Research Article

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L^p Hardy's identities and inequalities for Dunkl operators

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Abstract: The main purpose of this article is to establish the L^p Hardy's identities and inequalities for Dunkl operator on any finite balls and the entire space $\mathbb{R}^N$. We also prove Hardy's identities and inequalities on certain domains with distance function to the boundary $\partial\Omega$. In particular, we use the notion of Bessel pairs introduced in Ghoussoub and Moradifam to extend Hardy's identities for the classical gradients obtained by Lam et al., Duy et al., Flynn et al. to Dunkl gradients introduced by Dunkl. Our Hardy's identities with explicit Bessel pairs significantly improve many existing Hardy's inequalities for Dunkl operators.

Keywords: p-Bessel pair, Dunkl operator, Hardy's identity, Hardy's inequality

MSC 2020: 46E35, 33C52, 26D10

1 Introduction

1.1 Hardy inequalities for the classical gradients

We first recall the classical L^p Hardy's inequality of the following form:

$$\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \geq \left|\frac{N-p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x, \tag{1.1}$$

for $N \geq 2, 1 \leq p < N$, and $u \in C_0^\infty(\mathbb{R}^N)$. The constant $\left|\frac{N-p}{p}\right|^p$ is sharp and is never attained by nontrivial functions. It is worth mentioning that Hardy-type inequalities and their variants play crucial roles in many areas of mathematics, such as analysis, probability, and partial differential equations (PDEs). We refer the interested readers to standard monographs on the topic [4,22,28,33,34,38,40,45], among many others.

In general, remainder terms might be added to improve Hardy's inequalities, due to the non-existence of extremal functions. For the first time in the literature, Brézis and Vázquez [9] investigated the improvements of Hardy's inequality on the bounded domain by adding nonnegative terms to the potential $\left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$. Besides, Maz'ya studied in [40] a refinement of both Sobolev and Hardy inequalities and established the following result:

Theorem. (Maz'ya [40]). *Let $N + k > 2, 2 < q < \frac{2(N+k)}{N+k-2}$, and $\gamma = -1 + (N+k)\left(\frac{1}{2} - \frac{1}{q}\right)$. There exists $c > 0$ such that*

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$$\int_{\mathbb{R}^{N+k}} |\nabla u|^2 \mathrm{d}x - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^{N+k}} \frac{|u|^2}{|y|^2}\mathrm{d}x \geq c\left(\int_{\mathbb{R}^{N+k}} |u|^q |y|^{\gamma q}\mathrm{d}x\right)^{\frac{2}{q}}$$

for all $u \in C_0^\infty(\mathbb{R}^{N+k})$ *subject to the condition* $u(0,z) = 1$ *in the case* $N = 1$. *Here*, $x = (y,z) \in \mathbb{R}^N \times \mathbb{R}^k$.

Their results have motivated many researchers to work on the remainders of Hardy's inequality. We refer interested readers to [5,6,14,15], to name a few. In [26], Frank and Seiringer used the notion of ground state representations to set up the following improved Hardy-type inequality: if the weighted p-Laplace equation

$$-\operatorname{div}(A(x)|\nabla\varphi(x)|^{p-2}\nabla\varphi) = B(x)\varphi(x)^{p-1}$$

has a positive solution φ, then

$$\int A(x)|\nabla u|^p \mathrm{d}x \geq \int B(x)|u|^p \mathrm{d}x.$$

Recently, Duy et al. [19] proved a general symmetrization principle for Hardy-type inequalities with non-radial weights of the form $A(|x|)x^P$.

Theorem. [19] *Let* $p > 1$, $0 < R \leq \infty$, A, *and* B *be positive functions on* $(0,R)$. *Then the following are equivalent*:

$$(A)\quad \int_{B_R^*} A(|x|)|\nabla u|^p x^P \mathrm{d}x \geq \int_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x \quad \text{for all } u \in C_0^\infty(B_R^*)$$

$$(B)\quad \int_{B_R^*} A(|x|)|\mathcal{R}u|^p x^P \mathrm{d}x \geq \int_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x \quad \text{for all } u \in C_0^\infty(B_R^*)$$

$$(C)\quad \int_{B_R^*} A(|x|)|\nabla u|^p x^P \mathrm{d}x \geq \int_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x \quad \text{for all radial } u \in C_0^\infty(B_R^*).$$

Here $x^P = |x_1|^{P_1}\ldots|x_N|^{P_N}$, $P_1 \geq 0,\ldots,P_N \geq 0$, is the monomial weight, and $\mathbb{R}_*^N = \{(x_1,\ldots,x_N) \in \mathbb{R}^N : x_i > 0 \ \text{ whenever } P_i > 0\}$, $B_R^* = B_R \cap \mathbb{R}_*^N$, and $\mathcal{R} := \frac{x}{|x|}\cdot\nabla$.

We also refer to the articles [41] and [1], in which Muckenhoupt pair has been used to study the necessary and sufficient conditions for the validity of the Hardy inequality in one-dimensional space. In order to study the L^2 Hardy-type inequalities with radial weights, Ghoussoub and Moradifam [27] first formulated and applied the notion of Bessel pairs in L^2 sense and proved that if $(r^{N-1}A, r^{N-1}B)$ is a one-dimensional Bessel pair on $(0,R)$, $0 < R \leq \infty$, then

$$\int_{B_R} A(|x|)|\nabla u|^2 \mathrm{d}x \geq \int_{B_R} B(|x|)|u|^2 \mathrm{d}x \tag{1.2}$$

for all $u \in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. Here, the Bessel pair (A,B) is defined if the ordinary differential equation (ODE)

$$(A(r)y')' + B(r)y = 0$$

has a positive solution on $(0,R)$. It is also worth mentioning that $(\tilde{A},\tilde{B}) := (r^{N-1}A, r^{N-1}B)$ is a one-dimensional Bessel pair – or simply called a Bessel pair, if and only if (A,B) is a n-dimensional Bessel pair, in the sense that

$$(r^{N-1}A(r)y')' + r^{N-1}B(r)y = 0$$

has a positive solution.

Recently, the notion of Bessel pairs has been revisited and used to establish Hardy's identities and inequalities, which sharpened Hardy's inequalities in the literature. Lam et al. proved [36,37] the following L^2 Hardy identities and inequalities.

Theorem. [36,37] *Let $0 < R \le \infty$, A, and B be positive C^1 functions on $(0, R)$. Assume that for some $\alpha \in \mathbb{R}$, $\Delta d(x) - \frac{\alpha-1}{d(x)}$ exists on $\{0 < d(x) < R\}$ in the sense of distribution, and $(r^{\alpha-1}A, r^{\alpha-1}B)$ is a Bessel pair on $(0, R)$. Then, for $u \in C_0^\infty(\{0 < d(x) < R\})$:*

$$\int\limits_{0<d(x)<R} A(d(x))|\nabla u(x)|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} B(d(x))|u(x)|^2 \mathrm{d}x$$

$$= \int\limits_{0<d(x)<R} A(d(x))\varphi^2(d(x)) \left| \nabla\left(\frac{u(x)}{\varphi(d(x))}\right) \right|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} A(d(x))|u(x)|^2 \left[\Delta d(x) - \frac{\alpha-1}{d(x)}\right] \frac{\varphi'(d(x))}{\varphi(d(x))} \mathrm{d}x$$

and

$$\int\limits_{0<d(x)<R} A(d(x))|\nabla d(x)\cdot\nabla u(x)|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} B(d(x))|u(x)|^2 \mathrm{d}x$$

$$= \int\limits_{0<d(x)<R} A(d(x))\varphi^2(d(x)) \left| \nabla d(x)\cdot\nabla\left(\frac{u(x)}{\varphi(d(x))}\right) \right|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} A(d(x))|u(x)|^2 \left[\Delta d(x) - \frac{\alpha-1}{d(x)}\right] \frac{\varphi'(d(x))}{\varphi(d(x))} \mathrm{d}x.$$

Here, φ is the positive solution of

$$(r^{\alpha-1}A(r)y'(r))' + r^{\alpha-1}B(r)y(r) = 0$$

on the interval $(0, R)$.

