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Research Article

Patrizia Pucci* and Yiwei Ye

Existence of nontrivial solutions for critical Kirchhoff-Poisson systems in the Heisenberg group

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Abstract: This article is devoted to the study of the combined effects of logarithmic and critical nonlinearities for the Kirchhoff-Poisson system

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u\ln|u|^2 + |u|^2 u & \text{in } \Omega, \\ -\Delta_H\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_H is the Kohn-Laplacian operator in the first Heisenberg group $\mathbb{H}^1$, Ω is a smooth bounded domain of $\mathbb{H}^1$, $q \in (2\theta, 4)$, $\mu \in \mathbb{R}$, and $\lambda > 0$ are some real parameters. Under suitable assumptions on the Kirchhoff function M, which cover the degenerate case, we prove the existence of nontrivial solutions for the above problem when $\lambda > 0$ is sufficiently large. Moreover, our results are new even in the Euclidean case.

Keywords: Heisenberg group, Kirchhoff-Poisson system, critical growth, logarithmic nonlinearity, concentration-compactness principle

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1 Introduction and main results

Consider the following Kirchhoff-Poisson system with logarithmic and critical nonlinearity in the Heisenberg group:

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u\ln|u|^2 + |u|^2 u & \text{in } \Omega, \\ -\Delta_H\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_H is the Kohn-Laplacian operator in the first Heisenberg group $\mathbb{H}^1$, $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $q \in (2\theta, 4)$ and θ is given by condition (M_2) below, and $\mu \in R$ and $\lambda > 0$ are real parameters. Let us

*** Corresponding author: Patrizia Pucci,** Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, 06123 Perugia, Italy, e-mail: patrizia.pucci@unipg.it
Yiwei Ye: School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, P. R. China, e-mail: yeyiwei2011@126.com

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put, for simplicity, $\mathbb{R}_0^+ = [0,\infty)$ and $\mathbb{R}^+ = (0,\infty)$. Concerning the Kirchhoff term M, we assume that $M \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ satisfies the following:

(M_1) For any $\tau > 0$, there exists $m_0 = m_0(\tau) > 0$ such that $M(t) \geq m_0$ for $t \geq \tau$.

(M_2) There exists $\theta \in [1,2)$ such that $\theta\widehat{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(s)ds$.

(M_3) There exists $m_1 > 0$ such that $M(t) \geq m_1 t^{\theta-1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.

A typical example is given by

$$M(t) = a + bt^{\theta-1}, \quad a, b \geq 0, \quad a + b > 0, \quad \theta \geq 1.$$

When M is of this type, problem (1.1) is called nondegenerate if $a > 0$, and degenerate if $a = 0$.

In recent years, geometric analysis in the Heisenberg group has become one of the most active and exciting research fields. This is because the Heisenberg group plays a crucial role in several branches of mathematics, such as representation theory, harmonic analysis, complex variables, quantum mechanics, and partial differential equations, see [6,8,13,14,21,24]. For example, [24] deals with multiplicity for entire solutions to a quasilinear equation in the Heisenberg group $\mathbb{H}^n$, depending on a real parameter λ. It proves that for any $\lambda > 0$, there exist infinitely many solutions $(u_k)_k$ with negative critical values that tend to zero as $k \to \infty$. In [21], the authors study the existence of entire solutions for the critical quasilinear elliptic systems in the Heisenberg group, involving (p, q) operators. The proof relies on the adaptation of Lions' concentration-compactness principle in the vectorial Heisenberg context and variational methods. See also [4,22,23] for the related results.

In the Euclidean setting, the existence and multiplicity of solutions for the nonlocal problems (i.e., Kirchhoff-type problems, Schrödinger-Poisson systems, and so on) have been widely studied, and many recent interesting results are obtained. We just quote, for example, [7,15,26,27]. Xiang et al. [26] consider the fractional nondegenerate Kirchhoff equations with logarithmic nonlinearity

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u = h(x)|u|^{\theta p-2}u\ln|u| + \lambda|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $s \in (0,1)$, $p \in (1, N/s)$, $\theta \in (1, p_s^*/p)$, Ω is a bounded domain with Lipschitz boundary of $\mathbb{R}^N$, and $h \in C(\overline{\Omega})$ is a sign-changing function. By applying the Nehari manifold approach, they prove the existence of two local least energy solutions for any exponent $q \in (1, \theta p)$ and $\lambda > 0$ small enough. However, the existence and multiplicity results of solutions of Kirchhoff-Poisson systems in the Heisenberg group are very few, see [2,16–18]. The Kirchhoff-Poisson system with critical nonlinearity of the form

$$\begin{cases} -\left(a - b\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u + |u|^2u & \text{in } \Omega, \\ -\Delta_H\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega \end{cases}$$

has been studied by Liu et al. [18], where the authors discussed the cases $q \in (1,2)$ and $q \in (2,4)$ and proved the existence and multiplicity results under suitable assumptions on μ and λ. Very recently, Liang and Pucci [16] have investigated the following Kirchhoff-Poisson system in the nondegenerate case

