Research Article

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Regularity of degenerate *k*-Hessian equations on closed Hermitian manifolds

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Abstract: In this article, we are concerned with the existence of weak $C^{1,1}$ solution of the k-Hessian equation on a closed Hermitian manifold under the optimal assumption of the function in the right-hand side of the equation. The key points are to show the weak $C^{1,1}$ estimates. We prove a Cherrier-type inequality to obtain the C^0 estimate, and the complex Hessian estimate is proved by using an auxiliary function, which was motivated by Hou et al. and Tosatti and Weinkove. Our result generalizes the Kähler case proved by Dinew et al.

Keywords: degenerate k-Hessian equations, optimal regularity, Hermitian manifold

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1 Introduction

Let (M, g) be a closed Hermitian manifold of complex dimension $n \geq 2$, and ω be the corresponding Hermitian form. We are concerned with the following degenerate k-Hessian equation on (M, ω):

$$\begin{cases} C_n^k \chi_u^k \wedge \omega^{n-k} = f\omega^n, \quad \sup\limits_M u = 0 \\ \chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u, \end{cases} \tag{1.1}$$

where χ is a smooth real (1, 1) form in $\Gamma_k(M)$ cone and $C_n^k = \frac{n!}{k!(n-k)!}$.

We are interested in the existence problem of equation (1.1) when f is nonnegative, not identical to zero, and $f^{\frac{1}{k-1}} \in C^{1,1}$. We want to seek the weak $C^{1,1}$ solution, which means that the solution of (1.1) satisfies

$$\sup_M |u| + \sup_M |\nabla u| + \sup_M |\triangle u| \leq C, \tag{1.2}$$

where C is a uniform constant depending only on $(M, \omega), \chi, n, k$ and $|f^{\frac{1}{k-1}}|_{C^{1,1}}$. By Sobolev embedding theorem, weak $C^{1,1}$ implies $C^{1,\alpha}$.

The Hessian equations are important fully nonlinear elliptic equations. They arise naturally from many interesting geometric problems in complex geometry, as illustrated by [8–10,17,18,21,22,24,28] and references therein. For the nondegenerate case (i.e., $f > 0$), there have been plenty of important results. In particular, when $k = n$, equation (1.1) is the famous complex Monge-Ampère equation, which was solved by Tung Yau [28] on a compact Kähler manifold, and it is now known as the Calabi-Yau theorem. On general Hermitian manifolds, the complex Monge-Ampère equation was solved by Cherrier [3] in the case of two dimensions and Tosatti and Weinkove [23] for arbitrary dimensions. The gradient and second-order estimate were proved by Guan and Li [10] and Zhang [30].

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When $2 \le k \le n$, on a compact Kähler manifold, Hou et al. [12] proved the following important second-order estimate:

$$\max|\partial\bar{\partial}u|_g \le C(1 + \max|\nabla u|_g^2). \tag{1.3}$$

Based on the above complex Hessian estimate (1.3), Dinew and Kolodziej [6] proved the gradient estimate and got the existence result on a compact Kähler manifold when $\chi = \omega$. Later, Sun [19] generalized the result for general $\chi \in \Gamma_k$. The Hermitian case was proved by Székelyhidi [21] for general $\chi \in \Gamma_k(M)$ and by Zhang [30] for $\chi = \omega$ independently. Based on the smooth results in [21] and [30], Kolodziej and Cuong Nguyen [14] proved the existence of the weak solution of the k-Hessian equation on compact Hermitian manifolds.

For the degenerate case, there are lots of results on degenerate fully nonlinear equations. We first recall some results in the real Euclidean case. Let f be the right-hand side function of the equation satisfying $f^{\frac{1}{n-1}} \in C^{1,1}$, Guan et al. [11] proved the $C^{1,1}$ solution of the real Monger-Ampère equation with Dirichlet boundary value condition. The power $\frac{1}{n-1}$ is optimal by a counterexample given by Jia Wang [26]. However, for the real k-Hessian equation, by assuming $f^{\frac{1}{k}} \in C^{1,1}$, Krylov [15,16] and Ivochina et al. [13] proved the $C^{1,1}$ existence result. Wang and Xu [25] got the same result with a slightly weaker condition on f. Xu [27] proved the optimal $C^{1,1}$ estimate for solutions to the Christoffel-Minkowski problem.

When $k = n$ and (M, ω) is a compact Kähler manifold, Blocki [1] proved the weak $C^{1,1}$ estimate. For the k-Hessian equation on a compact Kähler manifold and $\chi = \omega$, weak $C^{1,1}$ estimate has been recently proved by Dinew et al. [7]. Moreover, they gave counterexamples to show that the power $\frac{1}{k-1}$ is optimal in both real and complex settings. Chu and McCleerey [4] proved the $C^{1,1}$ regularity result by assuming $f^{\frac{1}{k}} \in C^2$.

In this article, we generalize Dinew et al.'s result [7] to the compact Hermitian case with $\chi \in \Gamma_k(M)$. We state the main result as follows.

Theorem 1.1. *Let (M, g) be a closed Hermitian manifold of complex dimension $n \ge 2$ and $\chi \in \Gamma_k(M)$. Let f be a nonnegative function on M, not identical to zero, and $f^{\frac{1}{k-1}} \in C^{1,1}$. Then there exists a positive constant c and weak $C^{1,1}(\chi, k)$-subharmonic function u solving*

$$C_n^k \chi_u^k \wedge \omega^{n-k} = cf\omega^n, \quad \sup_M u = 0, \tag{1.4}$$

where we call u is (χ, k)-subharmonic, if $\chi + \sqrt{-1}\partial\bar{\partial}u$ is k-nonnegative in the sense of currents (*see* Definition 2.6 *in* [14]).

By approximation, the above theorem follows from the following *a priori* estimates.

Theorem 1.2. *Let (M, g) be a closed Hermitian manifold of complex dimension $n \ge 2$ and $\chi \in \Gamma_k(M)$. Let f be a positive smooth function on M. Let u be the solution of the k-Hessian equation*:

$$\begin{cases} C_n^k \quad \chi_u^k \wedge \omega^{n-k} = f\omega^n, \quad \sup_M u = 0, \\ \chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u \in \Gamma_k(M). \end{cases} \tag{1.5}$$

Then, there exists a uniform constant C depending only on $(M, \omega), \chi, n, k$, and $|f^{\frac{1}{k-1}}|_{C^{1,1}}$ such that

$$\sup_M|u| + \sup_M|\nabla u| + \sup_M|\partial\bar{\partial}u| \le C. \tag{1.6}$$

Remark 1.3. According to the counterexample by Dinew et al. [7], the assumption $f^{\frac{1}{k-1}} \in C^{1,1}$ is optimal.

