Research Article

Zhen-Feng Jin, Hong-Rui Sun*, and Jianjun Zhang

Existence of ground state solutions for critical fractional Choquard equations involving periodic magnetic field

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Abstract: In this paper, we consider the following critical fractional magnetic Choquard equation:

$$\begin{aligned}\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u &= \varepsilon^{\alpha-N}\left(\int\limits_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}y\right)|u|^{2^*_{s,\alpha}-2}u\\ &\quad+\varepsilon^{\alpha-N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,|u(y)|^2)}{|x-y|^\alpha}\mathrm{d}y\right)f(x,|u|^2)u \quad \text{in } \mathbb{R}^N,\end{aligned}$$

where $\varepsilon > 0$, $s \in (0,1)$, $\alpha \in (0,N)$, $N > \max\{2\mu + 4s, 2s + \alpha/2\}$, $2^*_{s,\alpha} = \frac{2N-\alpha}{N-2s}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $(-\Delta)^s_A$ stands for the fractional Laplacian with periodic magnetic field A of $C^{0,\mu}$-class with $\mu \in (0,1]$ and V is a continuous potential and allows to be sign-changing. Under some mild assumptions imposed on V and f, we establish the existence of at least one ground state solution.

Keywords: Choquard equation, fractional magnetic Laplacian, ground state solution, critical, Hardy-Littlewood-Sobolev inequality

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1 Introduction

This paper is concerned with the following fractional Choquard equation involving magnetic field and critical nonlinearity:

$$\begin{aligned}\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u &= \varepsilon^{\alpha-N}\left(\int\limits_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}y\right)|u|^{2^*_{s,\alpha}-2}u\\ &\quad+\varepsilon^{\alpha-N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,|u(y)|^2)}{|x-y|^\alpha}\mathrm{d}y\right)f(x,|u|^2)u \quad \text{in } \mathbb{R}^N,\end{aligned}\tag{1.1}$$

*** Corresponding author: Hong-Rui Sun,** School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P. R. China, e-mail: hrsun@lzu.edu.cn
Zhen-Feng Jin: School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P. R. China, e-mail: jinzhf15@lzu.edu.cn
Jianjun Zhang: College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, P. R. China, e-mail: zhangjianjun09@tsinghua.org.cn

where ε is a small positive parameter, $s \in (0, 1)$, $\alpha \in (0, N)$, $N > \max\{2\mu + 4s, 2s + \alpha/2\}$, and $2^*_{s,\alpha} = \frac{2N-\alpha}{N-2s}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. The fractional magnetic Laplacian $(-\Delta)^s_A$ has been introduced in [16,24] with motivations falling into the framework of the general theory of Lévy processes, and up to normalization constants, is defined for any $u \in C^\infty_c(\mathbb{R}^N, \mathbb{C})$ as follows:

$$(-\Delta)^s_A u(x) := \lim_{r\to 0} \int\limits_{\mathbb{R}^N\setminus B_r(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}} dy,$$

where $B_r(x)$ denotes the ball in $\mathbb{R}^N$ of center at x and radius $r > 0$, which can be considered as the fractional counterpart of the classical magnetic Laplacian $\left(\frac{1}{i}\nabla - A\right)^2$. If $A \equiv 0$, the operator $(-\Delta)^s_A$ becomes the celebrated fractional Laplacian $(-\Delta)^s$, which has been widely used in different subjects, for instance, the thin obstacle problem, ecology, finance, and anomalous diffusion. For a comprehensive discussion of the properties and applications of the fractional Laplacian $(-\Delta)^s$ for more details, the readers can refer to the guide [19]. Here, the potential functions A, V satisfy

(A) $A \in C^{0,\mu}(\mathbb{R}^N, \mathbb{R}^N)$ with $\mu \in (0, 1]$ is a $\mathbb{Z}^N$-periodic vector potential. That is, for all $x \in \mathbb{R}^N$, it holds that $A(x + y) = A(x)$, $\forall y \in \mathbb{Z}^N$;

(V1) $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $V(x) \leqslant \bar{V} := \max_{x\in\mathbb{R}^N} V(x) \in (0, \infty)$, and there exists a constant $\zeta_0 > 0$ such that

$$[u]^2_{A_\varepsilon} + \int\limits_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 dx \geqslant \zeta_0 \int\limits_{\mathbb{R}^N} [\bar{V} - V_\varepsilon(x)]|u|^2 dx \quad \text{for all } u \in H^s_\varepsilon,$$

where $A_\varepsilon(x) := A(\varepsilon x)$ and $V_\varepsilon(x) := V(\varepsilon x)$;

(V2) $V(x) = V_{\mathcal{P}}(x) - W(x)$, where $V_{\mathcal{P}} \in C(\mathbb{R}^N, \mathbb{R})$ is $\mathbb{Z}^N$-periodic, and $W \in L^{\frac{N}{2s}}(\mathbb{R}^N, \mathbb{R})$ with $W(x) > 0$, $\forall x \in \mathbb{R}^N$;

and the nonlinearity f fulfills

(F1) $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$, and there exist $C_0 > 0$ and p_1, p_2 with $\frac{2N-\alpha}{N} < p_1 \leqslant p_2 < 2^*_{s,\alpha}$, such that for any $t \in \mathbb{R}$

$$|f(x, t)| \leqslant C_0\left(|t|^{\frac{p_1-2}{2}} + |t|^{\frac{p_2-2}{2}}\right) \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(F2) $t \mapsto f(x, t)$ is nondecreasing on $\mathbb{R}$ for every $x \in \mathbb{R}^N$;

(F3)

$$\lim_{|t|\to+\infty} \frac{F(x, t)}{|t|^{\frac{2^*_{s,\alpha} - \frac{2s}{N-2s}}{2}}} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^N,$$

where $F(x, t) = \int_0^t f(x, s) ds$;

(F4) $f(x, t)$ is $\mathbb{Z}^N$-periodic with respect to $x \in \mathbb{R}^N$.

The following nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\alpha} dy\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

seems to arise from the work of Pekar [34] for the modeling of quantum polaron and was mentioned in [26] that Choquard used this equation to study a certain approximation to Hartree-Fock theory of one component plasma. When $\alpha = 1$, $p = 2$, $V \in L^\infty(\mathbb{R}^3)$ is $\mathbb{Z}^3$-periodic and 0 lies in the gap of the spectrum of $-\Delta + V(x)$, Buffoni et al. [10] proved the existence of a nontrivial solution for (1.2) by using variational methods and Lyapunov-Schmidt reduction, and Ackermann [1] obtained the existence and multiplicity of solutions for (1.2) by applying an abstract critical point theorem. Wu et al. [39] considered the following Choquard equation with lower critical exponent:

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{\frac{2N-\alpha}{N}}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{\frac{-\alpha}{N}} u + f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $\alpha \in (0, N)$, $N \geqslant 1$ and V satisfies (V_1) and (V_2). In their paper, the condition (V_2) introduced in [18] is applied to obtain an equivalent norm of $H^1(\mathbb{R}^N)$. The nonlinear perturbation f satisfies suitable growth including $(F4)$. The authors proved that there exists a ground state solution of above problem by applying variational methods and Lions' concentration compactness principle. Bueno et al. [9] considered the following magnetic Choquard equation with upper critical exponent

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{2_\alpha^*-2} u + \lambda \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{p-2} u \quad \text{in } \mathbb{R}^N,$$

where $\alpha \in (0, N)$, $2_\alpha^* = \frac{2N-\alpha}{N-2}$, $p \in \left(\frac{2N-\alpha}{N}, 2_\alpha^*\right)$, $N \geqslant 3$, $\lambda > 0$, A and V satisfy conditions (A) and (V_2), respectively. The authors obtained the existence of at least one ground state solution for the aforementioned problem, provided that p belongs to some intervals that depend on N and λ. Problems like or similar to (1.2) have been extensively studied in recent years by many authors, and the readers can refer to [6,11,12,14, 20,21,28,25,30,31,33,35] and the references therein.

On the other hand, nonlinear Choquard equation of the type

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\alpha-N}\left(\int_{\mathbb{R}^N} \frac{G(y, u(y))}{|x-y|^\alpha} \mathrm{d}y\right) g(x,u) \quad \text{in } \mathbb{R}^N \tag{1.3}$$

has also received much attention recently. When $s = 1$, $\alpha \in (0, N)$, $N \geqslant 3$, and $G(x, t) = |t|^p$ with $p \in \left[\frac{2N-\alpha}{N}, 2_\alpha^*\right)$, Moroz and Van Schaftingen [32] testified that (1.3) has a family of solutions concentrating to the local minimum of V for small ε by a novel nonlocal penalization technique, provided that the external potential $V \in C(\mathbb{R}^N, [0, \infty))$ satisfies some additional assumptions at infinity. When $s = 1$, $N = 3$, $\alpha \in (0, 3)$, $G(x, t) = Q(x)H(t)$ and H has a critical growth, Alves et al. [3] obtained the existence and multiplicity of solutions for (1.3) and characterized the concentration behavior for small ε, provided that $Q, V \in C(\mathbb{R}^3, (0, \infty))$ satisfy some additional assumptions at infinity. When $s \in (0, 1)$, $N > 2s$, $\alpha \in (0, 2s)$, V is a continuous potential function and satisfies the following local conditions, which was introduced in [17]:
(V′) $\inf_{x\in\mathbb{R}^3} V(x) > 0$, and there is a bounded open domain Ω such that

$$V_0 := \inf_{\Omega} V(x) < \min_{\partial\Omega} V(x),$$

and g is a superlinear continuous function with subcritical growth. Ambrosio [7] investigated the multiplicity and concentration of positive solutions for problem (1.3) by using the penalization method and the Ljusternik-Schnirelmann theory. For $\varepsilon > 0$ small, Yang and Zhao [40] investigated the existence, multiplicity, and concentration behavior of positive solutions for problem (1.3), provided that $s \in (0, 1)$, $N = 3$, $\alpha \in (0, 3)$, V satisfies condition (V') and $g(x, t) = |t|^{2_{s,\alpha}^*-2} t + \frac{1}{2_{s,\alpha}^*} f(t)$, where the nonlinear perturbation f satisfies suitable growth including (F_1)–(F_3). For more results about Choquard equation (1.3), the readers can refer to [2,4,5,13,29] and the references therein.