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \phi|u|^{q-2}u = h(\xi, u) + \lambda|u|^2u & \text{in } \Omega, \\ -\Delta_H\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Besides some other conditions, they assume that $q \in (1,2)$ and $M(t) \geq m_0 > 0$ for all $t \in \mathbb{R}_0^+$, and that there exists $\theta \in (2q, 4)$ such that $0 < \theta H(\xi, t) \leq h(\xi, t)t$ for $x \in \Omega$ and $|t| \geq T$, and they prove a multiplicity result when λ is sufficiently small. To the best of our knowledge, there are no results concerning the existence and multiplicity of solutions of the Kirchhoff-Poisson system (1.1) with logarithmic and critical nonlinearities in the Heisenberg group, even in the Euclidean case.

Inspired by the aforementioned works, we are interested in the study of the combined effects of logarithmic and critical nonlinearities for system (1.1) in the Heisenberg group. To this aim, let us recall that in [10], Folland and Stein introduced the Hilbert space $S_0^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the inner product $\langle u, v\rangle := \int_\Omega \nabla_H u \nabla_H v d\xi$, with Hilbertian corresponding norm

$$\|u\| = \|u\|_{S_0^1(\Omega)} = \left(\int_\Omega |\nabla_H u|^2 d\xi\right)^{\frac{1}{2}}.$$

The embedding $S_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1, Q^*]$, and the embedding is compact if and only if $s \in [1, Q^*)$, where $Q^* := \frac{2Q}{Q-2} = 4$ is the critical exponent in $\mathbb{H}^1$. The best Sobolev constant

$$S = \inf_{u \in S_0^1(\mathbb{H}^1)\setminus\{0\}} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi}{\left(\int_{\mathbb{H}^1} |u|^4 d\xi\right)^{\frac{1}{2}}} \tag{1.2}$$

is achieved by the C^∞ function $U(x, y, t) = c_0[(1 + x^2 + y^2)^2 + t^2]^{\frac{1}{2}}$, where $c_0 > 0$ is a constant (see [12]).

The main result of the article is the following.

Theorem 1.1. *Assume that* (M_1)–(M_3) *are satisfied and* $\mu < S|\Omega|^{-\frac{1}{2}}$, *where* S *is the best Sobolev constant given by* (1.2). *Then there exists* $\lambda^* > 0$ *such that problem* (1.1) *has a nontrivial solution for any* $\lambda > \lambda^*$.

Remark 1.1. The features of Theorem 1.1 are as follows:
(i) the presence of the logarithmic term;
(ii) the presence of the critical nonlinearity, which contributes to the lack of compactness; and
(iii) the fact that the result includes the degenerate case, which corresponds to the Kirchhoff function M vanishing at zero.

We point out that the degenerate case is rather appealing, not only from a mathematical point of view but also in applications. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. It is treated in famous well-known articles in Kirchhoff theory, see [9]. In addition, let us note that although the Kohn-Laplacian Δ_H and the classical Laplacian Δ have similar properties, the similarities may be misleading (see [11]). Moreover, the critical exponent $Q^* = 4$ in $\mathbb{H}^1$, while $2^* = 6$ in $\mathbb{R}^3$, which causes some obstacles in proving the compactness. In order to overcome these difficulties, we use the concentration-compactness principle in the Heisenberg group and carefully analyze the competing nonlinear terms to prove that the $(PS)_c$ condition holds at suitable levels of c.

The article is organized as follows. In Section 2, we present some preliminaries on the Heisenberg group functional setting and prove the local Palais-Smale condition. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

We briefly recall some definitions and notations on the Heisenberg group. For a complete treatment, we refer to [5,11].

