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Research Article

Liping Wang and Chunyi Zhao*

Concentrations for nonlinear Schrödinger equations with magnetic potentials and constant electric potentials

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Abstract: This article studies point concentration phenomena of nonlinear Schrödinger equations with magnetic potentials and constant electric potentials. The existing results show that a common magnetic field has no effect on the locations of point concentrations, as long as the electric potential is not a constant. This article finds out the role of the magnetic fields in the locations of point concentrations when the electric potential is a constant.

Keywords: magnetic Schrödinger equations, concentrations, constant electric potentials

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1 Introduction

The magnetic Schrödinger equation in $\mathbb{R}^N$ is given by

$$i\varepsilon\frac{\partial\psi}{\partial t} = (i\varepsilon\nabla + \boldsymbol{A}(x))^2\psi + Q(x)\psi - |\psi|^{p-1}\psi, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$ and i is the imaginary unit. Here the function ψ is complex-valued. The vector $\boldsymbol{A} = (A_1, A_2, \ldots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ denotes the magnetic potential and $Q : \mathbb{R}^N \to \mathbb{R}$ represents the electric potential. The magnetic Laplacian $(i\varepsilon\nabla + \boldsymbol{A})^2$ is defined by

$$(i\varepsilon\nabla + \boldsymbol{A})^2\psi := -\varepsilon^2\Delta\psi + 2i\varepsilon\boldsymbol{A}\cdot\nabla\psi + |\boldsymbol{A}|^2\psi + i\varepsilon\psi\nabla\cdot\boldsymbol{A}.$$

The vector $\boldsymbol{A}$ models the presence in some quantum model of the magnetic field $\boldsymbol{B}$, which is given by

$$\boldsymbol{B} = \begin{cases} \partial_1 A_2 - \partial_2 A_1, & \text{for } N = 2, \\ \operatorname{curl}\boldsymbol{A}, & \text{for } N = 3, \\ (\partial_j A_k - \partial_k A_j)_{N\times N}, & \text{for } N > 3. \end{cases}$$

For the discussion of this operator, one may refer to [16].

Equation (1) arises in various physical contexts such as Bose-Einstein condensates and nonlinear optics [17]; or plasma physics where one can simulate the interaction effect among many particles by introducing some nonlinear terms, see [19].

*** Corresponding author: Chunyi Zhao,** School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, 200241, Shanghai, China, e-mail: cyzhao@math.ecnu.edu.cn
Liping Wang: School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, 200241, Shanghai, China

From now on we consider standing wave solutions to problem (1), namely $\psi(t,x) = e^{i\lambda\varepsilon^{-1}t}u(x)$ for some complex-valued function $u(x)$. Substituting this ansatz into problem (1), $u(x)$ should satisfy the following nonlinear magnetic Schrödinger equation:

$$(i\varepsilon\nabla + \boldsymbol{A})^2 u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $V(x) = Q(x) + \lambda$. The potential $V(x)$ is then usually assumed to be smooth and satisfy

$$\inf_{\mathbb{R}^N} V(x) > 0.$$

Concerning nonlinear Schrödinger equations with the magnetic field, the pioneer work is by Esteban-Lions [11], in which they prove the existence of standing waves to (1) by a constrained minimization approach, in the case $V(x) = 1$ and for special classes of magnetic fields. For more results, one can refer to [1,13,14] and references therein. On the other hand, problem (2) seems to be a very interesting problem since the correspondence's principle establishes that classical mechanics is, roughly speaking, contained in quantum mechanics. The mathematical transition from quantum mechanics to classical mechanics can be formally described by letting the Planck constant $\varepsilon \to 0$, and thus the existence of solutions for ε small has physical interest. Standing waves for ε small are usually referred to as semi-classical bound states, see e.g., [13]. In the linear case, Helffer et al. [14] studied the asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit.

Concentrations play an important role in the study of problem (2). When $\boldsymbol{A} \equiv 0$, the pioneer work of concentrations to problem (2) in one dimension was carried out by Floer and Weistein [12]. They proved that if the electric potential $V(x)$ has a non-degenerate critical point, then $u(x)$ concentrates near this critical point as $\varepsilon \to 0$. This kind of solution with point concentrations is sometimes called peak solutions. After that, there are too many peak solution results for nonlinear Schrödinger problems to give a full reference list here.

The first concentration result for nonlinear magnetic Schrödinger equation (2) is given by Kurata [15]. Under some assumptions linking the magnetic field and electric potential, he showed that least energy solutions exist for every small ε and they exhibit peak concentrations at a global minimum point of V. Moreover, Kurata [15] mentioned that the magnetic potential $\boldsymbol{A}$ only contributes the phase factor of least energy solutions. Next, previous studies [4,7] proved that peak concentrations may also occur at any non-degenerate critical point, not necessarily a minimum of V, as $\varepsilon \to 0$ by a reduction method. Cingolani and Secchi [8] again show that there exist peak solutions concentrating at topologically nontrivial critical points of V by a penalization procedure. In the aforementioned papers, the presence of a magnetic field produce a phase in the complex wave, but does not influence the location of peaks. Later Cingolani et al. [6] used a variational approach and proved the existence of a multi-peak solution without any non-degenerate condition. But the locations of peaks are still near local minimum points of V. For more related results see [5,20] and references therein.

It is interesting to ask whether the magnetic field affects the locations of concentrations. Secchi and Squassina in [18] show that the magnetic potential $\boldsymbol{A}$ might perhaps affect the locations of concentration points for the three-dimensional magnetic Schrödinger equation. Nevertheless, in the particular but important case of power-type nonlinearities just as our case, they emphasize that the locations of peaks are independent of $\boldsymbol{A}$. DiCosmo and Van Schaftingen in [10] then consider a strong magnetic field case $\boldsymbol{A} = O(\varepsilon^{-2})$, where the interaction between the magnetic field and the electric field is thus comparable. In this case, they obtain solutions concentrating around global or local minima of a limit energy that depends on the electric potential and the magnetic field. However, their limit energy does not show the effect of magnetic fields explicitly. Later Bonheure et al. [2] proved the existence of semiclassical cylindrically symmetric solutions to three-dimensional problem (2) whose moduli concentrates around a circle, which is driven by the magnetic and electric potentials. Their result shows that the magnetic field really influences the locations of concentrations if it occurs around a locus. Actually, this is a high-dimensional concentration phenomenon, not a peak concentration. See also [21] for a curve concentration phenomenon under a weak magnetic potential without symmetric conditions.

Thus, a nature question arises that whether a common, neither weak nor strong, magnetic potential $\boldsymbol{A}$ influences the locations of peak concentrations to the nonlinear magnetic Schrödinger problem. What role does $\boldsymbol{A}$ play in this kind of concentration? To this aim, we assume the electric potential V to be a constant, otherwise $\boldsymbol{A}$ has no effect on the locations of peaks due to the aforementioned works [4,6,7,15].

Recently, Bonheure et al. [3] considers the ground state solution of magnetic Schrödinger equations with a constant electric potential and no Planck constant ε. Based on the assumption of a weak constant magnetic field, they show that the ground state solution is unique up to magnetic translations and rotations. Furthermore, the energy expansion of the ground state solution is also given in [3] as the constant magnetic field $\boldsymbol{B}$ is small enough.

In this article, we study the following nonlinear magnetic Schrödinger equation:

$$(\mathrm{i}\varepsilon\nabla + \boldsymbol{A}(x))^2 u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \tag{3}$$

Here $p > 1$ is Sobolev subcritical, and $\boldsymbol{A} = (A_1, \ldots, A_N)\colon \mathbb{R}^N \to \mathbb{R}^N$ is assumed in $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and smooth everywhere for simplicity. To state our main result, we introduce the Frobenius norm of a matrix $\mathcal{M} = (m_{ij})_{I\times J}$

$$\|\mathcal{M}\|_F = \left(\sum_{i=1}^{I}\sum_{j=1}^{J}|m_{ij}|^2\right)^{\frac{1}{2}}.$$

And denote by $w(y) = w(|y|)$ the unique radial real-valued solution of

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = \max_{\mathbb{R}^N} w > 0, \quad w(\pm\infty) = 0. \tag{4}$$

Now our main results are the following.

Theorem 1.1. *Assume that the Frobenius norm $\|\boldsymbol{B}\|_F$ of the magnetic field $\boldsymbol{B}$ admits K local maximum (minimum) points $\{P_m\}_{m=1}^K$ (which may be degenerate) and K disjoint, closed and bounded regions $\{\Omega_m\}_{m=1}^K$ of $\mathbb{R}^N$ such that*

$$\|\boldsymbol{B}(P_m)\|_F = \max_{\Omega_m}\|\boldsymbol{B}\|_F > \max_{\partial\Omega_m}\|\boldsymbol{B}\|_F. \quad \left(\|\boldsymbol{B}(P_m)\|_F = \min_{\Omega_m}\|\boldsymbol{B}\|_F < \min_{\partial\Omega_m}\|\boldsymbol{B}\|_F.\right).$$

Then there exists an $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$, problem (3) admits a solution u_ε with the form

$$u_\varepsilon(x) = \sum_{m=1}^{K}\left[w\left(\frac{|x-\zeta_m^\varepsilon|}{\varepsilon}\right) + \varepsilon\Psi_m\left(\frac{x}{\varepsilon}\right)\right]\mathrm{e}^{\mathrm{i}\sigma_m^\varepsilon + \mathrm{i}\varepsilon^{-1}\boldsymbol{A}(\zeta_m^\varepsilon)\cdot x} + O(\varepsilon^2),$$

for some $(\sigma_1^\varepsilon, \ldots, \sigma_K^\varepsilon) \in [0, 2\pi]^K$, $\zeta_m^\varepsilon \in \Omega_m$. The definition of Ψ_m is given in (9).

For general critical points, i.e., not local extremum points, we also have the following result.

Theorem 1.2. *Assume that $p > \frac{3}{2}$ and $P_1, P_2, \ldots, P_K$, $K \geq 1$ are all non-degenerate critical points of $\|\boldsymbol{B}\|_F^2$. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, problem (3) admits a solution u_ε with the form*

$$u_\varepsilon(x) = \sum_{m=1}^{K}\left[w\left(\frac{|x-\zeta_m^\varepsilon|}{\varepsilon}\right) + \varepsilon\Psi_m\left(\frac{x}{\varepsilon}\right)\right]\mathrm{e}^{\mathrm{i}\sigma_m^\varepsilon + \mathrm{i}\varepsilon^{-1}\boldsymbol{A}(\zeta_m^\varepsilon)\cdot x} + O(\varepsilon^2),$$

for some $(\sigma_1^\varepsilon, \ldots, \sigma_K^\varepsilon) \in [0, 2\pi]^K$ and $\zeta_m^\varepsilon = P_m + o(1)$. The definition of Ψ_m is given in (9).

Remark 1.3. Theorems 1.1 and 1.2 clearly show the effect of the magnetic vector $\boldsymbol{A}$, or precisely the magnetic field $\boldsymbol{B}$, in driving the locations of peak concentrations if the electric potential is constant. This is a new phenomenon.

Remark 1.4. Theorem 1.1 holds for any subcritical $p > 1$. And Theorem 1.2 holds only for the subcritical $p > \frac{3}{2}$. The reason is that the critical point of the function deduced by the energy functional is the local extremum point in Theorem 1.1, which may be obtained by direct comparison. While in Theorem 1.2, $P_1, P_2, \dots, P_K$ may not be extremum points any more. Thus, we have to study the derivatives of the energy functional. Naturally, the corresponding estimates should have higher accuracy for which the better regularity of the nonlinear term $|u|^{p-1}u$ is required.

