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Research Article

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A general method to study the convergence of nonlinear operators in Orlicz spaces

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Abstract: We continue the work started in a previous article and introduce a general setting in which we define nets of nonlinear operators whose domains are some set of functions defined in a locally compact topological group. We analyze the behavior of such nets and detect the fairest assumption, which are needed for the nets to converge with respect to the uniform convergence and in the setting of Orlicz spaces. As a consequence, we give results of convergence in this frame, study some important special cases, and provide graphical representations.

Keywords: Orlicz spaces, modular convergence, Kantorovich sampling-type operators, Kantorovich convolution-type operators, Kantorovich Mellin-type operators, estimates, pointwise convergence, uniform convergence

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1 Introduction

In [56], a family of nets of linear operators acting on functions defined in locally compact topological group was introduced, and some results concerning convergence of such nets in Orlicz spaces were established. We continue here a programmatic effort to extend the results of previous papers [52–56] to the case where the operators are nonlinear. Namely, if H, G are locally compact topological groups, with (left-invariant) Haar measures μ_H and μ_G, respectively, [44], $(h_w)_{w>0}$ is a family of homeomorphisms $h_w : H \to h_w(H) \subset G$ and if $(\chi_w)_{w>0}$ is a family of kernels $\chi_w : G \times \mathbb{R} \to \mathbb{R}$ satisfying certain assumptions, we introduce the family of operators $(T_w)_{w>0}$ acting on (a subset of) $M(G)$ (the set of measurable functions $f : G \to \mathbb{R}$), defined as

$$T_w f(z) = \int_H \chi\Big(z - h_w(t), L_{h_w(t)} f\Big) \mathrm{d}\mu_H(t), \tag{1}$$

where $(L_{h_w(t)})_{t\in H, w>0}$ is a family of operators $L_{h_w(t)} : M(G) \to \mathbb{R}$.

The general form of (1) allows us to study many different kinds of operators introduced in the last few decades, both of linear and nonlinear type, and to understand to what extent different assumptions are needed in order to achieve the convergence of the aforementioned operators.

Among these operators, there are well-known generalized sampling series, introduced (as extensions of the Whittaker-Kotelnikov-Shannon sampling theorem [42,49,57] and [11,13–15,35,38,39,41]) and extensively studied since the 1980s by the German mathematician Butzer and his school at the RWTH Polytechnic of Aachen [2,4,5,16,17,19–21,40, 51,52]; the Kantorovich sampling series, introduced in 2007 in [3], then in [29,30] in the

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multidimensional setting (linear and nonlinear) and also developed with considerable applications in [7,8,22–24,27,28,31–34], in addition to the classic and Mellin convolution operators in their standard and Kantorovich versions (see, e.g., [1]). The theory also includes the Durrmeyer sampling-type operators [9,26].

Beyond the mathematical significance of the theory involved in the analysis of (1), there are various relevant practical motivations which justify the introduction of the above nonlinear operators (1). One important example can be furnished in signal processing, when one has to describe some nonlinear transformations generated by signals that, during their filtering process, generate new frequencies. Other examples are related to various aspects concerning power electronics and wireless communications, where amplifiers introduce nonlinear distortions to their input signals, or in radiometric photography and CCD image sensors. To be a little bit more exhaustive, in radiometric photography the response relates the radiance at the input to the intensity: this response is clearly monotone increasing, but very often exhibits nonlinearity. Again, nonlinear phenomena occur whenever amplifier saturation introduces distortions to the input signal which are, in general, nonlinear.

One of the main difficulties in dealing with (1) is that, in the case of nonlinear sampling, one has to introduce a suitable notion of singularity. This notion has been first introduced by Musielak in [46], and then extended in, e.g., [10]. Another peculiarity is given by the necessity of imposing suitable (but not too much unmild) assumptions on the kernels in order for (1) to be well-defined and converge: this type of assumption is generally recognized in the literature to be a kind of generalized Lipschitz condition. It follows that the methods and the results obtained in the nonlinear framework are different from those achieved in the linear one. As a concrete consequence, the regularity properties which hold in the linear cases are no longer valid in the nonlinear ones. Another interesting property of (1) lies in the generality of the family $(L_{h_w(t)})_{t\in\mathbb{R}}$. This choice allows us to collect in one single setting both generalized sampling and sampling Kantorovich operators, and apart from this, discrete and integral operators, as we will see. Obviously, some assumptions must be made on such a family in order to imitate the effects of a family of sampling values. Least, but not last, the general framework of topological spaces: it allows one to deal with unidimensional and multidimensional operators, with discrete and integral operators, and permits one to define operators in which the base spaces are not necessarily abelian subgroups of $\mathbb{R}^n$.

The article is organized as follows: in Section 2, we recall some basic facts concerning topological groups and Haar measures, together with the notions of modular functional and Orlicz space over a topological group, focusing on the notions of convergence in such spaces. Section 3 deals with the basic assumptions which we consider in the treatment of (1). Section 4 contains all the main results of the article. The main objective is to achieve a convergence theorem in Orlicz spaces: we will proceed by first studying the convergence in the sup-norm for bounded uniformly continuous functions, then we analyze the behavior of (1) in Orlicz spaces when f is continuous and with compact support. Finally, the main Theorem 4.7, where we give a result of convergence in a Orlicz space $L^{\varphi}(G)$ for functions belonging to a subset $\mathcal{Y}$ of $L^{\eta}(G)$, using the modular continuity of $T_w : \mathcal{Y} \to L^{\varphi}(G)$ given in Theorem 4.5. It is worth emphasizing the importance of studying the convergence in Orlicz spaces: they include L^p-spaces, interpolation spaces ($L^{\alpha}\ln^{\beta}L$-spaces, [12,50]), exponential spaces [37], and others which have a crucial importance in many fields. Section 5 is devoted to several examples of operators that are included in the general theory; among them, there are sampling-type operators, convolution and Mellin convolution operators. Here we show that the assumptions made on the kernel functions are satisfied, sometimes with the addition of some further sufficient condition.

We conclude the article with some graphical representations showing the convergence for the operators $T_w f$ toward f in some of the cases previously discussed, including both the unidimensional and the multidimensional setting.

2 Preliminaries

We begin by stating some useful basic facts which we will use throughout the article. If $(H, +)$ is a locally compact Hausdorff topological group and if θ_H is its neutral element, there is a unique (up to a multiplicative constant) left (resp. right) translation invariant regular Haar measure μ_H (resp. ν_H). It follows that, if A is a Borel set contained in H, and if $-A$ denotes the set $-A = \{-a, |a \in A\}$, then $\mu_H(-A) = k\nu_H(A)$, for some constant $k > 0$. The group H is said to be unimodular if the right and left invariant Haar measures coincide. In this case, $\mu_H(A) = \mu_H(-A) = \nu_H(A)$ for every Borel set $A \subset H$. Compact groups, abelian groups, and discrete groups are very well-known examples of unimodular groups. From now on, we will denote by $\mathcal{B}$ a local base of the neutral element θ_H of H.

Let $(G, +)$ be a locally compact topological group with neutral element θ_G and (left-invariant) Haar measure μ_G. Let $M(G)$ denote the set of measurable functions $f : G \to \mathbb{R}$. By $C(G)$ (resp. $C_c(G)$) we denote the subset of $M(G)$ consisting of uniformly continuous and bounded (resp. uniformly continuous with compact support) functions $f : G \to \mathbb{R}$. As usual, the sets $C(G)$ and $C_c(G)$ are equipped with the standard $\|\cdot\|_\infty$-norm.

A continuous function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a φ-function if it satisfies the following conditions:

(i) $\varphi(0) = 0$ and $\varphi(x) > 0$ for every $x > 0$;
(ii) φ is non-decreasing on $\mathbb{R}_0^+$;
(iii) $\lim_{x\to+\infty}\varphi(x) = +\infty$.

Given a φ-function φ, the functional

$$I_G^\varphi(f) : M(G) \to [0, +\infty], \quad f \mapsto \int_G \varphi(|f(x)|)\mathrm{d}\mu_G(x)$$

is called the *modular functional* on $M(G)$. The Orlicz space $L^\varphi(G)$ is the subset of $M(G)$ consisting of those measurable functions $f \in M(G)$ such that

$$I_G^\varphi(\lambda f) < +\infty$$

for some $\lambda > 0$, equipped with the strong norm

$$\|f\|_\varphi := \inf\{\lambda > 0 | I_G^\varphi(f/\lambda) \leq 1\},$$

provided that φ is convex. This norm is called the *Luxemburg norm*. There is a more natural notion of convergence in $L^\varphi(G)$, weaker than the norm convergence, which is called the *modular convergence*. Namely, a sequence $(f_n)_{n\in\mathbb{N}} \subset L^\varphi(G)$ is said to modularly converge to a function $f \in L^\varphi(G)$ if there exists a number $\lambda > 0$ such that

$$\lim_{n\to+\infty} I_G^\varphi[\lambda(f_n - f)] = 0.$$

It is not hard to show that a sequence converges strongly in $L^\varphi(G)$ if and only if the above limit holds *for every* $\lambda > 0$. If the so-called Δ_2-*condition* holds, i.e., if there exists a number $M > 0$ such that

$$\frac{\varphi(2x)}{\varphi(x)} \leq M \quad \text{for every } x > 0,$$

then the strong convergence is equivalent to the modular convergence. The Δ_2-condition is in turn satisfied if and only if $L^\varphi(G) = E^\varphi(G)$, where

$$E^\varphi(G) := \{f \in L^\varphi(G) \mid I_G^\varphi(\lambda f) < +\infty \text{ for every } \lambda > 0\}.$$

Orlicz spaces arise as a generalization of L^p spaces, and indeed if $\varphi(x) = x^p$ ($p \in \mathbb{N}$), then $L^\varphi(G) = L^p(G)$. Since the function φ satisfies the Δ_2-condition, modular and Luxemburg convergences are equivalent in $L^p(G)$ and they are in turn equivalent to the standard L^p-norm. There are many examples of Orlicz spaces which are important in applications, like partial differential equation (PDE) and functional analysis. For instance, the so-called exponential spaces [37] generated by the function $\varphi_\alpha(x) = \exp(x^\alpha) - 1$, where $\alpha > 0$. The Orlicz space $L^{\varphi_\alpha}(G)$ generated by φ_α furnishes a case in which modular convergence and Luxemburg convergence are not equivalent, since the function φ_α does not satisfy the Δ_2-condition above. However, there are many other examples of Orlicz spaces which are interesting: among these, we mention the so-called Zygmund (or interpolation) spaces, where the generating function is $\varphi_{\alpha,\beta}(x) = x^\alpha \ln^\beta(e + x)$, where

$\alpha \geq 1$ and $\beta > 0$ [12,50]. For more information concerning Orlicz spaces, the reader can refer to [10,43,45,47,48].