Let $\mathbb{H}^1$ be the Heisenberg group of topological dimension 3, that is, the Lie group where underlying manifold is R^3, endowed with the nonAbelian law

$$\tau : \mathbb{H}^1 \to \mathbb{H}^1, \quad \tau_\xi(\xi') = \xi \circ \xi',$$

where

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - xy'))$$

for all $\xi, \xi' \in \mathbb{H}^1$, with $\xi = (x, y, t)$ and $\xi' = (x', y', t')$. The inverse is given by $\xi^{-1} = -\xi$, and hence $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$. Consider the family of dilations on $\mathbb{H}^1$ defined by

$$\delta_s(\xi) = (sx, sy, s^2 t), \quad \forall \xi \in \mathbb{H}^1,$$

so $\delta_s(\xi \circ \xi') = \delta_s(\xi) \circ \delta_s(\xi')$ (see [21]). It is easy to check that the Jacobian determinate of the dilatations $\delta_s : \mathbb{H}^1 \to \mathbb{H}^1$ is a constant and equals to s^4. As a result, the number $Q = 4$ is the homogeneous dimension of $\mathbb{H}^1$. The Haar measure on $\mathbb{H}^1$ coincides with the Lebesgue measure on $\mathbb{R}^3$. It is invariant under left translations and Q-homogeneous with respect to dilations. Then

$$|B_H(\xi_0, r)| = \omega_Q r^Q,$$

where $B_H(\xi_0, r)$ is the Heisenberg ball of radius r centered at ξ_0, i.e.,

$$B_H(\xi_0, r) = \{\xi \in \mathbb{H}^1 : d_H(\xi_0, \xi) < r\}$$

and $\omega_Q = |B_H(0, 1)|$.

The Kohn-Laplacian Δ_H on $\mathbb{H}^1$ is defined as

$$\Delta_H u = \operatorname{div}_H(\nabla_H u),$$

where $\nabla_H u = (Xu, Yu)$. Indeed, the vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}$$

constitute a basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. It is well known that Δ_H is a degenerate elliptic operator, and the Bony maximum principle is satisfied (see [3]).

First, we consider the problem

$$\begin{cases} -\Delta_H \phi = u^2 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It follows from the Lax-Milgram theorem that for every $u \in S_0^1(\Omega)$, problem (2.1) has a unique solution $\phi_u \in S_0^1(\Omega)$. Moreover, by the maximum principle, $\phi_u \geq 0$ and $\phi_u > 0$ if $u \neq 0$. We give some properties of the solution ϕ_u, and the detailed proof can be found in [2].

Proposition 2.1. (see [2]) *Let $u \in S_0^1(\Omega)$ be fixed. The corresponding solution $\phi_u \in S_0^1(\Omega)$ of problem* (2.1), *has the properties*

(i) $\phi_u \geq 0$ *and* $\phi_{tu} = t^2\phi_u$ *for all* $t > 0$;

(ii) $\int_\Omega |\nabla_H \phi_u|^2 d\xi = \int_\Omega \phi_u u^2 d\xi \leq S^{-1}\|u\|^4_{\frac{8}{3}} \leq S^{-1}|\Omega|^{\frac{1}{2}} \int_\Omega |u|^4 d\xi$;

(iii) *let* $u_n \rightharpoonup u$ *in* $S_0^1(\Omega)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $S_0^1(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} u_n v d\xi = \int_\Omega \phi_u u v d\xi \quad \text{for all } v \in S_0^1(\Omega).$$

On $S_0^1(\Omega)$, we define the functional

$$J_\lambda(u) = \frac{1}{2}\widehat{M}(\|u\|^2) + \frac{\mu}{4}\int_\Omega \phi_u |u|^2 d\xi + \lambda \int_\Omega \left(\frac{2}{q^2}|u|^q - \frac{1}{q}|u|^q \ln|u|^2\right) d\xi - \frac{1}{4}\int_\Omega |u|^4 d\xi.$$

From Proposition 2.1, it is easy to check that the functional J_λ is well defined in $S_0^1(\Omega)$. Moreover, $J_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$ and

$$\langle J_\lambda'(u), v\rangle = M(\|u\|^2)\langle u, v\rangle + \mu\int_\Omega \phi_u uv\mathrm{d}\xi - \lambda\int_\Omega |u|^{q-2}\ln|u|^2 uv\mathrm{d}\xi - \int_\Omega |u|^2 uv\mathrm{d}\xi$$

for all $u, v \in S_0^1(\Omega)$. The critical points of J_λ correspond to the solutions of problem (1.1).