Motivated by the works [9,39,40], in the present paper, we focus our attention on the existence of ground state solutions for problem (1.1). We emphasize that the lack of compactness due to the fact that (1.1) contains the upper critical exponent $\frac{2N-\alpha}{N-2s}$ in the sense of Hardy-Littlewood-Sobolev inequality and the nonlocal nature of the fractional magnetic operator bring the main difficulties. To handle these difficulties, we first consider the problem (1.1) with periodic electric potential, that is, $V = V_{\mathcal{P}}$, and prove that the energy functional has a mountain pass geometry. Thus, we obtain a Cerami sequence at the mountain pass level (denoted by c_ε) by [22], and we can show the boundedness of this sequence by using the monotone condition (F_2). Next, we prove the existence of ground state solutions for problem (1.1) with periodic electric potential $V_{\mathcal{P}}$ (see Theorem 3.5) by using a fractional version of the Lions' concentration compactness principle ([36, Lemma 2.4]), the estimate of the level c_ε (see Lemma 3.4) and the periodicity condition on A, V, and f. Finally, taking advantage of the estimate of $c_\varepsilon > d_\varepsilon$ for all $\varepsilon > 0$, where d_ε is the mountain pass level of the energy functional associated with the original problem (1.1), we get the existence of ground state solutions for (1.1) by applying the concentration compactness argument once again. The main result can be stated as follows.

Theorem 1.1. *Let $s \in (0,1)$, $\alpha \in (0,N)$, and $N > \max\{2\mu + 4s, 2s + \alpha/2\}$. Assume that (A), (V_1), (V_2), and (F_1)–$(F4)$ hold. Then, there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, problem (1.1) has at least one ground state solution.*

The paper is organized as follows. In Section 2, we introduce the functional setting and we recall some useful lemmas for the fractional magnetic Sobolev spaces. In Section 3, we give the existence result for (1.1).

2 Preliminaries

In this section, we are devoted to some notations and preliminary results. Let $D^{s,2}(\mathbb{R}^N, \mathbb{R})$ denote as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ with respect to

$$[u]_s^2 := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

The fractional Sobolev space $H^s(\mathbb{R}^N, \mathbb{R})$ is defined as follows:

$$H^s(\mathbb{R}^N, \mathbb{R}) := \{u \in L^2(\mathbb{R}^N, \mathbb{R}) : [u]_s < \infty\}$$

endowed with the norm $\|u\|^2 := [u]_s^2 + \|u\|_{L^2}^2$; see [19] for more details.

The space $D_A^{s,2}(\mathbb{R}^N, \mathbb{C})$ is defined as follows:

$$D_A^{s,2}(\mathbb{R}^N, \mathbb{C}) := \left\{u \in L^{2_s^*}(\mathbb{R}^N, \mathbb{C}) : [u]_A^2 < \infty\right\},$$

where

$$[u]_A^2 := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \quad \text{and} \quad 2_s^* := \frac{2N}{N-2s}.$$

The fractional magnetic Sobolev space H_ε^s is defined as follows:

$$H_\varepsilon^s := \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) : [u]_{A_\varepsilon}^2 < \infty\right\}$$

under the scalar product

$$\langle u, v\rangle_{H_\varepsilon^s} := \mathfrak{Re} \iint\limits_{\mathbb{R}^{2N}} \frac{\left(u(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)} u(y)\right)\overline{\left(v(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)} v(y)\right)}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \mathfrak{Re} \int\limits_{\mathbb{R}^N} u\bar{v}\mathrm{d}x, \quad \forall u, v \in H_\varepsilon^s,$$

with the associated norm $\|u\|_{H_\varepsilon^s}^2 = \langle u, u\rangle_{H_\varepsilon^s}$, where $\mathfrak{Re}(w)$ denotes the real part of $w \in \mathbb{C}$ and $\bar{w}$ denotes its complex conjugate.

Lemma 2.1. *Assume that V satisfies (V_1). Then, there exist two constants $C_1, C_2 > 0$ such that*

$$C_1\|u\|_{H_\varepsilon^s}^2 \leqslant [u]_{A_\varepsilon}^2 + \int\limits_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x \leqslant C_2\|u\|_{H_\varepsilon^s}^2 \qquad \textit{for all } u \in H_\varepsilon^s.$$

Proof. From (V_1), we find that

$$[u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x = [u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} \bar{V}|u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} (\bar{V} - V_\varepsilon(x))|u|^2 \mathrm{d}x$$

$$\geqslant [u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} \bar{V}|u|^2 \mathrm{d}x - \frac{1}{\zeta_0}\left([u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x\right).$$

So, it holds that $C_1\|u\|_{H_\varepsilon^s}^2 \leqslant [u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x$.

On the other hand, from (V_1), we get that

$$[u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x \leqslant [u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} \bar{V}|u|^2 \mathrm{d}x \leqslant C_2\|u\|_{H_\varepsilon^s}^2.$$ □

Lemma 2.1 implies that

$$\|u\|_\varepsilon^2 := [u]_{A_\varepsilon}^2 + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2 \mathrm{d}x$$

is an equivalent norm of H_ε^s with the scalar product

$$\langle u, v\rangle_\varepsilon := \Re\iint_{\mathbb{R}^{2N}} \frac{\left(u(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)}u(y)\right)\overline{\left(v(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)}v(y)\right)}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \Re\int_{\mathbb{R}^N} V_\varepsilon(x)u\bar{v}\mathrm{d}x, \ \ \forall u, v \in H_\varepsilon^s.$$

Referring to [8,16], we recall the following useful properties for the space H_ε^s.

Lemma 2.2. (Diamagnetic inequality) *If $u \in H_\varepsilon^s$, then $|u| \in H^s(\mathbb{R}^N, \mathbb{R})$, and it holds that*

$$[|u|]_s \leqslant [u]_{A_\varepsilon}.$$

Lemma 2.3. (Magnetic Sobolev embedding) *The space H_ε^s is continuously embedded in $L^r(\mathbb{R}^N, \mathbb{C})$ for $r \in [2, 2_s^*]$, and compactly embedded in $L_{\mathrm{loc}}^r(\mathbb{R}^N, \mathbb{C})$ for $r \in [1, 2_s^*)$.*

Lemma 2.4. *If $u \in H^s(\mathbb{R}^N, \mathbb{R})$ and u has compact support, then $w = \mathrm{e}^{iA(0)\cdot x}u \in H_\varepsilon^s$.*

We use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} := \inf_{u\in D^{s,2}(\mathbb{R}^N,\mathbb{R})\setminus\{0\}} \frac{[u]_s^2}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2_{s,\alpha}^*}|u(x)|^{2_{s,\alpha}^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2_{s,\alpha}^*}}}.$$

From Lemma 2.2, we know that

$$S_{A_\varepsilon} := \inf_{u\in D_{A_\varepsilon}^{s,2}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{[u]_{A_\varepsilon}^2}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2_{s,\alpha}^*}|u(x)|^{2_{s,\alpha}^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2_{s,\alpha}^*}}} \geqslant S_{H,L}. \tag{2.1}$$

Lieb and Loss [27] introduced the following well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.5. *Let $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{t} + \frac{\alpha}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N, \mathbb{R})$, and $h \in L^r(\mathbb{R}^N, \mathbb{R})$. Then, there exists a sharp constant $C(t, N, \alpha, r)$, independent of f and h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \leqslant C(t, N, \alpha, r)\|f\|_{L^t}\|h\|_{L^r}. \tag{2.2}$$

If $t = r = \frac{2N}{2N-\alpha}$, *then*

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(N - \frac{\alpha}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-1+\frac{\alpha}{N}}.$$

In this case, there is equality in (2.2) *if and only if* $h = cf$ *for a constant* c *and*

$$f(x) = A(\gamma^2 + |x - a|^2)^{-(2N-\alpha)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$, *and* $a \in \mathbb{R}^N$.