From the point view of physics, the magnetic field $\boldsymbol{B}$ is essential, not the particular choice of magnetic potential $\boldsymbol{A}$. At the same time, there is a gauge invariance for the magnetic Laplacian correspondingly, that is, the magnetic field $\boldsymbol{B}$ is invariant under the transform of the potential $\boldsymbol{A} \to \boldsymbol{A} + \nabla f$. Also, it is easy to see that the energy of problem (3) is unchanged under the gauge invariance, with which our result coincides (see Proposition 4.3).

The proofs of main theorems are based on the reduction method. When the electric potential $V(x)$ has some non-degeneracy, we just need to make an approximation from the limit equation, which is enough to deduce the roles of function $V(x)$. But in order to see the roles of magnetic vector potential $\boldsymbol{A}$ in our case, we need an approximation up to order ε of problem (3), i.e., a second approximation have to be done. Finally, we mention that the solutions given in both Theorems 1.1 and 1.2 show simple peak concentrations. In the forthcoming article, we will deal with multi-bump solutions for constant electric potential.

The article is organized as follows. In Section 2, the ansatz and the estimate of the error are given. The corresponding nonlinear problem is solved in Section 3. In Section 4, the original problem is reduced to the finite dimension problem using variational reduction process and the expansion of the energy functional is shown. Finally, Theorems 1.1 and 1.2 are proved in Section 5.

Notations.
1. Constant $\beta = \min\{p-1, 1\}$.
2. The real part of $z \in \mathbb{C}$ will be denoted by $\mathrm{Re}\, z$.
3. The complex conjugate of $z \in \mathbb{C}$ will be denoted by $\bar{z}$.
4. C denotes a generic positive constant, which may be different from lines to lines.
5. Landau symbol $O(\varepsilon)$ is a generic function such that $|O(\varepsilon)| \le C\varepsilon$ and $o(\varepsilon)$ means that $\lim_{\varepsilon\to 0^+} \frac{|o(\varepsilon)|}{\varepsilon} = 0$.

2 Ansatz

In this section, we present the approximation of the problem and give the corresponding error estimate.

Recall in [15] that the problem

$$\Delta\tilde{w} - \tilde{w} + |\tilde{w}|^{p-1}\tilde{w} = 0, \quad \tilde{w} \in H^1(\mathbb{R}^N, \mathbb{C}),$$

possesses a unique ground state solution $\tilde{w}(y) = w(y)\mathrm{e}^{\mathrm{i}\sigma}$, $\forall\, \sigma \in [0, 2\pi]$ where $w(y)$ is the radial solution of problem (4). Thus, by the gauge invariance,

$$\widetilde{U}(x) = w\left(\frac{|x-\zeta|}{\varepsilon}\right)\mathrm{e}^{\mathrm{i}\sigma + \mathrm{i}\varepsilon^{-1}\boldsymbol{A}(\zeta)\cdot x}, \quad \forall \sigma \in [0, 2\pi],$$

is also the ground state solution to the constant magnetic potential problem

$$(\mathrm{i}\varepsilon\nabla + \boldsymbol{A}(\zeta))^2\widetilde{U} + \widetilde{U} - |\widetilde{U}|^{p-1}\widetilde{U} = 0, \quad \widetilde{U} \in H^1(\mathbb{R}^N, \mathbb{C}).$$

In the frame of large variable $y = x/\varepsilon$, the original problem (3) is equivalent to

$$(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2 u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \tag{5}$$

Therefore, the function

$$U(y) = w(|y-\zeta'|)\mathrm{e}^{\mathrm{i}\sigma + \mathrm{i}\boldsymbol{A}(\zeta)\cdot y}, \quad \zeta' = \zeta/\varepsilon,$$

formally approximates the solution at least near $y = \zeta'$. Now it is time to give the first approximation

$$W(y) = \sum_{m=1}^{K} U_m(y), \quad U_m(y) = w(|y - \zeta'_m|)e^{i\sigma_m + i\boldsymbol{A}(\zeta_m)\cdot y}, \quad \sigma_m \in [0, 2\pi], \tag{6}$$

where $\zeta_m \in \Omega_m$, $m = 1, 2, \dots, K$. Denote

$$\rho := \min_{1\le i\ne j\le K} \operatorname{dist}(\Omega_i, \Omega_j) > 0$$

and take a positive number δ such that $\delta \le \frac{1}{4}\rho$. Let $\widehat{R}(y)$ be the error caused by the first approximation W, which is

$$\widehat{R}(y) = (i\nabla + \boldsymbol{A}(\varepsilon y))^2 W + W - |W|^{p-1}W.$$

It is checked that for $|y - \zeta'_1| \le \frac{\delta}{\sqrt{\varepsilon}}$, $|U_m| \le |U_1|e^{-\frac{|\zeta'_m - \zeta'_1|}{3}}$, $m = 2, 3, \dots, K$ and

$$\begin{aligned}
\widehat{R}(y) &= \widetilde{R}(y) + O\Big(w(|y - \zeta'_1|)\max_{2\le m\le K} e^{-\frac{|\zeta_m-\zeta_1|}{3\varepsilon}} + w^{p-1}(|y - \zeta'_1|)\max_{2\le m\le K} e^{-\frac{|\zeta_m-\zeta_1|}{2\varepsilon}}\Big)\\
&= \widetilde{R}(y) + O(w(|y - \zeta'_1|) + w^{p-1}(|y - \zeta'_1|))e^{-\frac{\delta}{\varepsilon}},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{R}(y) &= (i\nabla + \boldsymbol{A}(\varepsilon y))^2 U_1 + U_1 - |U_1|^{p-1}U_1 = (i\nabla + \boldsymbol{A}(\varepsilon y))^2 U_1 - (i\nabla + \boldsymbol{A}(\zeta_1))^2 U_1\\
&= 2i(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1))\cdot\nabla U_1 + i\varepsilon(\nabla_x\cdot A)U_1 + (|\boldsymbol{A}(\varepsilon y)|^2 - |\boldsymbol{A}(\zeta_1)|^2)U_1\\
&= 2i(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1))\cdot[\nabla w(|y - \zeta'_1|) + i\boldsymbol{A}(\zeta_1)w(|y - \zeta'_1|)]e^{i\sigma_1 + i\boldsymbol{A}(\zeta_1)\cdot y} + i\varepsilon(\nabla_x\cdot \boldsymbol{A})U_1\\
&\quad + (|\boldsymbol{A}(\varepsilon y)|^2 - |\boldsymbol{A}(\zeta_1)|^2)w(|y - \zeta'_1|)e^{i\sigma_1 + i\boldsymbol{A}(\zeta_1)\cdot y}\\
&= [-2\boldsymbol{A}(\zeta_1)\cdot(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1)) + (|\boldsymbol{A}(\varepsilon y)|^2 - |\boldsymbol{A}(\zeta_1)|^2)]U_1 + i[2(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1))\cdot\nabla w(|y - \zeta'_1|)\\
&\quad + \varepsilon(\nabla_x\cdot A)w(|y - \zeta'_1|)]e^{i\sigma_1 + i\boldsymbol{A}(\zeta_1)\cdot y}\\
&= (|\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1)|)^2 w(|y - \zeta'_1|)e^{i\sigma_1 + i\boldsymbol{A}(\zeta_1)\cdot y} + i[2(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1))\cdot\nabla w(|y - \zeta'_1|)\\
&\quad + \varepsilon(\nabla_x\cdot \boldsymbol{A})w(|y - \zeta'_1|)]e^{i\sigma_1 + i\boldsymbol{A}(\zeta_1)\cdot y}.
\end{aligned} \tag{7}$$

As usual one writes $\zeta'_m = (\zeta'_{m,1}, \zeta'_{m,2}, \dots, \zeta'_{m,N})$. Direct calculation shows that

$$\begin{aligned}
|\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1)|^2 &= \sum_{i=1}^{N}(A_i(\varepsilon y) - A_i(\zeta_1))^2\\
&= \sum_{i=1}^{N}\Big(\varepsilon\nabla A_i(\zeta_1)\cdot(y - \zeta'_1) + \frac{\varepsilon^2}{2}(y - \zeta'_1)^{\perp}\cdot\nabla^2 A_i(\zeta_1)\cdot(y - \zeta'_1) + O(\varepsilon^3|y - \zeta'_1|^3)\Big)^2\\
&= \varepsilon^2\sum_{i,j,k=1}^{N}\partial_j A_i(\zeta_1)\partial_k A_i(\zeta_1)(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k}) + \varepsilon^3\sum_{i,j,k,\ell=1}^{N}\partial_j A_i(\zeta_1)\partial_{k\ell}A_i(\zeta_1)(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k})(y_\ell\\
&\quad - \zeta'_{1,\ell}) + O(\varepsilon^4|y - \zeta'_1|^4).
\end{aligned}$$

Similarly, it is verified that

$$
\begin{aligned}
(\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1))\cdot \nabla w(|y-\zeta_1'|) &= \sum_{i=1}^{N}(A_i(\varepsilon y) - A_i(\zeta_1))\partial_i w(|y-\zeta_1'|) \\
&= \varepsilon \sum_{i,j=1}^{N} \partial_j A_i(\zeta_1)(y_j - \zeta_{1,j}')\partial_i w(|y-\zeta_1'|) + \frac{\varepsilon^2}{2}\sum_{i,j,k=1}^{N}\partial_{jk}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')\partial_i w(|y-\zeta_1'|) \\
&\quad + \frac{\varepsilon^3}{6}\sum_{i,j,k,\ell=1}^{N}\partial_{jk\ell}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')\partial_i w(|y-\zeta_1'|) \\
&\quad + O(\varepsilon^4|y-\zeta_1'|^4|\nabla w(|y-\zeta_1'|)|).
\end{aligned}
$$

As for the term $\nabla_x\cdot \boldsymbol{A}(\varepsilon y)w(|y-\zeta_1'|)$, we expand it similarly to

$$
\begin{aligned}
\nabla_x\cdot \boldsymbol{A}(\varepsilon y)w(|y-\zeta_1'|) &= \nabla_x\cdot \boldsymbol{A}(\zeta_1)w(|y-\zeta_1'|) + \varepsilon\nabla(\nabla_x\cdot \boldsymbol{A})(\zeta_1)\cdot(y-\zeta_1')w(|y-\zeta_1'|) \\
&\quad + \frac{1}{2}\varepsilon^2(y-\zeta_1')^{\perp}\cdot\nabla^2(\nabla_x\cdot \boldsymbol{A})(\zeta_1)\cdot(y-\zeta_1')w(|y-\zeta_1'|) + O(\varepsilon^3|y-\zeta_1'|^3 w(|y-\zeta_1'|)) \\
&= \nabla_x\cdot \boldsymbol{A}(\zeta_1)w(|y-\zeta_1'|) + \varepsilon\sum_{i,j=1}^{N}\partial_{ij}A_i(\zeta_1)(y_j-\zeta_{1,j}')w(|y-\zeta_1'|) \\
&\quad + \frac{\varepsilon^2}{2}\sum_{i,j,k=1}^{N}\partial_{jki}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')w(|y-\zeta_1'|) + O(\varepsilon^3|y-\zeta_1'|^3 w(|y-\zeta_1'|)).
\end{aligned}
$$