Since for all $q \in (2\theta, 4)$ and $r \in (q, 4)$

$$\lim_{t\to 0}\frac{|t|^{q-1}\ln|t|^2}{|t|^{2\theta-1}} = 0 \quad \text{and} \quad \lim_{|t|\to\infty}\frac{|t|^{q-1}\ln|t|^2}{|t|^{r-1}} = 0,$$

for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{q-1}\ln|t|^2 \le \varepsilon|t|^{2\theta-1} + C_\varepsilon|t|^{r-1}. \tag{2.2}$$

Hence, if $u_n \rightharpoonup u$ in $S_0^1(\Omega)$, then the Vitali convergence theorem implies that

$$\int_\Omega |u_n|^q \ln|u_n|^2\mathrm{d}\xi \to \int_\Omega |u|^q\ln|u|^2\mathrm{d}\xi \quad \text{as } n\to\infty. \tag{2.3}$$

Given $c \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset S_0^1(\Omega)$ is a $(PS)_c$ *sequence* for the functional J_λ at the level c if $J_\lambda(u_n) \to c$ and $J_\lambda'(u_n) \to 0$ as $n\to\infty$. Moreover, J_λ is said to satisfy $(PS)_c$ *condition at the level* c if any $(PS)_c$ sequence possesses a strongly convergent subsequence in $S_0^1(\Omega)$. Let us prove

Lemma 2.1. *Assume that conditions (M_1)–(M_3) hold and $\mu < S|\Omega|^{-\frac{1}{2}}$. Then J_λ satisfies the $(PS)_c$ condition at any*

$$c\in I, \quad I := \left(0, \left(\frac{1}{2\theta} - \frac{1}{q}\right)m_1^{\frac{2}{2-\theta}}S^{\frac{2\theta}{2-\theta}}\right).$$

Proof. Let c be in I and let $(u_n)_n$ be a $(PS)_c$ sequence of J_λ, i.e.,

$$J_\lambda(u_n)\to c \quad \text{and} \quad J_\lambda'(u_n)\to 0 \tag{2.4}$$

as $n\to\infty$. Proposition 2.1 gives that

$$\int_\Omega |u|^4\mathrm{d}\xi - \mu\int_\Omega \phi_u|u|^2\mathrm{d}\xi \ge \begin{cases} \int_\Omega |u|^4\mathrm{d}\xi, & \text{if } \mu\le 0,\\ \left(1-\mu S^{-1}|\Omega|^{\frac{1}{2}}\right)\int_\Omega |u|^4\mathrm{d}\xi, & \text{if } 0<\mu<S|\Omega|^{-\frac{1}{2}}, \end{cases} \tag{2.5}$$

so that

$$\begin{aligned} c + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{q}\langle J_\lambda'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2\theta}-\frac{1}{q}\right)M(\|u\|^2)\|u\|^2 + \frac{2\lambda}{q^2}\int_\Omega |u_n|^q\mathrm{d}\xi + \left(\frac{1}{q}-\frac{1}{4}\right)\left(\int_\Omega |u_n|^4\mathrm{d}\xi - \mu\int_\Omega \phi_{u_n}|u_n|^2\mathrm{d}\xi\right) \\ &\ge \left(\frac{1}{2\theta}-\frac{1}{q}\right)m_1\|u_n\|^{2\theta}, \end{aligned} \tag{2.6}$$

by (M_2)–(M_3) and the fact that $q\in(2\theta,4)$. This implies that $(u_n)_n$ is bounded in $S_0^1(\Omega)$. Hence by [20] and Proposition 2.1, passing eventually to a subsequence, we may assume that for some $u\in S_0^1(\Omega)$

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } S_0^1(\Omega), \quad \phi_{u_n}\rightharpoonup \phi_u \text{ in } S_0^1(\Omega),\\ &u_n\to u \quad \text{in } L^s(\Omega), \text{ with } 1\le s<4,\\ &u_n\to u \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.7}$$

Now we claim that

$$\|u_n\|^2 \to \|u\|^2 \quad \text{as } n \to \infty, \tag{2.8}$$

implying that $u_n \to u$ in $S_0^1(\Omega)$ as $n \to \infty$.

