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Research Article

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Gradient estimate of the solutions to Hessian equations with oblique boundary value

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Abstract: In this paper, we study Hessian equations with the prescribed contact angle boundary value or oblique derivative boundary value and finally derive the a priori global gradient estimate for the admissible solutions.

Keywords: oblique derivative boundary value, prescribed contact angle boundary value, gradient estimate, Hessian equations

MSC code: 35J65

1 Introduction

In this paper, we consider the following Hessian equation with oblique boundary value,

$$\begin{cases} \sigma_k(u_{ij}) = f(x, u) & \text{in } \Omega, \\ G(x, Du) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n with smooth boundary and $f(x, t)$ and $G(x, \overrightarrow{p})$ are smooth functions defined, respectively, on $\Omega \times \mathbf{R}$ and $\overline{\Omega} \times \mathbf{R}^n$. We mainly study two general but important cases of $G(x, Du)$, one is the prescribed contact angle boundary value problem and the other is the oblique derivative boundary value problem. The topic in this paper is also concentrated on the global gradient estimate, which would be one step forward to conclude the existence of the solution to problem (1).

Hessian equations including Laplace equations and Monge-Ampère equations as their special cases with various boundary values are in no doubt an interesting subject in recent years, and many topics in differential geometry, convex geometry, and optimal transport have close relations with these kind of elliptic equations. For the given boundary value, one may first be interested in the existence of the solution. In general, it is necessary to obtain the $C^{2,\alpha}$ estimate to conclude the existence of the solution. For instance, when the boundary value is of the Dirichlet type, one can refer to [1–3] for the existence results. For the Neumann boundary value, Trudinger [4] considered the special domain case and obtained the existence result. Also, he conjectured in [4] that one can solve the problem in sufficiently smooth uniformly convex domains. Recently, Ma and Qiu [22] gave a positive answer to this problem and solved the Neumann problem of k-Hessian equations in uniformly convex domains. Chen and Zhang [14] considered the Hessian quotient equation and also derived the existence results with the Neumann boundary condition.

Now, it is of natural interest to consider the existence of the solutions to Hessian equations with the other types of boundary value problems such as prescribed contact angle boundary value and oblique

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derivative boundary value. It seems to be a little more complicated for these kinds of boundary values. For instance, a necessary condition for the existence of the solution to Monge-Ampère equations was exhibited in [9,10]. Till now, there are only a few progress results on this topic. In [8], the oblique derivative boundary problems for Monge-Ampère equations were considered and the existence of the solutions to two-dimensional Monge-Ampère equations was derived, and the generalized solutions for general dimension Monge-Ampère equations were also considered. In [11–13], Urbas also derived some existence results for Monge-Ampère equations with the oblique derivative boundary value. For some augmented Hessian equations with oblique boundary value, Jiang and Trudinger in [5,6] considered the existence result. Wang [7] derived the interior gradient estimate of the solutions to k-curvature equations, and Deng and Ma [25] obtained the global gradient estimate for k-curvature equations with the prescribed contact angle boundary value. It is still open for the existence of the solutions to k-curvature equations and Hessian equations with prescribed contact angle or oblique derivative boundary value. In this paper, we make an attempt for this problem and finally will derive the global gradient estimate for admissible solutions to Hessian equations with these kinds of boundary conditions, which would be considered as a little step forward to the existence of the solutions to these interesting problems.

Gradient estimate of the solutions to various partial differential equations is an important and interesting issue in the study of P.D.E. Usually, it includes interior gradient estimate and global gradient estimate, which, respectively, have close relation to Liouville type results and the existence of the solution to P.D.E. One can refer to [1,3,7,8,14–16,18,20,23–25], and the references therein for more details.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminaries for the follow-up of the paper. In Section 3, we give the global gradient estimate of the solution for Hessian equations with the prescribed contact angle boundary value, and in Section 4, we come to deal with the oblique derivative boundary data case.

2 Notations and preliminaries

In this section, we list some notations and preliminaries that are necessary for the gradient estimate.

First, we denoted by $d(x) = \operatorname{dist}(x, \partial\Omega)$ the distance from x to $\partial\Omega$, the boundary of a bounded smooth domain Ω. As a known fact, $d(x)$ is also smooth near the boundary, such as on the annular domain $\Omega_{\mu_1} = \{x \in \Omega \mid d(x) \leq \mu_1\}$, where μ_1 is a positive constant related to the domain.

Second, we give some basic properties of elementary symmetric functions, denoted by $\sigma_k(\lambda)$ for $\lambda \in \mathbf{R}^n$, which could be found in [1,3].

We denoted by $\sigma_k(\lambda|i)$ the kth symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ by the kth symmetric function with $\lambda_i = \lambda_j = 0$. Then we have the following propositions.

