 4) \in [2, 4]$, one derives

$$\int_{B_1} |x|^b u^{q+1} \mathrm{d}x + \int_{B_1} |x|^a v^{p+1} \mathrm{d}x \le \int_{B_{r_0}} |x|^b u^{q+1} \mathrm{d}x + \int_{B_{r_0}} |x|^a v^{p+1} \mathrm{d}x \le C, \tag{40}$$

where $C = C(a, b, p, q)$ is a positive constant independent of the solution (u, v). Replacing (u, v) with $(2^{l\alpha}u(2^l x), 2^{l\beta}v(2^l x))$ in (40), we obtain

$$\int_{B_{2^l}} |x|^b u^{q+1} \mathrm{d}x + \int_{B_{2^l}} |x|^a v^{p+1} \mathrm{d}x \le C 2^{l(N-2-\alpha-\beta)}. \tag{41}$$

Let $l \to \infty$ in (41), we derive

$$\int_{\mathbb{R}^N} |x|^b u^{q+1} \mathrm{d}x + \int_{\mathbb{R}^N} |x|^a v^{p+1} \mathrm{d}x = 0,$$

hence

$$u \equiv 0 \quad \text{and} \quad v \equiv 0, \quad x \in \mathbb{R}^N,$$

a contradiction. Therefore, the case of $N = 5$ is established. □

4.2 Proof of Corollary 1

Proof. Case 1: $N = 4$. By Theorem B, for $p, q > 0$, $a, b \in \mathbb{R}$, if $\max\{a, b\} \le -2$ or $pq \le 1$ or $\max\{\alpha, \beta\} \ge N - 2$, then system (1) admits no positive upper solutions, where α and β have been defined in Section 1. From the definition of upper solution and Corollary 5, it follows that if $\max\{a, b\} \le -2$ or $pq \le 1$ or $\max\{\alpha, \beta\} \ge N - 2$, then system (1) has no nonnegative nontrivial solutions.

By [23, Theorem 3.5], if $a \le -2$ or $b \le -2$, $p, q > 0$, then system (1) admits no positive upper solutions. Corollary 5 yields that if $a \le -2$ or $b \le -2$, then system (1) has no nonnegative nontrivial solutions. This combined with Theorem 1 concludes the result in Corollary 1 for $N = 4$.

Case 2: $N = 5$. Similar to the case of $N = 4$. □

5 Conclusion

In this article, the Liouville-type theorem of the Hénon-Lane-Emden system is researched. We use the Rellich-Pohozaev inequality and some functional inequalities to partially confirm the Hénon-Lane-Emden conjecture through some analytical techniques. The parameter range of Liouville theorem for the Hénon-Lane-Emden system is extended in $N = 4$ and $N = 5$, which is significant for studying the structure of stars in hydrostatic equilibrium in astrophysics. Moreover, this work provides a new idea for the Liouville-type theorem of other related systems, such as polyharmonic Hénon-Lane-Emden system.

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