In fact, it follows from the concentration-compactness principle on the Heisenberg group (see [25, Lemma 3.5]) that there exist an at most countable set of distinct points $\{x_j\}_{j\in\Lambda} \subset \Omega$, nonnegative numbers $\{\omega_j\}_{j\in\Lambda}$, $\{\nu_j\}_{j\in\Lambda}$, and two positive Radon measures ω and ν in $\mathbb{H}^1$, with support in Ω, such that

$$\begin{aligned} &|\nabla_H u_n|^2 \mathrm{d}\xi \overset{*}{\rightharpoonup} \mathrm{d}\omega \quad \text{and} \quad |u_n|^4 \mathrm{d}\xi \overset{*}{\rightharpoonup} \mathrm{d}\nu \quad \text{in } \mathcal{M}(\mathbb{H}^1),\\ &\mathrm{d}\omega \ge |\nabla_H u|^2 \mathrm{d}\xi + \sum_{j\in\Lambda} \omega_j \delta_{x_j},\\ &\mathrm{d}\nu = |u|^4 \mathrm{d}\xi + \sum_{j\in\Lambda} \nu_j \delta_{x_j}, \end{aligned} \tag{2.9}$$

and

$$\omega_j \ge S\nu_j^{\frac{1}{2}}. \tag{2.10}$$

In order to prove (2.8), we proceed by steps.

Step 1. Fixed $j \in \Lambda$. Then, either $\omega_j = 0$ or

$$\omega_j \ge (m_1 S^2)^{\frac{1}{2-\theta}}. \tag{2.11}$$

For $\varepsilon > 0$ small, we set $\psi_{j,\varepsilon} \in C_0^\infty(B_H(\xi_j, \varepsilon))$ such that $0 \le \psi_{j,\varepsilon}(\xi) \le 1$, $\psi_{j,\varepsilon}(\xi) = 1$ in $B_H(\xi_j, \varepsilon/2)$, $\psi_{j,\varepsilon}(\xi) = 0$ in $\Omega\backslash B(\xi_j, \varepsilon)$, and $|\nabla_H \psi_{j,\varepsilon}| \le 2/\varepsilon$. Clearly, $(u_n\psi_{j,\varepsilon})_n$ is bounded in $S_0^1(\Omega)$, and so (2.4) implies that

$$\langle J_\lambda'(u_n), u_n\psi_{j,\varepsilon}\rangle \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\begin{aligned} M(\|u_n\|^2)\left(\int_\Omega |\nabla_H u_n|^2 \psi_{j,\varepsilon} \mathrm{d}\xi + \int_\Omega u_n \nabla_H u_n \nabla_H \psi_{j,\varepsilon} \mathrm{d}\xi\right) + \mu \int_\Omega \phi_{u_n} |u_n|^2 \psi_{j,\varepsilon} \mathrm{d}\xi \\ = \lambda \int_\Omega |u_n|^q \ln|u_n|^2 \psi_{j,\varepsilon} \mathrm{d}\xi + \int_\Omega |u_n|^4 \psi_{j,\varepsilon} \mathrm{d}\xi + o(1). \end{aligned} \tag{2.12}$$

By the dominated convergence theorem, we obtain that

$$\int_{B_H(x_j,\varepsilon)} |u_n|^q \ln|u_n|^2 \psi_{j,\varepsilon} \mathrm{d}\xi \to \int_{B_H(x_j,\varepsilon)} |u|^q \ln|u|^2 \psi_{j,\varepsilon} \mathrm{d}\xi$$

as $n \to \infty$, and then, by sending $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{B_H(x_j,\varepsilon)} |u_n|^q \ln|u_n|^2 \psi_{j,\varepsilon} \mathrm{d}\xi = 0. \tag{2.13}$$

Proposition 2.1(iii) gives

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} u_n u \mathrm{d}\xi = \int_\Omega \phi_u |u|^2 \mathrm{d}\xi.$$

Moreover, by (2.7),

$$\left|\int_\Omega \left(\phi_{u_n}|u_n|^2 - \phi_u u_n u\right)\mathrm{d}\xi\right| \le \int_\Omega |\phi_{u_n}||u_n||u_n - u|\mathrm{d}\xi \le |\phi_{u_n}|_4 |u_n|_{8/3} |u_n - u|_{8/3} \to 0.$$

Consequently, the last two limits give at once

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^2 d\xi = \int_\Omega \phi_u |u|^2 d\xi.$$

Thus,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^2\psi_{j,\varepsilon} d\xi = \lim_{\varepsilon\to 0}\int_{B_H(x_j,\varepsilon)} \phi_u|u|^2\psi_{j,\varepsilon} d\xi = 0. \tag{2.14}$$