Therefore, in the region $|y-\zeta_1'| < \frac{\delta}{\sqrt{\varepsilon}}$, we obtain that

$$
\begin{aligned}
\widetilde{R}(y)\mathrm{e}^{-\mathrm{i}\sigma_1 - \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} &= \varepsilon \mathrm{i}\left(2\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)(y_j-\zeta_{1,j}')\partial_i w(|y-\zeta_1'|) + \nabla_x\cdot \boldsymbol{A}(\zeta_1)w(|y-\zeta_1'|)\right) \\
&\quad + \varepsilon^2\sum_{i,j,k=1}^{N}\partial_j A_i(\zeta_1)\partial_k A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')w(|y-\zeta_1'|) \\
&\quad + \varepsilon^2\mathrm{i}\sum_{i,j,k=1}^{N}\partial_{jk}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')\partial_i w(|y-\zeta_1'|) + \varepsilon^2\mathrm{i}\sum_{i,j=1}^{N}\partial_{ij}A_i(\zeta_1)(y_j-\zeta_{1,j}')w(|y-\zeta_1'|) \\
&\quad + \varepsilon^3\sum_{i,j,k,\ell=1}^{N}\partial_j A_i(\zeta_1)\partial_{k\ell}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')w(|y-\zeta_1'|) \\
&\quad + \frac{\varepsilon^3}{3}\mathrm{i}\sum_{i,j,k,\ell=1}^{N}\partial_{jk\ell}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')\partial_i w(|y-\zeta_1'|) \\
&\quad + \frac{\varepsilon^3}{2}\mathrm{i}\sum_{i,j,k=1}^{N}\partial_{jki}A_i(\zeta_1)(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')w(|y-\zeta_1'|) \\
&\quad + O(\varepsilon^4|y-\zeta_1'|^4|w(|y-\zeta_1'|)|) + \mathrm{i}O(\varepsilon^4|y-\zeta_1'|^4|\nabla w(|y-\zeta_1'|)| + \varepsilon^4|y-\zeta_1'|^3|w(|y-\zeta_1'|)|).
\end{aligned}
$$

Note that the imaginary part of $\widetilde{R}(y)\mathrm{e}^{-\mathrm{i}\sigma_1-\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}$ is $O(\varepsilon)$. It is of less accuracy for later application. Complying with the guideline that the better approximation, the more possibility to obtain a solution, we should improve the accuracy of the approximation. To this purpose, we find that the real-valued function $\Psi_{1,ij}(y) = \frac{1}{2}(y_i-\zeta_{1,i}')(y_j-\zeta_{1,j}')w(|y-\zeta_1'|)(i\neq j)$ is the solution to

$$
-\Delta\Psi_{1,ij} + \Psi_{1,ij} - w^{p-1}(|y-\zeta_1'|)\Psi_{1,ij} = -2(y_j-\zeta_{1,j}')\partial_i w(|y-\zeta_1'|) = -2(y_i-\zeta_{1,i}')\partial_j w(|y-\zeta_1'|).
$$

Besides, $\Psi_{1,ii}(y) = \frac{1}{2}(y_i-\zeta_{1,i}')^2 w(|y-\zeta_1'|)$ satisfies the equation

$$
-\Delta\Psi_{1,ii} + \Psi_{1,ii} - w^{p-1}(|y-\zeta_1'|)\Psi_{1,ii} = -2(y_i-\zeta_{1,i}')\partial_i w(|y-\zeta_1'|) - w(|y-\zeta_1'|).
$$

Then obviously the function

$$
\Psi_1(y) = \mathrm{i}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{N}\partial_j A_i(\zeta_1)\Psi_{1,ij}(y) + \sum_{i=1}^{N}\partial_i A_i(\zeta_1)\Psi_{1,ii}(y)\right)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} = \mathrm{i}\left(\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)\Psi_{1,ij}(y)\right)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} := \mathrm{i}\psi_1(y)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}
$$

satisfies

$$\begin{aligned}&(\mathrm{i}\nabla + \boldsymbol{A}(\zeta_1))^2\Psi_1 + \Psi_1 - (p-1)|U_1|^{p-3}\operatorname{Re}(\overline{U}_1\Psi_1)U_1 - |U_1|^{p-1}\Psi_1\\ &\quad = \mathrm{i}\left[-2\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)(y_j - \zeta'_{1,j})\partial_i w(|y-\zeta'_1|) - \nabla_x\cdot\boldsymbol{A}(\zeta_1)w(|y-\zeta'_1|)\right]\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y},\end{aligned}$$

since $\operatorname{Re}(\overline{U}_1\Psi_1) = 0$. Moreover, the same computation may also be carried out in the region $|y-\zeta'_m| < \frac{\delta}{\sqrt{\varepsilon}}$ for $m = 2,\dots,K$. The ultimate approximation is then selected as

$$\mathcal{W}(y) = W(y) + \varepsilon\Psi(y), \tag{8}$$

where $\Psi(y) = \sum_{m=1}^{K}\Psi_m(y)$. Here

$$\Psi_m(y) = \mathrm{i}\left(\sum_{i,j=1}^{N}\partial_j A_i(\zeta_m)\Psi_{m,ij}(y)\right)\mathrm{e}^{\mathrm{i}\sigma_m + \mathrm{i}\boldsymbol{A}(\zeta_m)\cdot y} := \mathrm{i}\psi_m(y)\mathrm{e}^{\mathrm{i}\sigma_m + \mathrm{i}\boldsymbol{A}(\zeta_m)\cdot y} \tag{9}$$

and

$$\Psi_{m,ij}(y) = \frac{1}{2}(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j})w(|y-\zeta'_m|).$$

Now the approximation $\mathcal{W}$ is good enough to help us find a solution. Precisely, our aim is to find a solution with the form $\mathcal{W}(y) + \phi(y)$ of problem (3), where ϕ is a small perturbation and satisfies the equation

$$L\phi := (\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\phi + \phi - (p-1)|\mathcal{W}|^{p-3}\operatorname{Re}(\overline{\mathcal{W}}\phi)\mathcal{W} - |\mathcal{W}|^{p-1}\phi = -R(y) + N(\phi). \tag{10}$$

Here $R(y)$ denotes the error caused by $\mathcal{W}$, which is

$$R(y) = (\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\mathcal{W} + \mathcal{W} - |\mathcal{W}|^{p-1}\mathcal{W},$$

and the nonlinear term

$$N(\phi) = |\mathcal{W}+\phi|^{p-1}(\mathcal{W}+\phi) - |\mathcal{W}|^{p-1}\mathcal{W} - (p-1)|\mathcal{W}|^{p-3}\operatorname{Re}(\overline{\mathcal{W}}\phi)\mathcal{W} - |\mathcal{W}|^{p-1}\phi.$$

Obviously, with the notation $\beta = \min\{p-1, 1\}$,

$$|N(\phi)| \le C|\phi|^{1+\beta}.$$

To solve problem (10), it is important to estimate the error $R(y)$.

Proposition 2.1. *We have that*

$$\|R(y)\|_{L^2} \le C\varepsilon^2$$

and

$$\|\partial_{\zeta'_{m,k}}R(y)\|_{L^2} \le C\varepsilon^2, \quad \|\partial_{\sigma_m}R(y)\|_{L^2} \le C\varepsilon^2, \quad \forall m = 1,\dots,K, \quad k = 1,\dots,N.$$

Proof. First, we consider the domain $|y-\zeta'_m| \le \frac{\delta}{\sqrt{\varepsilon}}$, without loss of generality, say $m = 1$. Then

$$\begin{aligned}R(y) &= \widehat{R}(y) + \varepsilon[(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\Psi + \Psi - (p-1)|W|^{p-3}\operatorname{Re}(\overline{W}\Psi)W - |W|^{p-1}\Psi]\\ &\quad - [|W+\varepsilon\Psi|^{p-1}(W+\varepsilon\Psi) - |W|^{p-1}W - \varepsilon(p-1)|W|^{p-3}\operatorname{Re}(\overline{W}\Psi)W - \varepsilon|W|^{p-1}\Psi]\\ &= \widetilde{R}(y) - \varepsilon\mathrm{i}\left(2\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)(y_j-\zeta'_{1,j})\partial_i w(|y-\zeta'_1|) + (\nabla_x\cdot\boldsymbol{A})(\zeta_1)w(|y-\zeta'_1|)\right)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}\\ &\quad + \varepsilon[(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\Psi_1 - (\mathrm{i}\nabla + \boldsymbol{A}(\zeta_1))^2\Psi_1] + \varepsilon\left[(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\sum_{m=2}^{K}\Psi_m + \sum_{m=2}^{K}\Psi_m\right]\\ &\quad - \frac{p-1}{2}\varepsilon^2\psi_1^2(y)w^{p-2}(|y-\zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} - \mathrm{i}\frac{p-1}{2}\varepsilon^3\psi_1^3(y)w^{p-3}(|y-\zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}\\ &\quad + O\Big(\varepsilon^4|y-\zeta'_1|^8|w^p(|y-\zeta'_1|) + \mathrm{e}^{-\frac{\delta}{\varepsilon}}w(|y-\zeta'_1|) + \mathrm{e}^{-\frac{\delta}{\varepsilon}}w^{p-1}(|y-\zeta'_1|)\Big).\end{aligned}$$

Obviously, we have that

$$\varepsilon^2\psi_1(y)^2 w^{p-2}(|y-\zeta_1'|) = \varepsilon^2 w^{p-2}(|y-\zeta_1'|)\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)\Psi_{1,ij}(y)\sum_{k,\ell=1}^{N}\partial_\ell A_k(\zeta_1)\Psi_{1,k\ell}(y)$$
$$= \frac{\varepsilon^2}{4}w^p(|y-\zeta_1'|)\sum_{i,j,k,\ell=1}^{N}\partial_j A_i(\zeta_1)\partial_\ell A_k(\zeta_1)(y_i-\zeta_{1,i}')(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')$$

and

$$\varepsilon\left|\left[(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\sum_{m=2}^{K}\Psi_m + \sum_{m=2}^{K}\Psi_m\right]\right| = O\left(\sum_{m=2}^{K} w(|y-\zeta_m'|)\right) = O\left(\mathrm{e}^{-\frac{\delta}{\varepsilon}}w(|y-\zeta_1'|)\right).$$

Moreover, it is checked as in (7) that

$$\begin{aligned}(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\Psi_1 - (\mathrm{i}\nabla + \boldsymbol{A}(\zeta_1))^2\Psi_1 &= 2\mathrm{i}(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}(\zeta_1))\cdot\nabla\Psi_1 + \mathrm{i}\varepsilon(\nabla_x\cdot\boldsymbol{A}(\varepsilon y))\Psi_1 + (|\boldsymbol{A}(\varepsilon y)|^2 - |\boldsymbol{A}(\zeta_1)|^2)\Psi_1\\ &= -[2(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}(\zeta_1))\cdot\nabla\psi_1 + \varepsilon(\nabla_x\cdot\boldsymbol{A}(\varepsilon y))\psi_1]e^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} + \mathrm{i}|\boldsymbol{A}(\varepsilon y)\\ &\quad - \boldsymbol{A}(\zeta_1)|^2\psi_1 e^{\mathrm{i}\sigma_1+\mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}.\end{aligned} \tag{11}$$