3 Existence of ground state solution

In this section, we study the existence of ground state solution for problem (1.1). Using the change of variable $u(x) \mapsto u(\varepsilon x)$, we can see that problem (1.1) is equivalent to

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x)u = \left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}y \right) |u|^{2^*_{s,\alpha}-2}u + \left(\int\limits_{\mathbb{R}^N} \frac{F(\varepsilon y, |u(y)|^2)}{|x-y|^\alpha} \mathrm{d}y \right) f(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Initially, we consider the following nonlocal problem with periodic potential associated with (3.1)

$$(-\Delta)^s_{A_\varepsilon} u + V_{\mathcal{P},\varepsilon}(x)u = \left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}y \right) |u|^{2^*_{s,\alpha}-2}u + \left(\int\limits_{\mathbb{R}^N} \frac{F(\varepsilon y, |u(y)|^2)}{|x-y|^\alpha} \mathrm{d}y \right) f(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

where $V_{\mathcal{P},\varepsilon}(x) := V_{\mathcal{P}}(\varepsilon x)$.

Combining (V_1) with (V_2), by a similar discussion as in Lemma 2.1, we get that

$$\|u\|^2_{\mathcal{P},\varepsilon} := [u]^2_{A_\varepsilon} + \int\limits_{\mathbb{R}^N} V_{\mathcal{P},\varepsilon}(x)|u|^2 \mathrm{d}x$$

is an equivalent norm of H^s_ε with the scalar product

$$\begin{aligned}\langle u, v\rangle_{\mathcal{P},\varepsilon} := \Re\mathfrak{e} \iint\limits_{\mathbb{R}^{2N}} \frac{\left(u(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)}u(y)\right)\overline{\left(v(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)}v(y)\right)}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ + \Re\mathfrak{e} \int\limits_{\mathbb{R}^N} V_{\mathcal{P},\varepsilon}(x)u\bar{v}\mathrm{d}x, \quad \forall u, v \in H^s_\varepsilon.\end{aligned}$$

The associated energy functional $J_{\mathcal{P},\varepsilon} : H^s_\varepsilon \to \mathbb{R}$ for problem (3.2) is defined as follows:

$$\begin{aligned}J_{\mathcal{P},\varepsilon}(u) = \frac{1}{2}\|u\|^2_{\mathcal{P},\varepsilon} - \frac{1}{2 \cdot 2^*_{s,\alpha}} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \\ - \frac{1}{4} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(\varepsilon x, |u(x)|^2)F(\varepsilon y, |u(y)|^2)}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y.\end{aligned}$$

Under our assumptions and using Lemma 2.5, we get that $J_{\mathcal{P},\varepsilon}$ is well defined on H^s_ε and belongs to $C^1(H^s_\varepsilon, \mathbb{R})$. Thus, u is a weak solution of problem (3.2) if and only if u is a critical point of the functional $J_{\mathcal{P},\varepsilon}$. Next, we prove that $J_{\mathcal{P},\varepsilon}$ has the mountain pass geometry.

Lemma 3.1. *Assume that* (A), (V_1), (V_2), (F_1), *and* (F_2) *hold. Then, the energy functional* $J_{\mathcal{P},\varepsilon}$ *satisfies the following properties*:
(*i*) *there exist* $\beta, \rho > 0$ *such that* $J_{\mathcal{P},\varepsilon}(u) \geqslant \beta$ *when* $\|u\|_{\mathcal{P},\varepsilon} = \rho$;
(*ii*) *there exists* $e \in H^s_\varepsilon$ *such that* $\|e\|_{\mathcal{P},\varepsilon} > \rho$ *and* $J_{\mathcal{P},\varepsilon}(e) < 0$.

Proof. (i) Using Lemma 2.5, (F_1) and Lemma 2.3, we get that

$$\begin{aligned} J_{\mathcal{P},\varepsilon}(u) &\geqslant \frac{1}{2}\|u\|^2_{\mathcal{P},\varepsilon} - \frac{C(N,\alpha)}{2\cdot 2^*_{s,\alpha}}\|u\|^{2\cdot 2^*_{s,\alpha}}_{L^{2^*_s}} - \frac{C(N,\alpha)}{4}\|F(\varepsilon x,|u|^2)\|^2_{L^{\frac{2N}{2N-\alpha}}} \\ &\geqslant \frac{1}{2}\|u\|^2_{\mathcal{P},\varepsilon} - \frac{C(N,\alpha)}{2\cdot 2^*_{s,\alpha}}\|u\|^{2\cdot 2^*_{s,\alpha}}_{L^{2^*_s}} - C_1\frac{C(N,\alpha)}{4}\left(\|u\|^{2p_1}_{L^{\frac{2Np_1}{2N-\alpha}}} + \|u\|^{2p_2}_{L^{\frac{2Np_2}{2N-\alpha}}}\right) \\ &\geqslant \frac{1}{2}\|u\|^2_{\mathcal{P},\varepsilon} - \frac{C(N,\alpha)}{2\cdot 2^*_{s,\alpha}}\|u\|^{2\cdot 2^*_{s,\alpha}}_{\mathcal{P},\varepsilon} - C_1\frac{C(N,\alpha)}{4}(\|u\|^{2p_1}_{\mathcal{P},\varepsilon} + \|u\|^{2p_2}_{\mathcal{P},\varepsilon}). \end{aligned}$$

Thus (i) holds if we take $\|u\|_{\mathcal{P},\varepsilon} = \rho$ sufficiently small.

(ii) Clearly, (F_1) and (F_2) imply that

$$f(x,t)t \geqslant F(x,t) \geqslant 0 \quad \text{for } x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{3.3}$$

Fix $u_0 \in H^s_\varepsilon \backslash \{0\}$, by (3.3), we obtain that

$$J_{\mathcal{P},\varepsilon}(tu_0) \leqslant \frac{t^2}{2}\|u_0\|^2_{\mathcal{P},\varepsilon} - \frac{t^{2\cdot 2^*_{s,\alpha}}}{2\cdot 2^*_{s,\alpha}}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u_0(x)|^{2^*_{s,\alpha}}|u_0(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \to -\infty \quad \text{as } t \to \infty,$$

which implies that we can take $e = t_0u_0$ for some $t_0 > 0$ such that $\|e\|_{\mathcal{P},\varepsilon} > \rho$ and $J_{\mathcal{P},\varepsilon}(e) < 0$. Thus, (ii) holds. □

Let c_ε denote the mountain pass level of $J_{\mathcal{P},\varepsilon}$ by

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} J_{\mathcal{P},\varepsilon}(\gamma(t)),$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H^s_\varepsilon) : \gamma(0) = 0, J_{\mathcal{P},\varepsilon}(\gamma(1)) < 0\}$, and we have the following characterization of the mountain pass level c_ε of $J_{\mathcal{P},\varepsilon}$.

Lemma 3.2. *Assume that* (A), (V_1), (V_2), (F_1), *and* (F_2) *hold. Then, for any* $u \in H^s_\varepsilon\backslash\{0\}$, *there exists a unique* $t_u > 0$ *such that* $t_uu \in \mathcal{N}_{\mathcal{P},\varepsilon}$, *where* $\mathcal{N}_{\mathcal{P},\varepsilon} := \{u \in H^s_\varepsilon\backslash\{0\} : \langle J'_{\mathcal{P},\varepsilon}(u), u\rangle = 0\}$ *is the Nehari manifold associated with* (3.2). *Moreover,*

$$c_\varepsilon = \inf_{u\in\mathcal{N}_{\mathcal{P},\varepsilon}} J_{\mathcal{P},\varepsilon}(u) = \inf_{u\in H^s_\varepsilon\backslash\{0\}}\max_{t\geqslant 0} J_{\mathcal{P},\varepsilon}(tu). \tag{3.4}$$

Proof. Let $u \in H^s_\varepsilon\backslash\{0\}$ and define $h_u : \mathbb{R}^+ \to \mathbb{R}$ as $h_u(t) = J_{\mathcal{P},\varepsilon}(tu)$, then $h_u \in C(\mathbb{R}^+, \mathbb{R})$. By (F_1) and (F_2), it is easy to see that $h_u(t) > 0$ when t is small, and $h_u(t) \to -\infty$ as $t \to \infty$. So, there exists $t_u > 0$ such that

$$h_u(t_u) = \max_{t\geqslant 0} h_u(t) \quad \text{and} \quad h'_u(t_u) = 0,$$

which implies that $t_uu \in \mathcal{N}_{\mathcal{P},\varepsilon}$. Next, we claim that t_u is unique.

Indeed, we may assume without loss of generality that there exists $\tau_u > t_u > 0$ such that $h'_u(t_u) = h'_u(\tau_u) = 0$. Thus, we have that

$$\begin{aligned} \|u\|^2_{\mathcal{P},\varepsilon} &= t_u^{2(2^*_{s,\alpha}-1)}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(\varepsilon y,|t_uu(y)|^2)f(\varepsilon x,|t_uu(x)|^2)|u(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &= \tau_u^{2(2^*_{s,\alpha}-1)}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(\varepsilon y,|\tau_uu(y)|^2)f(\varepsilon x,|\tau_uu(x)|^2)|u(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.5}$$

While, by $\tau_u > t_u > 0$, (F_1), and (F_2), we get that

$$t_u^{2(2^*_{s,\alpha}-1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y < \tau_u^{2(2^*_{s,\alpha}-1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,|t_uu(y)|^2)f(\varepsilon x,|t_uu(x)|^2)|u(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,|\tau_uu(y)|^2)f(\varepsilon x,|\tau_uu(x)|^2)|u(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y,$$

which contradict with the second equality in (3.5). So, t_u is unique.