As a consequence, the authors obtained

Theorem. [36,37] *Let $0 < R \le \infty$, A, and B be positive C^1 functions on $(0, R)$. Assume that for some $\alpha \in \mathbb{R}$, $\varphi'(d(x))\left(\Delta d(x) - \frac{\alpha-1}{d(x)}\right) \le 0$ in $\{0 < d(x) < R\}$ in the sense of distribution, and $(r^{\alpha-1}A, r^{\alpha-1}B)$ is a Bessel pair on $(0, R)$. Then, for $u \in C_0^\infty(\{0 < d(x) < R\})$:*

$$\int\limits_{0<d(x)<R} A(d(x))|\nabla u(x)|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} B(d(x))|u(x)|^2 \mathrm{d}x \ge \int\limits_{0<d(x)<R} A(d(x))\varphi^2(d(x)) \left| \nabla\left(\frac{u(x)}{\varphi(d(x))}\right) \right|^2 \mathrm{d}x$$

and

$$\int\limits_{0<d(x)<R} A(d(x))|\nabla d(x)\cdot\nabla u(x)|^2 \mathrm{d}x - \int\limits_{0<d(x)<R} B(d(x))|u(x)|^2 \mathrm{d}x$$

$$\ge \int\limits_{0<d(x)<R} A(d(x))\varphi^2(d(x)) \left| \nabla d(x)\cdot\nabla\left(\frac{u(x)}{\varphi(d(x))}\right) \right|^2 \mathrm{d}x.$$

Here, φ is the positive solution of

$$(r^{\alpha-1}A(r)y'(r))' + r^{\alpha-1}B(r)y(r) = 0$$

on the interval $(0, R)$.

Subsequently, Duy et al. introduced in [19] the notion of L^p-Bessel pairs for all $p > 1$ and established numerous L^p-Hardy's identities and inequalities, which improved Hardy's inequalities in the literature. This notion of p-Bessel pairs has been further elaborated by Flynn et al. in their work [24], where the authors proved L^p Hardy identities and inequalities with p-Bessel pairs on domains $\Omega \subset \mathbb{R}^n$ with respect to the distance and mean distance to the boundary. Hardy-type inequalities have also been established on hyperbolic spaces [7,8], Cartan-Hadamard manifolds [25], and other related Riemannian manifolds. We begin by recalling the following definition given in [19] (see also [24]).

Definition 1.1. (A, B) is a p-Bessel pair *on* $(0, R)$ if the ODE $(A(r)|y'|^{p-2}y')' + B(r)|y|^{p-2}y = 0$ has a positive solution on $(0, R)$.

For $p > 1$, we also denote

$$\begin{aligned} C_p(x, y) &= |x|^p - |x-y|^p - p\,|x-y|^{p-2}(x-y)\cdot y \\ &= |x|^p + (p-1)|x-y|^p - p\,|x-y|^{p-2}(x-y)\cdot x. \end{aligned}$$

Duy et al. [19] proved the following L^p Hardy identities with monomial weights:

Theorem. (Duy et al. [19]) *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A, and B be positive C^1 functions on $(0, R)$. If $(r^{D-1}A, r^{D-1}B)$ is a p-Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R^*\backslash\{0\})$:*

$$\int\limits_{B_R^*} A(|x|)|\nabla u|^p x^P \mathrm{d}x - \int\limits_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x = \int\limits_{B_R^*} A(|x|) C_p\left(\nabla u, \varphi\nabla\left(\frac{u}{\varphi}\right)\right) x^P \mathrm{d}x$$

and

$$\int\limits_{B_R^*} A(|x|)|\mathcal{R}u|^p x^P \mathrm{d}x - \int\limits_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x = \int\limits_{B_R^*} A(|x|) C_p\left(\mathcal{R}u, \varphi\mathcal{R}\left(\frac{u}{\varphi}\right)\right) x^P \mathrm{d}x.$$

Here, $\varphi = \varphi_{p,D;A,B;R}$ is the positive solution of

$$(r^{D-1}A(r)|y'|^{p-2}y')' + r^{D-1}B(r)|y|^{p-2}y = 0$$

on $(0, R)$. $|x|^P = |x_1|^{P_1}|\cdots x_N|^{P_N}$ is the monomial weight, where $P_1 \geq 0, \ldots, P_N \geq 0$, and $D = N + P_1 + \cdots + P_N$.

Moreover, Flynn et al. established in [24] L^p Hardy identities and inequalities with p-Bessel pairs on domains in Euclidean space with respect to the distance and mean distance to the boundary. Let $d_\Omega(x)$ be the distance from x to $\partial\Omega$.

Theorem. (Flynn et al. [24]) *Let $1 < p < \infty$, let $\Omega \subsetneq \mathbb{R}^N$ has inradius $0 < \rho \leq \infty$, and let $0 < R \leq \infty$ be such that $\rho < R$ when $\rho < \infty$ and $R = \infty$ when $\rho = \infty$. Suppose (V, W) is a p-Bessel pair on $(0, R)$ with positive solution φ. Then, for $u \in C_0^\infty(\Omega)$, there holds*

$$\begin{aligned} &\int\limits_\Omega V(d_\Omega(x))|\nabla u(x)|^p \mathrm{d}x - \int\limits_\Omega W(d_\Omega(x))|u(x)|^p \mathrm{d}x \\ &= \int\limits_\Omega V(d_\Omega(x)) C_p\left(\nabla u(x), \varphi(d_\Omega(x))\nabla\left(\frac{u(x)}{\varphi(d_\Omega(x))}\right)\right) - \Delta d_\Omega, V \circ d_\Omega \left|\frac{\varphi' \circ d_\Omega}{\varphi \circ d_\Omega}\right|^{p-2} \frac{\varphi' \circ d_\Omega}{\varphi \circ d_\Omega}|u|^p \mathrm{d}x \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\int\limits_\Omega V(d_\Omega(x))|\nabla d_\Omega(x)\cdot\nabla u(x)|^p \mathrm{d}x - \int\limits_\Omega W(d_\Omega(x))|u(x)|^p \mathrm{d}x \\ &= \int\limits_\Omega V(d_\Omega(x)) C_p\left(\nabla d_\Omega(x)\cdot u(x), \varphi(d_\Omega(x))\nabla d_\Omega(x)\cdot\nabla\left(\frac{u(x)}{\varphi(d_\Omega(x))}\right)\right)\mathrm{d}x \\ &\quad - \Delta d_\Omega, V \circ d_\Omega \left|\frac{\varphi' \circ d_\Omega}{\varphi \circ d_\Omega}\right|^{p-2} \frac{\varphi' \circ d_\Omega}{\varphi \circ d_\Omega}|u|^p \mathrm{d}x, \end{aligned} \tag{1.4}$$

where $\Delta d_\Omega, \cdot$ denotes the pairing with the distributional Laplacian of d_Ω.

Another way to improve Hardy-type inequalities is to replace the usual gradient ∇ by $\frac{x}{|x|}$, and $\frac{x}{|x|}\cdot\nabla$ is just the radial derivative. In particular, for polar coordinates $(r,\sigma)=\left(|x|,\frac{x}{|x|}\right)$, we have $\frac{x}{|x|}\cdot\nabla u(x)=\partial_r u(r\sigma)$. Actually, the Hardy inequalities with radial derivative have been studied by many authors, see [19,32,35,44], to name a few. Moreover, Hardy's identities and inequalities on Carnot groups (including the well-known Heisenberg group) and on hyperbolic spaces and Cartan-Hadamard manifolds have been established by Flynn et al. [23,25].

1.2 Hardy's inequalities for Dunkl gradients

During recent years, there has been a rapid development in the area of special functions with reflection symmetries and harmonic analysis related to root systems. The motivation for this subject comes from the theory of Riemannian symmetric spaces, whose spherical functions can be written as multi-variable special functions depending on certain discrete sets of parameters. Dunkl operators are an important tool in the study of special functions with reflection symmetries.

In general, these are commuting differential-difference operators, associated with a finite reflection group $G(R)$ on Euclidean spaces. The class of such operators, now often called "rational" Dunkl operators, was first introduced by Dunkl [16]. Besides harmonic analysis and special functions, this theory also has deep and fruitful interactions with algebra, mathematical physics, and probability theory. Interested readers are referred to many references for surveys on Dunkl theory. For example, see [18,43] about rational Dunkl theory, and [42] about trigonometric Dunkl theory, and [20] about integrable systems related to Dunkl theory. We shall denote the first-order Dunkl operator along a vector $\xi\in\mathbb{R}^N$ by T_ξ, the Dunkl gradient by ∇_k, and the Dunkl divergence by div_k (see Section 2 for details and more notations).

Several functional and geometric inequalities have been developed in a Dunkl setting. For examples, Gorbachev et al. proved Pitts inequalities for the fractional Dunkl operators [29]. Anoop and Parui proved the Hardy inequality and fractional Hardy inequality [3] by looking for a solution and the potential function in some differential equations. More precisely, they established the following.