3 Basic assumptions

We introduce here the operators we deal with, together with the assumptions we need both for the operators to make sense and to obtain the desired convergence results. Note that not all the assumptions we are going to list will be needed simultaneously, and we will indicate when and where some specific assumptions will be needed.

So, let H and G be two locally compact Hausdorff topological groups with regular (left-invariant) Haar measures μ_H and μ_G, respectively (see [44]). Assume that there exists a family of homeomorphisms $(h_w)_{w>0} : H \to h_w(H) \subset G$. Now, fix $w > 0$ and let $(L_{h_w(t)})_{t\in H}$ be a family of operators $L_{h_w(t)} : M(G) \to \mathbb{R}$. Furthermore, let $(\chi_w)_{w>0} \subset L^1(G)$ be a family of kernel functions $\chi_w : G \times \mathbb{R} \to \mathbb{R}$, i.e., for every $w > 0$, $\chi_w(\cdot, u) \in M(G)$ for every $u \in \mathbb{R}$, $\chi_w(z, \cdot)$ is continuous for every $z \in G$, and $\chi_w(z, 0) = 0$ for every $z \in G$. For $w > 0$, we define the operator $T_w : \mathrm{Dom}(T_w) \to M(G)$ as

$$T_w f := z \mapsto \int_H \chi_w(z - h_w(t), L_{h_w(t)} f)\mathrm{d}\mu_H(t), \tag{2}$$

where the domain $\mathrm{Dom}(T_w) \subset M(G)$ is defined as the subset of $M(G)$ consisting of the functions $f \in M(G)$ such that the integral in (2) exists for a.e. $z \in G$ and $T_w f \in M(G)$.

Our aim is to study the properties and the convergence of the family $(T_w f)_{w>0}$ as $w \to +\infty$. As it is quite obvious, some assumptions must be made in order to guarantee both the consistency of the domain $\mathrm{Dom}(T_w)$ and the convergence of the family $(T_w f)_{w>0}$. In the next sections, we provide several examples of operators which satisfy all the assumptions we are going to state in the following lines.

Let us introduce some notations which will be useful. If $w > 0$, $z \in G$, and $A \subset G$ is a measurable set, we define

$$\begin{aligned} A_{w,z} &:= \{t \in H | z - h_w(t) \in A\}, \\ A_w &:= \{t \in H | h_w(t) \in A\}, \\ \Upsilon_w(A) &:= \mu_H(A_w). \end{aligned}$$

Let us start by analyzing the family $(\chi_w)_{w>0}$. We require that

(χ1) the map $t \mapsto \chi_w(z - h_w(t), u) \in L^1(H)$ for every $z \in G$, $w > 0$, and $u \in \mathbb{R}$;

(χ2) the family $(\chi_w)_{w>0}$ is a family of (C_w, ψ)-*Lipschitz kernels*, i.e., there exist a family of measurable functions $C_w : \mathbb{R} \to \mathbb{R}_0^+$ and a continuous φ-function ψ such that

$$|\chi_w(z, u) - \chi_w(z, v)| \leq C_w(z)\psi(|u - v|),$$

for every $z \in G$, $u, v \in \mathbb{R}$, and $w > 0$.

Note that this assumption, together with the fact that $\chi_w(z, 0) = 0$ for every $z \in G$, implies that

$$|\chi_w(z, u)| \leq C_w(z)\psi(|u|),$$

for every $z \in G$, $u \in \mathbb{R}$, and $w > 0$.

(χ3) for every $n \in \mathbb{N}$ and $w > 0$

$$\mathcal{S}_w^{(n)}(z) := \sup_{\frac{1}{n}<u<n} \left| \frac{1}{u} \int_H \chi_w(z - h_w(t), u)\mathrm{d}\mu_H(t) - 1 \right| \to 0$$

as $w \to +\infty$, uniformly with respect to $z \in G$.

We will have to specify further the nature of the family $(\chi_w)_{w>0}$ by restricting the possible choices of the functions C_w. We state here all the assumptions we need in this article. These assumptions, as remarked earlier, will not be used simultaneously, and we always list the ones which we need to achieve a certain result.

(C1) $C_w \in L^1(G)$ and $\|C_w\|_1 \le \Gamma < +\infty$ for every $w > 0$;

(C2) there exists an absolute constant $M > 0$ such that

$$M := \sup_{z\in G} \int_H C_w(z - h_w(t))\,\mathrm{d}\mu_H(t) < +\infty,$$

for every $w > 0$;

(C3) if $w > 0$, $z \in G$, and $B \in \mathcal{B}$,

$$\lim_{w\to+\infty} \int_{H\setminus B_{w,z}} C_w(z - h_w(t))\,\mathrm{d}\mu_H(t) = 0,$$

uniformly with respect to $z \in G$;

(C4) for every $\varepsilon > 0$ and for every compact set $K \subset G$, there exists a symmetric compact set $C \subset G$ (C obviously depends on the choice of ε and K) such that, if $t \in K_w$ (i.e., $h_w(t) \in K$), then

$$\int_{G\setminus C} \Upsilon_w(K) C_w(z - h_w(t))\,\mathrm{d}\mu_G(z) < \varepsilon,$$

for sufficiently large $w > 0$.

Next, let us pause on the family $(L_{h_w(t)})_{t\in H}$. We assume that the family $(L_{h_w(t)})_{t\in H}$ is chosen in such a way that:

(L1) when restricted to $L^\infty(G)$, the operators $L_{h_w(t)}$ are *uniformly bounded*: if $\tilde{L}_{h_w(t)}$ denotes the restriction of $L_{h_w(t)}$ to L^∞, i.e., $\tilde{L}_{h_w(t)} : L^\infty(G) \to \mathbb{R}$, then there exists a positive constant Υ such that

$$\|\tilde{L}_{h_w(t)}\| := \sup_{\|f\|_\infty \le 1} |L_{h_w(t)} f| < \Upsilon < +\infty,$$

for every $w > 0$ and $t \in H$;

(L2) the family $(L_{h_w(t)})_{t\in H}$ *preserves the continuity* of f in the sense that if $f \in C(G)$, for every $\varepsilon > 0$ there exists $\overline{w}(\varepsilon) > 0$ and a set $B \in \mathcal{B}$ such that if $z - h_w(t) \in B_\varepsilon$ and $w > \overline{w}$, then $|L_{h_w(t)} f - f(z)| < \varepsilon$;

(L3) let φ be a φ-function. If $f \in C_c(G)$ and $\mathrm{Supp}(f) = K_1 \subset G$, then there exist a compact set $K \supset K_1$ and a positive number α such that for every $\varepsilon > 0$ there exists $\overline{w} > 0$ with

$$\int_{H\setminus K_w} \varphi\big(\alpha \cdot \psi\big(|L_{h_w(t)} f|\big)\big)\,\mathrm{d}\mu_H(t) < \varepsilon,$$

for every $w > \overline{w}$, where ψ is the φ-function of condition (χ_2).

Before formulating the next condition (L4), we need to introduce the following growth condition which relates the composition of the two φ-functions φ and ψ to a φ-function η.

(H) Let φ be a φ-function. There exists a φ-function η such that for every $\lambda \in (0, 1)$, there exists a constant $C_\lambda \in (0, 1)$ such that

$$\varphi(C_\lambda \psi(u)) \le \eta(\lambda u),$$

for every $u \ge 0$, where ψ is the φ-function of condition (χ_2) (see, e.g., [10]).

Now, (L4) reads as follows:

(L4) there exists a subspace $\mathcal{Y} \subset L^\eta(G)$ (being η the φ-function of condition (H)) with $\mathcal{Y} \supset C_c^\infty(G)$ and such that for every $f \in \mathcal{Y}$ and $\lambda > 0$, there exist two constants $c = c(f, \lambda, \eta) > 0$ and $\beta = \beta(f, \eta) > 0$ (note that β **does not** depend on λ, but only on f and η) such that

$$\limsup_{w\to+\infty}\|C_w\|_1 \int_H \eta(\lambda(|L_{h_w(t)}f|))\mathrm{d}\mu_H(t) \le cI^\eta(\lambda\beta f).$$

Remark 3.1.

(a) Note that assumption (L3) could have been formulated directly with the integrand function η instead of φ, that is,

$$\int_{H\setminus K_w} \eta(\lambda(|L_{h_w(t)}f|))\mathrm{d}\mu_H(t) < \varepsilon,$$

for some constant $\lambda > 0$, but we preferred to leave it with the function φ, precisely to highlight where condition (H) is really needed.

(b) Condition (H) is quite common when working with nonlinear operators in the setting of Orlicz spaces. As an example, let $\varphi(u) = u^p$ and $\psi(u) = u^{q/p}$ $(1 \le q \le p < +\infty)$: then we can take $\eta(u) = u^q$ and $C_\lambda = \lambda^{q/p}$.

(c) By slightly modifying assumption (L1), one can include cases when $L_{h_w(t)}$ are not necessarily linear, but however possess the property of being uniformly-Lipschitz. This permits one to include operators $T_w f$ which are (in some sense) perturbations of linear ones, thus allowing us to analyze also operators where some kind of jitter and round-off errors appear (see [54,56] for examples of this kind; note that, in [54], for the operators $\mathcal{V}_w^{(4)}$ one should have assumed $L_{h_w(t)}$ to be uniformly Lipschitz). Instead of dealing with this more general case, we have preferred to limit our examination to the case when $L_{h_w(t)}$ are linear, due exclusively to clarity and smoothness of the exposition.