By a similar discussion as in [38, Theorem 4.2], we have (3.4). So we omit it here. □

Lemma 3.3. *Assume that (V_1), (F_1), and (F_2) hold. Then, for all $t \geqslant 0$ and $u \in H^s_\varepsilon$, we get that*

$$J_{\mathcal{P},\varepsilon}(u) \geqslant J_{\mathcal{P},\varepsilon}(tu) + \frac{1-t^2}{2}\langle J'_{\mathcal{P},\varepsilon}(u),u\rangle + \left(\frac{1-t^2}{2}+\frac{t^{2\cdot 2^*_{s,\alpha}}-1}{2\cdot 2^*_{s,\alpha}}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

In particular, if $u \in \mathcal{N}_{\mathcal{P},\varepsilon}$, then $J_{\mathcal{P},\varepsilon}(u) = \max_{t\geqslant 0}J_{\mathcal{P},\varepsilon}(tu)$.

Proof. By direct computation, we have

$$\begin{aligned}
&J_{\mathcal{P},\varepsilon}(u) - J_{\mathcal{P},\varepsilon}(tu) - \frac{1-t^2}{2}\langle J'_{\mathcal{P},\varepsilon}(u),u\rangle - \left(\frac{1-t^2}{2}+\frac{t^{2\cdot 2^*_{s,\alpha}}-1}{2\cdot 2^*_{s,\alpha}}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&= \frac{1-t^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,|u(y)|^2)f(\varepsilon x,|u(x)|^2)|u(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x,t^2|u(x)|^2)F(\varepsilon y,t^2|u(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&\quad - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x,|u(x)|^2)F(\varepsilon y,|u(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&=: g(t),
\end{aligned}$$

and

$$g'(t) = t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,t^2|u(y)|^2)f(\varepsilon x,t^2|u(x)|^2) - F(\varepsilon y,|u(y)|^2)f(\varepsilon x,|u(x)|^2)}{|x-y|^\alpha}|u(x)|^2\mathrm{d}x\mathrm{d}y.$$

So, in view of (F_1) and (F_2), we get that $g'(t) \leqslant 0$ for $0 < t < 1$; $g'(1) = 0$; $g'(t) \geqslant 0$ for $t > 1$. Thus, we get that $g(t) \geqslant g(1) = 0,\ \forall t \geqslant 0$. □

Lemma 3.4. *Let $s \in (0,1)$, $\alpha \in (0,N)$ and $N > \max\{2\mu + 4s, 2s + \alpha/2\}$. Assume that (A), (V_1), (V_2), (F_1), and (F_2) hold. Then, there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,*

$$c_\varepsilon < \frac{N+2s-\alpha}{2(2N-\alpha)}(S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Proof. From [15], we know that the best Sobolev constant S_s for the embedding $D^{s,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*_s}(\mathbb{R}^N,\mathbb{R})$ is obtained by the family of functions:

$$U_\varrho(x) := \varrho^{\frac{2s-N}{2}}U_0\left(\frac{x}{\varrho}\right),$$

where $\varrho > 0$ and $U_0(x) := \frac{\alpha_{N,s}}{(1+|x|^2)^{(N-2s)/2}}$. The normalizing constant $\alpha_{N,s}$ depends only on N and s and is suitably chosen such that $U_\varrho(x)$ solves the equation:

$$(-\Delta)^s u = |u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^N,$$

and verifies the equality

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|U_\varrho(x)-U_\varrho(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^N}|U_\varrho|^{2_s^*}\mathrm{d}x=S_s^{\frac{N}{2s}}.$$

Furthermore,

$$S_{H,L}=\frac{S_s}{C(N,\alpha)^{\frac{1}{2^*_{s,\alpha}}}}. \tag{3.6}$$

Let $\phi:\mathbb{R}^N\to[0,1]$ be a smooth function such that

$$\phi(x)=\begin{cases}1 & \text{in } B_\delta,\\ 0 & \text{in } \mathbb{R}^N\backslash B_{2\delta},\end{cases}$$

where B_δ denotes the ball in $\mathbb{R}^N$ of center at origin and radius δ. We define, for any $\varrho>0$,

$$u_\varrho(x):=\phi(x)U_\varrho(x).$$

From [37, Proposition 21, Proposition 22] and [40, Lemma 4.6], we get that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varrho(x)-u_\varrho(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\leqslant S_s^{\frac{N}{2s}}+O(\varrho^{N-2s}), \tag{3.7}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varrho(x)|^{2^*_{s,\alpha}}|u_\varrho(y)|^{2^*_{s,\alpha}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y\geqslant C(N,\alpha)^{\frac{N}{2s}}S_{H,L}^{\frac{2N-\alpha}{2s}}-O(\varrho^{\frac{2N-\alpha}{2}}), \tag{3.8}$$

$$\int_{\mathbb{R}^N}|u_\varrho^2(x)|\mathrm{d}x\geqslant C_s\varrho^{2s}+O(\varrho^{N-2s})\quad \text{if } N>4s, \tag{3.9}$$

and

$$\int_{B_\delta}\int_{B_\delta}\frac{|U_\varrho(x)|^{q}|U_\varrho(y)|^{q}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y=O(\varrho^{2N-\alpha-q(N-2s)})\quad \text{if } q<2^*_{s,\alpha}. \tag{3.10}$$

Let $\tilde{u}_\varrho(x):=\mathrm{e}^{iA(0)\cdot x}u_\varrho(x)$, we have $|\tilde{u}_\varrho|=u_\varrho$ and $\tilde{u}_\varrho\in H^s_\varepsilon$ by Lemma 2.4. From Lemma 3.2, there exists a unique $t_{\tilde{u}_\varrho}>0$ (denoted by t_ϱ for simplicity) such that $t_\varrho\tilde{u}_\varrho\in\mathcal{N}_{\mathcal{P},\varepsilon}$.

Claim 1. There exist two constants $A_1, A_2>0$, which are independent of ϱ such that $A_1<t_\varrho<A_2$.

Indeed, by $t_\varrho\tilde{u}_\varrho\in\mathcal{N}_{\mathcal{P},\varepsilon}$, we have

$$\|t_\varrho\tilde{u}_\varrho\|^2_{\mathcal{P},\varepsilon}=t_\varrho^{2\cdot 2^*_{s,\alpha}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\tilde{u}_\varrho(x)|^{2^*_{s,\alpha}}|\tilde{u}_\varrho(y)|^{2^*_{s,\alpha}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,|t_\varrho\tilde{u}_\varrho(y)|^2)f(\varepsilon x,|t_\varrho\tilde{u}_\varrho(x)|^2)|t_\varrho\tilde{u}_\varrho(x)|^2}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y. \tag{3.11}$$

From (3.11) and (F_1), we obtain that

$$\begin{aligned}\|\tilde{u}_\varrho\|^2_{\mathcal{P},\varepsilon}=t_\varrho^{2(2^*_{s,\alpha}-1)}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\tilde{u}_\varrho(x)|^{2^*_{s,\alpha}}|\tilde{u}_\varrho(y)|^{2^*_{s,\alpha}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y,|t_\varrho\tilde{u}_\varrho(y)|^2)f(\varepsilon x,|t_\varrho\tilde{u}_\varrho(x)|^2)|\tilde{u}_\varrho(x)|^2}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y\to 0\quad \text{if } t_\varrho\to 0,\end{aligned}$$

which contradicts with the definition of $\tilde{u}_\varrho$. Thus, there exists a constant $A_1>0$ independent of ϱ such that $A_1<t_\varrho$.

On the other hand, by (3.8), (3.11), Lemma 2.5, and (F_1), we get that

$$\begin{aligned}
Ct_\varrho^{2\cdot 2^*_{s,\alpha}} &\leqslant t_\varrho^{2\cdot 2^*_{s,\alpha}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}_\varrho(x)|^{2^*_{s,\alpha}}|\tilde{u}_\varrho(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&= \|t_\varrho\tilde{u}_\varrho\|^2_{\mathcal{P},\varepsilon} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(\varepsilon y, |t_\varrho\tilde{u}_\varrho(y)|^2)f(\varepsilon x, |t_\varrho\tilde{u}_\varrho(x)|^2)|t_\varrho\tilde{u}_\varrho(x)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\leqslant Ct_\varrho^2 + C(N,\alpha)\|F(\varepsilon x, |t_\varrho\tilde{u}_\varrho(x)|^2)\|_{L^{\frac{2N}{2N-\alpha}}}\|f(\varepsilon x, |t_\varrho\tilde{u}_\varrho(x)|^2)|t_\varrho\tilde{u}_\varrho(x)|^2\|_{L^{\frac{2N}{2N-\alpha}}} \\
&\leqslant C\left(t_\varrho^2 + t_\varrho^{2p_1} + t_\varrho^{2p_2}\right).
\end{aligned}$$

Thus, there exists a constant $A_2 > 0$ independent of ϱ such that $t_\varrho < A_2$. So, Claim 1 holds.