According to the Liouville theorem by Dinew and Kolodziej [6], it is sufficient to prove the C^0-estimate and the second-order estimate of Hou-Ma-Wu type (1.3). We prove the C^0-estimate by establishing the

Cherrier-type inequality. The Hou-Ma-Wu-type estimate (1.3) can be established by the auxiliary function which is motivated by Hou et al. [12] and Tosatti and Weinkove [24]. We also need a lemma by Dinew et al. [7] which follows from an elementary lemma by Blocki [1].

The rest of the article is organized as follows. In Section 2, we prove the Cherrier-type inequality and thus obtain the C^0 estimate. In Section 3, we prove the Hou-Ma-Wu-type estimate. In Section 4, we show the existence of the weak $C^{1,1}$ solution of the k-Hessian equation.

2 C^0-estimate

Lemma 2.1. *Let u be a solution of* Theorem 1.2. *Then, there exists a constant C depending only on* (M, ω), n, k, *and* $\sup_M f$ *such that*

$$\sup_M |u| \leq C.$$

Remark 2.2. Székelyhidi [21] proved the C^0 estimate by the ABP estimate method. We proved the $\chi = \omega$ case in [29] by establishing a Cherrier-type inequality. Based on our key lemma in [29], Sun [20] also proved the C^0 estimate.

Motivated by [29], we prove the C^0 estimate by establishing a Cherrier-type inequality. The Cherrier inequality was first proved by Cherrier [3] for the complex Monge-Ampère equation. Compared to [29], the difficulty here is that χ is only k-positive.

We first recall the definition and some inequalities about the k-Hessian operator. The kth elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1\leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

Let $A^{1,1}(M, \mathbb{R})$ be the space of smooth real (1, 1) forms on M. For $\chi \in A^{1,1}(M, \mathbb{R})$, we define

$$\sigma_k(\chi) = C_n^k \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

Definition 2.3. Let

$$\Gamma_k \coloneqq \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \quad j = 1, \ldots, k\}. \tag{2.1}$$

Similarly, we define Γ_k on M as follows:

$$\Gamma_k(M) \coloneqq \{\chi \in A^{1,1}(M, \mathbb{R}) : \sigma_j(\chi) > 0, \quad j = 1, \ldots, k\}. \tag{2.2}$$

Furthermore, $\sigma_r(\lambda|i_1 \ldots i_l)$ stands for the rth symmetric function with $\lambda_{i_1} = \cdots = \lambda_{i_l} = 0$.

The following lemma which is similar to Lemma 2.4 in [29] plays an important role during the proof of the Cherrier-type inequality.

Lemma 2.4. *There exists a positive constant C depending only on* (M, ω), k, *and* n *such that*

$$\sum_{i=0}^{k-2} \left| \frac{\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_u^i \wedge T_{n-i-1}}{\omega^n} \right| \leq C \sum_{i=0}^{k-2} \frac{\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_u^i \wedge \omega^{n-i-1}}{\omega^n}, \tag{2.3}$$

where T_{n-i-1} *are combinations of* $\omega, \partial\omega, \partial\bar{\partial}\omega, \chi, \partial\chi, \partial\bar{\partial}\chi$ *and are real* $(n-i-1, n-i-1)$ *forms.*

For the C^0 estimate, it suffices to show the following inequality. Here, our proof is simpler than that in [29].

Lemma 2.5. *There exist constants p_0 and C depending only on $(M,\omega), n, k$, and $\sup_M f$ such that, for any $p \geq p_0$:*

$$\int_M |\partial e^{-\frac{p}{2}u}|_g^2 \omega^n \leq Cp \int_M e^{-pu}\omega^n.$$

Proof. On the one hand, by equation (1.5), we have

$$\chi_u^k \wedge \omega^{n-k} - \chi^k \wedge \omega^{n-k} = \frac{f}{C_n^k}\omega^n - \chi^k \wedge \omega^{n-k} \leq C_0\omega^n,$$

where $C_0 = \sup_M f$. On the other hand,

$$\chi_u^k \wedge \omega^{n-k} - \chi^k \wedge \omega^{n-k} = (\chi_u^k - \chi^k) \wedge \omega^{n-k} = \sqrt{-1}\partial\bar{\partial}u \wedge \alpha, \tag{2.4}$$

where $\alpha = \sum_{i=1}^k \chi_u^{k-i} \wedge \chi^{i-1} \wedge \omega^{n-k}$.

Multiplying e^{-pu} in (2.4) and integrating by parts, we have

$$\begin{aligned} C_0 \int_M e^{-pu}\omega^n &\geq \int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}u \wedge \alpha \\ &= -\int_M \partial e^{-pu}\sqrt{-1}\bar{\partial}u \wedge \alpha + \int_M e^{-pu}\sqrt{-1}\bar{\partial}u \wedge \partial\alpha \\ &= p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \alpha - \frac{1}{p}\int_M \sqrt{-1}\bar{\partial}e^{-pu} \wedge \partial\alpha \\ &= p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \alpha + \frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar{\partial}\partial\alpha \\ &= A + B, \end{aligned} \tag{2.5}$$

where we write

$$\begin{aligned} A &= p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\chi_u^{i-1} \wedge \chi^{k-i}) \wedge \omega^{n-k}, \\ B &= \frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar{\partial}\partial\alpha. \end{aligned}$$

Since $\chi \in \Gamma_k$, there exists a sufficiently small constant $\varepsilon > 0$ such that $\chi - \varepsilon\omega \in \Gamma_k$. Then, we have

$$\begin{aligned} &\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\chi_u^{i-1} \wedge \chi^{k-i}) \wedge \omega^{n-k} - \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\varepsilon^{k-i}\chi_u^{i-1} \wedge \omega^{n-i}) \\ &= \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\chi_u^{i-1} \wedge (\chi^{k-i} - (\varepsilon\omega)^{k-i})) \wedge \omega^{n-k} \\ &= \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k \left(\chi_u^{i-1} \wedge (\chi - \varepsilon\omega) \wedge \sum_{j=0}^{k-i-1} (\varepsilon^j\omega^j \wedge \chi^{k-i-1-j})\right) \wedge \omega^{n-k} \\ &\geq 0, \end{aligned}$$

where the last inequality follows from Proposition 2.1 in [2]. Thus, we have

$$\begin{aligned} A &= p\int_M \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\chi_u^{i-1} \wedge \chi^{k-i}) \wedge \omega^{n-k} \\ &\geq p\varepsilon^k \int_M \sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \sum_{i=1}^k (\chi_u^{i-1} \wedge \omega^{n-i}). \end{aligned} \tag{2.6}$$

To prove the lemma, we want to use term A to control term B.