4 Convergence results

This section contains the main results of the article. We now briefly describe the road we will follow to prove the main theorem concerning the convergence of the operators $T_w f$ to f in Orlicz spaces. We will proceed step by step, by first proving their convergence in the L^∞-norm when $f \in C(G)$, and then the modular convergence when $f \in C_c(G)$. Next, we will apply a density result, together with a modular continuity property for the involved operators, to achieve the main theorem concerning the modular convergence in the subspace $\mathcal{Y} \subset L^\eta(G)$ of assumption (L4).

We first prove the following proposition.

Proposition 4.1. *Suppose that* (L1), (χ_1), (χ_2), *and* (C2) *are valid. Let* $f \in L^\infty(G)$. *Then the operator* T_w *is well defined for every* $w > 0$ *and in fact*

$$|T_w f(z)| \le M\psi(\Upsilon\|f\|_\infty).$$

In particular, T_w *maps* $L^\infty(G)$ *into* $L^\infty(G)$, *for every* $w > 0$.

Proof. Let $f \in L^\infty(G)$. Then

$$\begin{aligned}|T_w f(z)| &\le \int_H |\chi_w(z - h_w(t), L_{h_w(t)}f)|\mathrm{d}\mu_H(t)\\ &\le \int_H |C_w(z - h_w(t))|\cdot\psi(|L_{h_w(t)}f|)\mathrm{d}\mu_H(t)\\ &\le M\psi(\Upsilon\|f\|_\infty).\end{aligned}$$

□

We now prove the first result concerning the convergence of T_w as $w \to +\infty$.

Theorem 4.2. *Suppose that* $(L1)$, $(L2)$, (χ_1)–(χ_3), *and* $(C1)$–$(C3)$ *are valid. Then each* T_w *maps* $C(G)$ *into* $L^\infty(G)$, *for every* $w > 0$, *and*

$$\lim_{w\to+\infty} \|T_w f - f\|_\infty = 0.$$

Proof. The fact that $C(G)$ is mapped to $L^\infty(G)$ via T_w is an easy consequence of Proposition 4.1. Now, let $f \in C(G)$. Then

$$\begin{aligned}
|T_w f(z) - f(z)| &= \left| \int_H \chi_w(z - h_w(t), L_{h_w(t)} f) \mathrm{d}\mu_H(t) - f(z) \right| \\
&\leq \left| \int_H \chi_w(z - h_w(t), L_{h_w(t)} f) \mathrm{d}\mu_H(t) - \int_H \chi_w(z - h_w(t), f(z)) \mathrm{d}\mu_H(t) \right| \\
&\quad + \left| \int_H \chi_w(z - h_w(t), f(z)) \mathrm{d}\mu_H(t) - f(z) \right| \\
&\leq \int_H C_w(z - h_w(t)) \cdot \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t) + \left| \int_H \chi_w(z - h_w(t), f(z)) \mathrm{d}\mu_H(t) - f(z) \right| \\
&=: I_1 + I_2.
\end{aligned}$$

Let us first analyze the summand

$$I_1 = \int_H C_w(z - h_w(t)) \cdot \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t).$$

Fix $\varepsilon > 0$, and let $B \in \mathcal{B}$ be such that, if $z - h_w(t) \in B$, then $|L_{h_w(t)} f - f(z)| < \varepsilon$ for sufficiently large $w > 0$ (see (L2)). Now

$$\begin{aligned}
I_1 &= \int_{B_{w,z}} C_w(z - h_w(t)) \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t) + \int_{H \setminus B_{w,z}} C_w(z - h_w(t)) \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t) \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}$$

Using (C2), we have

$$I_{1,1} = \int_{B_{w,z}} C_w(z - h_w(t)) \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t) \leq M\psi(\varepsilon).$$

Moreover,

$$\begin{aligned}
I_{1,2} &= \int_{H \setminus B_{w,z}} C_w(z - h_w(t)) \psi\Big(|L_{h_w(t)} f - f(z)|\Big) \mathrm{d}\mu_H(t) \\
&\leq \psi(\Upsilon\|f\|_\infty + \|f\|_\infty) \int_{H \setminus B_{w,z}} C_w(z - h_w(t)) \mathrm{d}\mu_H(t).
\end{aligned}$$

Using the property (C3), we have that

$$\lim_{w\to+\infty} I_{1,2} = 0,$$

uniformly with respect to $z \in G$. This, together with the facts that ε is arbitrary and ψ is continuous, implies that the summand I_1 converges to 0 as $w \to +\infty$, uniformly with respect to $z \in G$.

It remains to discuss the behavior of

$$I_2 = \left| \int_H \chi_w(z - h_w(t), f(z))\mathrm{d}\mu_H(t) - f(z) \right|.$$

Let us fix again $\varepsilon > 0$, and choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and $\sup_{z\in G}|f(z)| \leq n$. Let $A_n = \{z \in G | 0 < |f(z)| \leq 1/n\}$. We can write

$$\begin{aligned} I_2 &\leq \left| \int_H \chi_w(z - h_w(t), f(z)\mathbf{1}_{G\setminus A_n}(z))\mathrm{d}\mu_H(t) - f(z)\mathbf{1}_{G\setminus A_n}(z) \right| + \left| \int_H \chi_w(z - h_w(t), f(z)\mathbf{1}_{A_n}(z))\mathrm{d}\mu_H(t) - f(z)\mathbf{1}_{A_n}(z) \right| \\ &\leq \mathcal{S}_w^{(n)}(z)\|f\|_\infty + \left| \int_H \chi_w(z - h_w(t), f(z)\mathbf{1}_{A_n}(z))\mathrm{d}\mu_H(t) - f(z)\mathbf{1}_{A_n}(z) \right| \\ &=: I_{2,1} + I_{2,2}, \end{aligned}$$

where the symbol $\mathbf{1}_A$ denotes the characteristic function of a measurable set $A \subset G$. Using (χ_3), we easily have that $I_{2,1} \to 0$ as $w \to +\infty$, uniformly with respect to $z \in G$. Concerning $I_{2,2}$, we have

$$I_{2,2} \leq \int_H C_w(z - h_w(t))\cdot \psi\Big(|f(z)|\mathbf{1}_{A_n}(z)\Big)\mathrm{d}\mu_H(t) + |f(z)|\mathbf{1}_{A_n}(z) \leq M\psi\left(\frac{1}{n}\right) + \frac{1}{n} \leq M\psi(\varepsilon) + \varepsilon.$$

Since ε is chosen at will, the above estimation for $I_{2,2}$ and the behavior of ψ imply that $I_2 \to 0$ as $w \to +\infty$, uniformly with respect to $z \in G$. Putting all the information together, we have $|T_w f(z) - f(z)| \to 0$ as $w \to +\infty$, uniformly with respect to $z \in G$, hence $\|T_w f - f\|_\infty \to 0$ as $w \to +\infty$, as desired. □

We now move to a first result of convergence in Orlicz spaces. It is a general fact that the assumptions which we have used to prove the uniform convergence for continuous functions alone are not sufficient to prove the convergence in Orlicz spaces, even when dealing with functions in $C_c(G)$. In fact, what we can prove is the following.

Theorem 4.3. *Suppose that* $(L1)$–$(L3)$, (χ_1)–(χ_3), *and* $(C1)$–$(C4)$ *are valid. Let φ be a convex φ-function and $f \in C_c(G)$. Then for $\lambda < \alpha/M$ (being α the constant of* $(L3)$*), we have*

$$\lim_{w\to+\infty} I^\varphi(\lambda(T_w f - f)) = 0, \tag{3}$$

i.e., $T_w f$ converges modularly to f in $L^\varphi(G)$.

Proof. Theorem 4.2 tells us that

$$\lim_{w\to+\infty} \|T_w f - f\|_\infty = 0. \tag{4}$$

We claim that, if $\lambda < \frac{\alpha}{M}$, the Vitali convergence theorem can be applied to the family $\varphi(|T_w f(\cdot) - f(\cdot)|)_{w>0}$. This will in turn imply immediately the statement of the theorem.

To prove the assertion, let $K_1 = \mathrm{Supp} f$, $K \supset K_1$, and $\alpha > 0$ be such that (L3) holds. For every fixed $\varepsilon > 0$, we use (C4) to find a symmetric compact set $C \subset G$ such that

$$\int_{G\setminus C} \Upsilon_w(K)\cdot C_w(z - h_w(t))\mathrm{d}\mu_H(t) < \varepsilon,$$

for every $t \in K_w$ and for every sufficiently large $w > 0$.

Now, we use Jensen's inequality together with the Fubini-Tonelli theorem to infer that

$$\begin{aligned}
I &:= \int\limits_{G\setminus C} \varphi(\lambda|T_w f(z)|)\mathrm{d}\mu_G(z) \\
&= \int\limits_{G\setminus C} \varphi\left(\lambda \left| \int\limits_H \chi_w(z-h_w(t), L_{h_w(t)}f)\mathrm{d}\mu_H(t) \right| \right)\mathrm{d}\mu_G(z) \\
&\le \int\limits_{G\setminus C} \varphi\left(\lambda \int\limits_H C_w(z-h_w(t))\cdot \psi\big(|L_{h_w(t)}f|\big)\mathrm{d}\mu_H(t)\right)\mathrm{d}\mu_G(z) \\
&\le \frac{1}{M\Upsilon_w(K)} \int\limits_H \left(\varphi\Big(\lambda M\psi\big(|L_{h_w(t)}f|\big)\Big) \int\limits_{G\setminus C} C_w(z-h_w(t))\Upsilon_w(K)\mathrm{d}\mu_G(z)\right)\mathrm{d}\mu_H(t) \\
&\le \frac{1}{M\Upsilon_w(K)} \left[\int\limits_{K_w} \cdots + \int\limits_{H\setminus K_w} \cdots \right] =: I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &= \frac{1}{M\Upsilon_w(K)} \int\limits_{K_w} \left(\varphi\Big(\lambda M\psi\big(|L_{h_w(t)}f|\big)\Big) \int\limits_{G\setminus C} C_w(z-h_w(t))\Upsilon_w(K)\mathrm{d}\mu_G(z)\right)\mathrm{d}\mu_H(t) \\
&\le \frac{1}{M\Upsilon_w(K)} \cdot \Upsilon_w(K)\varphi(\lambda M\psi(\Upsilon\|f\|_\infty))\cdot \varepsilon = \frac{\varphi(\lambda M\psi(\Upsilon\|f\|_\infty))}{M}\cdot\varepsilon.
\end{aligned}$$