Let us estimate them one by one in (11) for $y \neq \zeta_1'$. Note that

$$\begin{aligned}(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}(\zeta_1))\cdot\nabla\psi_1 &= \sum_{k=1}^{N}(A_k(\varepsilon y)-A_k(\zeta_1))\partial_k\psi_1\\ &= \varepsilon\sum_{k=1}^{N}\nabla A_k(\zeta_1)\cdot(y-\zeta_1')\partial_k\psi_1 + \frac{\varepsilon^2}{2}\sum_{k=1}^{N}(y-\zeta_1')^{\perp}\cdot\nabla^2 A_k(\zeta_1)\cdot(y-\zeta_1')\partial_k\psi_1 + O(\varepsilon^3|y-\zeta_1'|^3|\nabla\psi_1|)\\ &= \varepsilon\sum_{k,\ell=1}^{N}\partial_\ell A_k(\zeta_1)(y_\ell-\zeta_{1,\ell}')\left[\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)\partial_k\Psi_{1,ij}\right]\\ &\quad + \frac{\varepsilon^2}{2}\sum_{k,\ell,s=1}^{N}\partial_{\ell s}A_k(\zeta_1)(y_\ell-\zeta_{1,\ell}')(y_s-\zeta_{1,s}')\left[\sum_{i,j=1}^{N}\partial_j A_i(\zeta_1)\partial_k\Psi_{1,ij}\right] + O(\varepsilon^3|y-\zeta_1'|^5 w(|y-\zeta_1'|)).\end{aligned}$$

It is easy to see that (δ_{ik} is the Kronecker symbol) for $y \neq \zeta_1'$,

$$\partial_k\Psi_{1,ij}(y) = \frac{1}{2}\left[(\delta_{ik}(y_j-\zeta_{1,j}') + \delta_{jk}(y_i-\zeta_{1,i}'))w(|y-\zeta_1'|) + (y_i-\zeta_{1,i}')(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')\frac{w'(|y-\zeta_1'|)}{|y-\zeta_1'|}\right].$$

Thus, we conclude that

$$\begin{aligned}(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}(\zeta_1))\cdot\nabla\psi_1 &= \frac{\varepsilon}{2}\sum_{j,k,\ell=1}^{N}\partial_\ell A_k(\zeta_1)[\partial_j A_k(\zeta_1)+\partial_k A_j(\zeta_1)](y_j-\zeta_{1,j}')(y_\ell-\zeta_{1,\ell}')w(|y-\zeta_1'|)\\ &\quad + \frac{\varepsilon}{2}\sum_{i,j,k,\ell=1}^{N}\partial_\ell A_k(\zeta_1)\partial_j A_i(\zeta_1)(y_i-\zeta_{1,i}')(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')\frac{w'(|y-\zeta_1'|)}{|y-\zeta_1'|}\\ &\quad + \frac{\varepsilon^2}{4}\sum_{j,k,\ell,s=1}^{N}\partial_{\ell s}A_k(\zeta_1)[\partial_j A_k(\zeta_1)+\partial_k A_j(\zeta_1)](y_j-\zeta_{1,j}')(y_\ell-\zeta_{1,\ell}')(y_s-\zeta_{1,s}')w(|y-\zeta_1'|)\\ &\quad + \frac{\varepsilon^2}{4}\sum_{i,j,k,\ell,s=1}^{N}\partial_{\ell s}A_k(\zeta_1)\partial_j A_i(\zeta_1)(y_i-\zeta_{1,i}')(y_j-\zeta_{1,j}')(y_k-\zeta_{1,k}')(y_\ell-\zeta_{1,\ell}')\\ &\quad \times (y_s-\zeta_{1,s}')\frac{w'(|y-\zeta_1'|)}{|y-\zeta_1'|} + O(\varepsilon^3|y-\zeta_1'|^5 w(|y-\zeta_1'|)).\end{aligned} \tag{12}$$

Also, it can be obtained that

$$
\begin{aligned}
(\nabla_x \cdot \boldsymbol{A}(\varepsilon y))\psi_1 = \frac{1}{2}(\nabla_x \cdot \boldsymbol{A})(\zeta_1)\sum_{i,j=1}^{N} \partial_j A_i(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})w(|y - \zeta'_1|) \\
+ \frac{\varepsilon}{2}\sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_1)\partial_{\ell k} A_\ell(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k})w(|y - \zeta'_1|) \\
+ O(\varepsilon^2 |y - \zeta'_1|^4 w(|y - \zeta'_1|))
\end{aligned}
$$

and

$$
|\boldsymbol{A}(\varepsilon y) - \boldsymbol{A}(\zeta_1)|^2 \psi_1 = O(\varepsilon^2 |y - \zeta'_1|^4 w(|y - \zeta'_1|)).
$$

Now (11) can be expanded as

$$
\begin{aligned}
&(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2 \Psi_1 - (\mathrm{i}\nabla + \boldsymbol{A}(\zeta_1))^2 \Psi_1 \\
&= -\varepsilon \sum_{j,k,\ell=1}^{N} \partial_\ell A_k(\zeta_1)[\partial_j A_k(\zeta_1) + \partial_k A_j(\zeta_1)](y_j - \zeta'_{1,j})(y_\ell - \zeta'_{1,\ell})w(|y - \zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad - \varepsilon \sum_{i,j,k,\ell=1}^{N} \partial_\ell A_k(\zeta_1)\partial_j A_i(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k})(y_\ell - \zeta'_{1,\ell})\frac{w'(|y - \zeta'_1|)}{|y - \zeta'_1|}\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad - \frac{\varepsilon}{2}(\nabla_x \cdot \boldsymbol{A}(\zeta_1))\sum_{i,j=1}^{N} \partial_j A_i(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})w(|y - \zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad - \frac{\varepsilon^2}{2}\sum_{j,k,\ell,s=1}^{N} \partial_{\ell s} A_k(\zeta_1)[\partial_j A_k(\zeta_1) + \partial_k A_j(\zeta_1)](y_j - \zeta'_{1,j})(y_\ell - \zeta'_{1,\ell})(y_s - \zeta'_{1,s})w(|y - \zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad - \frac{\varepsilon^2}{2}\sum_{i,j,k,\ell,s=1}^{N} \partial_{\ell s} A_k(\zeta_1)\partial_j A_i(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k})(y_\ell - \zeta'_{1,\ell})(y_s - \zeta'_{1,s})\frac{w'(|y - \zeta'_1|)}{|y - \zeta'_1|}\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad - \frac{\varepsilon^2}{2}\sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_1)\partial_{\ell k} A_\ell(\zeta_1)(y_i - \zeta'_{1,i})(y_j - \zeta'_{1,j})(y_k - \zeta'_{1,k})w(|y - \zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y} \\
&\quad + [O(\varepsilon^3 |y - \zeta'_1|^5 + \varepsilon^3 |y - \zeta'_1|^4) + \mathrm{i}O(\varepsilon^2 |y - \zeta'_1|^4)]w(|y - \zeta'_1|)\mathrm{e}^{\mathrm{i}\sigma_1 + \mathrm{i}\boldsymbol{A}(\zeta_1)\cdot y}.
\end{aligned}
$$

Therefore, we have, in $|y - \zeta'_m| \le \frac{\delta}{\sqrt{\varepsilon}}$, that

$$
R(y)\mathrm{e}^{-\mathrm{i}\sigma_m - \mathrm{i}\boldsymbol{A}(\zeta_m)\cdot y} := R_{m,1}(y) + \mathrm{i}R_{m,2}(y),
$$

where

$$
\begin{aligned}
R_{m,1}(y) = {} & \varepsilon^2 \sum_{i,j,k=1}^{N} \partial_j A_i(\zeta_m)\partial_k A_i(\zeta_m)(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k}) w(|y - \zeta'_m|) \\
& - \varepsilon^2 \sum_{j,k,\ell=1}^{N} \partial_\ell A_k(\zeta_m)[\partial_j A_k(\zeta_m) + \partial_k A_j(\zeta_m)](y_j - \zeta'_{m,j})(y_\ell - \zeta'_{m,\ell}) w(|y - \zeta'_m|) \\
& - \varepsilon^2 \sum_{i,j,k,\ell=1}^{N} \partial_\ell A_k(\zeta_m)\partial_j A_i(\zeta_m)(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k})(y_\ell - \zeta'_{m,\ell}) \frac{w'(|y - \zeta'_m|)}{|y - \zeta'_m|} \\
& - \frac{\varepsilon^2}{2}(\nabla_x \cdot \boldsymbol{A}(\zeta_m)) \sum_{i,j=1}^{N} \partial_j A_i(\zeta_m)(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j}) w(|y - \zeta'_m|) \\
& - \frac{(p-1)\varepsilon^2}{8} \sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_m)\partial_\ell A_k(\zeta_m)(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k})(y_\ell - \zeta'_{m,\ell}) w^p(|y - \zeta'_m|) \\
& + \varepsilon^3 \sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_m)\partial_{k\ell} A_i(\zeta_m)(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k})(y_\ell - \zeta'_{m,\ell}) w(|y - \zeta'_m|) \\
& - \frac{\varepsilon^3}{2} \sum_{j,k,\ell,s=1}^{N} \partial_{\ell s} A_k(\zeta_m)[\partial_j A_k(\zeta_m) + \partial_k A_i(\zeta_m)](y_j - \zeta'_{m,j})(y_\ell - \zeta'_{m,\ell})(y_s - \zeta'_{m,s}) w(|y - \zeta'_m|) \\
& - \frac{\varepsilon^3}{2} \sum_{i,j,k,\ell,s=1}^{N} \partial_{\ell s} A_k(\zeta_m)\partial_j A_i(\zeta_m)(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k})(y_\ell - \zeta'_{m,\ell})(y_s - \zeta'_{m,s}) \frac{w'(|y - \zeta'_m|)}{|y - \zeta'_m|} \\
& - \frac{\varepsilon^3}{2} \sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_m)\partial_{\ell k} A_\ell(\zeta_m)(y_i - \zeta'_{m,i})(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k}) w(|y - \zeta'_m|) \\
& + O(\varepsilon^4 |y - \zeta'_m|^4 + \varepsilon^4 |y - \zeta'_m|^5) w(|y - \zeta'_m|) + O(\varepsilon^4 |y - \zeta'_m|^8 w^p(|y - \zeta'_m|))
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
R_{m,2}(y) = {} & \varepsilon^2 \sum_{i,j,k=1}^{N} \partial_{jk} A_i(\zeta_m)(y_j - \zeta'_{m,j})(y_k - \zeta'_{m,k})\partial_i w(|y - \zeta'_m|) + \varepsilon^2 \sum_{i,j=1}^{N} \partial_{ij} A_i(\zeta_m)(y_j - \zeta'_{m,j}) w(|y - \zeta'_m|) \\
& + O(\varepsilon^3 |y - \zeta'_m|^4 + \varepsilon^3 |y - \zeta'_m|^2) w(|y - \zeta'_m|) + O(\varepsilon^4 |y - \zeta'_m|^8) w^p(|y - \zeta'_m|).
\end{aligned}
$$

Hence, we obtain the estimate

$$
\sum_{m=1}^{K} \int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta'_m)} |R(y)|^2 \mathrm{d}y \le C\varepsilon^2.
$$

As for the domain $|y - \zeta'_m| \ge \frac{\delta}{\sqrt{\varepsilon}}$, $\forall m = 1, 2, \ldots, K$, using the asymptotic behavior of $w(|y - \zeta'_m|)$, it is easy to see that

$$
\int_{\mathbb{R}^N \setminus \bigcup_{m=1}^{K} B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta'_m)} |R(y)|^2 \mathrm{d}y \le C\mathrm{e}^{-\frac{\delta}{\sqrt{\varepsilon}}}.
$$

The result for $R(y)$ is concluded.