Theorem. (Anoop and Parui [3]) *Let w be a positive radial function and V be a function satisfying $-\Delta_k w+Vw\geq 0$ in $\mathbb{R}^N$, then for $u\in C_0^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}(|\nabla_k u|^2+V|u|^2)h_k^2(x)\mathrm{d}x\geq\int_{\mathbb{R}^N}|\nabla_k(w^{-1}u)|^2w^2h_k^2(x)\mathrm{d}x. \tag{1.5}$$

If u is G-invariant, the equality of (1.5) holds. We note that such an identity is actually a special case of our result. The inequalities were also established for the half space and the cone by the authors. However, under the Dunkl setting, the improvements for the remainder term in Hardy-type inequalities have not been studied explicitly.

The main purpose of this article is to derive Hardy-type identities in the setting of Dunkl operators, using the notion of p-Bessel pair for Dunkl operators. This is actually an improvement of Hardy-type inequalities because as an additional nonnegative term is added to the right-hand side of (1.1). We also discuss the Hardy identities and inequalities on the G-invariant domains.

The article is organized as follows. In the rest of this section, we give our main results with several applications. In Sction 2, we provide a brief and necessary introduction to Dunkl theory, which will be used throughout the article. The proofs of our main theorems are discussed in Section 3.

Definition 1.2. Let $y(r)$ be a radial function defined on a ball of radius R centered at the origin, where $r=|x|$ is the Euclidean norm of x. (A,B) is an N-dimensional p-Bessel pair for Dunkl operators on $(0,R)$ if

$$\mathrm{div}_k(A(r)|\nabla_k y(r)|^{p-2}\nabla_k y(r))+B(r)|y(r)|^{p-2}y(r)=0$$

has a positive solution on $(0,R)$.

We next point out the relationship between the Dunkl p-Bessel pair and the classic p-Bessel pair in the following corollary.

Corollary 1.1. *(A, B) is an N-dimensional p-Bessel pair for Dunkl operators on $(0, R)$ if and only if it is a d_k-dimensional p-Bessel pair in the usual derivative sense, i.e., if*

$$(r^{d_k-1}A(r)|y'|^{p-2}y')' + r^{d_k-1}B(r)|y|^{p-2}y = 0$$

has a positive solution on $(0, R)$, where $d_k = N + 2\gamma_k$ and $\gamma_k = \sum_{\alpha\in R^+}k(\alpha)$.

It is worth noting that (A, B) is an N-dimensional p-Bessel pair if and only if $(r^{N-1}A, r^{N-1}B)$ is a *one*-dimensional Bessel pair, in which case we have immediately:

Corollary 1.2. *(A, B) is an N-dimensional p-Bessel pair for Dunkl operators on $(0, R)$ if $(r^{d_k-1}A, r^{d_k-1}B)$ is a (1-dimensional) p-Bessel pair for usual derivatives.*

1.3 Hardy's identities and inequalities on balls and $\mathbb{R}^N$

Now we are ready to present the following L^p Hardy identity:

Theorem 1.1. *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A, and B be positive C^1 functions. If (A, B) is an N-dimensional p-Bessel pair for Dunkl operator on $(0, R)$, then for all G-invariant $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$, we have*

$$\int_{B_R} A(|x|)|\nabla_k u|^p \mathrm{d}\mu_k - \int_{B_R} B(|x|)|u|^p \mathrm{d}\mu_k = \int_{B_R} A(|x|)C_p\left(\nabla_k u, w\nabla_k \frac{u}{w}\right)\mathrm{d}\mu_k, \tag{1.6}$$

where w is the positive solution of

$$\mathrm{div}_k(A(r)|\nabla_k y(r)|^{p-2}\nabla_k y(r)) + B(r)|y(r)|^{p-2}y(r) = 0.$$

Obviously, this result implies the following Hardy-type inequality:

Theorem 1.2. *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A, and B be positive C^1 functions. If (A, B) is an N-dimensional p-Bessel pair for Dunkl operator on $(0, \infty)$, then for all G-invariant $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$, we have*

$$\int_{B_R} A(|x|)|\nabla_k u|^p \mathrm{d}\mu_k \geq \int_{B_R} B(|x|)|u|^p \mathrm{d}\mu_k. \tag{1.7}$$

It is also worth mentioning that when $p = 2$, $C_2(x, y) = |y|^2$. Hence, we obtain the following L^2 Hardy identity and inequality for Dunkl operators as a consequence of Theorem 1.1.

Corollary 1.3. *Let $N \geq 1$, $0 < R \leq \infty$, A, and B be positive C^1 functions. If (A, B) is an N-dimensional L^2-Bessel pair for Dunkl operator on $(0, R)$, then for all G-invariant $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$, we have*

$$\int_{B_R} A(|x|)|\nabla_k u|^2 \mathrm{d}\mu_k - \int_{B_R} B(|x|)|u|^2 \mathrm{d}\mu_k = \int_{B_R} A(|x|)\left| w(|x|)\nabla_k\left(\frac{u}{w}\right)\right|^2 \mathrm{d}\mu_k \tag{1.8}$$

and

$$\int_{B_R} A(|x|)|\nabla_k u|^2 \mathrm{d}\mu_k \geq \int_{B_R} B(|x|)|u|^2 \mathrm{d}\mu_k. \tag{1.9}$$

We also provide some other consequences of Theorem 1.1, by giving some explicit p-Bessel pairs.

The simplest Bessel pair on $(0,\infty)$ is $(1, \left(\frac{N-p}{p}\right)^p r^{-p})$, with positive solution $w(r) = r^{-\frac{N-p}{p}}$. Therefore, we recover the classical L^p Hardy identity and inequality for Dunkl operators:

Corollary 1.4. *Let $N \ge 2$, $1 \le p < \infty$, and $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ be G-invariant, we have*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p \mathrm{d}\mu_k - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}\mu_k = \int_{\mathbb{R}^N} C_p\Big(\nabla_k u, |x|^{-\frac{d_k-p}{p}} \nabla\Big(u|x|^{\frac{d_k-p}{p}}\Big)\Big) \mathrm{d}\mu_k$$

and

$$\int_{\mathbb{R}^N} |\nabla_k u|^p \mathrm{d}\mu_k \ge \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}\mu_k,$$

where the constant in the inequality is sharp.

Instead of proving this corollary, we present and prove a more general version as follows:

Corollary 1.5. *Let $N \ge 2$, $1 < p < \infty$, $\varepsilon \in \mathbb{R}$, and $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ be G-invariant, we have*

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{p\varepsilon} |\nabla_k u|^p \mathrm{d}\mu_k - \left|\varepsilon + \frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} |x|^{p(\varepsilon-1)} |u|^p \mathrm{d}\mu_k
&= \int_{\mathbb{R}^N} |x|^{p\varepsilon} C_p\Big(\nabla_k u, |x|^{-\frac{d_k-p}{p}-\varepsilon} \nabla\Big(u|x|^{\frac{d_k-p}{p}+\varepsilon}\Big)\Big) \mathrm{d}\mu_k \\
&= \int_{\mathbb{R}^N} C_p\Big(|x|^{\varepsilon} \nabla_k u, |x|^{-\frac{d_k-p}{p}} \nabla\Big(u|x|^{\frac{d_k-p}{p}+\varepsilon}\Big)\Big) \mathrm{d}\mu_k
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |x|^{p\varepsilon} |\nabla_k u|^p \mathrm{d}\mu_k \ge \left|\varepsilon + \frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} |x|^{p(\varepsilon-1)} |u|^p \mathrm{d}\mu_k,$$

where the constant in the inequality is sharp.

Proof. We first check $\Big(r^{d_k-1} r^{p\varepsilon}, \left|\varepsilon + \frac{D_k - p}{p}\right|^p r^{d_k-1} r^{p(\varepsilon-1)}\Big)$ is a p-Bessel pair on $(0,\infty)$, with $w = r^{-\frac{d_k-p}{p}-\varepsilon}$. Since $w'(r) = -\Big(\frac{d_k-p}{p} + \varepsilon\Big) r^{-\frac{d_k-p}{p}-\varepsilon-1}$, then

$$\begin{aligned}
(r^{d_k-1} r^{p\varphi} |w'(r)|^{p-2} w'(r))' &= -\left|\varepsilon + \frac{d_k - p}{p}\right|^{p-2} \left(\varepsilon + \frac{d_k - p}{p}\right)\Big(r^{d_k-1+p\varepsilon-\left(\frac{d_k-p}{p}+\varepsilon+1\right)(p-1)}\Big)' \\
&= -\left|\varepsilon + \frac{d_k - p}{p}\right|^{p-2} \left(\varepsilon + \frac{d_k - p}{p}\right)\Big(r^{\frac{d_k-p}{p}+\varepsilon}\Big)' \\
&= -\left|\varepsilon + \frac{d_k - p}{p}\right|^{p} r^{\frac{d_k-p}{p}+\varepsilon-1} \\
&= -\left|\varepsilon + \frac{d_k - p}{p}\right|^{p} r^{d_k-1} r^{p(\varepsilon-1)} r^{-\left(\frac{d_k-p}{p}+\varepsilon\right)(p-1)} \\
&= -\left|\varepsilon + \frac{d_k - p}{p}\right|^{p} r^{d_k-1} r^{p(\varepsilon-1)} |w|^{p-2} w.
\end{aligned}$$