Direct manipulation gives the following results:

$$\begin{aligned}\bar\partial\partial\alpha &= \sum_{i=1}^{k}\chi_u^{i-1}\wedge\sqrt{-1}\bar\partial\partial\omega^{n-i} + \sum_{i=2}^{k}(i-1)\chi_u^{i-2}\wedge\sqrt{-1}\bar\partial\partial\chi\wedge\omega^{n-i}\\ &\quad + \sum_{i=2}^{k}(i-1)\chi_u^{i-2}\wedge(\bar\partial\chi\wedge\partial\omega^{n-i}-\partial\chi\wedge\bar\partial\omega^{n-i}) + \sum_{i=3}^{k}(i-1)(i-2)\chi_u^{i-2}\wedge\bar\partial\chi\wedge\partial\chi\wedge\omega^{n-i}\\ &=: \sum_{i=1}^{k}\chi_u^{i-1}\wedge T_{n-i+1},\end{aligned}$$

where every T_{n-i+1} is independent of u and p and thus has a uniform bound.

Now, we estimate every term in B. For any $1 \le i \le k$, we have

$$\begin{aligned}\int_M e^{-pu}\chi_u^{i-1}\wedge T_{n-i+1} &= -p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_u^{i-2}\wedge T_{n-i+1} + \int_M e^{-pu}\chi_u^{i-2}\wedge\chi\wedge T_{n-i+1}\\ &\quad + \frac{1}{p}\int_M e^{-pu}\chi_u^{i-2}\wedge\sqrt{-1}\bar\partial\partial T_{n-i+1} + \frac{(i-2)}{p}\int_M e^{-pu}\chi_u^{i-3}\wedge(\sqrt{-1}\partial\bar\partial\chi\wedge T_{n-i+1}\\ &\quad + \sqrt{-1}\partial\chi\wedge\bar\partial T_{n-i+1} - \sqrt{-1}\bar\partial\chi\wedge\partial T_{n-i+1})\\ &\quad + \frac{(i-2)(i-3)}{p}\int_M e^{-pu}\chi_u^{i-4}\wedge\sqrt{-1}\partial\chi\wedge\bar\partial\chi\wedge T_{n-i+1}.\end{aligned} \tag{2.7}$$

From the above equality, we claim that for any $1 \le i \le k$, there exists a uniform constant C independent of p such that

$$\left|\int_M e^{-pu}\chi_u^{i-1}\wedge T_{n-i+1}\right| \le Cp\sum_{j=1}^{i-1}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_u^{j-1}\wedge\omega^{n-j} + C\int_M e^{-pu}\omega^n. \tag{2.8}$$

When $i = 1$, the claim obviously holds. Assuming that the claim holds for any $i \le m$, we need to show it also holds for $i = m + 1$, where $m + 1 \le k$. Indeed, by combining equality (2.7) with the induction assumption, we obtain

$$\begin{aligned}\left|\int_M e^{-pu}\chi_u^{m}\wedge T_{n-m}\right| &\le -p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_u^{m-1}\wedge T_{n-m} + Cp\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\sum_{j=1}^{m-1}\chi_u^{j-1}\wedge\omega^{n-j}\\ &\quad + C\int_M e^{-pu}\omega^n\\ &\le Cp\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\sum_{j=1}^{m}\chi_u^{j-1}\wedge\omega^{n-j} + C\int_M e^{-pu}\omega^n,\end{aligned}$$

where we have used Lemma 2.4 in the last inequality.

By (2.8), we have

$$B \ge -C\sum_{i=1}^{k-1}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_u^{i-1}\wedge\omega^{n-i} - C\int_M e^{-pu}\omega^n.$$

Combining the above inequality with (2.5) and (2.6), we obtain

$$(p\varepsilon^k - C)\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\sum_{i=1}^{k}(\chi_u^{i-1}\wedge\omega^{n-i}) \le C\int_M e^{-pu}\omega^n.$$

Now, we take $p_0 = 2\varepsilon^{-k}C + 1$, then, for any $p \ge p_0$,

$$p^2 \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \le 2p\varepsilon^{-k} C \int_M e^{-pu}\omega^n. \tag{2.9}$$

Therefore, we obtain the conclusion

$$\int_M |\partial e^{-\frac{p}{2}u}|_g^2 \omega^n = \frac{np^2}{4}\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \le \frac{np\varepsilon^{-k}C}{2}\int_M e^{-pu}\omega^n. \tag{2.10}$$

□

3 Second-order estimate

In this section, we prove the second-order estimate motivated by the auxiliary function in [29], which was first used by [24] on compact Hermitian manifolds. To obtain the second-order estimate depending only on $\left|f^{\frac{1}{k-1}}\right|_{C^{1,1}}$, we need a lemma by Dinew et al. [7] which follows from an elementary lemma by Blocki [1].

Lemma 3.1. [7]. *Let (M, ω) be a compact Hermitian manifold of dimension n and f be a positive smooth function. Then, there exists a uniform constant M_0 depending only on (M, ω), n, and $\left|f^{\frac{1}{k-1}}\right|_{C^{1,1}}$ such that*

$$|\log f|_{C^{1,1}} \le M_0 f^{-\frac{1}{k-1}}. \tag{3.1}$$

Now, we are ready to prove the complex Hessian estimate.

Theorem 3.2. *Let u be a solution of* Theorem 1.2. *Then, there exists a uniform constant C depending only on (M, ω), n, k, and $\left|f^{\frac{1}{k-1}}\right|_{C^{1,1}}$ such that*

$$\sup_M |\partial\bar{\partial} u|_g \le C\left(1 + \sup_M |\nabla u|_g^2\right). \tag{3.2}$$

Proof. We use the covariant derivative with respect to the Chern connection and denote $w_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$ and let $\xi \in T^{1,0}M$, $|\xi|_g^2 = 1$.