Moreover, applying the Heisenberg polar coordinates (see [19]), we deduce that

$$\int_{B_H(x_j,\varepsilon)} d\xi = \int_{B_H(0,\varepsilon)} d\xi = |B_H(0,1)|\varepsilon^4,$$

and then, using the Hölder inequality,

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n\nabla_H u_n \nabla_H \psi_{j,\varepsilon} d\xi\right| &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{B(x_j,\varepsilon)}|\nabla_H u_n|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{B(x_j,\varepsilon)}|u_n|^2|\nabla_H\psi_{j,\varepsilon}|^2 d\xi\right)^{\frac{1}{2}}\\ &\le C\lim_{\varepsilon\to 0}\left(\int_\Omega |u|^2|\nabla_H\psi_{j,\varepsilon}|^2 d\xi\right)^{\frac{1}{2}}\\ &\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)}|u|^4 d\xi\right)^{\frac{1}{4}}\left(\int_{B(x_j,\varepsilon)}|\nabla_H\psi_{j,\varepsilon}|^4 d\xi\right)^{\frac{1}{4}} = 0.\end{aligned} \tag{2.15}$$

Hence, combining (2.12)–(2.15) and condition (M_3), we obtain the key inequality

$$\nu_j \ge m_1\omega_j^\theta,$$

which, jointly with (2.10), yields that either $\omega_j = 0$ or ω_j verifies (2.11).

Step 2. Estimate (2.11) cannot occur, and hence $\omega_j = 0$ for all j.

Indeed, if (2.11) holds, then by (2.6),

$$c = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{q}\langle J_\lambda'(u_n), u_n\rangle\right) \ge \lim_{n\to\infty}\left(\frac{1}{2\theta}-\frac{1}{q}\right)m_1\left(\int_\Omega|\nabla_H u_n|^2\psi_{j,\varepsilon}d\xi\right)^\theta = \left(\frac{1}{2\theta}-\frac{1}{q}\right)m_1\left(\int_\Omega \psi_{j,\varepsilon}d\mu\right)^\theta,$$

and so, letting $\varepsilon\to 0$,

$$c \ge \left(\frac{1}{2\theta}-\frac{1}{q}\right)m_1\omega_j^\theta \ge \left(\frac{1}{2\theta}-\frac{1}{q}\right)m_1^{\frac{2}{2-\theta}}S^{\frac{2\theta}{2-\theta}} \notin I,$$

which is impossible.

Step 3. Claim (2.8) holds true.

Since j is arbitrary in *Step 1*, we deduce that $\omega_j = 0$ for all $j\in\Lambda$. As a consequence, from (2.9), it follows that

$$\int_\Omega |u_n|^4 d\xi \to \int_\Omega |u|^4 d\xi \quad \text{as } n\to\infty. \tag{2.16}$$

Let $\lim_{n\to\infty}\|u_n\|^2 = A$. If $A = 0$, i.e., $u_n \to 0$ in $S_0^1(\Omega)$, then using (2.7), (2.3), and the fact $M(0) = 0$, we see that

$$c + o(1) = \left(J_\lambda(u_n) - \frac{1}{4}\langle J'_\lambda(u_n), u_n\rangle\right)$$
$$= \frac{1}{2}\widehat{M}(\|u_n\|^2) - \frac{1}{4}M(\|u_n\|^2)\|u_n\|^2 + \frac{2\lambda}{q^2}\int_\Omega |u_n|^q \mathrm{d}\xi + \lambda\left(\frac{1}{4} - \frac{1}{q}\right)\int_\Omega |u_n|^q \ln|u_n|^2 \mathrm{d}\xi = o(1).$$

This is impossible because $c > 0$. Hence $A > 0$ and so Proposition 2.1, (2.3), (2.16), and the fact that $\langle J'_\lambda(u_n), u_n\rangle = o(1)$ yield

$$M(\|u_n\|^2)\|u_n\|^2 = -\mu\int_\Omega \phi_u|u|^2\mathrm{d}\xi + \lambda\int_\Omega |u|^q \ln|u|^2\mathrm{d}\xi + \int_\Omega |u|^4\mathrm{d}\xi + o(1). \tag{2.17}$$

Since $\langle J'_\lambda(u_n), v\rangle = o(1)$ for any $v \in S^1_0(\Omega)$, one sees by (2.7) that

$$M(A)\langle u, v\rangle = -\mu\int_\Omega \phi_u uv\mathrm{d}\xi + \lambda\int_\Omega |u|^{q-2}uv \ln|u|^2\mathrm{d}\xi + \int_\Omega |u|^2 uv\mathrm{d}\xi. \tag{2.18}$$

Therefore, (2.17) and (2.18), with $v = u$, give at once that $M(A)\|u\|^2 = M(\|u_n\|^2)\|u_n\|^2 + o(1)$. Hence, (M_1) proves claim (2.8) and completes the proof. □

3 Proof of Theorem 1.1

Now we are in a position to prove Theorem 1.1, and we assume that the hypotheses of Theorem 1.1 are satisfied. We need the mountain pass theorem in the following version.