Then by Corollary 1.2, $\left(r^{p\varepsilon}, \left|\varepsilon + \frac{D_k - p}{p}\right|^p r^{p(\varepsilon-1)}\right)$ is an N-dimensional p-Bessel pair for Dunkl operator, which proves the identity and inequality. In order to see that the constant $\left|\varepsilon + \frac{d_k - p}{p}\right|$ is sharp, we check the radial function

$$f(r) = \begin{cases} c^{-1} & r \le 1 \\ c^{-1} r^c & r > 1, \end{cases}$$

where $c = -\varepsilon - \frac{d_k - p}{p}$. Then, we have

$$\frac{\int_{\mathbb{R}^N} |x|^{p\varepsilon} |\nabla_k f|^p \mathrm{d}\mu_k}{\int_{\mathbb{R}^N} |x|^{p(\varepsilon-1)} |f|^p \mathrm{d}\mu_k} = \frac{\int_0^\infty r^{-\frac{p}{N} - d_k + p} \mathrm{d}r}{\int_0^\infty c^p r^{p(\varepsilon-1)} r^{-\frac{p}{N} - d_k + p} \mathrm{d}r} = \left|\varepsilon + \frac{d_k - p}{p}\right|^p.$$

Letting $\varepsilon = 0$, the above identity and inequality reduce to Corollary 1.4. □

By using 2-Bessel pair $\left(r^{2-d_k}, r^{2-d_k} \frac{z_0^2}{R^2}\right)$ on $(0, R)$, $R > 0$, we obtain the following Poincáre-type identity and weighted Sobolev inequality:

Corollary 1.6. *Let $R > 0$ and $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ be G-invariant, we have*

$$\int_{B_R} \frac{|\nabla_k u|^2}{|x|^{d_k-2}} \mathrm{d}\mu_k - \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} \mathrm{d}\mu_k = \int_{B_R} \frac{J_0^2\left(\frac{z_0}{R}|x|\right)}{|x|^{d_k-2}} \left|\nabla_k \left(\frac{u}{J_0\left(\frac{z_0}{R}|x|\right)^2}\right)\right|^2 \mathrm{d}\mu_k$$

and

$$\int_{B_R} \frac{|\nabla_k u|^2}{|x|^{d_k-2}} \mathrm{d}\mu_k \ge \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^{d_k-2}} \mathrm{d}\mu_k. \tag{1.10}$$

Moreover, the optimizer of the inequality (1.10) *exists and takes the form $J_0\left(\frac{z_0}{R}|x|\right)$. Here, $z_0 \approx 2.4048$ is the first zero of Bessel function $J_0(z)$.*

Proof. We recognize that $\left(r^{d_k-1} r^{2-d_k}, r^{d_k-1} r^{2-d_k} \frac{z_0^2}{R^2}\right)$ is a 2-Bessel pair on $(0, R)$ with $w(r) = J_0\left(\frac{z_0}{R} r\right)$, for any $R > 0$. This is due to $\left(r^{-\lambda}, \left(\frac{n-\lambda-2}{2}\right)^2 r^{-\lambda-2} + \frac{z_0^2}{R^2} r^{-\lambda}\right)$ being an n-dimensional 2-Bessel pair [27] on $(0, R)$, $R > 0$, for $0 \le \lambda < n - 2$, $n \ge 3$, with the same positive solution. Hence, $\left(r^{2-d_k}, r^{2-d_k} \frac{z_0^2}{R^2}\right)$ is an N-dimensional 2-Bessel pair for the Dunkl operator, and the identity and inequality follow. Besides, since $J_0(s) \sim 1$, for $s \ll 1$, we have

$$\int_{B_R} \frac{|J_0\left(\frac{z_0}{R}|x|\right)|^2}{|x|^{d_k-2}} \mathrm{d}\mu_k \sim \int_0^R \left|J_0\left(\frac{z_0}{R} r\right)\right|^2 h_k^2(r) r \mathrm{d}r \sim \left(\int_0^R |J_0(s)|^4 s^2 ds\right)^{1/2} \left(\int_0^R |h_k(|x|)|^4 \mathrm{d}x\right)^{1/2} < \infty.$$

Therefore, $J_0\left(\frac{z_0}{R}|x|\right)$ is the optimizer for the weighted Sobolev inequality (1.10). □

In fact, if $0 \le \lambda \le d_k - 2$, $0 \le \alpha \le \frac{d_k - \lambda - 2}{2}$, and z_α is the first zero of the Bessel function of the first kind J_α, then the couple $\left(r^{d_k-1} r^{-\lambda}, r^{d_k-1} r^{-\lambda}\left[\left(\frac{(d_k-\lambda-2)^2}{4} - \alpha^2\right) r^{-2} + \frac{z_\alpha^2}{R^2}\right]\right)$ is a 2-Bessel pair on $(0, R)$, $R > 0$, with $w(r) = r^{\frac{2-d_k+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R} r\right)$. We obtain the following Brézis-Vázquez-type identity and inequality:

Corollary 1.7. *Let $R > 0$, $0 \le \alpha \le \frac{d_k-\lambda-2}{2}$, and $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$ be G-invariant, we have*

$$\int_{B_R} \frac{|\nabla_k u|^2}{|x|^\lambda} - \left(\frac{(d_k-\lambda-2)^2}{4} - \alpha^2\right)\frac{|u|^2}{|x|^{\lambda+2}} \mathrm{d}\mu_k$$

$$\ge \frac{z_\alpha^2}{R^2}\int_{B_R} \frac{|u|^2}{|x|^\lambda}\mathrm{d}\mu_k + \int_{B_R} \left| |x|^{\frac{2-d_k}{2}} J_\alpha\left(\frac{z_\alpha}{R}\right)\nabla_k\left(\frac{u}{|x|^{\frac{2-d_k+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right)}\right)\right|^2 \mathrm{d}\mu_k$$

and

$$\int_{B_R} \frac{|\nabla_k u|^2}{|x|^\lambda} - \left(\frac{(d_k-\lambda-2)^2}{4} - \alpha^2\right)\frac{|u|^2}{|x|^{\lambda+2}} \mathrm{d}\mu_k \ge \frac{z_\alpha^2}{R^2}\int_{B_R} \frac{|u|^2}{|x|^\lambda}\mathrm{d}\mu_k.$$

It is clear that when $\alpha = 0$ and $d_k = N$ so that $\nabla_k = \nabla$, we recover the inequality proved by Vázquez and Zuazua [46] with weights:

$$\int_{B_R} \frac{|\nabla u|^2}{|x|^\lambda} - \left(\frac{N-\lambda-2}{2}\right)^2 \frac{|u|^2}{|x|^{\lambda+2}}\mathrm{d}x \ge \frac{z_\alpha^2}{R^2}\int_{B_R}\frac{|u|^2}{|x|^\lambda}\mathrm{d}x. \tag{1.11}$$

1.4 Hardy's identities and inequalities on domains

Now we consider the Hardy's identity and inequality on domain $\Omega \subset \mathbb{R}^N$ with distance $d(x) = d(x, \partial\Omega)$ from the point x to the boundary $\partial\Omega$. These inequalities have been studied extensively in the case of usual derivatives. The classical Hardy-type inequality on domain takes on the form:

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \ge C(\Omega)\int_\Omega \frac{|x|^2}{d(x)^2}\mathrm{d}x. \tag{1.12}$$

When $0 \in \Omega$ and $d(x) = |x|$ is the distance to the origin, it is well-known that $C(\Omega) = \left(\frac{N-2}{2}\right)^2$ is optimal for $N \ge 3$. Moreover, the equality is never attained by nontrivial functions in $H_0^1(\Omega)$.

In their seminal article, Brézis and Vázquez [9] first investigated the improvement of Hardy's inequality on bounded domains by adding nonnegative terms, when they studied the stability of a certain singular solutions of a nonlinear elliptic equation.