As for the estimates of $\partial_{\zeta'_{m,k}} R$ and $\partial_{\sigma_m} R$, one may check it similarly. □

3 The linear problem and the nonlinear problem

This section is devoted to the invertibility of the linear operator L in order to solve problem (10):

$$
L\phi = (\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))^2\phi + \phi - (p-1)|\mathcal{W}|^{p-3}\mathrm{Re}(\overline{\mathcal{W}}\phi)\mathcal{W} - |\mathcal{W}|^{p-1}\phi = -R(y) + N(\phi).
$$

Let H be the Hilbert space as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product

$$(u, v) = \operatorname{Re} \int_{\mathbb{R}^N} (\mathrm{i}\nabla u + \boldsymbol{A}(\varepsilon y)u)\overline{(\mathrm{i}\nabla v + \boldsymbol{A}(\varepsilon y)v)} + u\bar{v}.$$

The norm deduced by the above scalar product is equivalent to the usual norm of $H^1(\mathbb{R}^N, \mathbb{C})$ due to the boundness of $|\boldsymbol{A}(x)|$, see [7]. In $|y - \zeta'_m| \le \frac{\delta}{\sqrt{\varepsilon}}$, the operator L formally looks like

$$(\mathrm{i}\nabla + \boldsymbol{A}(\zeta_m))^2\phi + \phi - (p-1)|U_m|^{p-3}\operatorname{Re}(\overline{U}_m\phi)U_m - |U_m|^{p-1}\phi,$$

which is not invertible. Precisely, the null space of this limit operator is

$$\operatorname{span}_{\mathbb{R}}\{Z_{m,0}, Z_{m,1}, \dots, Z_{m,N}\},$$

where

$$Z_{m,0} = \mathrm{i}w(|y - \zeta'_m|)\mathrm{e}^{\mathrm{i}\sigma_m + \mathrm{i}\boldsymbol{A}(\zeta_m)\cdot y} = \mathrm{i}U_m \quad \text{and} \quad Z_{m,i} = \frac{\partial w(|y - \zeta'_m|)}{\partial \zeta'_{m,i}}\mathrm{e}^{\mathrm{i}\sigma_m + \mathrm{i}\boldsymbol{A}(\zeta_m)\cdot y}, \quad 1 \le i \le N.$$

The symbol $\operatorname{span}_{\mathbb{R}}$ means the linear combinations on real numbers, see for instance [7,8]. Therefore, we study the following linear problem with $h \in L^2(\mathbb{R}^N, \mathbb{C})$

$$\begin{cases} L\phi = h + \sum_{i=0}^{N}\sum_{m=1}^{K} c_{m,i}\chi_m Z_{m,i}, \\ \operatorname{Re} \int_{\mathbb{R}^N} \chi_m \overline{Z}_{m,i}\phi = 0, \quad i = 0, 1, \dots, N, \quad m = 1, \dots, K, \end{cases} \tag{14}$$

where $\chi_m(y) = \chi(|y - \zeta'_m|)$ is a smooth cut-off function on the large ball $B_R(\zeta'_m)$, satisfying $\chi(s) = 1$ for $|s| \le R$ and $\chi(s) = 0$ for $|s| \ge R + 1$.

Next, we prove the following invertibility proposition, which is the main result in this section.

Proposition 3.1. *The linear problem* (14) *admits a unique solution* $(\phi, c_{m,i}) = (\widetilde{T}(h), c_{m,i})$, $i = 0, 1, \dots, N$, $m = 1, \dots, K$ *satisfying*

$$\|\phi\|_{H^2} = \|\widetilde{T}(h)\|_{H^2} \le C\|h\|_{L^2}, \quad |c_{m,i}| \le C\|h\|_{L^2}.$$

Before giving the proof, it is necessary to obtain an *a priori* estimate.

Lemma 3.2. *If* $(\phi, c_{m,i})$ *is a solution of problem* (14), *then*

$$\|\phi\|_{H^2} \le C\|h\|_{L^2}, \quad |c_{m,i}| \le C\|h\|_{L^2}.$$

Proof. The proof is very standard and we here prove it briefly for the completion. First, we test the equation (14) by $\overline{Z}_{\ell,j}$, $1 \le \ell \le K$, $0 \le j \le N$ and obtain that

$$\langle L\phi, \overline{Z}_{\ell,j}\rangle = \operatorname{Re} \int_{\mathbb{R}^N} h\overline{Z}_{\ell,j} + c_{\ell,j}\int_{\mathbb{R}^N} |Z_{\ell,j}|^2 + O\bigg(\mathrm{e}^{-\frac{\delta}{\varepsilon}}\sum_{i=0}^{N}\sum_{m=1}^{K}|c_{m,i}|\bigg). \tag{15}$$

Note that

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^N} (\mathrm{i}\nabla + A(\varepsilon y))\phi\overline{(\mathrm{i}\nabla + A(\varepsilon y))Z_{\ell,j}} &= \operatorname{Re} \int_{\mathbb{R}^N} (\mathrm{i}\nabla + A(\varepsilon y))Z_{\ell,j}\overline{(\mathrm{i}\nabla + A(\varepsilon y))\phi} \\ &= \operatorname{Re} \int_{\mathbb{R}^N} (\mathrm{i}\nabla + A(\zeta_\ell))Z_{\ell,j}\overline{(\mathrm{i}\nabla + A(\zeta_\ell))\phi} + O(\varepsilon)\|\phi\|_{H^1}, \end{aligned}$$

and

$$\left|\int_{\mathbb{R}^N} h\overline{Z}_{\ell,j}\right| \le C\|h\|_{L^2}.$$

Thus, it holds from (15) and the equation of $\overline{Z}_{\ell,j}$ that

$$c_{\ell,j} = O\Big(\varepsilon\|\phi\|_{H^1} + \|h\|_{L^2}\Big). \tag{16}$$

Next we will prove $\|\phi\|_{H^1} \le C\|h\|_{L^2}$ by contradiction. Suppose that for some sequence $\{\varepsilon_n\}$, there always exist $\{\phi_n\}$ and $\{h_n\}$ such that

$$\|\phi_n\|_{H^1} = 1 \quad \text{and} \quad \|h_n\|_{L^2} = o(1) \quad \text{as } \varepsilon_n \to 0.$$

Testing (14) against $\eta_m\varphi(\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}))$ where $\eta_m(y) \equiv 1$ in $|y - \zeta_m'| < \frac{\delta}{\sqrt{\varepsilon}}$ and $\eta_m(y) \equiv 0$ in $|y - \zeta_m'| > \frac{2\delta}{\sqrt{\varepsilon}}$, one can obtain that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^N}(\mathrm{i}\nabla + A(\varepsilon_n y))\phi_n\overline{(\mathrm{i}\nabla + A(\varepsilon_n y))\eta_m\varphi} + \mathrm{Re}\int_{\mathbb{R}^N}\phi_n\eta_m\overline{\varphi} - (p-1)\,\mathrm{Re}\int_{\mathbb{R}^N}|\mathcal{W}|^{p-3}(\mathrm{Re}(\overline{\mathcal{W}}\phi_n))\mathcal{W}\eta_m\overline{\varphi}\\
&-\mathrm{Re}\int_{\mathbb{R}^N}|\mathcal{W}|^{p-1}\phi_n\eta_m\overline{\varphi} = \mathrm{Re}\int_{\mathbb{R}^N}h_n\eta_m\overline{\varphi} + \sum_{i=0}^{N}c_{m,i}\,\mathrm{Re}\int_{\mathbb{R}^N}\chi_m Z_{m,i}\overline{\varphi}.
\end{aligned}$$

Note that $\phi_n \rightharpoonup \phi$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ up to a subsequence. Thus, dominated convergence theorem tells us that

$$\mathrm{Re}\int_{\mathbb{R}^N}(\mathrm{i}\nabla + A(\zeta_m))\phi\overline{(\mathrm{i}\nabla + A(\zeta_m))\varphi} + \mathrm{Re}\int_{\mathbb{R}^N}\phi\overline{\varphi} - (p-1)\,\mathrm{Re}\int_{\mathbb{R}^N}|U_m|^{p-3}(\mathrm{Re}(\overline{U}_m\phi))U_m\overline{\varphi} - \mathrm{Re}\int_{\mathbb{R}^N}|U_m|^{p-1}\phi\overline{\varphi} = 0.$$

This means that ϕ is a solution of

$$(\mathrm{i}\nabla + \boldsymbol{A}(\zeta_m))^2\phi + \phi - (p-1)|U_m|^{p-3}\,\mathrm{Re}(\overline{U}_m\phi)U_m - |U_m|^{p-1}\phi = 0 \quad \text{in } \mathbb{R}^N.$$

Then one obtains that $\phi = 0$ from the orthogonal conditions, which further implies that

$$\phi_n \to 0 \quad \text{a.e. in } B_R(\zeta_m'), \quad \forall R > 0, \quad m = 1, 2, \ldots, K. \tag{17}$$

On the other hand, note that

$$\int_{\mathbb{R}^N}|(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))\phi_n|^2 + \int_{\mathbb{R}^N}|\phi_n|^2 - (p-1)\int_{\mathbb{R}^N}|\mathcal{W}|^{p-3}(\mathrm{Re}(\overline{\mathcal{W}}\phi_n))^2 - \int_{\mathbb{R}^N}|\mathcal{W}|^{p-1}|\phi_n|^2 = \mathrm{Re}\int_{\mathbb{R}^N}\bar{h}_n\phi_n = o(1). \tag{18}$$

From (17) and the exponential decay of $|U_m|$, we obviously have

$$\int_{\mathbb{R}^N}|\mathcal{W}|^{p-1}|\phi_n|^2 = \sum_{m=1}^{K}\int_{B_R(\zeta_m')}|U_m|^{p-1}|\phi_n|^2 + \int_{\mathbb{R}^N\setminus\cup_{m=1}^K B_R(\zeta_m')}|\mathcal{W}|^{p-1}|\phi_n|^2 + O(\varepsilon) = O(\mathrm{e}^{-R}) + o(1).$$

So is $\int_{\mathbb{R}^N}|\mathcal{W}|^{p-3}(\mathrm{Re}(\overline{\mathcal{W}}\phi_n))^2$, which toobtainher with (18) shows that

$$\int_{\mathbb{R}^N}|\phi_n|^2 = O(\mathrm{e}^{-R}) + o(1) \quad \text{and} \quad \int_{\mathbb{R}^N}|(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))\phi_n|^2 = O(\mathrm{e}^{-R}) + o(1).$$

Finally, it is derived that

$$\begin{aligned} O(\mathrm{e}^{-R}) + o(1) &= \int_{\mathbb{R}^N} |(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))\phi_n|^2 \\ &= \int_{\mathbb{R}^N} |\nabla\phi_n|^2 + \int_{\mathbb{R}^N} |\boldsymbol{A}(\varepsilon_n y)|^2|\phi_n|^2 + 2\mathrm{Re}\int_{\mathbb{R}^N} \mathrm{i}\boldsymbol{A}(\varepsilon_n y)\cdot\nabla\phi_n\phi_n \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\phi_n|^2 - \int_{\mathbb{R}^N} |\boldsymbol{A}(\varepsilon_n y)|^2|\phi_n|^2 = \frac{1}{2}\|\phi_n\|_{H^1} + O(\mathrm{e}^{-R}) + o(1), \end{aligned}$$

since $\boldsymbol{A}(x)$ is bounded. This leads to a contradiction to $\|\phi_n\|_{H^1} = 1$. Hence,

$$\|\phi\|_{H^1} \leq C\|h\|_{L^2}.$$

Finally, the regularity theory gives $\|\phi\|_{H^2} \leq C\|h\|_{L^2}$. □

Proof of Proposition 3.1. Denote the Hilbert space

$$\mathcal{H} = \left\{\phi \in H^1(\mathbb{R}^N) \,\middle|\, \mathrm{Re}\int_{\mathbb{R}^N} \chi_m\phi\overline{Z}_{m,i} = 0, \quad i = 0, \dots, N, \ m = 1, \dots, K\right\}$$

with the equivalent inner product introduced at the beginning of this section. Problem (14) expressed in weak form is equivalent to finding a $\phi \in \mathcal{H}$, such that

$$(\phi, v) - \mathrm{Re}\int_{\mathbb{R}^N} [(p-1)|\mathcal{W}|^{p-3}\mathrm{Re}(\overline{\mathcal{W}}\phi)\mathcal{W} + |\mathcal{W}|^{p-1}\phi]\bar{v} = \mathrm{Re}\int_{\mathbb{R}^N} h\bar{v}, \quad \forall v \in \mathcal{H}.$$