Claim 2. For $N > 2\mu + 4s$, it holds that

$$\lim_{\varepsilon\to 0}[\tilde{u}_\varrho]^2_{A_\varepsilon} = [u_\varrho]^2 \quad \text{uniformly in } \varrho \in (0,1].$$

Indeed, the proof of Claim 2 is motivated by the work of Ambrosio and d'Avenia [8]. By direct computation, we have

$$\begin{aligned}
[\tilde{u}_\varrho]^2_{A_\varepsilon} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\mathrm{e}^{iA(0)\cdot x}u_\varrho(x) - \mathrm{e}^{i(x-y)\cdot A_\varepsilon\left(\frac{x+y}{2}\right)}\mathrm{e}^{iA(0)\cdot y}u_\varrho(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u_\varrho(x)-u_\varrho(y)) + u_\varrho(y)\left(1-\mathrm{e}^{i\left[A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right]\cdot(x-y)}\right)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&= [u_\varrho]^2 + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_\varrho^2(y)\left|1-\mathrm{e}^{i\left[A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right]\cdot(x-y)}\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\quad + 2\mathfrak{Re}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\varrho(x)-u_\varrho(y))u_\varrho(y)\left(1-\mathrm{e}^{-i\left[A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right]\cdot(x-y)}\right)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&=: [u_\varrho]^2 + X_\varepsilon + 2Y_\varepsilon.
\end{aligned} \tag{3.12}$$

From Hölder inequality, we have $|Y_\varepsilon| \leqslant [u_\varrho]\sqrt{X_\varepsilon}$. Thus, it is enough to show that $X_\varepsilon \to 0$ as $\varepsilon \to 0$ to deduce that Claim 2 holds.

Let us observe that for $0 < \kappa < \frac{\mu}{1+\mu}$, we get

$$\begin{aligned}
X_\varepsilon &= \int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{|x-y|>\varepsilon^{-\kappa}} \frac{\left|1-\mathrm{e}^{i\left[A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right]\cdot(x-y)}\right|^2}{|x-y|^{N+2s}}\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{|x-y|<\varepsilon^{-\kappa}} \frac{\left|1-\mathrm{e}^{i\left[A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right]\cdot(x-y)}\right|^2}{|x-y|^{N+2s}}\mathrm{d}x =: X_\varepsilon^1 + X_\varepsilon^2.
\end{aligned} \tag{3.13}$$

From $|\mathrm{e}^{it}-1|^2 \leqslant 4$ for all $t \in \mathbb{R}$, $\varrho \in (0,1]$ and $N > 2\mu + 4s$, we can see that

$$\begin{aligned}
X_\varepsilon^1 &\leqslant C\int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{\varepsilon^{-\kappa}}^{\infty} \tau^{-1-2s}\mathrm{d}\tau \\
&= C\varepsilon^{2\kappa s}\int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&= C\varepsilon^{2\kappa s}\alpha_{N,s}^2 \int_{\mathbb{R}^N} \phi^2(y)\frac{\varrho^{N-2s}}{(1+|y|^2)^{N-2s}}\mathrm{d}y \\
&\leqslant C\varepsilon^{2\kappa s}\varrho^{2s}\int_0^\infty \frac{t^{N-1}}{(1+t^2)^{N-2s}}\mathrm{d}t \\
&\leqslant C\varepsilon^{2\kappa s}.
\end{aligned}
$$

Next, we consider X_ε^2. Since $|\mathrm{e}^{it}-1|^2 \leqslant t^2$ for all $t\in\mathbb{R}$, $A\in C^{0,\mu}(\mathbb{R}^N,\mathbb{R}^N)$, and $|x+y|^2\leqslant 2(|x-y|^2+4|y|^2)$, we have

$$
\begin{aligned}
X_\varepsilon^2 &\leqslant \int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{|x-y|<\varepsilon^{-\kappa}} \frac{\left|A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)\right|^2}{|x-y|^{N+2s-2}}\mathrm{d}x \\
&\leqslant C\varepsilon^{2\mu}\int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{|x-y|<\varepsilon^{-\kappa}} \frac{|x+y|^{2\mu}}{|x-y|^{N+2s-2}}\mathrm{d}x \\
&\leqslant C\varepsilon^{2\mu}\left(\int_{\mathbb{R}^N} u_\varrho^2(y)\mathrm{d}y \int_{|x-y|<\varepsilon^{-\kappa}} \frac{1}{|x-y|^{N+2s-2-2\mu}}\mathrm{d}x + \int_{\mathbb{R}^N}|y|^{2\mu}u_\varrho^2(y)\mathrm{d}y\int_{|x-y|<\varepsilon^{-\kappa}}\frac{1}{|x-y|^{N+2s-2}}\mathrm{d}x\right) \\
&=: C\varepsilon^{2\mu}(X_\varepsilon^{2,1}+X_\varepsilon^{2,2}).
\end{aligned}
\tag{3.15}
$$

From $\varrho\in(0,1]$ and $N>2\mu+4s$, we get that

$$
X_\varepsilon^{2,1} \leqslant C\int_{\mathbb{R}^N}u_\varrho^2(y)\mathrm{d}y\int_0^{\varepsilon^{-\kappa}}\tau^{1+2\mu-2s}\mathrm{d}\tau = C\varepsilon^{-2\kappa(1+\mu-s)}\int_{\mathbb{R}^N}u_\varrho^2(y)\mathrm{d}y \leqslant C\varepsilon^{-2\kappa(1+\mu-s)},
\tag{3.16}
$$

and

$$
\begin{aligned}
X_\varepsilon^{2,2} &\leqslant C\int_{\mathbb{R}^N}|y|^{2\mu}u_\varrho^2(y)\mathrm{d}y\int_0^{\varepsilon^{-\kappa}}\tau^{1-2s}\mathrm{d}\tau \\
&= C\varepsilon^{-2\kappa(1-s)}\int_{\mathbb{R}^N}|y|^{2\mu}u_\varrho^2(y)\mathrm{d}y \\
&= C\varepsilon^{-2\kappa(1-s)}\alpha_{N,s}^2\int_{\mathbb{R}^N}|y|^{2\mu}\phi^2(y)\frac{\varrho^{N-2s}}{(1+|y|^2)^{N-2s}}\mathrm{d}y \\
&\leqslant C\varepsilon^{-2\kappa(1-s)}\varrho^{2(\mu+s)}\int_0^\infty\frac{t^{N+2\mu-1}}{(1+t^2)^{N-2s}}\mathrm{d}t \\
&\leqslant C\varepsilon^{-2\kappa(1-s)}.
\end{aligned}
\tag{3.17}
$$

Taking into account (3.12)–(3.17) and $0<\kappa<\frac{\mu}{1+\mu}$, we can see that Claim 2 holds.

Now we estimate $J_{\mathcal{P},\varepsilon}(t_\varrho\tilde{u}_\varrho)$. Note that

$$
\begin{aligned}
J_{\mathcal{P},\varepsilon}(t_\varrho\tilde{u}_\varrho) &= \frac{t_\varrho^2}{2}[\tilde{u}_\varrho]_{A_\varepsilon}^2 - \frac{t_\varrho^{2\cdot 2_{s,\alpha}^*}}{2\cdot 2_{s,\alpha}^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\tilde{u}_\varrho(x)|^{2_{s,\alpha}^*}|\tilde{u}_\varrho(y)|^{2_{s,\alpha}^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{t_\varrho^2}{2}\int_{\mathbb{R}^N}V_{\mathcal{P},\varepsilon}(x)|\tilde{u}_\varrho|^2\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x,|t_\varrho\tilde{u}_\varrho(x)|^2)F(\varepsilon y,|t_\varrho\tilde{u}_\varrho(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y
\end{aligned}
\tag{3.18}
$$

$$= \left(\frac{t_\varrho^2}{2}[\tilde{u}_\varrho]^2_{A_\varepsilon} - \frac{t_\varrho^{2\cdot 2^*_{s,\alpha}}}{2\cdot 2^*_{s,\alpha}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varrho(x)|^{2^*_{s,\alpha}}|u_\varrho(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \right)$$
$$+ \left(\frac{t_\varrho^2}{2}\int_{\mathbb{R}^N} V_{\mathcal{P},\varepsilon}(x)|u_\varrho|^2\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(\varepsilon x, |t_\varrho u_\varrho(x)|^2)F(\varepsilon y, |t_\varrho u_\varrho(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \right) := E_1 + E_2.$$

Combining (3.6)–(3.8) with Claim 2, and by a similar discussion as in the proof of [40, Lemma 4.7 (4.15)], we can see that there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$,

$$E_1 \leqslant \frac{N+2s-\alpha}{2(2N-\alpha)}(S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}} + O(\varrho^{N-2s}) + O(\varrho^{\frac{2N-\alpha}{2}}) + o_\varepsilon(1). \tag{3.19}$$

From (F3), (3.9), (3.10), and Claim 1, by a similar discussion as in the proof of [40, Lemma 4.7 (4.16), (4.17)], we have that for any $A_0 > 0$,

$$E_2 \leqslant (C - C_1A_0^2)\varrho^{2s}. \tag{3.20}$$

Finally, using (3.18)–(3.20) and Claim 2, we get that for any $\varepsilon \in (0, \varepsilon^*)$,

$$J_{\mathcal{P},\varepsilon}(t_\varrho\tilde{u}_\varrho) \leqslant \frac{N+2s-\alpha}{2(2N-\alpha)}(S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}} + (C - C_1A_0^2)\varrho^{2s} + O(\varrho^{N-2s}) + O(\varrho^{\frac{2N-\alpha}{2}}) + o_\varepsilon(1).$$