Theorem. (Brézis and Vázquez [9]) *For any bounded domain $\Omega \subset \mathbb{R}^N$, $N \ge 2$ and any $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega |\nabla u|^2\mathrm{d}x - \left(\frac{N-2}{2}\right)^2\int_\Omega \frac{|u|^2}{|x|^2}\mathrm{d}x \ge z_0^2\omega_N^{\frac{2}{N}}|\Omega|^{-\frac{2}{N}}\int_\Omega |u|^2\mathrm{d}x,$$

where $z_0 \approx 2.4048$ is the first zero of the Bessel function $J_0(z)$, and ω is the volume of the unit ball in $\mathbb{R}^N$. The constant $z_0^2\omega_N^{\frac{2}{N}}|\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball but is not achieved in $H_0^1(\Omega)$, as shown in (1.11) *for $\lambda = 0$.*

The geometry of the domain Ω plays an important role. Later, Lam et al. [36,37] studied the improved Hardy's inequalities with Bessel pairs for the general distance function $d(x)$ with $|\nabla d| = 1$. The distance function can be the distance to a point, the boundary of the domain, or a surface with codimension α. In Velicu [47] proved the Dunkl equivalent of the original Hardy's inequality for G-invariant convex domains Ω. More precisely, it was proved that (1.12) holds true when ∇ is replaced by ∇_k:

$$\int_\Omega |\nabla_k u|^2\mathrm{d}\mu_k \ge \frac{1}{4}\int_\Omega \frac{|x|^2}{d(x)^2}\mathrm{d}\mu_k.$$

In spirit of aforementioned works, we are able to obtain an improved Hardy's inequality with the distance function to the boundary of the G-invariant domain, using the notion of Bessel pair. We first present some information about the distance function:

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $\partial\Omega \neq \varnothing$, and $d(x) = d(x, \partial\Omega)$ be the Euclidean distance from x to the boundary, then*
(1) $|\nabla d| = 1$ *wherever is defined.*
(2) $\Delta d(x) \le 0$ *in the sense of distribution if Ω is convex.*
(3) $d(x)$ *is G-invariant and $\nabla_k d(x) = \nabla d(x)$ if Ω is G-invariant.*

Proof. We only prove (3) here. Let $x \in \Omega$, $g \in G$, there exists a sequence $\{y_n\} \in \partial\Omega$ such that

$$|y_n - x| \to d(x), \quad n \to \infty.$$

Since Ω is G-invariant, $(gy_n) \in \partial\Omega$, then

$$|y_n - x| = |gy_n - gx| \ge d(gx).$$

On the other hand,

$$|y_n - x| = |g^{-1}y_n - g^{-1}x| \ge d(g^{-1}x).$$

Thus, $d(x) = d(gx)$ and $\nabla_k d(x) = \nabla d(x)$ follows by definition. □

Theorem 1.3. *If $\Omega \subset \mathbb{R}^N$ is G-invariant and* diam $\Omega < 2R$, $0 < R < \infty$, *(A, B) is an N-dimensional 2-Bessel pair for Dunkl operator on $(0, R)$, and $d(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$, assume on $(0, R)$*

$$\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\varphi'(d(x)) \le 0 \textit{ in the sense of distribution},$$

then for $u \in C_0^\infty(0 < d(x) < R)$,

$$\int_\Omega A(d(x))|\nabla_k u|^2 \mathrm{d}\mu_k - \int_\Omega B(d(x))|u(x)|^2 \mathrm{d}\mu_k \int_\Omega A(d(x))\varphi^2(d(x))\left|\nabla_k\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2 \mathrm{d}\mu_k$$
$$- \int_\Omega A(d(x))|u(x)|^2\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\frac{\varphi'(d(x))}{\varphi(d(x))}\mathrm{d}\mu_k,$$

where φ is the positive solution of

$$(r^{\alpha-1}A(r)y'(r))' + r^{\alpha-1}B(r)y(r) = 0$$

on the interval $(0, R)$.

As a direct consequence, we obtain

Theorem 1.4. *If $\Omega \subset \mathbb{R}^N$ is G-invariant, and* diam $\Omega < 2R$, $0 < R < \infty$, *(A, B) is an N-dimensional p-Bessel pair for Dunkl operator on $(0, R)$, and $d(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$, assume on $(0, R)$*

$$\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\varphi'(d(x)) \le 0 \textit{ in the sense of distribution},$$

then for $u \in C_0^\infty(0 < d(x) < R)$, we have

$$\int_\Omega A(d(x))|\nabla_k u|^2 \mathrm{d}\mu_k \ge \int_\Omega B(d(x))|u(x)|^2 \mathrm{d}\mu_k$$
$$\int_\Omega A(d(x))\varphi^2(d(x))\left|\nabla_k\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2 \mathrm{d}\mu_k.$$

In particular,

$$\int_\Omega A(d(x))|\nabla_k u|^2 \mathrm{d}\mu_k \geq \int_\Omega B(d(x))|u(x)|^2 \mathrm{d}\mu_k.$$

Indeed, most of the Bessel pairs mentioned earlier can be applied to Theorems 1.3 and 1.4. We will not list all of them but only one that is of the Brézis-Vázquez and Brézis-Marcus types.

Corollary 1.8. *If* $\Omega \subset B(R) \subset \mathbb{R}^N$ *is G-invariant, and* $\operatorname{diam}\Omega < R$, $0 < R < \infty$, $0 \leq \lambda \leq d_k - 2$, *then for* $u \in C_0^\infty(0 < d(x) < R)$, *we have*

$$\begin{aligned}\int_\Omega \frac{|\nabla_k u|^2}{d(x)^\lambda} - \frac{(d_k-\lambda-2)^2}{4}\int_\Omega \frac{|u|^2}{d(x)^{\lambda+2}}\mathrm{d}\mu_k \\ = \frac{z_0^2}{R^2}\int_{B_R}\frac{|u|^2}{d(x)^\lambda}\mathrm{d}\mu_k + \int_\Omega \left| d(x)^{\frac{2-d_k}{2}} J_0\left(\frac{z_0}{R}d(x)\right)\nabla_k\left(\frac{u}{d(x)^{\frac{2-d_k+\lambda}{2}}J_0\left(\frac{z_0}{R}d(x)\right)}\right)\right|^2 \mathrm{d}\mu_k \\ - \int_\Omega \frac{1}{d(x)^\lambda}|u(x)|^2\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\left[-\frac{d_k-2\lambda}{2d(x)} + \frac{z_0}{R}\frac{J_0'\left(\frac{z_0}{R}d(x)\right)}{J_0\left(\frac{z_0}{R}d(x)\right)}\right]\mathrm{d}\mu_k.\end{aligned}$$

Proof. Note that for $R > 0$, $\left(r^{-\lambda}, r^{-\lambda}\left[\left(\frac{(d_k-\lambda-2)^2}{4}\right)r^{-2} + \frac{z_0^2}{R^2}\right]\right)$ is an N-dimensional 2-Bessel pair for Dunkl operator on $(0, R)$ with $\varphi = r^{\frac{2-d_k+\lambda}{2}}J_0(\frac{rz_0}{R})$. The result follows by applying Theorem 1.3 to the Bessel pair $\left(r^{-\lambda}, r^{-\lambda}\left[\left(\frac{(d_k-\lambda-2)^2}{4}\right)r^{-2} + \frac{z_0^2}{R^2}\right]\right)$. □

1.5 Connection with monomial weights

Now we consider the simplest family of weight functions, which is defined by

$$h_k^2(x) = \prod_{i=1}^N |x_i|^{k_i}, \quad x \in \mathbb{R}^N,$$

for $k_i \geq 0$, $1 \leq i \leq N$. Clearly, they are invariant under sign change, that is, invariant under the group $\mathbb{Z}_2^N$. Hence, this family of weight functions leads us to the definition of monomial weights:

$$x^P = |x_1|^{P_1}\cdots|x_n|^{P_n},$$

where $P = (P_1, \dots, P_n)$ is a vector in $\mathbb{R}^n$ with $P_i \geq 0$ for $i = 1, \dots, n$.

Actually, this type of problems appeared when Cabré and Ros-Oton studied in [11], which was motivated by an open question raised by Brézis and Vázquez [9] and Brézis [10], concerning the regularity of a certain PDE. Cabré and Ros-Oton studied this question in the space that they call double revolution, which consists of domains of $\mathbb{R}^n$ that are invariant under rotations of the first m variables and the last $n - m$ variables. Cabré and Ros-Oton later generalized the idea with multiple axial symmetries and set up the general monomial weights in [12].

Recall in the case of $\mathbb{Z}_2^N$, and with weight function k_i replaced by the scalar A_i, the Dunkl operators take the following form:

$$T_i f(x) = \partial_i f(x) + A_i \frac{f(x) - f(\sigma_i x)}{x_i},$$

where $\sigma_i x = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N)$. Thus,

$$\int_{B_R} A(|x|)|\nabla_k u|^p x^P \mathrm{d}x - \int_{B_R} B(|x|)|u|^p x^P \mathrm{d}x = \int_{B_R} A(|x|) C_p\left(\nabla_k u, w\nabla_k \frac{u}{w}\right) x^P \mathrm{d}x.$$

In particular, if f is axial symmetric, i.e., $f(x_i) = f(-x_i)$, $i = 1, \dots, n$, so that $\nabla_k f = \nabla f$. Let $B_R^* = B_R \cap \mathbb{R}_*^N$, where $\mathbb{R}_*^N = \{(x_1 \cdots ,x_N) \in \mathbb{R}^N : x_i > 0 \quad \text{whenever } A_i > 0\}$, then we partially recover the L^p Hardy identity with nonradial weights of the form $A(|x|)x^P$ in [19].