We consider the following auxiliary function

$$H(x, \xi) = \log\left(w_{k\bar{l}}\xi^k\bar{\xi}^l\right) + c_0 \log\left(g^{k\bar{l}} w_{p\bar{l}} w_{k\bar{q}} \xi^p \bar{\xi}^q\right) + \varphi(|\nabla u|_g^2) + \psi(u),$$

where φ and ψ are given by

$$\varphi(s) = -\frac{1}{2}\log\left(1 - \frac{s}{2K}\right), \quad 0 \le s \le K - 1,$$
$$\psi(t) = -A\log\left(1 + \frac{t}{2L}\right), \quad -L + 1 \le t \le 0,$$

with

$$K = \sup_M |\nabla u|_g^2 + 1, \quad L = \sup_M |u| + 1, \quad A := 2L(C_0 + 1),$$

where C_0 is a constant to be determined later and c_0 is a small positive constant depending only on n, which was chosen similar to that in [29]. We have

$$\frac{1}{2K} \ge \varphi' \ge \frac{1}{4K} > 0, \quad \varphi'' = 2(\varphi')^2 > 0. \tag{3.3}$$

$$\frac{A}{L} \ge -\psi' \ge \frac{A}{2L} = C_0 + 1, \quad \psi'' \ge \frac{2\varepsilon_0}{1 - \varepsilon_0}(\psi')^2, \quad \text{for all } \varepsilon_0 \le \frac{1}{2A + 1}. \tag{3.4}$$

Suppose $H(x,\xi)$ attains its maximum at the point x_0 in the direction ξ_0, then we choose local coordinates $\left\{\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^n}\right\}$ near x_0 such that

$$g_{i\bar{j}}(x_0)=\delta_{ij},\quad u_{i\bar{j}}=u_{i\bar{i}}(x_0)\delta_{ij},$$
$$\lambda_i=w_{i\bar{i}}(x_0)=1+u_{i\bar{i}}(x_0)\quad \text{with } \lambda_1\geq\dots\geq\lambda_n.$$

Similar to [29], we can prove

$$H(x_0,\xi)\leq H\left(x_0,\frac{\partial}{\partial z^1}\right),\quad \forall\xi\in T^{1,0}M,\ |\xi|_g^2=1,\ \sum_{i,j}w_{i\bar{j}}(x_0)\xi^i\xi^{\bar{j}}>0,$$

by choosing c_0 small enough. We extend ξ_0 near x_0 as

$$\xi_0=\left(g_{1\bar{1}}\right)^{-\frac{1}{2}}\frac{\partial}{\partial z^1}.$$

Consider the function

$$Q(x)=H(x,\xi_0)=\log\left(g_{1\bar{1}}^{-1}w_{1\bar{1}}\right)+c_0\log\left(g_{1\bar{1}}^{-1}g^{k\bar{l}}w_{1\bar{l}}w_{k\bar{1}}\right)+\varphi(|\partial u|_g^2)+\psi(u).$$

Then, Q attains its maximum at x_0 in the direction $\frac{\partial}{\partial z_1}$. We will manipulate $F^{i\bar{j}}Q_{i\bar{j}}$ at x_0 to derive the estimate. By the manipulation in [29], at x_0, we have

$$Q_i=(1+2c_0)\frac{w_{1\bar{1}i}}{w_{1\bar{1}}}+\varphi_i+\psi_i=0 \tag{3.5}$$

and

$$Q_{i\bar{i}}=(1+2c_0)\frac{w_{1\bar{1}i\bar{i}}}{w_{1\bar{1}}}+\frac{c_0}{w_{1\bar{1}}^2}\sum_{p\neq1}(|w_{1\bar{p}i}|^2+|w_{1\bar{p}\bar{i}}|^2)-(1+2c_0)\frac{|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2}+(**)_{i\bar{i}}+\varphi_{i\bar{i}}+\psi_{i\bar{i}}, \tag{3.6}$$

where $(**)_{i\bar{i}}$ is given as follows:

$$\begin{aligned}(**)_{i\bar{i}}={}&\frac{2}{w_{1\bar{1}}}\sum_{j\neq1}\mathrm{Re}\left(w_{j\bar{1}i}\xi^k{}_{\bar{i}}+w_{1\bar{j}i}\bar{\xi}^j_i\right)+\xi^1{}_{i\bar{i}}+\bar{\xi}^1_{i\bar{i}}+\sum_{j=1}^n\frac{w_{j\bar{j}}}{w_{1\bar{1}}}\left(|\xi^j{}_{\bar{i}}|^2+|\xi^j{}_i|^2\right)\\&+\frac{2c_0}{w_{1\bar{1}}}\sum_{p\neq1}\mathrm{Re}\left(w_{p\bar{1}i}\xi^p{}_{\bar{i}}+w_{p\bar{1}\bar{i}}\xi^p{}_i\right)+\sum_{p\neq1}\frac{2c_0w_{p\bar{p}}}{w_{1\bar{1}}^2}\mathrm{Re}\left(w_{1\bar{p}i}\bar{\xi}^p{}_{\bar{i}}+w_{p\bar{1}i}\xi^p{}_{\bar{i}}\right)\\&+\frac{2c_0w_{p\bar{p}}^2}{w_{1\bar{1}}^2}(|\xi^p{}_i|^2+|\xi^p{}_i|^2)+c_0\left(\xi^1{}_{i\bar{i}}+\bar{\xi}^1_{i\bar{i}}\right).\end{aligned}$$

For this term $(**)_{i\bar{i}}$, we have the following estimate:

$$(**)_{i\bar{i}}\geq-\frac{c_0}{2w_{1\bar{1}}^2}\sum_{p\neq1}(|w_{1\bar{p}i}|^2+|w_{1\bar{p}\bar{i}}|^2)-C_1, \tag{3.7}$$

where C_1 is a positive constant depending only on (M,ω).

Let

$$F=\log\sigma_k(\chi_u).$$

We denote by

$$F^{i\bar{j}}=\frac{\partial F}{\partial w_{i\bar{j}}},\quad F^{i\bar{j},p\bar{q}}=\frac{\partial^2F}{\partial w_{i\bar{j}}\partial w_{p\bar{q}}}.$$

Then, at x_0, we have

$$F^{i\bar{j}}=\delta_{ij}F^{i\bar{i}}=\frac{\sigma_{k-1}(\lambda|i)}{\sigma_k(\lambda)}\delta_{ij},\quad \sum_{i=1}^nF^{i\bar{i}}w_{i\bar{i}}=\frac{\sum_{i=1}^n\sigma_{k-1}(\lambda|i)\lambda_i}{\sigma_k(\lambda)}=k, \tag{3.8}$$

$$\mathcal{F} := \sum_{i=1}^{n} F^{i\bar{i}} = \frac{\sum_{i=1}^{n}\sigma_{k-1}(\lambda|i)}{\sigma_k(\lambda)} = (n-k+1)\frac{\sigma_{k-1}(\lambda)}{\sigma_k(\lambda)} \geq k\left(\frac{C_n^k}{n}\right)^{\frac{1}{k-1}}\frac{\sigma_1^{\frac{1}{k-1}}(\lambda)}{\sigma_k^{\frac{1}{k-1}}(\lambda)} \geq c_{n,k}\lambda_1^{\frac{1}{k-1}}f^{-\frac{1}{k-1}}, \tag{3.9}$$

where we use the following Maclaurin's inequality:

$$\frac{\sigma_k(\lambda)/C_n^k}{\sigma_{k-1}(\lambda)/C_n^{k-1}} \leq \left(\frac{\sigma_k(\lambda)/C_n^k}{\sigma_1(\lambda)/C_n^1}\right)^{\frac{1}{k-1}}.$$