Then, from Riesz representation theorem equation (14) is equivalent to

$$\phi - T(\phi) = \tilde{h}, \quad \text{in } \mathcal{H},$$

where T is a compact operator on $\mathcal{H}$. Based on Proposition 3.1, Fredholm alternative tells us the unique existence of ϕ. And $c_{m,i}$ can be given by ϕ using integration. Their estimates were given in the aforementioned proposition. □

Also for $\phi = \widetilde{T}(h)$, it is important for later purposes to understand the differentiability of the operator $\widetilde{T}$ with respect to ζ_j' and σ_j, $j = 1, \dots, K$. Recall that ϕ satisfies the equation

$$L\phi = (\mathrm{i}\nabla + A(\varepsilon y))^2\phi + \phi - (p-1)|\mathcal{W}|^{p-3}\mathrm{Re}(\overline{\mathcal{W}}\phi)\mathcal{W} - |\mathcal{W}|^{p-1}\phi = h + \sum_{i=0}^{N}\sum_{m=1}^{K} c_{m,i}\chi_m Z_{m,i}.$$

Thus, for $k = 1, \dots, N$,

$$\begin{aligned} L\big(\partial_{\zeta_{j,k}'}\phi\big) &= (\mathrm{i}\nabla + A(\varepsilon y))^2\big(\partial_{\zeta_{j,k}'}\phi\big) + \partial_{\zeta_{j,k}'}\phi - (p-1)|\mathcal{W}|^{p-3}\mathrm{Re}\big(\overline{\mathcal{W}}\partial_{\zeta_{j,k}'}\phi\big)\mathcal{W} - |\mathcal{W}|^{p-1}\partial_{\zeta_{j,k}'}\phi \\ &= O\big(|\mathcal{W}|^{p-2}|\phi||\partial_{\zeta_{j,k}'}\mathcal{W}|\big) + \sum_{i=0}^{N} c_{j,i}\partial_{\zeta_{j,k}'}(\chi_j Z_{j,i}) + \sum_{m=1}^{K}\sum_{i=0}^{N}\big(\partial_{\zeta_{j,k}'}c_{m,i}\big)\chi_m Z_{m,i}. \end{aligned}$$

Moreover, the derivative of the orthogonal condition is

$$\begin{aligned} \mathrm{Re}\int_{\mathbb{R}^N} \chi_m\overline{Z}_{m,i}\big(\partial_{\zeta_{j,k}'}\phi\big) &= 0, \quad \text{for } m \neq j, \\ \mathrm{Re}\int_{\mathbb{R}^N} \chi_j\overline{Z}_{j,i}\big(\partial_{\zeta_{j,k}'}\phi\big) &= -\mathrm{Re}\int_{\mathbb{R}^N} \partial_{\zeta_{j,k}'}(\chi_j\overline{Z}_{j,i})\phi. \end{aligned}$$

Set $\varphi = \partial_{\zeta_{j,k}'}\phi + \sum_{i=0}^{N} b_{jk,i}\chi_j Z_{j,i}$ and

$$b_{jk,i} = \operatorname{Re} \int_{\mathbb{R}^N} \partial_{\zeta'_{j,k}} (\chi_j \overline{Z}_{j,i}) \phi \Bigg/ \int_{\mathbb{R}^N} \chi_j |Z_{j,i}|^2 .$$

Note that $|b_{jk,i}| \leq C\|\phi\|_{L^2} \leq C\|h\|_{L^2}$. Then φ satisfies all the orthogonal conditions in (14), and direct computations show that

$$L\varphi = \sum_{i=0}^{N} b_{jk,i} L(\chi_j Z_{j,i}) + O\big(|\mathcal{W}|^{p-2}|\phi||\partial_{\zeta'_{j,k}} \mathcal{W}|\big) + \sum_{i=0}^{N} c_{j,i} \partial_{\zeta'_{j,k}} (\chi_j Z_{j,i}) + \sum_{m=1}^{K} \sum_{i=0}^{N} \big(\partial_{\zeta'_{j,k}} c_{m,i}\big) \chi_m Z_{m,i}.$$

With Lemma 3.2 in hand,

$$\|\varphi\|_{H^2} \leq C \sum_{i=0}^{N} \|b_{jk,i} L(\chi_j Z_{j,i})\|_{L^2} + C\||\mathcal{W}|^{p-2} \phi \partial_{\zeta'_{j,k}} \mathcal{W}\|_{L^2} + C \sum_{i=0}^{N} \|c_{j,i} \partial_{\zeta'_{j,k}} (\chi_j Z_{j,i})\|_{L^2} \leq C\|h\|_{L^2}.$$

Therefore, we conclude that

$$\|\partial_{\zeta'_{j,k}} \widetilde{T}(h)\|_{H^2} = \|\partial_{\zeta'_{j,k}} \phi\|_{H^2} \leq \|\varphi\|_{H^2} + \sum_{i=0}^{N} \|b_{jk,i} \chi_j Z_{j,i}\|_{H^2} \leq C\|h\|_{L^2}.$$

The same process may be carried out for $\partial_{\sigma_j} \phi$. Based on the above discussion, the following proposition holds obviously.

Proposition 3.3. *For the unique solution $\phi = \widetilde{T}(h)$ in Proposition* 3.1, *it holds that*

$$\|\partial_{\zeta'_{m,i}} \widetilde{T}(h)\|_{H^2} \leq C\|h\|_{L^2}, \quad \|\partial_{\sigma_m} \widetilde{T}(h)\|_{H^2} \leq C\|h\|_{L^2}, \quad \forall m = 1, \ldots, K, \ i = 1, \ldots, N.$$

Now we can deal with the following nonlinear problem:

$$\begin{cases} L\phi = -R(y) + N(\phi) + \displaystyle\sum_{i=0}^{N} \sum_{m=1}^{K} c_{m,i} \chi_m Z_{m,i}, \\ \operatorname{Re} \displaystyle\int_{\mathbb{R}^N} \chi_m \overline{Z}_{m,i} \phi = 0, \quad i = 0, \ldots, N, m = 1, \ldots, K. \end{cases} \tag{19}$$

Proposition 3.4. *The nonlinear problem* (19) *admits a unique solution ϕ satisfying*

$$\|\phi\|_{H^2} = O(\varepsilon^2).$$

Moreover, $(\boldsymbol{\sigma}, \boldsymbol{\zeta}') \to \phi$ is of class C^1 for $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_K)$, $\boldsymbol{\zeta}' = (\zeta'_1, \ldots, \zeta'_K)$, and

$$\|\partial_{\zeta'_{m,i}} \phi\|_{H^2} = O(\varepsilon^{2\beta}) \quad \textit{and} \quad \|\partial_{\sigma_m} \phi\|_{H^2} = O(\varepsilon^{2\beta}), \quad \forall m = 1, \ldots K, i = 1, \ldots, N.$$

Proof. Recall that $\beta = \min\{p-1, 1\}$. The proof is based on the contraction mapping theorem. First, for a large enough number $\gamma_0 > 0$, we set

$$\mathcal{S} = \{\phi \in \mathcal{H} | \|\phi\|_{H^2} \leq \gamma_0 \varepsilon^2\}.$$

In terms of the operator $\widetilde{T}$ defined in Proposition 3.1, the nonlinear problem (19) is transferred to solving

$$\phi = \widetilde{T}(-R(y) + N(\phi)) := \mathcal{A}(\phi),$$

which means to find a fixed point of the operator $\mathcal{A}$.

First, the operator $\mathcal{A}$ is from $\mathcal{S}$ to itself. In fact,

$$\|\mathcal{A}(\phi)\|_{H^2} = \|\widetilde{T}(-R(y) + N(\phi))\|_{H^2} \leq C\|R(y)\|_{L^2} + C\|N(\phi)\|_{L^2} \leq C\varepsilon^2 + C\|\phi\|_{H^2}^{1+\beta} \leq \gamma_0 \varepsilon^2.$$

Next the operator $\mathcal{A}$ is a contraction mapping, since

$$\begin{aligned}\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{H^2} &= \|\widetilde{T}(N(\phi_1) - N(\phi_2))\|_{H^2} \le C\|N(\phi_1) - N(\phi_2)\|_{L^2}\\ &\le C(\|\phi_1\|_{H^2}^{\beta} + \|\phi_2\|_{H^2}^{\beta})\|\phi_1 - \phi_2\|_{H^2}.\end{aligned}$$

Thus, $\mathcal{A}$ has a unique fixed point in $\mathcal{S}$, which is the unique solution of problem (19).

Next we come to $\partial_{\zeta'_{m,i}}\phi$ and $\partial_{\sigma_m}\phi$. The C^1-regularity in ζ'_m and σ_m is guaranteed by the implicit function theorem. One may refer to the proof of Lemma 4.1 in [7]. In order to estimate the differentiability, we follow the procedure in [9]. It is easy to see

$$\partial_{\zeta'_{m,i}}\phi = \partial_{\zeta'_{m,i}}\widetilde{T}(-R(y) + N(\phi)) + \widetilde{T}\big(-\partial_{\zeta'_{m,i}}R(y) + \partial_{\zeta'_{m,i}}N(\phi)\big). \tag{20}$$

Note that $|\partial_{\zeta'_{m,i}}N(\phi)| = O(|\phi|^{\beta}|\partial_{\zeta'_{m,i}}w| + |\phi|^{\beta}|\partial_{\zeta'_{m,i}}\phi|)$. So we obtain

$$\|\partial_{\zeta'_{m,i}}N(\phi)\|_{L^2} \le C\|\phi\|_{H^2}^{\beta} + C\|\phi\|_{H^2}^{\beta}\|\partial_{\zeta'_{m,i}}\phi\|_{H^2}.$$

Thus, (20) and Proposition 3.3 lead to

$$\|\partial_{\zeta'_{m,i}}\phi\|_{H^2} \le C\|R(y)\|_{L^2} + C\|N(\phi)\|_{L^2} + C\|\partial_{\zeta'_{m,i}}R(y)\|_{L^2} + C\|\partial_{\zeta'_{m,i}}N(\phi)\|_{L^2} \le C\varepsilon^{2\beta}.$$

The estimate for $\|\partial_{\sigma_m}\phi\|_{H^2}$ may be obtained by the same process. □

4 Variational reduction

According to the above discussion, the remaining thing is to let $c_{m,i} = 0$ in the nonlinear problem (19) in order to make $\mathcal{W} + \phi$ be a solution of the original problem. It can be done by the variational reduction process.