Theorem 1.5. *Let $N \geq 1, p > 1, 0 < R \leq \infty$, A, and B be positive C^1 functions. If $(r^{D-1}A, r^{D-1}B)$ is a (one-dimensional) p-Bessel pair on $(0, R)$, then for $\mathbb{Z}_2^N$-invariant $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, we have*

$$\int_{B_R^*} A(|x|)|\nabla u|^p x^P \mathrm{d}x - \int_{B_R^*} B(|x|)|u|^p x^P \mathrm{d}x = \int_{B_R^*} A(|x|) C_p\left(\nabla u, w\nabla \frac{u}{w}\right) x^P \mathrm{d}x. \tag{1.13}$$

Proof. Indeed, one notes that B_R^* is "almost" the same B_R in the following sense:

Set $x = r\omega$, where $r = |x|$ and $\omega = (\omega_1, \dots, \omega_n)$ be the unit vector in $\partial B_1^A = \partial B_1 \cap \mathbb{R}_*^n$, denote $D = N + A_1 + \dots + A_N$, then we obtain

$$x^A = (r\omega)^A = (r\omega_1)^{A_1} \cdots (r\omega_n)^{A_n}$$

and

$$x^A \mathrm{d}x = (r\omega)^A r^{n-1} \mathrm{d}r \mathrm{d}S_\omega = r^{D-1}\omega^A \mathrm{d}r \mathrm{d}S_\omega,$$

where $\mathrm{d}S_\omega$ denotes the surface measure on ∂B_1^A. Hence,

$$\int_{B_R^*} A(|x|)|\nabla u|^p x^A \mathrm{d}x = \int_{\omega \in \partial B_1^A} \int_0^R A(|x|)|\nabla u|^p \mathrm{d}r \mathrm{d}S_\omega = \left(\int_{\omega \in \partial B_1^A} \omega^A \mathrm{d}S_\omega \right) \int_0^R A(|x|)|\nabla u|^p r^{D-1} \mathrm{d}r$$

and

$$\int_{B_R^*} B(|x|)|u|^p x^A \mathrm{d}x = \int_{\omega \in \partial B_1^A} \int_0^R B(|x|)|u|^p \mathrm{d}r \mathrm{d}S_\omega = \left(\int_{\omega \in \partial B_1^A} \omega^A \mathrm{d}S_\omega \right) \int_0^R B(|x|)|u|^p r^{D-1} \mathrm{d}r.$$

Moreover, Cabré and Ros-Oton showed that the first integral can be computed explicitly [12]. Indeed, $\int_{\omega \in \partial B_1^A} \omega^A \mathrm{d}S_\omega = Dm(B_1^A)$, where

$$m(B_1^A) = \int_{B_1^A} x^A \mathrm{d}x = \frac{\Gamma\left(\frac{A_1+1}{2}\right) \cdots \Gamma\left(\frac{A_n+1}{2}\right)}{2^k \Gamma\left(1 + \frac{D}{2}\right)}.$$

Therefore, (1.13) is obtained by similar argument to the proof of Theorem 1.1. □

For many other geometric and functional inequalities with general monomial weights, the interested reader is referred to [12,19,21,48].

2 Preliminaries of Dunkl theory

The purpose of this section is to introduce the theory of Dunkl operators. We refer the interested reader to [2,13] for general Dunkl theory. For a background on reflection group and root systems, the reader is referred to [30,31].

2.1 Root systems and reflection groups

The basic ingredients of the theory of Dunkl operators are root system and reflection groups, acting on some Euclidean space $\mathbb{R}^N$. Let $\langle x, y\rangle = \sum_{i=1}^N x_i y_i$ denote the standard Euclidean inner product and $|x| := \sqrt{\langle x, x\rangle}$. For $\alpha \in \mathbb{R}^N\backslash\{0\}$, we denote by σ_α the reflection with respect to the hyperplane $\langle\alpha\rangle^\perp$ as follows:

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha, x\rangle}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^N.$$

Definition 2.1. A (reduced) root system is a finite set $R \subset \mathbb{R}^N\backslash\{0\}$ such that
(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$; (Reducedness)
(2) $\sigma_\alpha(R) = R$ for all $\alpha \in R$.

The set $\{u^\perp : u \in R\}$ is a finite set of hyperplanes; hence, there exists $u_0 \in \mathbb{R}^N$ such that $\langle u, u_0\rangle \neq 0$ for all $u \in R$. With respect to u_0, a root system can be written as the disjoint union of R_+ and R_-, where $R_+ := \{v \in R : \langle v, u_0\rangle > 0\}$ and $R_- := -R_+$. For simplicity, we assume that R is normalized in the sense that $\langle\alpha, \alpha\rangle = 2$ for all $\alpha \in R$.

Definition 2.2. For a given (reduced) root system, the reflection group $G = G(R)$ is a finite group generated by all the reflections σ_α.

Definition 2.3. A function $k : R \to \mathbb{R}_{\geq 0}$ on the root system is called a multiplicity function if it is invariant under the nature action of G on R, i.e., $k(v) = k(u)$ whenever there exists $g \in G$ such that $ug = v$.

Definition 2.4. Fix a reflection group G and a multiplicity function k, and we define the G-invariant homogeneous weight function $h_k(x)$ of degree $\gamma_k := \sum_{\alpha\in R^+} k(\alpha)$ by

$$h_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{k(\alpha)}.$$

Throughout the article, we assume that $k(\alpha) \geq 0$ and denote $d_k = N + 2\gamma_k$.

2.2 Dunkl operators

Definition 2.5. Given a multiplicity function k, then for $\xi \in \mathbb{R}^N$, the Dunkl operator T_ξ is defined by

$$T_\xi f(x) = \partial_\xi f(x) + E_\xi f(x), \quad f \in C^1(\mathbb{R}^N),$$

where the difference part $E_\xi f(x) = \sum_{\alpha\in R_+} k(\alpha)\langle x, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle\alpha, x\rangle}$ and ∂_ξ denotes the usual directional derivative corresponding to ξ. For the standard basis, we denote $T_j = T_{e_j}$.

We see that if $k(\alpha) = 0$, then Dunkl operators deduce to the differential operator. It is also easy to verify that T_i is the first-order differential-difference operator. In the case of $\mathbb{Z}_2^N$, the Dunkl operators take on the formula

$$T_i f(x) = \partial_i f(x) + k_i \frac{f(x) - f(\sigma_i x)}{x_i},$$

where $\sigma_i x = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_N)$.

The natural spaces to study the Dunkl operators are the weighted spaces $L^p(\mu_k)$, where $\mathrm{d}\mu_k(x) = h_k^2(x)\mathrm{d}x$. We list some important properties that Dunkl operators possess as follows:

(1) Dunkl operators commute,

$$T_iT_j = T_jT_i, \quad 1 \le i, j \le N.$$

(2) For $f, g \in C^1(\mathbb{R}^N)$, and at least one of them is G-invariant,

$$T_i(fg) = T_i(f)g + fT_i(g).$$

(3) Skew-adjointness,

$$\int_{\mathbb{R}^N} T_i f(x)g(x)\mathrm{d}\mu_k(x) = -\int_{\mathbb{R}^N} f(x)T_i g(x)\mathrm{d}\mu_k(x).$$

One can define the Dunkl gradient by $\nabla_k = (T_1, T_2, \dots, T_N)$, and Dunkl divergence by $\operatorname{div}_k = \sum_{j=1}^N T_j$. The Dunkl operators can also be used to define an analog of the Laplace operator, called the Dunkl Laplacian, by $\Delta_k = \sum_{j=1}^N T_j^2$, which can be expressed as

$$\Delta_k f(x) = \Delta_0 f(x) + \sum_{\alpha\in R_+} 2k(\alpha)\frac{\langle \nabla_0 f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \sum_{\alpha\in R_+} 2k(\alpha)\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2},$$

where Δ_0 denotes the usual Euclidean Laplacian on $\mathbb{R}^N$.