Moreover, choosing $\lambda < \frac{\alpha}{M}$, we have

$$I_2 \le \frac{1}{M\Upsilon_w(K)} \cdot \varepsilon \cdot \|C_w\|_1 \Upsilon_w(K) \le \frac{\Gamma}{M}\varepsilon,$$

for every sufficiently large $w > 0$. Putting all these information together, we have that for every $\lambda < \frac{\alpha}{M}$ the integral

$$I \le \left[\frac{\varphi(\lambda M\psi(\Upsilon\|f\|_\infty)) + \Gamma}{M}\right]\cdot\varepsilon,$$

for sufficiently large $w > 0$. Now, if $C \subset G$ is a measurable set with $\mu_G(C) < +\infty$, we can write, by Proposition 4.1,

$$\int\limits_C \varphi(\lambda|T_w f(z)|)\mathrm{d}\mu_G(z) \le \mu_G(C)\varphi(\lambda M\psi(\Upsilon\|f\|_\infty)).$$

This implies that for every $\varepsilon > 0$ it suffices to take

$$\delta \le \frac{\varepsilon}{\varphi(\lambda M\psi(\Upsilon\|f\|_\infty))}$$

to have

$$\int\limits_C \varphi(\lambda|T_w f(z)|)\mathrm{d}\mu_G(z) \le \varepsilon,$$

whenever

$$\mu_G(C) \le \delta.$$

All these facts imply that the integrals

$$\int_{(.)} \varphi(\lambda|T_w f(z) - f(z)|)\mathrm{d}\mu_G(z)$$

are equi-absolutely continuous, hence the claim is proved, i.e., the Vitali convergence theorem can be applied to the family $\varphi(|T_w f(\cdot) - f(\cdot)|)_{w>0}$ whenever $\lambda < \frac{\alpha}{M}$. □

Note that if the family $(L_{h_w(t)})$ satisfies $L_{h_w(t)}f = 0$ whenever $t \notin K_w$, then assumption (L3) is no more needed in the proof of Theorem 4.3, hence no choice of α (and hence λ) has to be done in order to obtain the proof. This shows that, in this case, the operators $T_w f$ converge to f in the stronger Luxemburg norm. We state this observation as a corollary.

Corollary 4.4. *Suppose that* (*L*1), (*L*2), (χ_1)–(χ_3), *and* (*C*1)–(*C*4) *are valid. Let* φ *be a convex* φ*-function and* $f \in C_c(G)$. *Assume that*

$$L_{h_w(t)}f = 0, \ \text{ whenever } \ t \notin K_w,$$

where $K \subset G$ *is a compact set with* $\mathrm{Supp} f = K_1 \subset K \subset G$. *Then*

$$\lim_{w\to+\infty} \|T_w f - f\|_\varphi = 0.$$

Let us now try to extend the convergence to a subset which is larger than $C_c(G)$. Let $f \in L^\varphi(G)$ (with φ convex) and let us estimate ($\lambda > 0$)

$$\begin{aligned} I^\varphi(\lambda T_w f) &\leq \frac{1}{M}\int_H \left(\varphi\Big(M\lambda\psi\big(|L_{h_w(t)}f|\big)\Big)\int_G C_w(z - h_w(t))\mathrm{d}\mu_G(z)\right)\mathrm{d}\mu_H(t) \\ &\leq \frac{\|C_w\|_1}{M}\int_H \varphi\Big(M\lambda\psi\big(|L_{h_w(t)}f|\big)\Big)\mathrm{d}\mu_H(t), \end{aligned} \tag{5}$$

by using again Jensen's inequality and the Fubini-Tonelli theorem. It appears to be clear now that if we want to obtain a result of convergence for a function $f \in L^\varphi(G)$, the last inequality in (5) has to be compared to a modular $I^\eta(\beta f)$, for some φ-function η and $\beta > 0$, because this will imply at least the modular continuity of the operator. This is exactly the purpose of the assumptions (H) and (L4). In particular, we have the following.

Theorem 4.5. *Let* (*H*), (*L*4), (χ_1), (χ_2), (*C*1), *and* (*C*2) *be valid. Then for every* $f \in \mathcal{Y} \subset L^\eta(G)$ ($\mathcal{Y}$ *being the space of assumption* (*L*4)), *and* $\lambda \in (0, 1)$ *there exists a constant* $\zeta > 0$ *such that*

$$I^\varphi(\zeta T_w f) \leq \frac{c}{M} I^\eta(\lambda\beta f),$$

for sufficiently large $w > 0$. *In particular, if* $w > 0$ *is sufficiently large,*

$$T_w : \mathcal{Y} \to L^\varphi(G).$$

Proof. Choose a constant $\zeta > 0$, and use the above estimate (5) to obtain

$$I^\varphi(\zeta T_w f) \leq \frac{\|C_w\|_1}{M}\int_H \varphi\Big(M\zeta\psi\big(|L_{h_w(t)}f|\big)\Big)\mathrm{d}\mu_H(t).$$

Now, with (L4) in mind and by using (*H*), take $\lambda \in (0, 1)$ and choose $\zeta > 0$ such that $\zeta M \leq C_\lambda$. We can rewrite the above inequality as

$$\begin{aligned} I^\varphi(\zeta T_w f) &\leq \frac{\|C_w\|_1}{M}\int_H \varphi\Big(C_\lambda\psi\big(|L_{h_w(t)}f|\big)\Big)\mathrm{d}\mu_H(t) \\ &\leq \frac{\|C_w\|_1}{M}\int_H \eta\big(\lambda|L_{h_w(t)}f|\big)\mathrm{d}\mu_H(t) \leq \frac{c}{M} I^\eta(\lambda\beta f), \end{aligned}$$

for sufficiently large $w > 0$. □

Note that the inequality in Theorem 4.5 can be rewritten by considering two functions $f, g \in \mathcal{Y}$; and in fact, reasoning as above one has

$$I^{\varphi}(\zeta(T_w f - T_w g)) \le \frac{c}{M} I^{\eta}(\lambda\beta(f-g)), \tag{6}$$

which gives the modular continuity of the operators T_w.

We are approaching to a result of convergence when $f \in \mathcal{Y}$. Before, we state a well-known density result (see [6] for a proof).

Lemma 4.6. *For every φ-function φ, the set $C_c^{\infty}(G)$ is dense in $L^{\varphi}(G)$ with respect to the modular convergence.*

We can now state and prove the main theorem.

Theorem 4.7. *Let (H), (L1)–(L4), (χ_1)–(χ_3), and (C1)–(C4) be valid. Let φ be a convex φ-function. Let $f \in \mathcal{Y} \cap L^{\varphi}(G)$. Then there exists a constant $\zeta > 0$ such that*

$$\lim_{w\to+\infty} I^{\varphi}(\zeta(T_w f - f)) = 0,$$

that is, $T_w f$ converges modularly to f as $w \to +\infty$.

Proof. Let $f \in \mathcal{Y} \cap L^{\varphi}(G)$, and fix $\varepsilon > 0$. By Lemma 4.6, there exists a function $g \in C_c^{\infty}(G)$ and a number $\lambda_1 > 0$ such that

$$I^{\varphi+\eta}(\lambda_1(f-g)) < \varepsilon.$$

This clearly implies that also

$$I^{\varphi}(\lambda_1(f-g)) < \varepsilon \text{ and } I^{\eta}(\lambda_1(f-g)) < \varepsilon.$$

Now, let β be the constant of assumption (L4), corresponding to the function $f-g$, and take $\bar{\lambda} = \min\{\lambda_1, \lambda_1/\beta\}$. It follows that

$$I^{\varphi}(\bar{\lambda}(f-g)) < \varepsilon \text{ and } I^{\eta}(\bar{\lambda}\beta(f-g)) < \varepsilon.$$

Moreover, from Theorem 4.3, one has

$$I^{\varphi}(\lambda(T_w g - g)) < \varepsilon,$$

for every $\lambda < \frac{\alpha}{M}$ and for sufficiently large $w > 0$. Choose

$$\zeta \le \min\left\{\frac{\alpha}{3M}, \frac{\bar{\lambda}}{3}, \frac{C_{\bar{\lambda}}}{3M}\right\}.$$

Then, if $w > 0$ is sufficiently large, by Theorem 4.5, we have

$$\begin{aligned} I^{\varphi}(\zeta(T_w f - f)) &\le I^{\varphi}(3\zeta(T_w f - T_w g)) + I^{\varphi}(3\zeta(T_w g - g)) + I^{\varphi}(3\zeta(g-f)) \\ &\le \frac{c}{M} I^{\eta}(\bar{\lambda}\beta(f-g)) + I^{\varphi}(3\zeta(T_w g - g)) + I^{\varphi}(\bar{\lambda}(g-f)) \\ &\le \left(2 + \frac{c}{M}\right)\varepsilon. \end{aligned}$$

The theorem is proved, since $\varepsilon > 0$ was chosen at will. □

Remark 4.8.

(1) Orlicz spaces are of fundamental importance in many fields, such as PDEs and interpolation theory. As already mentioned, besides including the Lebesgue spaces, other well-known examples of Orlicz spaces are the so-called interpolation (or Zygmund) spaces and the exponential spaces. In a little more detail, Zygmund spaces are generated by the φ-functions

$$\varphi_{\alpha,\beta}(u) = u^\alpha \ln^\beta(e + u),$$

where $\alpha \geq 1$ and $\beta > 0$, which arise in connection with the Hardy-Littlewood maximal functions. For information, see [12,50]. Another example is furnished by the exponential spaces, generated by the functions $\varphi_\alpha(u) = e^{u^\alpha} - 1$, where $\alpha > 0$. The interested reader can see, for instance, [37].
Therefore, Theorem 4.7 includes, as particular cases, approximation result for L^p, Zygmund and exponential spaces. Note that exponential spaces do not satisfy the Δ_2-condition and hence in this instance we have a modular convergence theorem which cannot be reduced to a strong convergence theorem by using the Luxemburg norm, as it happens in the cases of L^p or $L^\alpha \ln^\beta L$ spaces.