Furthermore,

$$F^{i\bar{j},p\bar{q}} = \begin{cases} \dfrac{\sigma_{k-2}(\lambda|ip)}{\sigma_k(\lambda)} - \dfrac{\sigma_{k-1}(\lambda|i)\sigma_{k-1}(\lambda|p)}{\sigma_k^2(\lambda)}, & \text{if } i=j,\ p=q,\ i\neq p; \\ -\dfrac{\sigma_{k-2}(\lambda|ip)}{\sigma_k(\lambda)}, & \text{if } i=q,\ p=j,\ i\neq p; \\ 0, & \text{otherwise}. \end{cases} \tag{3.10}$$

At the maximum point x_0, by (3.6) and (3.7), we have

$$\begin{aligned} 0 \geq F^{i\bar{j}}Q_{i\bar{j}} &= F^{i\bar{i}}Q_{i\bar{i}} \\ &\geq (1+2c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}w_{1\bar{1}i\bar{i}}}{w_{1\bar{1}}} + \frac{c_0}{2}\sum_{i=1}^{n}\sum_{p\neq 1}\frac{F^{i\bar{i}}|w_{1\bar{p}i}|^2}{w_{1\bar{1}}^2} \\ &\quad - (1+2c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \psi'\sum_{i=1}^{n}F^{i\bar{i}}u_{i\bar{i}} + \psi''\sum_{i=1}^{n}F^{i\bar{i}}|u_i|^2 \\ &\quad + \varphi''\sum_{i=1}^{n}F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 + \varphi'\sum_{i,p=1}^{n}F^{i\bar{i}}(|u_{p\bar{i}}|^2 + |u_{pi}|^2) \\ &\quad + \varphi'\sum_{i,p=1}^{n}F^{i\bar{i}}\left(u_{p\bar{i}i}u_{\bar{p}} + u_{\bar{p}i\bar{i}}u_p\right) - C_1\mathcal{F} \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{IV} - C_1\mathcal{F}. \end{aligned} \tag{3.11}$$

First, we can estimate the term I_1 as follows. By covariant derivative formula, we obtain

$$\sum_{i=1}^{n}F^{i\bar{i}}w_{1\bar{1}i\bar{i}} = \sum_{i=1}^{n}F^{i\bar{i}}w_{i\bar{i}1\bar{1}} + \sum_{i=1}^{n}F^{i\bar{i}}(w_{1\bar{1}} - w_{i\bar{i}})R_{i\bar{i}1\bar{1}} + \sum_{i=1}^{n}F^{i\bar{i}}\left(2\sum_{p=1}^{n}\mathrm{Re}(T_{1i}^{p}w_{p\bar{1}\bar{i}}) - \sum_{p=1}^{n}|T_{1i}^{p}|^2 w_{p\bar{p}}\right). \tag{3.12}$$

Differentiating the equation $F(\chi_u) = \log f$, we obtain

$$\sum_{i=1}^{n}F^{i\bar{i}}w_{i\bar{i}1\bar{1}} = (\log f)_{1\bar{1}} - \sum_{i,j,p,q=1}^{n}F^{i\bar{j},p\bar{q}}w_{i\bar{j}1}w_{p\bar{q}\bar{1}}. \tag{3.13}$$

By (3.12) and (3.13), we can estimate I_1 as follows:

$$\begin{aligned} \mathrm{I}_1 &= (1+2c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}w_{1\bar{1}i\bar{i}}}{w_{1\bar{1}}} \\ &\geq -(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar{j},p\bar{q}}w_{i\bar{j}1}w_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1+2c_0)\sum_{i}^{n}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^{1}w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right) \\ &\quad + \underbrace{2(1+2c_0)\sum_{i=1}^{n}F^{i\bar{i}}\mathrm{Re}\left(\sum_{p\neq 1}\frac{T_{1i}^{p}w_{p\bar{1}\bar{i}}}{w_{1\bar{1}}}\right)}_{\text{bad term}} + (1+2c_0)\frac{(\log f)_{1\bar{1}}}{w_{1\bar{1}}} - C_2\mathcal{F}, \end{aligned} \tag{3.14}$$

where $C_2 = n\sup_M(|\mathrm{Rm}|_g + |T|_g^2)$. The above bad term can be controlled by I_2 as follows:

$$
\begin{aligned}
&\frac{c_0}{2}\sum_{i=1}^{n}\sum_{p\neq 1}\frac{F^{i\bar i}|w_{1\bar p i}|^2}{w_{1\bar 1}^2} + I_2 \\
&= \frac{c_0}{2}\sum_{i=1}^{n}\sum_{p\neq 1}\frac{F^{i\bar i}|w_{1\bar p i}|^2}{w_{1\bar 1}^2} + 2(1+2c_0)\sum_{i=1}^{n}F^{i\bar i}\mathrm{Re}\left(\sum_{p\neq 1}\frac{T_{1i}^{p}w_{p\bar 1\bar i}}{w_{1\bar 1}}\right) \\
&= \frac{c_0}{2}\sum_{i=1}^{n}F^{i\bar i}\sum_{p\neq 1}\left|\frac{w_{1\bar p i}}{w_{1\bar 1}} + \frac{(1+2c_0)}{c_0}T_{1i}^{p}\right|^2 - \frac{2(1+2c_0)^2}{c_0}\sum_{i=1}^{n}\sum_{p\neq 1}F^{i\bar i}|T_{1i}^{p}|^2 \\
&\geq -\frac{2(1+2c_0)^2}{c_0}\sum_{i=1}^{n}\sum_{p\neq 1}F^{i\bar i}|T_{1i}^{p}|^2 \\
&\geq -C_3\mathcal{F},
\end{aligned}
$$

where $C_3 = nc_0^{-1}2(1+2c_0)^2\sup_M|T|_g^2$. Thus, we obtain

$$
\begin{aligned}
\mathrm{I}_1 + \mathrm{I}_2 &\geq -(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar j,p\bar q}w_{i\bar j 1}w_{p\bar q\bar 1}}{w_{1\bar 1}} + 2(1+2c_0)\sum_{i}^{n}F^{i\bar i}\mathrm{Re}\left(\frac{T_{1i}^{1}w_{1\bar 1\bar i}}{w_{1\bar 1}}\right) - (C_2+C_3)\mathcal{F} - (1+2c_0)\frac{(\log f)_{1\bar 1}}{w_{1\bar 1}} \\
&\geq -(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar j,p\bar q}w_{i\bar j 1}w_{p\bar q\bar 1}}{w_{1\bar 1}} + 2(1+2c_0)\sum_{i}^{n}F^{i\bar i}\mathrm{Re}\left(\frac{T_{1i}^{1}w_{1\bar 1\bar i}}{w_{1\bar 1}}\right) - C_4\mathcal{F} - M_0 f^{-\frac{1}{k-1}},
\end{aligned}
\tag{3.15}
$$

where we use (3.1) and $C_4 = C_2 + C_3$.