Without loss of generality, we may choose fixed $\varrho > 0$ small and $A_0 > 0$ large such that

$$(C - C_1A_0^2)\varrho^{2s} + O(\varrho^{N-2s}) + O(\varrho^{\frac{2N-\alpha}{2}}) < 0,$$

since $N > \max\{2\mu + 4s, 2s + \alpha/2\}$. Thus, we obtain that there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$,

$$c_\varepsilon \leqslant J_{\mathcal{P},\varepsilon}(t_\varrho\tilde{u}_\varrho) < \frac{N+2s-\alpha}{2(2N-\alpha)}(S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}}.$$ □

In view of Lemmas 3.1, 3.3, and 3.4, we can establish an existence result of the ground state solution for problem (3.2). More precisely, we obtain:

Theorem 3.5. *Let $s \in (0,1)$, $\alpha \in (0,N)$ and $N > \max\{2\mu + 4s, 2s + \alpha/2\}$. Assume that (A), (V_1), (V_2), and (F_1)–(F_4) hold. Then, there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, problem* (3.2) *has at least one ground state solution.*

Proof. From Lemma 3.1, there exists a Cerami sequence $\{u_n\} \subset H^s_\varepsilon$ of $J_{\mathcal{P},\varepsilon}$ at the mountain pass level c_ε by [22, Theorem (1)] (denoted by $(C)_{c_\varepsilon}$ sequence for simplicity). That is, there exists a sequence $\{u_n\} \subset H^s_\varepsilon$ satisfying

$$J_{\mathcal{P},\varepsilon}(u_n) \to c_\varepsilon, \ \|J'_{\mathcal{P},\varepsilon}(u_n)\|(1 + \|u_n\|_{\mathcal{P},\varepsilon}) \to 0 \quad \text{as } n \to \infty. \tag{3.21}$$

Clearly, by (3.21), we can prove that $\{u_n\}$ is also a $(PS)_{c_\varepsilon}$ sequence of $J_{\mathcal{P},\varepsilon}$, provided that we have got the boundedness $\{u_n\}$ in H^s_ε. We break the proof into the following Steps.

Step 1. We claim that $\{u_n\}$ is bounded in H^s_ε and there exists $u \in H^s_\varepsilon$, up to a subsequence, such that

$$u_n \rightharpoonup u \text{ in } H^s_\varepsilon,\ u_n \to u \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C}) \quad \text{and} \quad u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty. \tag{3.22}$$

Indeed, suppose by contradiction, we may have that $\|u_n\|_{\mathcal{P},\varepsilon} \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{\mathcal{P},\varepsilon}}$, then $\{v_n\}$ is bounded in H^s_ε. Up to a subsequence, we obtain that there exists $v \in H^s_\varepsilon$ such that $v_n \rightharpoonup v$ in H^s_ε. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n|^2\mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [36, Lemma 2.4], $v_n \to 0$ in $L^p(\mathbb{R}^N, \mathbb{C})$ for $p \in (2, 2^*_s)$. Fix $R := (1 + 2c_\varepsilon)^{1/2}$, by Lemma 2.5 and (F_1), we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x,|Rv_n(x)|^2)F(\varepsilon y,|Rv_n(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \leqslant C(N,\alpha)\|F(\varepsilon x,|Rv_n(x)|^2)\|^2_{L^{\frac{2N}{2N-\alpha}}}$$
$$\leqslant C(N,\alpha)\|C_0\Big(|Rv_n|^{p_1}+|Rv_n|^{p_2}\Big)\|^2_{L^{\frac{2N}{2N-\alpha}}}$$
$$\leqslant C_0C(N,\alpha)\left(R^{2p_1}\|v_n\|^{2p_1}_{L^{\frac{2Np_1}{2N-\alpha}}}+R^{2p_2}\|v_n\|^{2p_2}_{L^{\frac{2Np_2}{2N-\alpha}}}\right)\to 0 \text{ as } n\to\infty.$$

Let $\theta_n := \frac{R}{\|u_n\|_{\mathcal{P},\varepsilon}}$, then $\theta_n \to 0$ as $n\to\infty$. By (3.21) and Lemma 3.3, it holds that

$$\begin{aligned}
c_\varepsilon + o_n(1) &= J_{\mathcal{P},\varepsilon}(u_n)\\
&\geqslant J_{\mathcal{P},\varepsilon}(\theta_n u_n) + \frac{1-\theta_n^2}{2}\langle J'_{\mathcal{P},\varepsilon}(u_n),u_n\rangle\\
&\quad + \left(\frac{1-\theta_n^2}{2}+\frac{\theta_n^{2\cdot 2^*_{s,\alpha}}-1}{2\cdot 2^*_{s,\alpha}}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_{s,\alpha}}|u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&= \frac{\theta_n^2}{2}\|u_n\|^2_{\mathcal{P},\varepsilon} + \frac{1-\theta_n^2}{2}\langle J'_{\mathcal{P},\varepsilon}(u_n),u_n\rangle\\
&\quad - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x,|Rv_n(x)|^2)F(\varepsilon y,|Rv_n(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&\quad + \left(\frac{2^*_{s,\alpha}-2^*_{s,\alpha}\theta_n^2-1}{2\cdot 2^*_{s,\alpha}}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_{s,\alpha}}|u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&\geqslant \frac{R^2}{2}-\frac{1}{4}+\left(\frac{2^*_{s,\alpha}-2^*_{s,\alpha}\theta_n^2-1}{2\cdot 2^*_{s,\alpha}}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_{s,\alpha}}|u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&\geqslant c_\varepsilon + \frac{1}{4} + o_n(1).
\end{aligned}$$

This contradiction shows $\delta > 0$. So, there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ such that $\int_{B_{1+\sqrt{N}}(y_n)}|v_n|^2\mathrm{d}x > \frac{\delta}{2}$. Define $\tilde{v}_n(x) = v_n(x+y_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)}|\tilde{v}_n|^2\mathrm{d}x > \frac{\delta}{2}. \tag{3.23}$$

Let $\tilde{u}_n(x) := u_n(x+y_n)$, then $\tilde{v}_n(x) = \frac{\tilde{u}_n(x)}{\|u_n\|_{\mathcal{P},\varepsilon}}$ and $\{\tilde{v}_n\}$ is bounded in H^s_ε. Up to a subsequence, there exists $\tilde{v}\in H^s_\varepsilon$ such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in H^s_ε, $\tilde{v}_n\to\tilde{v}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$, for $p\in[2,2^*_s)$, and $\tilde{v}_n\to\tilde{v}$ a.e. in $\mathbb{R}^N$. From (3.23), then $\tilde{v}\neq 0$. Thus, there exists a set $A_1\subset B_{1+\sqrt{N}}(0)$, which has positive Lebesgue measure such that, for $x\in A_1$,

$$\begin{aligned}
&\tilde{v}(x)\neq 0,\\
&|\tilde{u}_n(x)| = \|u_n\|_{\mathcal{P},\varepsilon}|\tilde{v}_n(x)|\to+\infty,\\
&\frac{|\tilde{u}_n(x)|^{2^*_{s,\alpha}}}{\|u_n\|_{\mathcal{P},\varepsilon}} = |\tilde{u}_n(x)|^{2^*_{s,\alpha}-1}|\tilde{v}_n(x)|\to+\infty.
\end{aligned}\tag{3.24}$$

So, by (3.21), (3.3), Fatou's lemma, and (3.24), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c_\varepsilon+o_n(1)}{\|u_n\|^2_{\mathcal{P},\varepsilon}}\\
&= \lim_{n\to\infty}\frac{J_{\mathcal{P},\varepsilon}(u_n)}{\|u_n\|^2_{\mathcal{P},\varepsilon}}\\
&= \lim_{n\to\infty}\frac{1}{\|u_n\|^2_{\mathcal{P},\varepsilon}}\left(\frac{1}{2}\|u_n\|^2_{\mathcal{P},\varepsilon}-\frac{1}{2\cdot 2^*_{s,\alpha}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_{s,\alpha}}|u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right.
\end{aligned}$$

$$- \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon x, |u_n(x)|^2)F(\varepsilon y, |u_n(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\Bigg)$$

$$\leqslant \frac{1}{2} - \frac{1}{2\cdot 2^*_{s,\alpha}}\int_{A_1}\int_{A_1}\liminf_{n\to\infty}\frac{|\tilde{u}_n(x)|^{2^*_{s,\alpha}}|\tilde{u}_n(y)|^{2^*_{s,\alpha}}}{\|u_n\|^2_{\mathcal{P},\varepsilon}|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \to -\infty \quad \text{as } n\to\infty,$$

which is impossible. Thus, $\{u_n\}$ is bounded in H^s_ε and (3.22) holds.

Step 2. We claim that u in (3.22) is a weak solution of (3.2).