(2) The theory exposed in this article gives a unified approach in studying the convergence of several class of (linear and) nonlinear operators, both of discrete and integral kind. There are other families of operators which can be studied in this setting; for instance one can introduce "time-jitter" errors and some types of round-off errors and similar perturbations (see, e.g., [54,56]).

5 Examples

There are a lot of well-known examples of operators T_w, which can be expressed in the form (2). We list here some of the most representative ones.

(1) If $G = \mathbb{R}$, $H = \mathbb{Z}$, then $d\mu_H$ is the counting measure. It follows that (2) becomes a series, namely

$$T_w f(x) := \sum_{k\in\mathbb{Z}} \chi_w(z - h_w(t), L_{h_w(t)} f).$$

This kind of series is a non-linear version of the *sampling-type series*, and in fact, according to the form of the operators $L_{h_w(t)}$, many different forms of sampling series arise. As an example, let $L_{h_w(t)} f = f(h_w(t))$, and take $h_w(t) = t_k / w$, where $(t_k)_{k\in\mathbb{Z}}$ is an increasing sequence of real numbers such that

$$\lim_{k\to\pm\infty} t_k = \pm\infty, \quad \delta < \Delta_k := t_{k+1} - t_k < \Delta, \ (\delta, \Delta > 0).$$

Then the aforementioned operator becomes

$$T_w^{(1)} f(x) = \sum_{k\in\mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}, f\left(\frac{t_k}{w}\right)\right), \tag{7}$$

which is a nonlinear version of the *generalized sampling series* [4,5,10,14,20,21].

(2) Let G, H and $h_w(t)$ be as in the previous example. Set

$$L_{h_w(t)} f = \frac{w}{\Delta_k} \int_{t_w/w}^{t_{k+1}/w} f(z)dz.$$

In this case, we obtain

$$T_w^{(2)} f(x) := \sum_{k\in\mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(z)dz\right), \tag{8}$$

which is a nonlinear version of the *Kantorovich sampling series* [53]. Actually, the average values $L_{h_w(t)} f$ can be replaced by different kinds of averages, giving rise to operators which reduce time-jitter errors [16,17,54,56].

(3) Let H, G and $h_w(t)$ be as above. Choose a function $\rho \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \rho(u)du = 1.$$

Set

$$L_{h_w(t)} f := w \int_{\mathbb{R}} \rho(wu - t_k) f(u) \mathrm{d}u.$$

We obtain the series

$$T_w^{(3)} f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w}, w \int_{\mathbb{R}} \rho(wu - t_k) f(u) \mathrm{d}u \right), \tag{9}$$

which is a nonlinear version of the *Durrmeyer generalized sampling series* [26].

(4) Now, take $H = G = \mathbb{R}$ and let $\mathrm{d}\mu_H(t) = \mathrm{d}t$ be the Lebesgue measure. We can retrieve some interesting integral operators, as follows: if $h_w(t) = t$ and $L_{h_w(t)} f = L_t f = f(t)$, we obtain

$$T_w^{(4)} f(x) = \int_{\mathbb{R}} \chi_w(x - t, f(t)) \mathrm{d}t, \tag{10}$$

which is a nonlinear convolution integral operator (see, e.g., [18]).
If, for instance, $h_w(t) = t$ as before, and

$$L_t f := \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) \mathrm{d}u,$$

then we obtain

$$T_w^{(5)} f(x) = \int_{\mathbb{R}} \chi_w \left(x - t, \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) \mathrm{d}u \right) \mathrm{d}t, \tag{11}$$

which is a Kantorovich version of a nonlinear convolution operator.

(5) Take now $H = G = \mathbb{R}^+$. In this case, the group operation is the product, and the unique Haar measure on $\mathbb{R}^+$ (up to multiplicative constants) is the logarithmic one: $\mathrm{d}\mu(t) = \frac{\mathrm{d}t}{t}$. We can set, for instance, $h_w(t) = t$, and $L_t f = f(t)$ as before. We obtain

$$T_w^{(6)} f(x) = \int_0^\infty \chi_w \left(\frac{x}{t}, f(t) \right) \mathrm{d}t. \tag{12}$$

This is a nonlinear version of the Mellin convolution operator.

There are many other types of operators which are included in the general form of (2), as for example a Kantorovich version of the operators $T_w^{(6)} f$. Furthermore, note that, for instance, time-jitter errors can be included [16,17,54,56], and the general form of the operators $L_{h_w(t)}$ allows one to include also some types of the so-called round-off errors [56] (see Remark 3.1 (c)).

Now, it is clear that all the examples above can be studied if the various assumptions stated in the previous section are satisfied. We will see that there is a huge amount of kernels for which the assumptions are satisfied. Let us illustrate all that in a more detailed way, by finding assumptions on the kernels in such a way that the examples above can be included (we will show these facts for only some of them).

Let us start with (7). We have

$$T_w^{(1)} f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w}, f\left(\frac{t_k}{w} \right) \right),$$

where $(t_k)_{k \in \mathbb{Z}}$ is an increasing sequence of real numbers such that

$$\lim_{k\to\pm\infty} t_k = \pm\infty, \quad \delta < \Delta_k := t_{k+1} - t_k < \Delta, \ (\delta, \Delta > 0).$$

It is straightforward to prove that $L_{t_k/w}f := f(t_k/w)$ satisfies (L1) and (L2), and in fact $\|\tilde{L}_{t_k/w}f\| = 1$, for every $k \in \mathbb{Z}$ and $w > 0$. Now, (L3) is satisfied as well, and in fact the stronger condition of Corollary 4.4 holds, which ensures the Luxemburg convergence for $f \in C_c(\mathbb{R})$. Concerning (L4), take any function η, let $\lambda > 0$, and write

$$\limsup_{w\to+\infty}\|C_w\|_1 \sum_{k\in\mathbb{Z}} \eta(\lambda|f(t_k/w)|) \le \limsup_{w\to+\infty}\frac{w\|C_w\|_1}{\delta}\sum_{k\in\mathbb{Z}}\frac{\Delta_k}{w}\eta(\lambda|f(t_k/w)|).$$

Now, the last sum in the right-hand side of the above inequality is a Riemann sum, hence

$$\limsup_{w\to+\infty}\sum_{k\in\mathbb{Z}}\frac{\Delta_k}{w}\eta(\lambda|f(t_k/w)|) = I^{\eta}(\lambda f),$$

whenever $f \in E^{\eta}(\mathbb{R}) \cap BV^{\eta}(\mathbb{R})$ (see [36]); here $BV^{\eta}(\mathbb{R})$ denotes the set of those functions such that $\eta(\lambda|f|) \in BV(\mathbb{R})$, for every $\lambda > 0$. It follows that

$$\limsup_{w\to+\infty}\|C_w\|_1 \sum_{k\in\mathbb{Z}} \eta(\lambda|f(t_k/w)|) \le cI^{\eta}(\lambda\beta f),$$

for $\beta = 1$ and $\frac{1}{\delta}\limsup_{w\to+\infty} w\|C_w\|_1 \le c < +\infty$. So, we have proved that (L4) is valid whenever $f \in \mathcal{Y} = E^{\eta}(\mathbb{R}) \cap BV^{\eta}(\mathbb{R})$, and

$$\limsup_{w\to+\infty} w\|C_w\|_1 < +\infty.$$

Therefore, we can state the following.

Theorem 5.1. *Let (χ_1)–(χ_3), $(C1)$–$(C4)$, and (H) be satisfied, together with the additional assumption*

$$\limsup_{w\to+\infty} w\|C_w\|_1 < +\infty.$$

Let $f \in E^{\eta}(\mathbb{R}) \cap BV^{\eta}(\mathbb{R}) \cap L^{\varphi}(\mathbb{R})$. Then there exists a number $\zeta > 0$ such that

$$\lim_{w\to+\infty} I^{\varphi}(\zeta(T_w^{(1)}f - f)) = 0.$$

Now, let us consider the operators $T_w^{(2)}f$. We have to study

$$T_w^{(2)}f(x) := \sum_{k\in\mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}, \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(z)\,dz\right), \ \Delta_k := t_{k+1} - t_k,$$

and test (and/or find conditions for) the validity of $(L1)$–$(L4)$. $(L1)$ is easily satisfied, since

$$\|L_{t_k/w}\| = 1.$$

For $(L2)$, let f be continuous at a point $x \in \mathbb{R}$. Let us fix $\varepsilon > 0$ and let $\xi > 0$ be such that $|f(x) - f(z)| < \varepsilon$, whenever $|x - z| < \xi$. Now, since

$$|L_{t_k/w}f - f(x)| \le \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} |f(z) - f(x)|\,dz,$$

by choosing $\overline{w} = \frac{2\Delta}{\xi}$ and $B_\varepsilon = B(0, \xi/2)$, we have

$$|L_{t_k/w}f - f(x)| \le \varepsilon,$$

since, for $w > \overline{w}$, $|t_{k+1}/w - t_k/w| < \xi/2$, and so $|x - z| < \xi$.

Now, we check ($L3$). Let $\mathrm{Supp}(f) = K_1 = [-a_1, a_1]$. Let $a = a_1 + \Delta$ and set $K = [-a, a]$. Clearly, if $w > 0$ and if $t_k/w \notin K$, then

$$\int_{t_k/w}^{t_{k+1}/w} |f(x)|\mathrm{d}x = 0,$$

i.e., the stronger assumption of Corollary 4.4 is valid. It remains to understand what ($L4$) stands for. For a convex φ-function η, we have, using Jensen's inequality,

$$\|C_w\|_1 \sum_{k\in\mathbb{Z}} \eta\left(\lambda\left(\left|\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(x)\mathrm{d}x\right|\right)\right) \leq \frac{w\|C_w\|_1}{\delta}\sum_{k\in\mathbb{Z}}\int_{t_k/w}^{t_{k+1}/w}\eta(\lambda|f(x)|)\mathrm{d}x = \frac{w\|C_w\|_1}{\delta}I^\eta(\lambda f). \tag{13}$$

This means that ($L4$) is satisfied with $\beta = 1$, $f \in \mathcal{Y} = L^\eta(\mathbb{R})$, and, as before,

$$\limsup_{w\to\infty} w\|C_w\|_1 < +\infty.$$

So, we can state now the following.