Next, we estimate terms III_2 and IV. Since

$$
u_{pi\bar i} = u_{i\bar i p} + T_{pi}^{i}u_{i\bar i} + u_q R_{i\bar i p\bar q}, \quad u_{\bar p i\bar i} = u_{i\bar p\bar i} = u_{i\bar i\bar p} - \overline{T_{ip}^{i}}u_{i\bar i}.
$$

Then, we have

$$
\sum_{i=1}^{n}F^{i\bar i}u_{pi\bar i} = \sum_{i=1}^{n}F^{i\bar i}u_{i\bar i p} + \sum_{i=1}^{n}F^{i\bar i}(T_{pi}^{i}u_{i\bar i} + u_q R_{i\bar i p\bar q}) = (\log f)_p + \sum_{i=1}^{n}F^{i\bar i}(T_{pi}^{i}u_{i\bar i} + u_q R_{i\bar i p\bar q} - \chi_{i\bar i p}), \tag{3.16}
$$

$$
\sum_{i=1}^{n}F^{i\bar i}u_{\bar p i\bar i} = \sum_{i=1}^{n}F^{i\bar i}u_{i\bar i\bar p} + \sum_{i=1}^{n}F^{i\bar i}\overline{T_{ip}^{i}}u_{i\bar i} = (\log f)_{\bar p} + \sum_{i=1}^{n}F^{i\bar i}(\overline{T_{ip}^{i}}u_{i\bar i} - \chi_{i\bar i\bar p}). \tag{3.17}
$$

Inserting (3.16) and (3.17) into IV, we obtain

$$
\begin{aligned}
\mathrm{IV} &= \varphi'\sum_{i,p=1}^{n}F^{i\bar i}\left(u_{pi\bar i}u_{\bar p} + u_{\bar p i\bar i}u_p\right) \\
&= 2\varphi'\sum_{i,p=1}^{n}F^{i\bar i}u_{i\bar i}\mathrm{Re}(u_{\bar p}T_{pi}^{i}) + \varphi'\sum_{p=1}^{n}\left(2\mathrm{Re}(u_{\bar p}(\log f)_p) + \sum_{i,p,q=1}^{n}u_{\bar p}u_q F^{i\bar i}R_{i\bar i p\bar q}\right) \\
&\geq 2\varphi'\sum_{i,p=1}^{n}F^{i\bar i}u_{i\bar i}\mathrm{Re}(u_{\bar p}T_{pi}^{i}) + 2\varphi'\sum_{i=1}^{n}\mathrm{Re}(u_{\bar i}(\log f)_i) - C\mathcal{F} \\
&\geq 2\varphi'\sum_{i,p=1}^{n}F^{i\bar i}u_{i\bar i}\mathrm{Re}(u_{\bar p}T_{pi}^{i}) - C_5\mathcal{F} - (k-1)f^{-\frac{1}{k-1}},
\end{aligned}
\tag{3.18}
$$

where we have used $(4K)^{-1} \leq \varphi' \leq (2K)^{-1}$, $K = \sup_M|\nabla u|^2 + 1$, and $C_5 = n^2\sup_M|\mathrm{Rm}|_g$. Then, we obtain

$$
\begin{aligned}
\mathrm{III}_2 + \mathrm{IV} &\geq \varphi'\sum_{i=1}^{n}F^{i\bar i}\left(|u_{i\bar i}|^2 + 2u_{i\bar i}\mathrm{Re}\left(\sum_{p=1}^{n}u_{\bar p}T_{pi}^{i}\right)\right) - C_5\mathcal{F} - (M_0+k-1)f^{-\frac{1}{k-1}} \\
&\geq \frac{3}{4}\varphi'\sum_{i=1}^{n}F^{i\bar i}|u_{i\bar i}|^2 - 4\varphi'\sum_{i=1}^{n}F^{i\bar i}\left|\sum_{p=1}^{n}u_p\overline{T_{pi}^{i}}\right|^2 - C_5\mathcal{F} - (M_0+k-1)f^{-\frac{1}{k-1}}
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{3}{4}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}|u_{i\bar{i}}|^2 - \left(C_5 + 4n\sup_M|T|_g^2\right)\mathcal{F} - (M_0 + k - 1)f^{-\frac{1}{k-1}}\\
&\geq \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - 15\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\chi_{i\bar{i}}^2 - (C_5 + 4n\sup_M|T|_g^2)\mathcal{F} - (M_0 + k - 1)f^{-\frac{1}{k-1}}\\
&\geq \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - \left(\sup_M|\chi|_g^2 + C_5 + 4n\sup_M|T|_g^2\right)\mathcal{F} - (M_0 + k - 1)f^{-\frac{1}{k-1}}\\
&\geq \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - C_6\mathcal{F} - C_6 f^{-\frac{1}{k-1}},
\end{aligned} \tag{3.19}$$

where we have used $(4K)^{-1} \leq \varphi' \leq (2K)^{-1}$, $K = \sup_M|\nabla u|^2 + 1$, and $C_6 = \sup_M|\chi|_g^2 + C_5 + 4n\sup_M|T|_g^2 + M_0 + k - 1$.

Since $\chi \in \Gamma_k(M)$, there exists a sufficiently small positive constant τ_0 such that $\chi - \tau_0\omega \in \Gamma_k(M)$. This gives $F^{i\bar{i}}\chi_{i\bar{i}} \geq \tau_0\mathcal{F}$. Since $\psi' < 0$ (see (3.4)), the estimate of the term II_2 in (3.11) is as follows:

$$\mathrm{II}_2 = \psi'\sum_{i=1}^{n}F^{i\bar{i}}u_{i\bar{i}} = \psi'\sum_{i=1}^{n}F^{i\bar{i}}\left(\lambda_i - \chi_{i\bar{i}}\right) = k\psi' - \psi'\sum_{i=1}^{n}F^{i\bar{i}}\chi_{i\bar{i}} \geq k\psi' - \tau_0\psi'\mathcal{F}. \tag{3.20}$$

Inserting (3.15), (3.19), and (3.20) into (3.11), we obtain

$$\begin{aligned}
0 \geq F^{i\bar{i}}Q_{i\bar{i}} \geq &-(1 + 2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar{j},p\bar{q}}w_{i\bar{j}1}w_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1 + 2c_0)\sum_{i=1}^{n}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^{1}w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right)\\
&- (1 + 2c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \varphi''\sum_{i=1}^{n}F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 + \psi''\sum_{i=1}^{n}F^{i\bar{i}}|u_i|^2\\
&+ \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 + (-\tau_0\psi' - C_7)\mathcal{F} - C_7 f^{-\frac{1}{k-1}} + k\psi',
\end{aligned} \tag{3.21}$$

where $C_7 = C_1 + C_4 + C_6$.