Proposition 3.1. (see [1]) *Let E be a real Banach space and the functional $\mathcal{I} \in C^1(E, \mathbb{R})$ satisfies $\mathcal{I}(0) = 0$ and*
(i) *there are constants $\rho, \alpha > 0$ such that* $\inf_{\|u\|=\rho}\mathcal{I} \ge \alpha$;
(ii) *there is $e \in E\backslash B_\rho$ such that $\mathcal{I}(e) < 0$.*

Let c be defined by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}(\gamma(t)), \quad \text{with}\,\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \mathcal{I}(\gamma(1)) < 0\}.$$

If $\mathcal{I}$ satisfies the $(PS)_c$ condition, then c is a critical value for $\mathcal{I}$ and $c \ge \alpha$.

Lemma 3.1. *The functional J_λ has a mountain pass geometry in $S^1_0(\Omega)$.*

Proof. From (M_2), (M_3), (2.2), and the Sobolev embedding inequality, we have

$$\begin{aligned} J_\lambda(u) &\ge \frac{1}{2\theta}M(\|u\|^2)\|u\|^2 + \frac{\mu}{4}\int_\Omega \phi_u|u|^2\mathrm{d}\xi - \frac{\lambda}{q}\int_\Omega (\varepsilon|u|^{2\theta} + C_\varepsilon|u|^r)\mathrm{d}\xi - \frac{1}{4}\int_\Omega |u|^4\mathrm{d}\xi \\ &\ge \left(\frac{1}{2\theta} - \frac{\lambda}{q}\varepsilon C_1\right)m_1\|u\|^{2\theta} - \frac{\lambda}{q}C_\varepsilon C_2\|u\|^r - C_3(|\mu| + 1)\|u\|^4. \end{aligned} \tag{3.1}$$

Thus, choosing $\varepsilon = q/4\lambda C_1\theta > 0$ and $\rho > 0$ small enough for all $u \in S^1_0(\Omega)$ with $\|u\| = \rho$, inequality (3.1) gives

$$J_\lambda(u) \ge \frac{1}{4\theta}m_1\rho^{2\theta} - \frac{\lambda}{q}C_\varepsilon C_2\rho^r - C_3(|\mu| + 1)\rho^4 \ge \alpha$$

for a suitable $\alpha > 0$ because $2\theta < r$. Observe that

$$2t^q - qt^q \ln|t|^2 \le 2 \quad \text{for all } t \in \mathbb{R}^+, \tag{3.2}$$

and for a fixed $t_0 > 0$, assumption (M_2) yields that

$$\widehat{M}(t) \le \frac{\widehat{M}(t_0)}{t_0^{\theta}} t^{\theta} = C_0 t^{\theta} \quad \text{for all } t \ge t_0. \tag{3.3}$$

Take $v \in S_0^1(\Omega)\backslash\{0\}$. Since, by (2.5),

$$\int_{\Omega} |v|^4 \mathrm{d}\xi - \mu \int_{\Omega} \phi_v |v|^2 \mathrm{d}\xi > 0 \quad \text{for all } \mu \in \left(-\infty, S|\Omega|^{-\frac{1}{2}}\right),$$

we deduce that

$$J_\lambda(tv) \le \frac{C_0}{2} t^{2\theta} \|v\|^{2\theta} + \frac{\lambda}{q^2}|\Omega| - \frac{t^4}{4}\left(\int_{\Omega} |v|^4 \mathrm{d}\xi - \mu \int_{\Omega} \phi_v |v|^2 \mathrm{d}\xi \right) \to -\infty \quad \text{as } t \to \infty$$

by (3.3), (3.2), and the fact that $\theta < 2$. Hence, putting $e = t_0 v$ with t_0 large enough, we obtain $J_\lambda(e) < 0$. This completes the proof. □

Proof of Theorem 1.1. By Lemmas 2.1 and 3.1 and Proposition 3.1, there exists a nontrivial critical point of J_λ at level

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], S_0^1(\Omega)) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0\}$, provided that

$$c_\lambda < \left(\frac{1}{2\theta} - \frac{1}{q}\right) m_1^{\frac{2}{2-\theta}} S^{\frac{2\theta}{2-\theta}}. \tag{3.4}$$

We claim that (3.4) holds true for all $\lambda > 0$ large enough.