Indeed, by (3.22) and [23, Lemma 2.3], we get that, for every $\psi \in H^s_\varepsilon$,

$$\begin{aligned}
\langle J'_{\mathcal{P},\varepsilon}(u_n), \psi\rangle &= \langle u_n, \psi\rangle_{\mathcal{P},\varepsilon} - \Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_{s,\alpha}}|u_n(x)|^{2^*_{s,\alpha}-2}u_n(x)\bar{\psi}(x)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\quad - \Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y, |u_n(y)|^2)f(\varepsilon x, |u_n(x)|^2)u_n(x)\bar{\psi}(x)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\to \langle u, \psi\rangle_{\mathcal{P},\varepsilon} - \Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{s,\alpha}}|u(x)|^{2^*_{s,\alpha}-2}u(x)\bar{\psi}(x)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\quad - \Re\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y, |u(y)|^2)f(\varepsilon x, |u(x)|^2)u(x)\bar{\psi}(x)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&= \langle J'_{\mathcal{P},\varepsilon}(u), \psi\rangle \quad \text{as } n\to\infty.
\end{aligned}$$

Hence, by (3.21) and the boundedness $\{u_n\}$ in H^s_ε, we obtain that $\langle J'_{\mathcal{P},\varepsilon}(u), \psi\rangle = 0$, $\forall \psi \in H^s_\varepsilon$. That is, u is a weak solution of (3.2).

Step 3. We prove the existence of ground state solution for (3.2).

Case 1. If $u \neq 0$, we obtain a nontrivial solution for problem (3.2). We claim that u is a ground state solution of (3.2).

From (3.21), Fatou's lemma and $u \in \mathcal{N}_{\mathcal{P},\varepsilon}$, we have

$$\begin{aligned}
c_\varepsilon &= \lim_{n\to\infty}\Big(J_{\mathcal{P},\varepsilon}(u_n) - \frac{1}{2}\langle J'_{\mathcal{P},\varepsilon}(u_n), u_n\rangle\Big) \\
&= \lim_{n\to\infty}\Bigg(\frac{N+2s-\alpha}{2(2N-\alpha)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_{s,\alpha}}|u_n(x)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y, |u_n(y)|^2)(2f(\varepsilon x, |u_n(x)|^2)|u_n(x)|^2 - F(\varepsilon x, |u_n(x)|^2))}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\Bigg) \\
&\geqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\liminf_{n\to\infty}\Bigg(\frac{N+2s-\alpha}{2(2N-\alpha)}\cdot\frac{|u_n(y)|^{2^*_{s,\alpha}}|u_n(x)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \\
&\quad + \frac{1}{4}\cdot\frac{F(\varepsilon y, |u_n(y)|^2)(2f(\varepsilon x, |u_n(x)|^2)|u_n(x)|^2 - F(\varepsilon x, |u_n(x)|^2))}{|x-y|^\alpha}\Bigg)\mathrm{d}x\mathrm{d}y \\
&= \frac{N+2s-\alpha}{2(2N-\alpha)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{s,\alpha}}|u(x)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(\varepsilon y, |u(y)|^2)(2f(\varepsilon x, |u(x)|^2)|u(x)|^2 - F(\varepsilon x, |u(x)|^2))}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&= J_{\mathcal{P},\varepsilon}(u) - \frac{1}{2}\langle J'_{\mathcal{P},\varepsilon}(u), u\rangle \\
&= J_{\mathcal{P},\varepsilon}(u).
\end{aligned}$$

So $J_{\mathcal{P},\varepsilon}(u) = c_\varepsilon$, which implies that u is a ground state solution of (3.2).

Case 2. If $u = 0$, we claim that

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \mathrm{d}x > 0. \tag{3.25}$$

Otherwise, by Lions' concentration compactness principle, $u_n \to 0$ in $L^p(\mathbb{R}^N, \mathbb{C})$ for $p \in (2, 2_s^*)$. From Lemma 2.5 and (F_1), we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(\varepsilon y, |u_n(y)|^2) f(\varepsilon x, |u_n(x)|^2)|u_n(x)|^2}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \to 0, \\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(\varepsilon x, |u_n(x)|^2) F(\varepsilon y, |u_n(y)|^2)}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \to 0. \end{aligned} \tag{3.26}$$

By (3.21) and (3.26), it holds that

$$\|u_n\|^2_{\mathcal{P},\varepsilon} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{s,\alpha}} |u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y + o_n(1) \quad \text{as } n \to \infty. \tag{3.27}$$

Since $\{u_n\}$ is bounded in H^s_ε, up to a subsequence, we may suppose that $\lim_{n\to\infty} \|u_n\|^2_{\mathcal{P},\varepsilon} = B_0 > 0$. Otherwise, we have $u_n \to 0$ in H^s_ε, which contradicts with (3.21) for $c_\varepsilon > 0$. Thus,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{s,\alpha}} |u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \to B_0 \quad \text{as } n \to \infty. \tag{3.28}$$

So, using (3.21) and (3.26)–(3.28), it holds that

$$c_\varepsilon = \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{s,\alpha}}\right) B_0 = \frac{N+2s-\alpha}{2(2N-\alpha)} B_0. \tag{3.29}$$

While, for every $u \in H^s_\varepsilon$, it holds that

$$\|u\|^2_{\mathcal{P},\varepsilon} \geqslant [u]^2_{A_\varepsilon} \geqslant S_{A_\varepsilon} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{s,\alpha}} |u(x)|^{2^*_{s,\alpha}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2^*_{s,\alpha}}},$$

which implies that $B_0 \geqslant (S_{A_\varepsilon})^{\frac{2N-\alpha}{N+2s-\alpha}}$. So, we deduce that

$$c_\varepsilon = \frac{N+2s-\alpha}{2(2N-\alpha)} B_0 \geqslant \frac{N+2s-\alpha}{2(2N-\alpha)} \left(S_{A_\varepsilon}\right)^{\frac{2N-\alpha}{N+2s-\alpha}} \geqslant \frac{N+2s-\alpha}{2(2N-\alpha)} (S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}},$$

which contradicts Lemma 3.4, which shows that, for any $\varepsilon \in (0, \varepsilon^*)$,

$$c_\varepsilon < \frac{N+2s-\alpha}{2(2N-\alpha)} (S_{H,L})^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Thus, (3.25) holds.

Without loss of generality, we may assume that there exist a constant r and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\{\varepsilon y_n\} \subset \mathbb{Z}^N$ and

$$\int_{B_r(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.30}$$

Clearly, $|y_n| \to \infty$ because of Lemma 2.3 and $u_n \rightharpoonup 0$ in H^s_ε. Let $w_n(x) := u_n(x + y_n)$. Combining (A), (V_2) with (F4), by direct computation, we get that

$$\|w_n\|_{\mathcal{P},\varepsilon} = \|u_n\|_{\mathcal{P},\varepsilon}, \quad J_{\mathcal{P},\varepsilon}(w_n) = J_{\mathcal{P},\varepsilon}(u_n), \quad \|J'_{\mathcal{P},\varepsilon}(w_n)\|(1 + \|w_n\|) \to 0 \quad \text{as } n \to \infty.$$

That is, $\{w_n\}$ is a $(C)_{c_\varepsilon}$ sequence of $J_{\mathcal{P},\varepsilon}$. From Step 1, we can see that $\{w_n\}$ is bounded in H^s_ε. Up to a subsequence, there exists $w \in H^s_\varepsilon$ such that, $w_n \rightharpoonup w$ in H^s_ε and $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$. We claim that $w \neq 0$.

Indeed, by (3.30), we have

$$0 < (\delta/2)^{\frac{1}{2}} < \|u_n\|_{L^2(B_r(y_n),\mathbb{C})} = \|w_n\|_{L^2(B_r(0),\mathbb{C})} \leqslant \|w_n - w\|_{L^2(B_r(0),\mathbb{C})} + \|w\|_{L^2(B_r(0),\mathbb{C})}.$$

Since $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$, we get $\|w_n - w\|_{L^2(B_r(0),\mathbb{R})} \to 0$ as $n \to \infty$. So $w \neq 0$. Applying the same arguments in Case 1, we can deduce that w is a ground state solution of problem (3.2). □

In the end, we show the proof of Theorem 1.1.

Proof of Theorem 1.1. Since problem (1.1) is equivalent to (3.1), we now establish the existence of ground state solutions for (3.1). Consider the associated energy functional $I_\varepsilon : H^s_\varepsilon \to \mathbb{R}$ for problem (3.1) denoted by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{2\cdot 2^*_{s,\alpha}} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{s,\alpha}}|u(y)|^{2^*_{s,\alpha}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y - \frac{1}{4}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(\varepsilon x, |u(x)|^2)F(\varepsilon y, |u(y)|^2)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

Let d_ε denote the mountain pass level of I_ε by

$$d_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)),$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H^s_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}$. By a similar discussion as Lemma 3.2, we can show that there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$ for all $u \in H^s_\varepsilon\backslash\{0\}$, where $\mathcal{N}_\varepsilon := \{u \in H^s_\varepsilon\backslash\{0\} : \langle I'_\varepsilon(u), u\rangle = 0\}$ is the Nehari manifold associated with (3.1). Moreover,

$$d_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) = \inf_{u\in H^s_\varepsilon\backslash\{0\}}\max_{t\geqslant 0} I_\varepsilon(tu).$$

Claim 1. $d_\varepsilon < c_\varepsilon$, for all $\varepsilon > 0$.