Note the energy functional of problem (5) is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\mathrm{i}\nabla u + \boldsymbol{A}(\varepsilon y)u|^2\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}.$$

Define

$$F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = E(\mathcal{W} + \phi)(\boldsymbol{\sigma}, \boldsymbol{\zeta}'),$$

then the existence of critical points to $E(u)$ may be reduced to find critical points of the finite dimensional function $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$.

Proposition 4.1. *If $(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ is a critical point of $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$, then $c_{m,i} = 0$ for all m, i.*

Proof. It is easy to see that

$$\begin{aligned}\partial_{\zeta'_{m,i}}F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') &= \partial_{\zeta'_{m,i}}E(\mathcal{W} + \phi) = E'(\mathcal{W} + \phi)\bigg[\frac{\partial\mathcal{W}}{\partial\zeta'_{m,i}} + \frac{\partial\phi}{\partial\zeta'_{m,i}}\bigg]\\ &= \sum_{j=0}^{N}\sum_{\ell=1}^{K}\operatorname{Re}\int_{\mathbb{R}^N} c_{\ell,j}\chi_\ell\overline{Z}_{\ell,j}\bigg[\frac{\partial U_m}{\partial\zeta'_{m,i}} + \varepsilon\frac{\partial\Psi_m}{\partial\zeta'_{m,i}} + \frac{\partial\phi}{\partial\zeta'_{m,i}}\bigg] = c_{m,i}\int_{\mathbb{R}^N}\chi_m|Z_{m,i}|^2 + o(1).\end{aligned}$$

Similarly, it is also true that

$$\partial_{\sigma_m}F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = c_{m,0}\int_{\mathbb{R}^N}\chi_m|Z_{m,0}|^2 + o(1).$$

Thus, $c_{m,i} = 0$ if $(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ is a critical point of F since the coefficient matrix of $c_{m,i}$ is diagonal dominant. □

Next we should calculate $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ in view of Proposition 4.1.

Proposition 4.2. *It holds that*

$$F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = E(\mathcal{W}) + O(\varepsilon^4),$$

and for any $m = 1, \dots, K$, $i = 1, \dots, N$,

$$\partial_{\zeta'_{m,i}} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = \partial_{\zeta'_{m,i}} E(\mathcal{W}) + O(\varepsilon^{2+2\beta}), \quad \partial_{\sigma_m} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = \partial_{\sigma_m} E(\mathcal{W}) + O(\varepsilon^{2+2\beta}).$$

Proof. Direct computation leads to

$$\begin{aligned} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') &= \frac{1}{2} \int_{\mathbb{R}^N} |\mathrm{i}\nabla(\mathcal{W} + \phi) + \boldsymbol{A}(\varepsilon y)(\mathcal{W} + \phi)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\mathcal{W} + \phi|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |\mathcal{W} + \phi|^{p+1} \\ &= E(\mathcal{W}) + \operatorname{Re} \int_{\mathbb{R}^N} (\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))\mathcal{W}\overline{(\mathrm{i}\nabla + \boldsymbol{A}(\varepsilon y))\phi} + \operatorname{Re} \int_{\mathbb{R}^N} \mathcal{W}\bar{\phi} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} |\mathrm{i}\nabla\phi + \boldsymbol{A}(\varepsilon y)\phi|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\phi|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |\mathcal{W} + \phi|^{p+1} + \frac{1}{p+1} \int_{\mathbb{R}^N} |\mathcal{W}|^{p+1}. \end{aligned}$$

Using equation (10) of ϕ and definitions of $R(y)$, $N(\phi)$ below it, we obtain from integration by parts that

$$\begin{aligned} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = E(\mathcal{W}) &+ \frac{1}{2} \int_{\mathbb{R}^N} \operatorname{Re}((R(y) + N(\phi))\bar{\phi}) - \int_{\mathbb{R}^N} \bigg[\frac{1}{p+1} |\mathcal{W} + \phi|^{p+1} - \frac{1}{p+1} |\mathcal{W}|^{p+1} - |\mathcal{W}|^{p-1} \operatorname{Re}(\mathcal{W}\bar{\phi}) \\ &- \frac{p-1}{2} |\mathcal{W}|^{p-3} (\operatorname{Re}(\mathcal{W}\bar{\phi}))^2 - \frac{1}{2} |\mathcal{W}|^{p-1} |\phi|^2 \bigg]. \end{aligned}$$

Then the proposition follows from Propositions 2.1 and 3.4 easily. By the computation in Proposition 3.4, it is easy to check that

$$\partial_{\zeta'_{m,i}} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = \partial_{\zeta'_{m,i}} E(\mathcal{W}) + O(\|\phi\|^{1+\beta}).$$

So is $\partial_{\sigma_m} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$. □

Since $E(\mathcal{W})$ is the main part of $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$, it is important to obtain the expression of $E(\mathcal{W})$. Elegant computation shows the following proposition.

Proposition 4.3. *It holds that for* ε *small enough,*

$$E(\mathcal{W}) = A_0 K + B_0 \varepsilon^2 \sum_{m=1}^{K} \sum_{i,j=1}^{N} (\partial_i A_j(\zeta_m) - \partial_j A_i(\zeta_m))^2 + O(\varepsilon^4).$$

Furthermore, the remainder term $O(\varepsilon^4)$ *also holds for the derivatives in* $\boldsymbol{\zeta}'$, $\boldsymbol{\sigma}$. *Here* $A_0 = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} w^{p+1}(|y|) \mathrm{d}y$ *and* $B_0 = \frac{1}{8} \int_{\mathbb{R}^N} y_1^2 w^2(|y|) \mathrm{d}y$ *are both universal positive constants.*

Proof. It is easy to see that

$$
\begin{aligned}
E(\mathcal{W}) &= \frac{1}{2}\int_{\mathbb{R}^N} |\mathrm{i}\nabla\mathcal{W} + \boldsymbol{A}(\varepsilon y)\mathcal{W}|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\mathcal{W}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |\mathcal{W}|^{p+1} \\
&= \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^N} R(y)\overline{\mathcal{W}} + \frac{p-1}{2(p+1)}\int_{\mathbb{R}^N} |\mathcal{W}|^{p+1} \\
&= \sum_{m=1}^{K}\mathrm{Re}\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}\left[\frac{1}{2}R(y)\overline{\mathcal{W}} + \frac{p-1}{2(p+1)}|\mathcal{W}|^{p+1}\right] + \mathrm{Re}\int_{\mathbb{R}^N\setminus\cup_{m=1}^{K}B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}\left[\frac{1}{2}R(y)\overline{\mathcal{W}}\right. \\
&\quad\left. + \frac{p-1}{2(p+1)}|\mathcal{W}|^{p+1}\right] \\
&= \sum_{m=1}^{K}\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}\left[\frac{R_{m,1}}{2}w(|y-\zeta_m'|) + \varepsilon\frac{R_{m,2}}{2}\psi_m(y) + \frac{p-1}{2(p+1)}|U_m|^{p+1} + \frac{(p-1)\varepsilon^2}{4}|U_m|^{p-1}|\Psi_m|^2\right] \\
&\quad + O(\varepsilon^4),
\end{aligned}
\tag{21}
$$

where ψ_m, Ψ_m are given in (9), and $R_{m,1}$, $R_{m,2}$ are defined in Proposition 2.1. First, by the oddness of the terms in order ε^3 in (13), it can be obtained that

$$
\begin{aligned}
\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} R_{m,1}w(|y-\zeta_m'|) &= \varepsilon^2\sum_{i,j,k=1}^{N}\partial_jA_i(\zeta_m)\partial_kA_i(\zeta_m)\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}(y_j-\zeta_{m,j}')(y_k-\zeta_{m,k}')w^2(|y-\zeta_m'|) \\
&\quad - \varepsilon^2\sum_{j,k,\ell=1}^{N}\partial_\ell A_k(\zeta_m)[\partial_jA_k(\zeta_m)+\partial_kA_j(\zeta_m)]\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}(y_j-\zeta_{m,j}')(y_\ell-\zeta_{m,\ell}')w^2(|y-\zeta_m'|) \\
&\quad - \varepsilon^2\sum_{i,j,k,\ell}^{N}\partial_\ell A_k(\zeta_m)\partial_jA_i(\zeta_m)\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}(y_i-\zeta_{m,i}')(y_j-\zeta_{m,j}')(y_k-\zeta_{m,k}')(y_\ell-\zeta_{m,\ell}') \\
&\quad \cdot\frac{w'(|y-\zeta_m'|)}{|y-\zeta_m'|}w(|y-\zeta_m'|) - \frac{\varepsilon^2}{2}(\nabla_x\cdot\boldsymbol{A}(\zeta_m))\sum_{i,j=1}^{N}\partial_jA_i(\zeta_m)\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}(y_i-\zeta_{m,i}')(y_j-\zeta_{m,j}')w^2(|y-\zeta_m'|) \\
&\quad - \frac{(p-1)\varepsilon^2}{8}\sum_{i,j,k,\ell}^{N}\partial_jA_i(\zeta_m)\partial_\ell A_k(\zeta_m)\int_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')}(y_i-\zeta_{m,i}')(y_j-\zeta_{m,j}')(y_k-\zeta_{m,k}')(y_\ell-\zeta_{m,\ell}')w^{p+1} \\
&\quad (|y-\zeta_m'|) + O(\varepsilon^4).
\end{aligned}
$$

Since integration by parts gives

$$
\begin{aligned}
\int_{\mathbb{R}^N} y_iy_jy_ky_\ell w(|y|)\frac{w'(|y|)}{|y|}\mathrm{d}y &= \frac{1}{2}\int_{\mathbb{R}^N} y_iy_jy_k(\partial_\ell w^2(|y|)) \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\delta_{i\ell}\delta_{jk}y_jy_kw^2(|y|) - \frac{1}{2}\int_{\mathbb{R}^N}\delta_{j\ell}\delta_{ik}y_iy_kw^2(|y|) - \frac{1}{2}\int_{\mathbb{R}^N}\delta_{k\ell}\delta_{ij}y_iy_jw^2(|y|),
\end{aligned}
$$

one obtains that

$$
\begin{aligned}
&\sum_{i,j,k,\ell=1}^{N}\partial_\ell A_k(\zeta_m)\partial_jA_i(\zeta_m)\int_{\mathbb{R}^N}(y_i-\zeta_{m,i}')(y_j-\zeta_{m,j}')(y_k-\zeta_{m,k}')(y_\ell-\zeta_{m,\ell}')\frac{w'(|y-\zeta_m'|)}{|y-\zeta_m'|}w(|y-\zeta_m'|) \\
&= -4B_0\sum_{i,j=1}^{N}[\partial_iA_i(\zeta_m)\partial_jA_j(\zeta_m) + (\partial_jA_i(\zeta_m))^2 + \partial_iA_j(\zeta_m)\partial_jA_i(\zeta_m)].
\end{aligned}
$$

It may be checked that

$$\begin{aligned}
&\int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} \frac{R_{m,1}}{2} w(|y-\zeta_m'|) \\
&= 2B_0\varepsilon^2 \Bigg\{ 2\sum_{i,j=1}^{N} (\partial_j A_i(\zeta_m))^2 - 2\sum_{i,j=1}^{N} \partial_j A_i(\zeta_m)(\partial_j A_i(\zeta_m) + \partial_i A_j(\zeta_m)) + \sum_{i,j=1}^{N} [\partial_i A_i(\zeta_m)\partial_j A_j(\zeta_m) \\
&\quad + (\partial_j A_i(\zeta_m))^2 + \partial_i A_j(\zeta_m)\partial_j A_i(\zeta_m)] - \left(\sum_{i=1}^{N} \partial_i A_i(\zeta_m)\right)^2 \Bigg\} - \frac{(p-1)\varepsilon^2}{16} \sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_m)\partial_\ell A_k(\zeta_m) \\
&\quad \times \int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} (y_i - \zeta_{m,i}')(y_j - \zeta_{m,j}')(y_k - \zeta_{m,k}')(y_\ell - \zeta_{m,\ell}') w^{p+1}(|y-\zeta_m'|) + O(\varepsilon^4).
\end{aligned} \tag{22}$$