3 Proofs

Proof of Corollary 1.1. Recall that we say a couple of C^1 functions (A, B) is an n-dimensional Bessel pair on $(0, R)$ for the usual derivatives, provided the ODE:

$$(r^{N-1}A(r)y')' + r^{N-1}y = 0 \qquad (\mathcal{B}_{A,B})$$

has a positive solution on $(0, R)$. Note that we can also write $(\mathcal{B}_{A,B})$ as

$$y''(r) + \left(\frac{N-1}{r} + \frac{A'(r)}{A(r)}\right)y'(r) + \frac{B(r)}{A(r)}y(r) = 0.$$

Now, for a radial function w, we have

$$\begin{aligned}
\operatorname{div}_k(A(r)|\nabla_k w|^{p-2}\nabla_k w) &= \sum_{j=1}^N T_j\left(A(r)|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right)\\
&= \sum_{j=1}^N (\partial_j + E_j)\left(A(r)|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right)\\
&= \sum_{j=1}^N\left((p-1)A(r)|w'(r)|^{p-2}w''(r)\left(\frac{x_j}{r}\right)^2 + A'(r)\left(\frac{x_j}{r}\right)^2|w'(r)|^{p-2}w'(r)\right.\\
&\quad \left. + A(r)|w'|^{p-2}w'(r)\left(\frac{1}{r} - \frac{1}{r^2}\frac{x_j^2}{r}\right)\right) + \frac{A(r)|w'(r)|^{p-2}w'(r)}{r}\sum_{j=1}^N E_j(x_j)\\
&= (p-1)A(r)|w'(r)|^{p-2}w''(r) + A'(r)|w'(r)|^{p-2}w'(r)\\
&\quad + \left(\frac{N-1+2\gamma_k}{r}\right)A(r)|w'(r)|^{p-2}w'(r).\\
&= (p-1)A(r)|w'(r)|^{p-2}w''(r) + A'(r)|w'(r)|^{p-2}w'(r) + \left(\frac{d_k-1}{r}\right)A(r)|w'(r)|^{p-2}w'(r).
\end{aligned}$$

Hence, (A, B) is a p-Bessel pair for Dunkl operator on $(0, R)$ if

$$(p-1)A(r)|y'(r)|^{p-2}y''(r) + A'(r)|y'(r)|^{p-2}y'(r) + \left(\frac{d_k-1}{r}\right)A(r)|y'(r)|^{p-2}y'(r) + B(r)|y(r)|^{p-2}y(r) = 0 \tag{3.1}$$

has a positive solution. Note that the aforementioned equation can be written as:

$$(p-1)|y'(r)|^{p-2}y''(r) + \left(\frac{A'(r)}{A(r)} + \frac{d_k-1}{r}\right)|y'(r)|^{p-2}y'(r) + \frac{B(r)}{A(r)}|y(r)|^{p-2}y(r) = 0,$$

so that

$$(r^{d_k-1}A(r)|y'|^{p-2}y')' + r^{d_k-1}B(r)|y|^{p-2}y = 0.$$

That is, (A, B) is a d_k-dimensional p-Bessel pair in the usual derivative sense. □

Proof of Theorem 1.1. Using polar coordinates, one has

$$\int_{B_R} A(|x|)|\nabla_k u|^p \mathrm{d}\mu_k(x) = \int_{B_R} A(|x|)|\nabla_k u|^p h_k^2(x)\mathrm{d}x = \int_{\mathbb{S}^{N-1}}\left(\int_0^R |\nabla_k u|^p r^{N-1}\mathrm{d}r\right)h_k^2(x)\mathrm{d}\omega.$$

The constant

$$c_N = \int_{\mathbb{S}^{N-1}} h_k^2(x)\mathrm{d}\omega$$

can be explicitly computed, see [17] for details. Therefore,

$$\int_{B_R} A(|x|)|\nabla_k u|^p \mathrm{d}\mu_k(x) = \left(\int_{\mathbb{S}^{N-1}} h_k^2(x)\mathrm{d}\omega\right)\left(\int_0^R A(r)|\nabla_k u|^p r^{N-1}\mathrm{d}r\right). \tag{3.2}$$

Letting $v = u/w$, where w is the positive radial solution in the definition of p-Bessel pair so that v is G-invariant, then we obtain the following:

$$\begin{aligned}\int_0^R A(r)|\nabla_k u|^p r^{N-1}\mathrm{d}r &= \int_0^R A(r)|w\nabla_k v + v\nabla_k w|^p r^{N-1}\mathrm{d}r\\ &= \int_0^R r^{N-1}A(r)|v|^p|\nabla_k w|r^{N-1} + pr^{N-1}A(r)|v\nabla_k w|^{p-2}vw\nabla_k w\cdot\nabla_k v\\ &\quad + r^{N-1}A(r)C_p\left(\nabla_k u, w\nabla_k\frac{u}{w}\right)\mathrm{d}r.\end{aligned} \tag{3.3}$$

Consider the first term in the right-hand side of (3.3),

$$\begin{aligned}\int_0^R r^{N-1}A(r)|v||^p\nabla_k w|^p\mathrm{d}r &= \int_0^R r^{N-1}A(r)|v|^p|\nabla_k w|^{p-2}\left(\sum_{j=1}^N T_j w T_j w\right)\mathrm{d}r\\ &= -\sum_{j=1}^N\int_0^R wT_j(r^{N-1}A(r)|v|^p|\nabla_k w|^{p-2}T_j w)\mathrm{d}r.\end{aligned}$$

By the definition of the Dunkl derivative,

$$\begin{aligned}T_j(r^{N-1}A(r)|v|^p|\nabla_k w|^{p-2}T_j w) &= (\partial_j + E_j)(r^{N-1}A(r)|v|^p|\nabla_k w|^{p-2}T_j w)\\ &= (p|v|^{p-1}\partial_j v)r^{N-1}A(r)|\nabla_0 w|^{p-2}\partial_j w + |v|^p\partial_j(r^{N-1}A(r)|v|^p|\nabla_0 w|^{p-2}\partial_j w)\\ &\quad + E_j\left(r^{N-1}A(r)|v|^p|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right).\end{aligned}$$

Since $r^{N-1}A(r)|w'(r)|^{p-2}\frac{w'(r)}{r}$ is radial,

$$E_j\left(r^{N-1}A(r)|v|^p|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right) = r^{N-1}A(r)|w'(r)|^{p-2}\frac{w'(r)}{r}E_j(|v|^p x_j).$$

Moreover,

$$E_j(|v|^p x_j) = \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{|v(x)|^p x_j - |v(\sigma_\alpha x)|^p \sigma_\alpha x_j}{\langle \alpha, x\rangle},$$

so that

$$\sum_{j=1}^N E_j(|v|^p x_j) = \sum_{\alpha\in R_+} k(\alpha)(|v|^p + |v(\sigma_\alpha x)|^p) = 2\sum_{\alpha\in R_+} k(\alpha)|v|^p,$$

since v is G-invariant. Hence,

$$\begin{aligned}\int_0^R r^{N-1}A(r)|v||\nabla_k w|^p \,\mathrm{d}r = -p &\int_0^R w|v|^{p-1}\nabla_0 v\cdot\nabla_0 w r^{N-1}A(r)|\nabla_0 w|^{p-2}\mathrm{d}r\\ &- \int_0^R w|v|^p \operatorname{div}_0(r^{N-1}A(r)|v|^p|\nabla_0 w|^{p-2}\nabla_0 w)\mathrm{d}r\\ &- 2\sum_{\alpha\in R_+} k(\alpha)\int_0^R r^{N-1}A(r)|w'(r)|^{p-2}w'(r)w(r)\frac{1}{r}|v|^p \mathrm{d}r.\end{aligned} \tag{3.4}$$

Recall that the Dunkl p-Laplacian is defined as

$$\Delta_{p,k}w = \operatorname{div}_k(|\nabla_k w|^{p-2}\nabla_k w).$$

Similarly,

$$\begin{aligned}\operatorname{div}_k(r^{N-1}A(r)|\nabla_k w|^{p-2}\nabla_k w) &= \sum_{j=1}^N T_j\left(r^{N-1}A(r)|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right)\\ &= \sum_{j=1}^N(\partial_j + E_j)\left(r^{N-1}A(r)|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right)\\ &= \sum_{j=1}^N\left[\partial_j\left(r^{N-1}A(r)|w'(r)|^{p-2}w'(r)\frac{x_j}{r}\right) + r^{N-1}A(r)|w'(r)|^{p-2}w'(r)\frac{1}{r}E_j(x_j)\right]\\ &= \sum_{j=1}^N \partial_j(r^{N-1}A(r)|\nabla_0 w|^{p-2}\nabla_0 w) + r^{N-1}A(r)\frac{|w'|^{p-2}w'}{r}\sum_{j=1}^N E_j(x_j)\\ &= \operatorname{div}_0(r^{N-1}A(r)|\nabla_0 w|^{p-2}\nabla_0 w) + r^{N-1}A(r)\frac{|w'|^{p-2}w'}{r}2\gamma_k.\end{aligned}$$