Indeed, let $u \in H^s_\varepsilon$ be a ground state solution of problem (3.2). Then, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. It follows from (V_2), Lemmas 3.2 and 3.3 that

$$0 < d_\varepsilon \leqslant I_\varepsilon(t_u u) < J_{\mathcal{P},\varepsilon}(t_u u) \leqslant \max_{t\geqslant 0} J_{\mathcal{P},\varepsilon}(tu) \leqslant J_{\mathcal{P},\varepsilon}(u) = c_\varepsilon.$$

Thus, Claim 1 holds.

Let $\{u_n\}$ be a $(C)_{d_\varepsilon}$ sequence of I_ε. Using a similar discussion as in the proof of Step 1 of Theorem 3.5, we can see that $\{u_n\}$ is bounded in H^s_ε, and there exists $u \in H^s_\varepsilon$ such that, up to a subsequence, $u_n \rightharpoonup u$ in H^s_ε. Arguing as in the proof of Case 1 in Theorem 3.5, u is a ground state solution of problem (3.1) if $u \neq 0$. In the following, we show that $u = 0$ does not hold. Otherwise, assume that $u_n \rightharpoonup 0$ in H^s_ε. From Lemma 2.3, $|u_n|^2 \rightharpoonup 0$ in $L^{\frac{N}{N-2s}}(\mathbb{R}^N, \mathbb{R})$. Since $W \in L^{\frac{N}{2s}}(\mathbb{R}^N, \mathbb{R})$, we obtain that

$$\lim_{n\to\infty}(J_{\mathcal{P},\varepsilon}(u_n) - I_\varepsilon(u_n)) = \lim_{n\to\infty}\int\limits_{\mathbb{R}^N} W(x)|u_n|^2\mathrm{d}x = 0. \tag{3.31}$$

Thus, $J_{\mathcal{P},\varepsilon}(u_n) \to d_\varepsilon$. Meanwhile, since $\{u_n\}$ is a $(C)_{d_\varepsilon}$ sequence of I_ε, we can verify that $\{u_n\}$ is also a $(PS)_{d_\varepsilon}$ sequence of I_ε due to the boundedness $\{u_n\}$ in H^s_ε. By Hölder inequality and (3.31), for any $\psi \in H^s_\varepsilon$, we get that

$$|\langle J'_{\mathcal{P},\varepsilon}(u_n) - I'_\varepsilon(u_n), \psi\rangle| = \left|\mathfrak{Re}\int\limits_{\mathbb{R}^N} W(x)u_n\bar{\psi}\mathrm{d}x\right| \leqslant C\left(\int\limits_{\mathbb{R}^N} W(x)|u_n|^2\mathrm{d}x\right)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $J'_{\mathcal{P},\varepsilon}(u_n) \to 0$, and $\{u_n\}$ is a $(C)_{d_\varepsilon}$ sequence of $J_{\mathcal{P},\varepsilon}$ thanks to the boundedness $\{u_n\}$ in H^s_ε. By the same argument as in the proof of Case 2 in Theorem 3.5 and the fact that $d_\varepsilon < c_\varepsilon$ for all $\varepsilon > 0$, we can deduce that there exists $w \in H^s_\varepsilon$ such that $J_{\mathcal{P},\varepsilon}(w) = d_\varepsilon$ and $J'_{\mathcal{P},\varepsilon}(w) = 0$. Moreover, there exists a unique $t_0 > 0$ such that $t_0 w \in \mathcal{N}_\varepsilon$. From the definition of d_ε, (V_2) and Lemma 3.3, we have that

$$d_\varepsilon \leqslant I_\varepsilon(t_0 w) < J_{\mathcal{P},\varepsilon}(t_0 w) \leqslant J_{\mathcal{P},\varepsilon}(w) = d_\varepsilon,$$

which is impossible. So $u = 0$ does not hold. The proof is complete. □

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References

[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), 423–443.

[2] C. O. Alves, G. M. Figueiredo, and M. Yang, *Multiple semiclassical solutions for a nonlinear Choquard equation with magnetic field*, Asymptot. Anal. **96** (2016), 135–159.

[3] C. O. Alves, F. Gao, M. Squassina, and M. Yang, *Singularly perturbed critical Choquard equations*, J. Differ. Equ. **263** (2017), 3943–3988.

[4] C. O. Alves and M. Yang, *Existence of semiclassical ground state solutions for a generalized Choquard equation*, J. Differ. Equ. **257** (2014), 4133–4164.

[5] C. O. Alves and M. Yang, *Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), 23–58.

[6] V. Ambrosio, *Concentration phenomena for a fractional Choquard equation with magnetic field*, Dyn. Partial Differ. Equ. **16** (2019), 125–149.

[7] V. Ambrosio, *Multiplicity and concentration results for a fractional Choquard equation via penalization method*, Potential Anal. **50** (2019), 55–82.

[8] V. Ambrosio and P. d'Avenia, *Nonlinear fractional magnetic Schrödinger equation: existence and multiplicity*, J. Differ. Equ. **264** (2018), 3336–3368.

[9] H. Bueno, N. da Hora Lisboa, and L. L. Vieira, *Nonlinear perturbations of a periodic magnetic Choquard equation with Hardy-Littlewood-Sobolev critical exponent*, Z. Angew. Math. Phys. **71** (2020), no. 143, 26.

[10] B. Buffoni, L. Jeanjean, and C. A. Stuart, *Existence of a nontrivial solution to a strongly indefinite semilinear equation*, Proc. Amer. Math. Soc. **119** (1993), 179–186.

[11] D. Cassani, J. Van Schaftingen, and J. Zhang, *Groundstates for Choquard type equations with Hardy-Littlewood-Sobolev lower critical exponent*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), 1377–1400.

[12] S. Chen and L. Xiao, *Existence of a nontrivial solution for a strongly indefinite periodic Choquard system*, Calc. Var. Partial Differ. Equ. **54** (2015), 599–614.

[13] S. Cingolani, S. Secchi, and M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 973–1009.

[14] R. Clemente, J. C. de Albuquerque, and E. Barboza, *Existence of solutions for a fractional Choquard-type equation in* $\mathbb{R}$ *with critical exponential growth*, Z. Angew. Math. Phys. **72** (2021), Paper no. 16, 13 pp.

[15] A. Cotsiolis and N. K. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl. **295** (2004), 225–236.

[16] P. d'Avenia and M. Squassina, *Ground states for fractional magnetic operators*, ESAIM Control Optim. Calc. Var. **24** (2018), 1–24.

[17] M. del Pino, and P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differ. Equ. **4** (1996), 121–137.

[18] Y. Deng, L. Jin, and S. Peng, *Solutions of Schrödinger equations with inverse square potential and critical nonlinearity*, J. Differ. Equ. **253** (2012), 1376–1398.
[19] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.
[20] F. Gao and M. Yang, *The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation*, Sci. China Math. **61** (2018), 1219–1242.
[21] Z. Gao, X. Tang, and S. Chen, *On existence and concentration behavior of positive ground state solutions for a class of fractional Schrödinger-Choquard equations*, Z. Angew. Math. Phys. **69** (2018), Paper no. 122, 21 pp.
[22] N. Ghoussoub and D. Preiss, *A general mountain pass principle for locating and classifying critical points*, Ann. Inst. H Poincaré Anal. Non Linéaire **6** (1989), 321–330.
[23] T. Guo and X. Tang, *Ground state solutions for nonlinear Choquard equations with inverse-square potentials*, Asymptot. Anal. **117** (2020), 141–160.
[24] T. Ichinose, *Magnetic relativistic Schrödinger operators and imaginary-time path integrals, mathematical physics, spectral theory and stochastic analysis*, in Operator Theory: Advances and Applications, vol. 232, Birkhäuser/Springer Basel AG, Basel, 2013, pp. 247–297.
[25] Q.. Li, K. Teng, and J. Zhang, *Ground state solutions for fractional Choquard equations involving upper critical exponent*, Nonlinear Anal. **197** (2020), 111846, 11.
[26] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies Appl. Math. **57** (1976/77), 93–105.
[27] E. H. Lieb, M. Loss, *Analysis*, 2nd edition, American Mathematical Society, Providence, 2001.
[28] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire. **1** (1984), 109–145.
[29] M. Liu and Z. W. Tang, *Pseudoindex theory and Nehari method for a fractional Choquard equation*, Pacific J. Math. **304** (2020), 103–142.
[30] L. Ma and L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), 455–467.
[31] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.
[32] V. Moroz, and J. Van Schaftingen, *Semi-classical states for the Choquard equation*, Calc. Var. Partial Differ. Equ. **52** (2015), 199–235.
[33] V. Moroz, and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), 773–813.
[34] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.
[35] D. Qin, V. D. Rădulescu, and X. Tang, *Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations*, J. Differ. Equ., **275** (2021), 652–683.
[36] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), Paper no. 031501, 17 pp.
[37] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.
[38] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
[39] Q. Wu, D. Qin, and J. Chen, *Ground states and non-existence results for Choquard type equations with lower critical exponent and indefinite potentials*, Nonlinear Anal. **197** (2020), Paper no. 111863, 20 pp.
[40] Z. Yang and F. Zhao, *Multiplicity and concentration behaviour of solutions for a fractional Choquard equation with critical growth*, Adv. Nonlinear Anal. **10** (2021), 732–774.