Also,

$$\begin{aligned}
\int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} |U_m|^{p-1}|\Psi_m|^2 &= \frac{1}{4}\sum_{i,j,k,\ell=1}^{N} \partial_j A_i(\zeta_m)\partial_\ell A_k(\zeta_m) \int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} (y_i - \zeta_{m,i}')(y_j - \zeta_{m,j}')(y_k - \zeta_{m,k}')(y_\ell - \zeta_{m,\ell}') w^{p+1}(|y-\zeta_m'|) \\
&\quad + O\left(e^{-\frac{\delta}{\varepsilon}}\right)
\end{aligned}$$

and

$$\int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} R_{m,2}\psi_m = O(\varepsilon^3)$$

by the oddness. Obviously, one has

$$\int\limits_{B_{\frac{\delta}{\sqrt{\varepsilon}}}(\zeta_m')} |U_m|^{p+1} = \int\limits_{\mathbb{R}^N} w^{p+1}(|y|)\mathrm{d}y + O\left(e^{-\frac{\delta}{\varepsilon}}\right).$$

Note that in (22) the term containing w^{p+1} is canceled with $\frac{p-1}{4}\varepsilon^2 \int\limits_{\mathbb{R}^N} |U_m|^{p-1}|\Psi_m|^2$. So we conclude, from (21), that

$$\begin{aligned}
E(\mathcal{W}) &= A_0 K + 2\varepsilon^2 B_0 \sum_{m=1}^{K}\sum_{i,j=1}^{N} [(\partial_j A_i(\zeta_m))^2 - \partial_j A_i(\zeta_m)\partial_i A_j(\zeta_m)] + O(\varepsilon^4) \\
&= A_0 K + \varepsilon^2 B_0 \sum_{m=1}^{K}\sum_{i,j=1}^{N} (\partial_j A_i(\zeta_m) - \partial_i A_j(\zeta_m))^2 + O(\varepsilon^4).
\end{aligned}$$

The last equality is due to the symmetry of indexes i and j. The remainder $O(\varepsilon^4)$ also holds for the derivatives of $E(\mathcal{W})$ in $(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ from directly checking the expressions of $R_{m,1}$ and $R_{m,2}$ in the proof of Proposition 2.1. □

5 Proof of the main theorems

This section devotes to the proof of main theorems.

Proof of Theorem 1.1. Propositions 4.2 and 4.3 mean

$$F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = A_0 K + B_0\varepsilon^2 \sum_{m=1}^{K} \|\boldsymbol{B}(\zeta_m)\|_F^2 + O(\varepsilon^4).$$

We shall show that F has a critical point under the assumption. Note that for any fixed $\boldsymbol{\zeta}'$, $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ is periodic in $\boldsymbol{\sigma} \in [0, 2\pi]^K$. So there always exists a $\boldsymbol{\sigma}(\boldsymbol{\zeta}')$ such that $\partial_{\boldsymbol{\sigma}} F(\boldsymbol{\sigma}(\boldsymbol{\zeta}'), \boldsymbol{\zeta}') = 0$. Next consider the configuration set $\boldsymbol{\Omega}' = \Omega_1' \times \Omega_2' \times \cdots \times \Omega_m'$ of $\boldsymbol{\zeta}' = (\zeta_1', \zeta_2', \ldots, \zeta_m')$, where $\Omega_m' = \varepsilon^{-1}\Omega_m$. Obviously,

$$\max_{\Omega'} F(\boldsymbol{\sigma}(\boldsymbol{\zeta}'), \boldsymbol{\zeta}') \geq A_0 K + B_0 \varepsilon^2 \sum_{m=1}^{K} \|\boldsymbol{B}(P_m)\|_F^2 + O(\varepsilon^4).$$

On the other hand, for any $\boldsymbol{\zeta}'$ on the boundary $\partial\Omega'$, i.e., at least $\zeta_1' \in \partial\Omega_1'$ without loss of generality, then $\|B(\zeta_1)\|_F^2 \leq \|B(P_1)\|_F^2 - \delta_0$ for some fixed small $\delta_0 > 0$. Thus, one finds that

$$F(\boldsymbol{\sigma}(\boldsymbol{\zeta}'), \boldsymbol{\zeta}')|_{\boldsymbol{\zeta}' \in \partial\boldsymbol{\Omega}'} \leq A_0 K + B_0 \varepsilon^2 \sum_{m=2}^{K} \|\boldsymbol{B}(P_m)\|_F^2 + B_0 \varepsilon^2 (\|\boldsymbol{B}(P_1)\|_F^2 - \delta_0) + O(\varepsilon^4).$$

Therefore, $\max_{\Omega'} F(\boldsymbol{\sigma}(\boldsymbol{\zeta}'), \boldsymbol{\zeta}') > \max_{\partial\Omega'} F(\boldsymbol{\sigma}(\boldsymbol{\zeta}'), \boldsymbol{\zeta}')$. It implies that $F(\boldsymbol{\sigma}, \boldsymbol{\zeta}')$ admits a critical point.

The same procedure can be carried out for the case of K local minimum points. Theorem 1.1 concludes from Proposition 4.1. □

Proof of Theorem 1.2. From Propositions 4.2 and 4.3, we see that

$$\nabla_{\zeta_m'} F(\boldsymbol{\sigma}, \boldsymbol{\zeta}') = B_0 \varepsilon^2 \nabla_{\zeta_m'} (\|\boldsymbol{B}(\zeta_m)\|_F^2) + O(\varepsilon^{2+2\beta}) = B_0 \varepsilon^3 \nabla_{\zeta_m} (\|\boldsymbol{B}(\zeta_m)\|_F^2) + O(\varepsilon^{2+2\beta}).$$

Assume $m = 1$ for simplicity. Choose $\zeta_1 = P_1 + \varepsilon^\alpha \xi_1$ where $0 < \alpha < 2\beta - 1$. Here the assumption $p > \frac{3}{2}$ is used to let $\beta = \min\{p - 1, 1\} > 1/2$. Then it is equivalent to find a $|\xi_1| \leq 1$ such that

$$0 = \nabla_{\zeta_1} (\|\boldsymbol{B}(\zeta_1)\|_F^2) + O(\varepsilon^{2\beta-1}) = \nabla_{\zeta_1} (\|\boldsymbol{B}(P_1)\|_F^2) + \varepsilon^\alpha \nabla^2_{\zeta_1\zeta_1} (\|\boldsymbol{B}(P_1)\|_F^2) \cdot \xi_1 + O(\varepsilon^{2\alpha}|\xi_1|^2) + O(\varepsilon^{2\beta-1}).$$

Thus, the nondegeneracy of the critical point P_1, together with the Brouwer fixed point theorem, leads to the existence of ξ_1 and also $|\xi_1| = o(1)$. Finally, the existence of critical point $\boldsymbol{\sigma}$ is guaranteed by the periodicity just like in the proof of Theorem 1.1. The proof is complete. □

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References

[1] G. Arioli and A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Arch. Ration. Mech. Anal. **170** (2003), no. 4, 277–295.
[2] D. Bonheure, S. Cingolani, and M. Nys, *Nonlinear Schrödinger equation: concentration on circles driven by an external magnetic field*, Calc. Var. Partial Differ. Equ. **55** (2016), no. 4, Art. 82, 33 pp.
[3] D. Bonheure, M. Nys, and J. Van Schaftingen, *Properties of ground states of nonlinear Schrödinger equations under a weak constant magnetic field*, J. Math. Pures Appl. **124** (2019), no. 9, 123–168.
[4] D. M. Cao and Z. W. Tang, *Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields*, J. Differ. Equ. **222** (2006), no. 2, 381–424.
[5] S. Cingolani and M. Clapp, *Intertwining semiclassical bound states to a nonlinear magnetic Schrödinger equation*, Nonlinearity **22** (2009), 2309–2331.
[6] S. Cingolani, L. Jeanjean, S. Secchi, *Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions*, ESAIM Control Optim. Calc. Var. **15** (2009), 653–675.
[7] C. Cingolani and S. Secchi, *Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields*, J. Math. Anal. Appl. **275** (2002), 108–130.
[8] C. Cingolani and S. Secchi, *Semiclassical states for NLS equations with magnetic potentials having polynomial growths*, J. Math. Phys.. **46** (2005), no. 5, 053503, 19 pp.

[9] M. delPino, M. Kowalczyk, and M. Musso, *Singular limits in Liouville-type equations*, Calc. Var. Partial Differ. Equ. 24 (2005), no. 1, 47–81.

[10] J. DiCosmo and J. Van Schaftingen, *Semiclassical stationary states for nonlinear Schrödinger equations under a strong external magnetic field*, J. Differ. Equ. **259** (2015), no. 2, 596–627.

[11] M. Esteban and P. L. Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, Partial differential equations and the calculus of variations*, In: Progress in Nonlinear Differential Equations and Their Application, 1, Vol. I, Birkhäuser Boston, Boston, MA, 1989, pp. 401–449.

[12] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), no. 3, 397–408.

[13] B. Helffer, *Semiclassical analysis for Schrödinger operators with magnetic wells, Quasiclassical methods* (*Minneapolis, MN, 1995*), The IMA Volumes in Mathematics and its Applications, vol. 95, Springer, New York, 1997, pp. 99–114.

[14] B. Helffer, *On spectral theory for Schrödinger operators with magnetic potentials. Spectral and scattering theory and applications*, Advanced Studies in Pure Mathematics, vol. 23, The Mathematical Society of Japan, Tokyo, 1994, pp. 113–141.

[15] K. Kurata, *Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields*, Nonlinear Anal. **41** (2000), no. 5–6, Ser. A: Theory Methods, 763–778.

[16] L. D. Landau and E. M. Lifshitz, *Quantum mechanics: non-relativistic theory*, Course of Theoretical Physics, Addison-Wesley Series in Advanced Physics, vol. 3, Pergamon Press, Ltd., London-Paris, for U.S.A. and Canada: Addison-Wesley Publishing Company, Inc., Reading, Mass, 1958, xii+515 pp.81.0X.

[17] D. L. Mills, *Nonlinear Optics: Basic Concepts*, Springer Science & Business Media, Berlin, Heidelberg, New York, 2012.

[18] S. Secchi and M. Squassina, *On the location of spikes for the Schrödinger equation with electromagnetic field*, Commun. Contemp. Math. **7** (2005), no. 2, 251–268.

[19] C. Sulem and P.-L. Sulem, Self-focusing and wave collapse, *Applied Mathematical Sciences*, vol. 139, Springer-Verlag, New York, 1999, xvi+350 pp.

[20] Z. W. Tang, *Multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields and critical frequency*, J. Differ. Equ. **245** (2008), no. 10, 2723–2748.

[21] L. P. Wang and C. Y. Zhao, *Concentration on curves for a nonlinear Schrödinger problem with electromagnetic potential*, J. Differ. Equ. **266** (2019), no. 8, 4800–4834.