Theorem 5.2. *Let* (χ_1)–(χ_3), $(C1)$–$(C4)$, *and* (H) *be satisfied, together with the additional assumption*

$$\limsup_{w\to+\infty} w\|C_w\|_1 < +\infty.$$

Then the operator $T_w^{(2)}$ *maps* $L^\eta(\mathbb{R})$ *into* $L^\varphi(\mathbb{R})$. *If* $f \in L^\varphi(\mathbb{R}) \cap L^\eta(\mathbb{R})$ $(L^{\varphi+\eta}(\mathbb{R}))$, *then* $T_w^{(2)}f$ *converges modularly to* f *in* $L^\varphi(\mathbb{R})$.

As an example, assume $\varphi(u) = u^p$ and $\psi(u) = u^{q/p}$ $(1 \leq q \leq p < +\infty)$: then $L^\varphi(\mathbb{R}) = L^p(\mathbb{R})$ and we can take $\eta(u) = u^q$ and $C_\lambda = \lambda^{q/p}$, so that $\mathcal{Y} = L^q(\mathbb{R})$.

The above theorem states that T_w maps $L^q(\mathbb{R})$ into $L^p(\mathbb{R})$ and that $T_w f$ converges to f in the L^p norm, whenever $f \in L^q(\mathbb{R}) \cap L^p(\mathbb{R})$. See [3,25,29–31,34] for other examples and applications of the Kantorovich sampling series.

Now we move our attention to an integral operator, namely $T_w^{(5)}f$, i.e.,

$$T_w^{(5)}f(x) = \int_{\mathbb{R}} \chi_w\left(x - t, \frac{w}{2}\int_{t-1/w}^{t+1/w} f(u)\mathrm{d}u\right)\mathrm{d}t.$$

Here

$$L_{h_w(t)}f = \frac{w}{2}\int_{t-1/w}^{t+1/w} f(z)\mathrm{d}z.$$

Again ($L1$) is satisfied, because

$$\|L_{h_w(t)}f\| = 1.$$

Arguing as in the example above, we can check that ($L2$) is valid as well. Indeed, let f be continuous at a point $x \in \mathbb{R}$. Choose $\varepsilon > 0$, and let $\xi > 0$ be such that $|f(x) - f(z)| < \varepsilon$ when $|x - z| < \xi$. Then, set $B = B(0, \xi/2)$ and $\overline{w} = 4/\xi$. If $w > \overline{w}$, $|x - t| < \xi/2$, and $z \in (t - 1/w, t + 1/w)$, then $|x - z| \leq |x - t| + |t - z| \leq \xi/2 + 2/w < \xi$, hence $|f(x) - f(z)| < \varepsilon$. It follows that

$$|L_{h_w(t)}f - f(x)| \leq \varepsilon.$$

Now, we check ($L3$). Let $f \in C_C(\mathbb{R})$ be such that $\mathrm{Supp}f = K_1 = [-a_1, a_1]$. Set $a = a_1 + 2$ and $K = [-a, a]$. Then it is easy to see that, also in this case, the stronger assumption of Corollary 4.4 holds, hence the convergence in the Luxemburg norm is ensured.

It remains to check (L4). Also in this case, let η be any convex φ-function. We have

$$\|C_w\|_1 \int_{\mathbb{R}} \eta\left(\lambda \cdot \left| \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u)\mathrm{d}u \right|\right)\mathrm{d}t \le \frac{w\|C_w\|_1}{2} \int_0^{2/w} \left[\int_{\mathbb{R}} \eta(\lambda|f(s+t-1/w)|)\mathrm{d}s \right]\mathrm{d}t$$
$$\le \frac{w\|C_w\|_1}{2} \int_0^{2/w} I^\eta(\lambda f)\mathrm{d}t = \|C_w\|_1 I^\eta(\lambda f).$$

This inequality means that $(L4)$ is valid whenever $f \in L^\eta(\mathbb{R})$, $\beta = 1$, and $\limsup_{w\to+\infty}\|C_w\|_1 \le c < +\infty$. We have proved the following.

Theorem 5.3. *Let (χ_1)–(χ_3), $(C_1)(C_4)$, and (H) hold. Assume further that*

$$\limsup_{w\to+\infty}\|C_w\|_1 < +\infty.$$

Then $T_w^{(5)}$ maps $L^\eta(\mathbb{R})$ into $L^\varphi(\mathbb{R})$. Moreover, if $f \in L^\eta(\mathbb{R}) \cap L^\varphi(\mathbb{R})$, then $T_w^{(5)}f$ converges modularly to f in $L^\varphi(\mathbb{R})$.

We finish this section by mentioning a case in which $(L3)$ is valid but Corollary 4.4 is not, namely a special case of the operators $T_w^{(3)}$,

$$T_w^{(3)}f(x) = \sum_{k\in\mathbb{Z}} \chi\left(nx-k, n\int_{\mathbb{R}} \rho(nu-k)f(u)\mathrm{d}u\right),$$

where $\chi \in L^1(\mathbb{R})$ (in this case $\chi_n(x) = \chi(nx)$,$t_k = k$, $w = n$, and $h_n(k) = k/n$.). It is easy to show that if $\mathrm{Supp} f = K$ and $[-a, a] \supset K$, then

$$L_{k/n}f = n\int_{-a}^{a} \rho(nu-k)f(u)\mathrm{d}u \ne 0$$

for $\frac{k}{n} \notin [-a, a]$, hence Corollary 4.4 does not hold. In this case, however, $(L3)$ is valid. For, let $f \in C_c(\mathbb{R})$ have as support the set $K_1 \supset [-a, a]$. It suffices to show that for every $\varepsilon > 0$ there exists a compact set $[-M_n, M_n] \supset K_1$ such that

$$\sum_{|k|>M_n} \varphi(\psi|L_{k/n}f|) = \sum_{|k|>M_n} \varphi\left(\psi\left(\left| n\int_{\mathbb{R}} \rho(nu-k)f(u)\mathrm{d}u \right|\right)\right) < \varepsilon$$

for all sufficiently large $n \in \mathbb{N}$. Set $\mu = \varphi \circ \psi$, where μ is assumed to be a convex φ-function (see, e.g., Remark 3.1 (b)). The summands above can be estimated by

$$\mu\left(\left| n\int_{\mathbb{R}} \rho(nu-k)f(u)\mathrm{d}u \right|\right) \le \frac{1}{\|\rho\|_1} \int_{\mathbb{R}} |\rho(t)\mu\left(\|\rho\|_1 f\left(\frac{k+t}{n}\right)\right)\mathrm{d}t.$$

Set

$$J_n = \sum_{k\in\mathbb{Z}} \mu\left(\|\rho\|_1 f\left(\frac{k+t}{n}\right)\right), \quad (n \in \mathbb{N}).$$

For fixed $n \in \mathbb{N}$, each J_n has a finite number of summands, namely those which are determined by the numbers $k \in \mathbb{Z}$ such that

$$-na - t < k < na - t.$$

Those values lie in an interval of length at most $2na$, and hence

$$J_n \le 2na\mu(\|\rho\|_1 \cdot \|f\|_\infty).$$

This shows that J_n converges totally, and

$$\frac{1}{\|\rho\|_1}\int_{\mathbb{R}} \left| \rho(t)| \cdot \left[\sum_{k\in\mathbb{Z}} \mu\left(\|\rho\|_1 f\left(\frac{k+t}{n} \right) \right) \right] \right| \mathrm{d}t \le 2na\mu(\|\rho\|_1 \cdot \|f\|_\infty).$$

This in turn implies that for every $\varepsilon > 0$ there exists a number $M_n \in \mathbb{N}$ such that

$$\sum_{|k|>M_n} \varphi(\psi(|L_{k/n}f|)) < \varepsilon,$$

i.e., $(L3)$ is valid with $\alpha = 1$.

Moreover, $(L1)$ is easily verified, and in fact

$$\|L_{k/n}f\|_\infty = \|\rho\|_1, \;\; k \in \mathbb{Z}, n \in \mathbb{N}.$$

To check $(L2)$, set

$$\rho_n(x) = n\rho(nx).$$

Then $\|\rho_n\|_1 = \|\rho\|_1$ for every $n \in \mathbb{N}$. Rewriting $L_{k/n}f$ we have

$$L_{k/n}f = \int_{\mathbb{R}} \rho_n\left(u - \frac{k}{n}\right) f(u)\mathrm{d}u,$$

which shows that $L_{k/n}$ is the restriction of the convolution

$$L_n f(z) = \int_{\mathbb{R}} \rho_n(u - z) f(u)\mathrm{d}u$$

to the values $z = k/n$. Let f be uniformly continuous and bounded. It can be proved (see [18]) that

$$\lim_{n\to+\infty} \|L_n f - f\|_\infty = 0,$$

and this implies that for every $\varepsilon > 0$ there must be a number $\bar{n} \in \mathbb{N}$ such that

$$|L_n f(z) - f(z)| \le \frac{\varepsilon}{2},$$

for every $n > \bar{n}$ and $z \in \mathbb{R}$, and hence by setting $z = \frac{k}{n}$,

$$\left| L_{k/n}f - f\left(\frac{k}{n}\right) \right| \le \frac{\varepsilon}{2}, \;\; k \in \mathbb{Z}, \;\; n > \bar{n}.$$

Now it is easy to show $(L2)$. Indeed, fix $n > \bar{n}$. There exists $\delta > 0$ such that, if $|x - k/n| < \delta$, then $|f(x) - f(k/n)| < \varepsilon/2$, hence if $|x - k/n| < \delta$, we have

$$|L_{k/n}f - f(x)| \le \varepsilon.$$

It remains to analyze $(L4)$, as usual. As before, setting $\rho_n(x) = n\rho(nx)$, we have

$$L_{k/n}f = \int_{\mathbb{R}} \rho_n\left(u - \frac{k}{n}\right) f(u)\mathrm{d}u.$$