To prove (3.4), we choose $v_0 \in S_0^1(\Omega)$ with $\|v_0\| = 1$. From the proof of Lemma 3.1, it is clear that $J_\lambda(tv_0) > 0$ for all $t > 0$ small enough and that $J_\lambda(tv_0) \to -\infty$ as $t \to \infty$. Hence, there is $t_\lambda > 0$ such that

$$J_\lambda(t_\lambda v_0) = \sup_{t \ge 0} J_\lambda(tv_0).$$

Moreover, $\langle J_\lambda'(t_\lambda v_0), t_\lambda v_0 \rangle = t_\lambda \frac{\mathrm{d}}{\mathrm{d}t} J_\lambda(tv_0)\Big|_{t=t_\lambda} = 0$, that is,

$$M(t_\lambda^2) t_\lambda^2 + \mu t_\lambda^4 \int_{\Omega} \phi_{v_0} |v_0|^2 \mathrm{d}\xi = \lambda \int_{\Omega} |t_\lambda v_0|^q \ln |t_\lambda v_0|^2 \mathrm{d}\xi + t_\lambda^4 \int_{\Omega} |v_0|^4 \mathrm{d}\xi. \tag{3.5}$$

It follows from (3.2), (3.3), and (M_2) that

$$C_0 \theta t_\lambda^{2\theta} \ge -\frac{2\lambda}{q}|\Omega| + t_\lambda^4 \left(\int_{\Omega} |v_0|^4 \mathrm{d}\xi - \mu \int_{\Omega} \phi_{v_0} |v_0|^2 \mathrm{d}\xi \right).$$

Since, by (2.5), $\int_{\Omega} |v_0|^4 \mathrm{d}\xi - \mu \int_{\Omega} \phi_{v_0} |v_0|^2 \mathrm{d}\xi > 0$ for all $\mu \in \left(-\infty, S|\Omega|^{-\frac{1}{2}}\right)$, the above inequality gives that $\{t_\lambda\}_{\lambda > 0}$ is bounded. Next, we claim that

$$t_\lambda \to 0 \quad \text{as } \lambda \to \infty. \tag{3.6}$$

Otherwise, there exists a sequence $(\lambda_n)_n$ with $\lambda_n \to \infty$ such that $t_{\lambda_n} \to t_0$ as $n \to \infty$ for some $t_0 > 0$. The dominated convergence theorem gives as $n \to \infty$,

$$\int_{\Omega} |t_{\lambda_n} v_0|^q \ln |t_{\lambda_n} v_0|^2 \mathrm{d}\xi \to \int_{\Omega} |t_0 v_0|^q \ln |t_0 v_0|^2 \mathrm{d}\xi,$$

and so

$$\lambda_n \int_\Omega |t_{\lambda_n} v_0|^q \ln |t_{\lambda_n} v_0|^2 \mathrm{d}\xi \to \infty.$$

Thanks to (3.5), this contradicts

$$\lim_{n\to\infty} \left(M\big(t_{\lambda_n}^2\big) t_{\lambda_n}^2 + \mu t_{\lambda_n}^4 \int_\Omega \phi_{v_0} |v_0|^2 \mathrm{d}\xi \right) = M(t_0^2) t_0^2 + \mu t_0^4 \int_\Omega \phi_{v_0} |v_0|^2 \mathrm{d}\xi (\in \mathbb{R})$$

and proves the latter claim (3.6).

Therefore, using (3.5), (3.6), and the fact that M is continuous at 0, we deduce at once that as $\lambda \to \infty$,

$$-\lambda \int_\Omega |t_\lambda v_0|^q \ln |t_\lambda v_0|^2 \mathrm{d}\xi = t_\lambda^4 \int_\Omega |v_0|^4 \mathrm{d}\xi - M(t_\lambda^2) t_\lambda^2 - \mu t_\lambda^4 \int_\Omega \phi_{v_0} |v_0|^2 \mathrm{d}\xi \to 0.$$

Moreover, also

$$\lambda \int_\Omega |t_\lambda v_0|^q \mathrm{d}\xi \to 0 \quad \text{as } \lambda \to \infty.$$

By the definition of J_λ, it follows that

$$J_\lambda(t_\lambda v_0) \to 0 \quad \text{as } \lambda \to \infty,$$

which implies that there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$,

$$c_\lambda \le \sup_{t \ge 0} J_\lambda(t v_0) = J_\lambda(t_\lambda v_0) < \left(\frac{1}{2\theta} - \frac{1}{q} \right) m_1^{\frac{2}{2-\theta}} S^{\frac{2\theta}{2-\theta}},$$

i.e., claim (3.4) holds true. The proof of Theorem 1.1 is now complete. □

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