Now, we take $\varepsilon = \frac{\delta}{4} \leq \frac{1}{16}$ and $\delta = \frac{1}{2A+1}$, where $A = 2L(C_0 + 1)$, $C_0 = \tau_0^{-1}(C_9 + 1)$, and $C_9 = C_7 + 9n^2\sup_M|T|_g^2$. We divide two cases to calculate the estimate.

Case 1: $\lambda_n < -\varepsilon\lambda_1$.

By (3.5), for $1 \leq i \leq n$, we have

$$-(1 + 2c_0)^2\left|\frac{w_{1\bar{1}i}}{w_{1\bar{1}}}\right|^2 = -|\varphi'|\nabla u|_i^2 + \psi' u_i|^2 \geq -2(\varphi')^2|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 - 2(\psi')^2|u_i|^2 = -\varphi''|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 - 2(\psi')^2|u_i|^2.$$

Then, we have

$$\begin{aligned}
&2(1 + 2c_0)\sum_{i\neq 1}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^{1}w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right) - (1 + 2c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \varphi''|\nabla u|_i^2|\nabla u|_{\bar{i}}^2\\
&\geq -(1 + 3c_0)\sum_{i=1}^{n}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - 9n^2|T|_g^2\mathcal{F} + \varphi''|\nabla u|_i^2|\nabla u|_{\bar{i}}^2\\
&\geq -(1 + 2c_0)^2\sum_{i=1}^{n}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - 9n^2|T|_g^2\mathcal{F} + \varphi''|\nabla u|_i^2|\nabla u|_{\bar{i}}^2\\
&\geq -2K(\psi')^2\mathcal{F} - 9n^2|T|_g^2\mathcal{F}\\
&\geq -2K(\psi')^2\mathcal{F} - C_8\mathcal{F},
\end{aligned} \tag{3.22}$$

where $C_8 = 9n^2\sup_M|T|_g^2$. Since $-\lambda_n \geq \varepsilon\lambda_1$, we have

$$\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 \geq \frac{1}{2}\varphi' F^{n\bar{n}}\lambda_n^2 > \frac{1}{2}\varphi'\varepsilon^2 F^{n\bar{n}}\lambda_1^2 \geq \frac{\varepsilon^2}{8nK}\lambda_1^2\mathcal{F}. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21) and by the concavity of $\log\sigma_k(\chi_u)$, we obtain

$$\begin{aligned}0 \geq \sum_{i=1}^{n} F^{i\bar{i}} Q_{i\bar{i}} &\geq \frac{\varepsilon^2}{8nK}\mathcal{F}\lambda_1^2 - 2K(\psi')^2\mathcal{F} + (-\tau_0\psi' - C_7 - C_8)\mathcal{F} - C_7 f^{-\frac{1}{k-1}} + k\psi' \\ &\geq \left(\frac{\varepsilon^2}{8nK}\lambda_1^2 - 2K(\psi')^2 - C_9\right)\mathcal{F} - C_9 f^{-\frac{1}{k-1}} + k\psi',\end{aligned} \tag{3.24}$$

where we use $\psi' < 0$ and $C_9 = C_7 + C_8$.

Inserting (3.9) into (3.24), we obtain

$$0 \geq \sum_{i=1}^{n} F^{i\bar{i}} Q_{i\bar{i}} \geq c_{n,k}\left(\frac{\varepsilon^2}{8nK}\lambda_1^2 - 2K(\psi')^2 - C_9\right)\lambda_1^{\frac{1}{k-1}} f^{-(k-1)} - C_9 f^{-\frac{1}{k-1}} - k|\psi'|. \tag{3.25}$$

From the above inequality, we have

$$\lambda_1 \leq M_{2,1}K, \tag{3.26}$$

where $M_{2,1}$ depends only on (M, ω), χ, n, k, and $\left|f^{\frac{1}{k-1}}\right|_{C^{1,1}}$.

Case 2: $\lambda_n > -\varepsilon\lambda_1$.

Let

$$I = \{i \in \{1, \ldots, n\} | \sigma_{k-1}(\lambda|i) \geq \varepsilon^{-1}\sigma_{k-1}(\lambda|1)\}.$$

Obviously, $i \in I$ if and only if $F^{i\bar{i}} > \varepsilon^{-1}F^{1\bar{1}}$. For $i \notin I$, by condition (3.5), we have

$$\begin{aligned}&-(1+2c_0)\sum_{i\notin I}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + 2(1+2c_0)\sum_{i\notin I}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^1 w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right) + \varphi''\sum_{i\notin I}F^{i\bar{i}}\,|\nabla u|^2{}_i|\nabla u|^2{}_{\bar{i}} \\ &\geq -(1+2c_0)^2\sum_{i\notin I}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - \frac{(1+2c_0)^2}{c_0}\sum_{i\notin I}F^{i\bar{i}}|T_{1i}^1|^2 + \varphi''\sum_{i\notin I}F^{i\bar{i}}\,|\nabla u|^2{}_i|\nabla u|^2{}_{\bar{i}} \\ &\geq -2(\psi')^2\sum_{i\notin I}F^{i\bar{i}}|u_i|^2 - 9n^2\sum_{i\notin I}F^{i\bar{i}}|T_{1i}^1|^2 \\ &\geq -2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} - C_8\mathcal{F}.\end{aligned} \tag{3.27}$$

Substituting the above inequality into (3.21), we have

$$\begin{aligned}0 \geq F^{i\bar{i}}Q_{i\bar{i}} \geq &-(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar{j},p\bar{q}}w_{i\bar{j}1}w_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1+2c_0)\sum_{i\in I}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^1 w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right) \\ &-(1+2c_0)\sum_{i\in I}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \varphi''\sum_{i\in I}F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2\Bigg) \\ &+\psi''\sum_{i=1}^{n}F^{i\bar{i}}|u_i|^2 + \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - 2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} \\ &+(-\tau_0\psi' - C_9)\mathcal{F} - k|\psi'| - C_9 f^{-\frac{1}{k-1}}.\end{aligned} \tag{3.28}$$