We have

$$\|C_n\|_1 \sum_{k\in\mathbb{Z}} \eta(\lambda|L_{k/n}f|) \le n\|C_n\|_1 \cdot \sum_{k\in\mathbb{Z}} \frac{1}{n}\eta\left(\lambda \left| \int_{\mathbb{R}} \rho_n\left(u - \frac{k}{n}\right) f(u)\mathrm{d}u \right|\right).$$

The quantity $n\|C_n\|_1$ equals $\|C\|_1$, and the right-hand summand of the above inequality is a Riemann sum of $\eta(\lambda|L_n f(z)|)$ for $z = k/n$, where

$$L_n f(z) = \int_{\mathbb{R}} \rho_n(u - z) f(u) \mathrm{d}u.$$

It follows that

$$\limsup_{n\to+\infty} \sum_{k\in\mathbb{Z}} \frac{1}{n}\eta\left(\lambda \left| \int_{\mathbb{R}} \rho_n\left(u - \frac{k}{n}\right) f(u)\mathrm{d}u \right|\right) = I^\eta(\lambda L_n f),$$

whenever $f \in E^\eta(\mathbb{R}) \cap BV^\eta(\mathbb{R})$ (see [36]). We claim that there is a constant $\beta > 0$, with β independent of $\lambda > 0$, such that

$$I^\eta(\lambda L_n f) \le I^\eta(\lambda\beta f),$$

which implies $(L4)$. For

$$\begin{aligned} I^\eta(\lambda L_n f) &\le \frac{1}{\|\rho_n\|_1} \int_{\mathbb{R}}\int_{\mathbb{R}} \eta(\lambda\|\rho_n\|_1 |f(u)|)\cdot|\rho_n(u - z)|\mathrm{d}u\mathrm{d}z \\ &\le \int_{\mathbb{R}} \eta(\lambda\|\rho_n\|_1 |f(u)|)\mathrm{d}u \\ &= \int_{\mathbb{R}} \eta(\lambda\|\rho\|_1 |f(u)|)\mathrm{d}u \\ &= I^\eta(\lambda\|\rho\|_1 f); \end{aligned}$$

this is satisfied with $\beta = \|\rho\|_1$ and hence $(L4)$ holds with $c = \|C\|_1$.

6 Some graphical representations

In this section, we illustrate with some plots the convergence of $T_w f$ to f in the L^p-norm as $w \to +\infty$, in some important cases. Some of them will be analogous to those depicted in the previous section, others will be different, with the aim of enumerating a whole panoply of operators whose properties can be analyzed using the results stated in this article. So, it is understood that, here, we will take $\varphi(x) = x^p$ for some $p \ge 1$. For computational purposes, it is convenient to take

$$\chi_w(x, u) = C(wx) g_w(u),$$

where $C \in L^1(\mathbb{R})$ and the family $(g_w)_{w>0}$ satisfies

(i) $g_w(u) \to u$ uniformly as $w \to +\infty$,

(ii) there exists a φ-function ψ such that $|g_w(u) - g_w(v)| \le \psi(|u - v|)$, for every $u, v > 0$. In the next examples, we choose

$$g_w(x) = \begin{cases} x^{1-1/w}, & 0 < x < 1 \\ x, & \text{otherwise}. \end{cases}$$

The choice of the function C can be made in various ways. For instance, when $G = \mathbb{R}$, we can take the Fejer kernel function

$$F(x) = \frac{1}{2}\operatorname{sin}c^2\left(\frac{x}{2}\right), \tag{14}$$

where

$$\operatorname{sin}c(x) = \begin{cases} \dfrac{\sin \pi x}{\pi x}, & x \in \mathbb{R}\setminus\{0\}, \\ 1, & x = 0. \end{cases}$$

However, many other kernel functions can be considered. For computational purposes, it will be convenient to take one which has compact support over $\mathbb{R}$. A very well-known kernel function of this sort is the B-spline function of order $n \in \mathbb{N}$, i.e.,

$$M_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\frac{n}{2}+x-1\right)_+^{n-1},$$

where the symbol $(\cdot)_+$ denotes the positive part. In order to obtain a faster convergence, it is useful to take some special linear combinations of B-splines [4]: we choose

$$M(x) = 4M_3(x) - 3M_4(x).$$

A graph of $M(x)$ is represented in Figure 1.

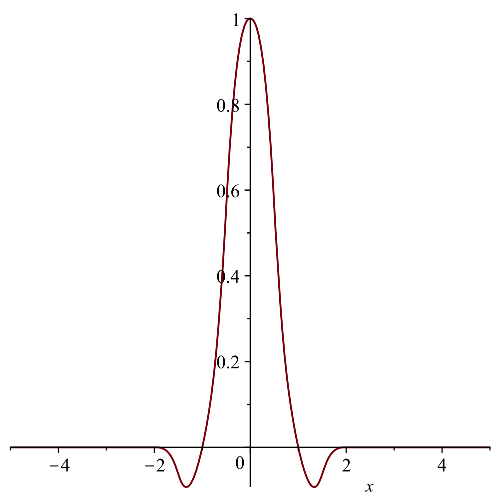

Figure 1: The graphs of the function $M(x)$.

In the next two examples, we will consider $C_w(x) = M(wx)$. Such a function and $\chi_w(x,u) = M(wx)g_w(u)$ satisfy all the assumptions which we have required throughout the article.

Let us start by considering operators of the form $T_w^{(2)}$ in the uniform-sampling case, i.e., when $t_k = k$. We have

$$T_w^{(2)}f(x) = \sum_{k\in\mathbb{Z}} M(wx-k)g_w\left(w\int_{k/w}^{(k+1)/w} f(u)\,du\right).$$

Let f be defined as

$$f(x) = \begin{cases} \dfrac{9}{x^2}, & x < -3 \\ x, & -3 \le x < -2 \\ -\dfrac{1}{2u}, & -2 \le x < 1 \\ \dfrac{3}{2}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x < 1 \\ -\dfrac{2}{u^5}, & x \ge 1. \end{cases}$$

Figure 2 illustrates the behavior of $T_w^{(2)}f(x)$ compared to that of $f(x)$ as $w = 5, 10, 20$.

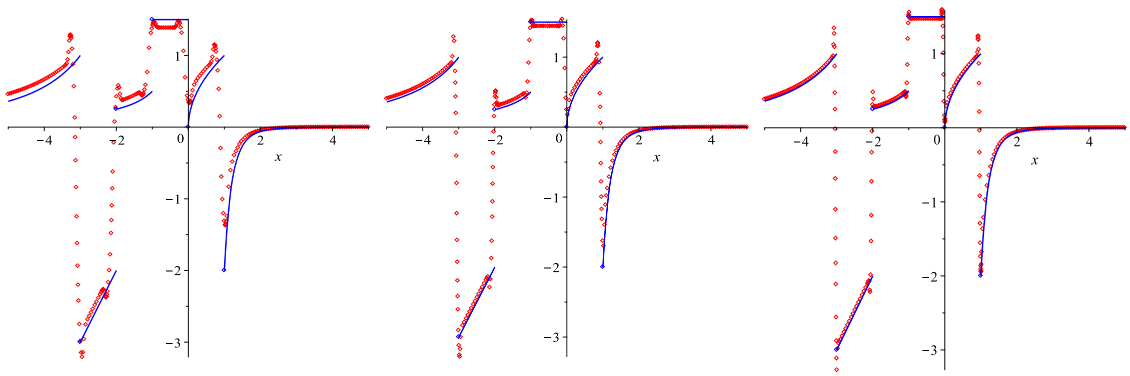

Figure 2: The graphs of the functions $T_5^{(2)}f(x)$, $T_{10}^{(2)}f(x)$, $T_{20}^{(2)}f(x)$ (red) compared to the graph of $f(x)$ (blue).

Next, we consider an operator of convolution-type. So, let $H = G = \mathbb{R}$, and set $h_w(t) = t$. Now, consider the operators (see [55,56])

$$T_w^{(5)}f(x) = \int_{\mathbb{R}} M(wx - t)g_w\left(\frac{w}{2}\int_{t-1/w}^{t+1/w} f(u)\mathrm{d}u\right)\mathrm{d}t.$$

Here

$$f(x) = \begin{cases} 3e^x, & x < -1 \\ -1, & -1 \le x < 0 \\ 2, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ -2e^{-x}, & x \ge 2. \end{cases}$$

Figure 3 shows the behavior of $T_w^{(5)}f(x)$ compared to that of $f(x)$ as $w = 5, 10, 20$.

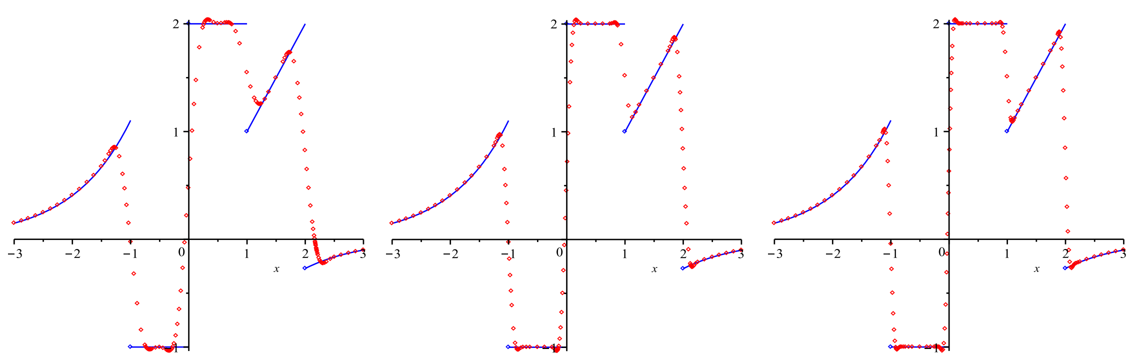

Figure 3: The graphs of the functions $T_5^{(5)}f(x)$, $T_{10}^{(5)}f(x)$, $T_{20}^{(5)}f(x)$ (red) compared to the graph of $f(x)$ (blue).