Thus, (3.4) can be written as

$$\begin{aligned}\int_0^R r^{N-1}A(r)|v|^p|\nabla_k w|^p \mathrm{d}r = -p&\int_0^R r^{N-1}A(r)w|v|^{p-1}\nabla_0 v\cdot\nabla_0 w|\nabla_0 w|^{p-2}\mathrm{d}r\\ &- \int_0^R w|v|^p\operatorname{div}_k(r^{N-1}A(r)|\nabla_k w|^{p-2}\nabla_k w)\mathrm{d}r.\end{aligned} \tag{3.5}$$

We then consider the second term in (3.3),

$$p\int_0^R r^{N-1}A(r)|v\nabla_k w|^{p-2}vw\nabla_k w\cdot\nabla_k v\mathrm{d}r = p\int_0^R r^{N-1}A(r)|v|^{p-1}|\nabla_k w|^{p-2}\nabla_0 w\cdot\nabla_0 v\mathrm{d}r$$
$$+\, p\int_0^R r^{N-1}A(r)vw\frac{w'(r)}{r}\left(\sum_{j=1}^N x_jE_j(v)\right)\mathrm{d}r.$$

Since v is G-invariant, we have

$$\sum_{j=1}^N x_jE_j(v) = \sum_{j=1}^N\sum_{\alpha\in R_+} k(\alpha)x_j\alpha_j\frac{|v(x)|-|v(\sigma_\alpha x)|}{\langle\alpha,x\rangle} = 0.$$

Hence,

$$p\int_0^R r^{N-1}A(r)|v\nabla_k w|^{p-2}vw\nabla_k w\cdot\nabla_k v = p\int_0^R r^{N-1}A(r)|v|^{p-1}|\nabla_k w|^{p-2}\nabla_0 w\cdot\nabla_0 v. \tag{3.6}$$

Substituting (3.3), (3.4), and (3.6) into (3.2) and using the definition of p-Bessel pair, we obtain

$$\begin{aligned}
&\int_{B_R} A(r)|\nabla_k u|^p\mathrm{d}\mu_k(x)\\
&= -\int_{\mathbb{S}^{N-1}}\int_0^R w|v|^p\,\mathrm{div}_k(r^{N-1}A(r)|\nabla_k w|^{p-2}\nabla_k w)\mathrm{d}\mu_k(r)\mathrm{d}S + \int_{\mathbb{S}^{N-1}}\int_0^R r^{N-1}A(r)C_p(\nabla_k u, w\nabla_k v)\mathrm{d}\mu_k(r)\mathrm{d}S\\
&= \int_{\mathbb{S}^{N-1}}\int_0^R w|v|^pB(r)|w|^{p-2}wr^{N-1}\mathrm{d}\mu_k(r)\mathrm{d}S + \int_{\mathbb{S}^{N-1}}\int_0^R A(r)C_p(\nabla_k u, w\nabla_k v)r^{N-1}\mathrm{d}\mu_k(r)\mathrm{d}S\\
&= \int_{B_R} B(r)|u|^p\mathrm{d}\mu_k(x) + \int_{B_R} A(r)C_p(\nabla_k u, w\nabla_k v)\mathrm{d}\mu_k(x).
\end{aligned}$$

That is,

$$\int_{B_R} A(r)|\nabla_k u|^p\mathrm{d}\mu_k(x) - \int_{B_R} B(r)|u|^p\mathrm{d}\mu_k(x) = \int_{B_R} A(r)C_p\left(\nabla_k u, w\nabla_k\frac{u}{w}\right)\mathrm{d}\mu_k(x), \tag{3.7}$$

where w is the positive solution of

$$\mathrm{div}_k(A(r)|\nabla_k y(r)|^{p-2}\nabla_k y(r)) + B(r)|y(r)|^{p-2}y(r) = 0$$

on $(0, R)$. □

Proof of Theorem 1.3. Define

$$Pu = \sqrt{A(d(x))}\,\nabla_k u - u\sqrt{A(d(x))}\,\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x).$$

We mention here that $T\varphi(d(x))$ is actually $T\varphi$ evaluated at $d(x)$, not the Dunkl derivative on the composite $\varphi(d(x))$. Then, the formal adjoint of P acting on u is

$$P^*u = -\mathrm{div}_k(\sqrt{A(d(x))}\,u) - u\sqrt{A(d(x))}\,\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x).$$

Thus,

$$\begin{aligned}
P^*Pu(x) &= -\operatorname{div}_k\left(A(d(x))\nabla_k u(x) - u(x)A((d(x)))\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k u\right)\\
&\quad - A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x)\cdot\left(\nabla_k u(x) - u(x)\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x)\right)\\
&= \operatorname{div}_k(A(d(x))\nabla_k u(x)) + \operatorname{div}_k\left(u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x)\right)\\
&\quad - A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k u(x)\cdot\nabla_k d(x) + u(x)A(d(x))\left(\frac{T\varphi(d(x))}{\varphi(d(x))}\right)^2|\nabla_k d(x)|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
&\operatorname{div}_k\left(u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k d(x)\right)\\
&= u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\Delta_k d(x) + \nabla_k d(x)\operatorname{div}_k\left(u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\right)\\
&= u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\Delta_k d(x) + u(x)TA(d(x))|\nabla_k d(x)|^2\frac{T\varphi(d(x))}{\varphi(d(x))} + A(d(x))\frac{\nabla_k d(x)}{\varphi^2(x)}\\
&\quad\times(T^2\varphi(d(x))\nabla_k d(x)u(x) + T\varphi(d(x))\nabla_k u(x))\varphi(x) - (T\varphi(d(x)))^2u(x)\nabla_k d(x)).
\end{aligned}$$

Hence,

$$\begin{aligned}
P^*Pu(x) &= -\operatorname{div}_k(A(d(x))\nabla_k u) + u(x)TA(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))} + u(x)A(d(x))\frac{T^2\varphi(d(x))}{\varphi(x)}\\
&\quad + u(x)A(d(x))\frac{T\varphi(d(x))}{\varphi(d(x))}\Delta_k d(x).
\end{aligned}$$

Now recall that A and φ are one-dimensional radial functions so that $T\varphi = \varphi'$ and the second-order Dunkl derivative takes on the form $T^2\varphi = \varphi'' + \frac{2k(x)}{x}\varphi'$. Then,

$$\begin{aligned}
&\int_\Omega P^*Pu(x)\overline{u(x)}\mathrm{d}\mu_k\\
&= \int_\Omega A(d(x))|\nabla_k u|^2 + \int_\Omega \frac{A(d(x))}{\varphi(d(x))}\left(\varphi'' + \left(\frac{2k(d(x))}{d(x)} + \frac{A'(d(x))}{A(d(x))} + \Delta_k d(x)\right)\varphi'\right)|u(x)|^2\mathrm{d}\mu_k\\
&= \int_\Omega A(d(x))|\nabla_k u|^2\mathrm{d}\mu_k - \int_\Omega B(d(x))|u(x)|^2\mathrm{d}\mu_k + \int_\Omega A(d(x))|u(x)|^2\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\frac{\varphi'(d(x))}{\varphi(d(x))}\mathrm{d}\mu_k.
\end{aligned}$$

We also note that

$$\begin{aligned}
\int_\Omega P^*Pu(x)\overline{u(x)}\mathrm{d}\mu_k &= \int_\Omega ||Tu(x)||^2\mathrm{d}\mu_k\\
&= \int_\Omega A(d(x))|\nabla_k u(x)|^2 + |u(x)|^2A(d(x))\left(\frac{T\varphi(d(x))}{\varphi(d(x))}\right)^2 - 2A(d(x))u(x)\frac{T\varphi(d(x))}{\varphi(d(x))}\nabla_k u(x)\\
&\qquad\cdot\nabla_k d(x)\mathrm{d}\mu_k\\
&= \int_\Omega A(d(x))\varphi^2(d(x))\left|\nabla_k\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2\mathrm{d}\mu_k.
\end{aligned}$$

Therefore,

$$\int_\Omega A(d(x))|\nabla_k u|^2 \mathrm{d}\mu_k - \int_\Omega B(d(x))|u(x)|^2 \mathrm{d}\mu_k \int_\Omega A(d(x))\varphi^2(d(x))\left|\nabla_k\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2 \mathrm{d}\mu_k$$
$$- \int_\Omega A(d(x))|u(x)|^2\left(\Delta_k d(x) + \frac{2k(d(x)) - \alpha + 1}{d(x)}\right)\frac{\varphi'(d(x))}{\varphi(d(x))}\mathrm{d}\mu_k. \qquad \square$$

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