Similar to the proof in [29], we can show that the following four terms on the first line and second lines of (3.28) are nonnegative

$$\begin{aligned}&-(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar{j},p\bar{q}}w_{i\bar{j}1}w_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1+2c_0)\sum_{i\in I}F^{i\bar{i}}\mathrm{Re}\left(\frac{T_{1i}^1 w_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\right) - (1+2c_0)\sum_{i\in I}\frac{F^{i\bar{i}}|w_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} \\ &+\varphi''\sum_{i\in I}F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 \geq 0.\end{aligned} \tag{3.29}$$

We also have

$$\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - 2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} \geq \frac{1}{4}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2, \tag{3.30}$$

where we assume $\frac{1}{4}\varphi' F^{1\bar{1}}\lambda_1^2 \geq 2\varepsilon^{-1}K(\psi')^2 F^{1\bar{1}}$, otherwise $\frac{1}{4}\varphi' F^{1\bar{1}}\lambda_1^2 \leq 2\varepsilon^{-1}K(\psi')^2 F^{1\bar{1}}$, and then

$$\lambda_1 \leq 8\varepsilon^{-\frac{1}{2}}|\psi'|K \leq 16\varepsilon^{-1}(C_0+1)K. \tag{3.31}$$

Inserting (3.29) and (3.30) into (3.28), we obtain

$$\begin{aligned}
0 \geq F^{i\bar{i}}Q_{i\bar{i}} &\geq \frac{1}{4}\varphi' \sum_{i=1}^{n} F^{i\bar{i}}\lambda_i^2 + (-\tau_0\psi' - C_9)\mathcal{F} - C_9 f^{-\frac{1}{k-1}} - k|\psi'| \\
&\geq (C_0\tau_0 - C_9)\mathcal{F} - C_9 f^{-\frac{1}{k-1}} - 2C_9(C_0+1) \\
&\geq \mathcal{F} - C_9 f^{-\frac{1}{k-1}} - 2k(C_0+1) \\
&\geq c_{n,k}\lambda_1^{\frac{1}{k-1}} f^{-\frac{1}{k-1}} - C_9 f^{-\frac{1}{k-1}} - 2k(C_0+1),
\end{aligned} \tag{3.32}$$

where we use $C_0 = \tau_0(C_9+1)$ and (3.9).

By (3.32), we obtain the following estimate:

$$\lambda_1 \leq c(n,k)^{1-k}\Big(C_9 + 2k(C_0+1)f^{\frac{1}{k-1}}\Big)^{k-1} \leq M_{2,2}. \tag{3.33}$$

Finally, combining the estimates (3.26) and (3.31) with (3.33), we obtain the desired estimate as follows:

$$\lambda_1 \leq M_2 K := (M_{2,1} + M_{2,2} + 16\varepsilon^{-1}(C_0+1))K. \qquad \square$$

4 Proof of Theorem 1.1

Since $f^{\frac{1}{k-1}} \in C^{1,1}$, there exists a sequence of positive smooth functions $\{f_i\}$ such that $f_i^{\frac{1}{k-1}} \to f^{\frac{1}{k-1}}$ in $C^{1,1}$ and $|f_i^{\frac{1}{k-1}}|_{C^2} \leq C$, where C is a uniform constant depending only on $|f^{\frac{1}{k-1}}|_{C^{1,1}}$. By the existence result in the nondegenerate case, there exist smooth functions u_i and constants c_i such that

$$\begin{cases} C_n^k \quad \chi_{u_i}^k \wedge \omega^{n-k} = c_i f_i \omega^n, \quad \sup\limits_M u_i = 0, \\ \chi_{u_i} = \chi + \sqrt{-1}\partial\bar{\partial}u_i \in \Gamma_k(M). \end{cases} \tag{4.1}$$

We next show c_i are uniformly bounded. Let x_0 be the maximum point of u_i. Then, we can obtain the uniform lower bound of c_i as follows:

$$c_i = \frac{\sigma_k(\chi_{u_i}(x_0))}{f_i(x_0)} \geq \frac{\sigma_k(\chi(x_0))}{\sup_M f_i} \geq \frac{\inf_M \sigma_k(\chi)}{C} > 0.$$

To obtain the uniform upper bound of c_i, we have

$$\begin{aligned}
c_i^{\frac{1}{k}} \int_M f_i^{\frac{1}{k}}\omega^n &= \int_M \sigma_k^{\frac{1}{k}}(\chi_{u_i})\omega^n \\
&\leq C_{n,k}\int_M (\mathrm{tr}_g\chi + \triangle_g u_i)\omega^n \\
&= C_{n,k}\Bigg(\int_M \mathrm{tr}_g\chi\omega^n + n\int_M \sqrt{-1}\partial\bar{\partial}u_i \wedge \omega^{n-1}\Bigg) \\
&= C_{n,k}\Bigg(\int_M \mathrm{tr}_g\chi\omega^n + \int_M u_i\sqrt{-1}\partial\bar{\partial}\omega^{n-1}\Bigg) \\
&\leq C\Bigg(1 + \int_M |u_i|\omega^n\Bigg),
\end{aligned} \tag{4.2}$$

where in the first inequality we have used the following Maclaurin's inequality:

$$\sigma_k^{\frac{1}{k}}(\chi_{u_i}) \le \frac{(C_n^k)^{\frac{1}{k}}}{n}\sigma_1(\chi_{u_i}) = C_{n,k}(\mathrm{tr}_g\chi + \triangle u_i).$$

Since $\chi_{u_i} \in \Gamma_k(M)$ and $\sup_M u_i = 0$, we have $\triangle u_i \ge -\mathrm{tr}_g\chi \ge -C$, and by [5], there exists a uniform constant C_1 such that

$$\int_M |u_i|\omega^n \le C_1. \tag{4.3}$$

Since f is not identical to zero, we have $\int_M f^{\frac{1}{k}} > 0$. By taking i sufficiently large, we have

$$\int_M f_i^{\frac{1}{k}} \ge \frac{1}{2}\int_M f^{\frac{1}{k}} > 0. \tag{4.4}$$

Inserting (4.3) and (4.4) into (4.2), we finally obtain the uniform upper bound of b_i as follows:

$$c_i \le \frac{2^k}{|f|_{L^{\frac{1}{k}}}}C(1 + C_1).$$

Therefore, we can apply the weak $C^{1,1}$ estimate to u_i such that

$$\sup_M|u_i| + \sup_M|\partial u_i| + \sup_M|\triangle u_i| \le C. \tag{4.5}$$

By taking a subsequence, we obtain the existence of weak $C^{1,1}$ solution to the degenerate k-Hessian equation.

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