In the next example, we will consider a nonlinear Mellin-type operator $T_w^{(6)}f(x)$. Let $H = G = \mathbb{R}^+$. We choose

$$C_w(x) = \begin{cases} wx^w, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

so that

$$T_w^{(6)}f(x) = \int_0^\infty C_w\left(\frac{x}{t}\right)g_w(f(t))\frac{\mathrm{d}t}{t}.$$

Here

$$f(x) = \begin{cases} 2x, & 0 \le x < 2 \\ 1, & 2 \le x \le 4 \\ \dfrac{-25}{x^3}, & x > 4. \end{cases}$$

Figure 4 shows how $T_w^{(6)} f(x)$ approaches $f(x)$ as $w = 10, 20, 30$.

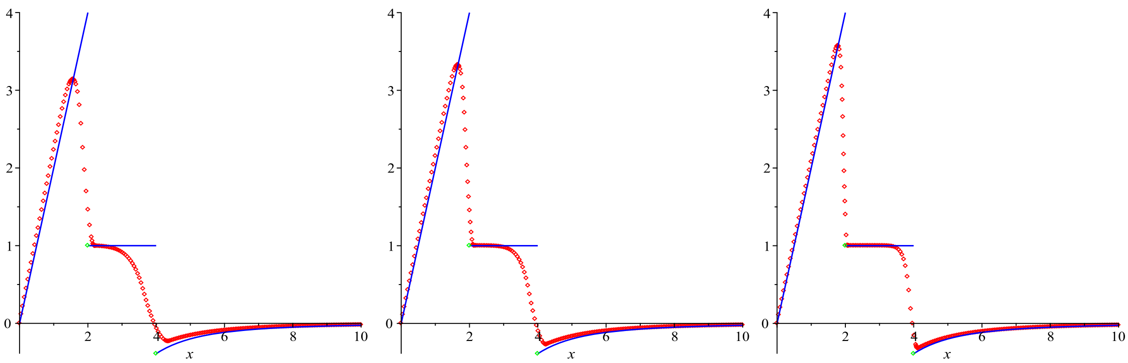

Figure 4: The graphs of the functions $T_{10}^{(6)} f(x)$, $T_{20}^{(6)} f(x)$, $T_{30}^{(6)} f(x)$ (red) compared to the graph of $f(x)$ (blue).

In our next example, the base spaces are circles. Let $H = G = S^1$. Functions defined on S^1 can be considered as periodic functions. For simplicity, let us use the usual identification $S^1 = [-\pi, \pi]$, where the points $-\pi, \pi$ are glued. A well-known kernel on S^1 is the Fejer periodic kernel, namely,

$$\mathcal{F}_w(x) = \frac{\sin^2\left(\frac{wx}{2}\right)}{2w\pi \sin^2\left(\frac{x}{2}\right)}, \quad x \in [-\pi, \pi] \setminus \{0\}$$

(see, e.g., [10]). We take

$$f(x) = \begin{cases} e^x, & -\pi < x < -1 \\ -1, & -1 \le x \le 0 \\ x, & 0 < x < 1 \\ 4 - e^{-x}, & 1 \le x \le \pi, \end{cases}$$

and consider the periodic version of the operators $T_w^{(4)}$, i.e., of the form

$$\overline{T}_w^{(4)} f(x) = \int_{-\pi}^{\pi} \mathcal{F}_w(x-t) g_w(f(t)) dt,$$

where $\chi_w(x, u) = F_w(x)\, g_w(u)$.

Figure 5 shows how $\overline{T}_w^{(4)} f(x)$ behaves as $w = 10, 15, 30$.

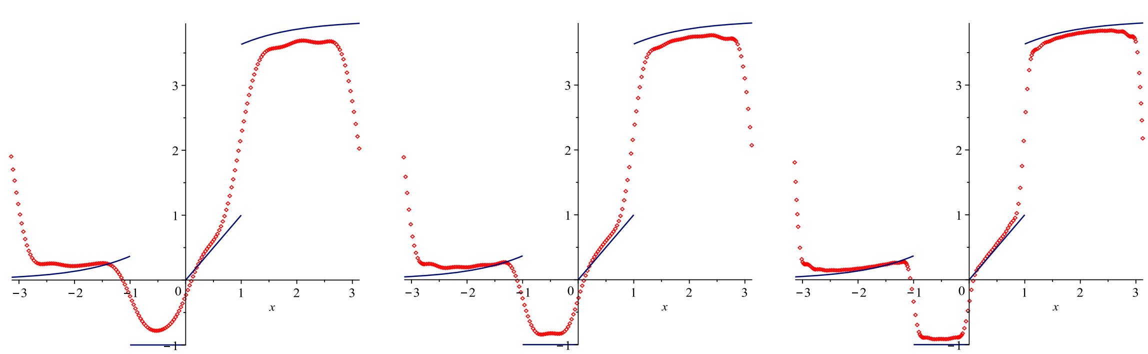

Figure 5: The graphs of the functions $\overline{T}_{10}^{(4)} f(x)$, $\overline{T}_{15}^{(4)} f(x)$, $\overline{T}_{30}^{(4)} f(x)$ (red) compared to the graph of $f(x)$ (blue).

Now, let us consider a Durrmeyer nonlinear generalized sampling series. Set $H = \mathbb{Z}$, $G = \mathbb{R}$, $h_w(t) = k/w$ and

$$L_{k/w} f = w \int_{\mathbb{R}} F(wu - k) f(u) \mathrm{d}u,$$

where, for example, $\rho(u) = F(u)$ is the Fejer kernel. If g_w is as above, we have the series

$$T_w^{(3)} f(x) = \sum_{k \in \mathbb{Z}} M(wx - k) g_w \left(w \int_{\mathbb{R}} F(wu - k) f(u) \mathrm{d}u \right).$$

We take

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & x < -1 \\ -1, & -1 \le x < 0 \\ 2, & 0 \le x < 2 \\ -\dfrac{3}{x^3}, & x \ge 2. \end{cases}$$

Figure 6 shows the behavior of $T_w^{(3)} f(x)$ when $w = 10, 20, 30$.

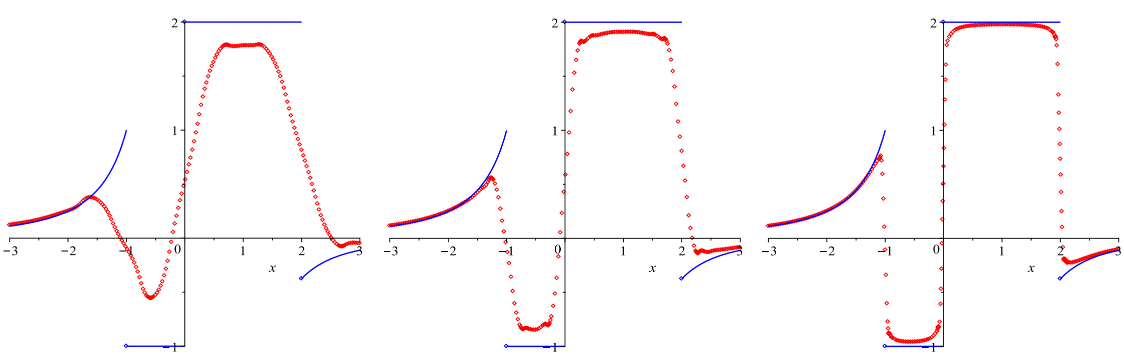

Figure 6: The graphs of the functions $T_{10}^{(3)} f(x)$, $T_{20}^{(3)} f(x)$, $T_{30}^{(3)} f(x)$ (red) compared to the graph of $f(x)$ (blue).

In our last example, we describe the behavior of a multidimensional nonlinear Kantorovich sampling operator. Let $H = \mathbb{Z}^2$ and $G = \mathbb{R}^2$. For $(k, n) \in \mathbb{Z}^2$, define $h_w(k, n) = (k/w, n/w) \in \mathbb{R}^2$. Then, define

$$L_{k,n} f = w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{n}{w}}^{\frac{n+1}{w}} f(u, v) \mathrm{d}u \mathrm{d}v.$$

We have to choose a suitable kernel. It turns out that a good choice is given by the following

$$C(x, y) = F(x) \cdot F(y),$$

where $F(u)$ is the Fejer kernel defined in (14). We then define

$$C_w(x, y) = C(wx, wy) = F(wx) \cdot F(wy),$$

and we are left with the operator (when g_w is as above)

$$\widetilde{T}_w f(x, y) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} F(wx - k) F(wy - n) \cdot g_w \left(w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{n}{w}}^{\frac{n+1}{w}} f(u, v) \mathrm{d}u \mathrm{d}v \right).$$

See [29] for information concerning multidimensional Kantorovich sampling operators. We take the function

$$f(x, y) = \begin{cases} 2 - x^2 - y^2, & (x, y) \in [0, 1] \times [0, 1] \\ 1 + x^2 + y^2, & (x, y) \in [1, 2] \times [1, 2] \\ 0, & \text{otherwise}. \end{cases}$$

The graphs in Figure 7 show the behavior of $\widetilde{T}_w f(x, y)$ for $w = 5, 10, 20$, respectively.

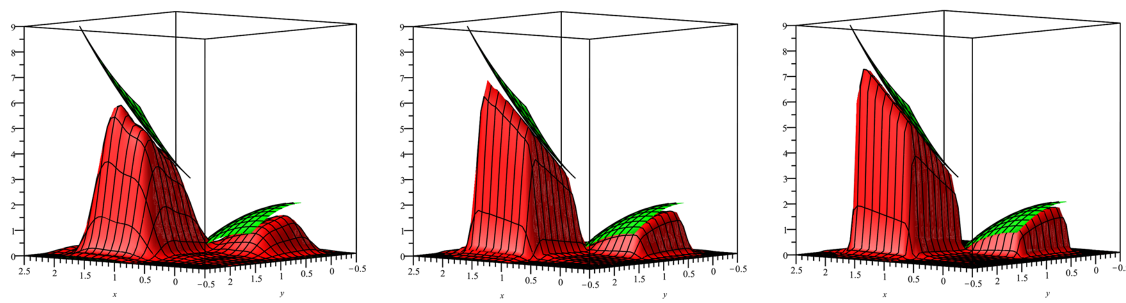

Figure 7: The graphs of the functions $\widetilde{T}_5 f(x, y)$, $\widetilde{T}_{10} f(x, y)$, $\widetilde{T}_{20} f(x, y)$ (red) compared to the graph of $f(x, y)$ (green).

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