Proposition 2.1. *Assume $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{R}^n$, and $k = 1, 2, \ldots, n$, then we have*

$$\begin{aligned} &\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i), \quad 1 \leq i \leq n, \\ &\sum_{i=1}^n \lambda_i\sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda), \\ &\sum_{i=1}^n \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda). \end{aligned} \tag{2}$$

Recall that the Garding's cone is defined as follows:

$$\Gamma_k = \{\lambda \in \mathbf{R}^n \mid \sigma_i(\lambda) > 0, \quad \forall 1 \leq i \leq k\}.$$

Proposition 2.2. *Assume $k \in \{1, 2, \ldots, n\}$ and $\lambda \in \Gamma_k$, suppose that*

$$\lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n,$$

then we have

$$\sigma_{k-1}(\lambda|n) \geq \cdots \geq \sigma_{k-1}(\lambda|k) \geq \cdots \geq \sigma_{k-1}(\lambda|1) > 0$$

and

$$\sigma_{k-1}(\lambda|k) \geq C(n,k)\sum_{i=1}^{n}\sigma_{k-1}(\lambda|i). \tag{3}$$

Remark that if the eigenvalues of (u_{ij}), denoted also by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, are located in Γ_k, then the equation in (1) is elliptic and we will call this kind of solution as "k-admissible" solution.

We also list the generalized Newton-MacLaurin inequality in the following, which includes the Newton inequality and the MacLaurin inequality as the special cases.

Proposition 2.3. *Assume $\lambda \in \Gamma_k$, and $k, l, r, s \in \{0, 1, 2, \ldots, n\}$ with $k > l \geq 0$, $r > s \geq 0$, $k \geq r$, $l \geq s$, we have*

$$\left[\frac{\frac{\sigma_k(\lambda)}{C_n^k}}{\frac{\sigma_l(\lambda)}{C_n^l}}\right]^{\frac{1}{k-l}} \leq \left[\frac{\frac{\sigma_r(\lambda)}{C_n^r}}{\frac{\sigma_s(\lambda)}{C_n^s}}\right]^{\frac{1}{r-s}}, \tag{4}$$

and the equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n > 0$.

As the last point of this section, we also state that the universal constant C during the whole paper may change from line to line.

3 Prescribed contact angle boundary data

In this section, we set out to obtain the gradient estimate of the admissible solution to Hessian equations with the prescribed contact angle boundary value. In a word, we will prove the following theorem.

Theorem 3.1. *Let Ω be a smooth bounded domain in $\mathbf{R}^n (n \geq 2)$ and u be the admissible solution to the following Hessian equations with the prescribed contact angle boundary value,*

$$\begin{cases} \sigma_k(u_{ij}) = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Assume that $f(x,t)$ is a positive smooth function defined on $\Omega \times \mathbf{R}$ with $f_t \geq 0$ and $\theta(x)$ is a smooth function defined on $\overline{\Omega}$ with $|\cos\theta| \leq 1 - b < 1$ for some positive constant b. ν is denoted to be the inward unit normal along $\partial\Omega$. Also we assume that we have already obtained the C^0 estimate as $|u| \leq M$. Then, there exists a positive constant $C = C(M, n, \Omega, b, |\theta|_{C^2(\overline{\Omega})}, |f|_{C^1(\Omega\times[-M,M])})$ such that

$$|Du| \leq C. \tag{6}$$

Proof. Due to [1], we have already known the interior gradient estimate, so we only need to obtain the gradient estimate near boundary, denoted by Ω_μ, where $\mu \leq \mu_1$ is a positive constant to be determined later.

Let $v = \sqrt{1+|Du|^2}$, $w = v + \sum_{l=1}^{n} u_l d_l \cos\theta$ and let $h(t)$, and τ be a smooth function and a positive constant, respectively, to be determined later. We choose the auxiliary function

$$\Phi = \log w + h(u) + \tau d.$$

Assume Φ achieves its maximum on the domain $\overline{\Omega_\mu}$ at the point x_0, according to the interior gradient estimate, we can only consider the following two cases.

Case I: $x_0 \in \partial\Omega$.

For convenience, we choose a coordinate around x_0 such that $\nu = \frac{\partial}{\partial x_n}$, assume $\frac{\partial}{\partial x_i}$ $(i = 1, 2, \ldots, n-1)$ are tangent to $\partial\Omega$. Under this coordinate, we have

$$\frac{\partial d}{\partial x_i} = 0, \quad \frac{\partial d}{\partial x_n} = 1, \quad \frac{\partial^2 d}{\partial x_n \partial x_\alpha} = 0, \quad \frac{\partial^2 d}{\partial x_i \partial x_j} = -\kappa_i \delta_{ij},$$

where $1 \le i, \ j < n-1$, $1 \le \alpha \le n$ and κ_i $(i = 1, 2, \ldots, n-1)$ are the principal curvatures of $\partial\Omega$ at x_0.

By the fact that x_0 is the maximum point on the boundary, we have

$$0 = \Phi_i = \frac{w_i}{w} + h' u_i + \tau d_i = \frac{w_i}{w} + h' u_i, \quad i = 1, 2, \ldots, n-1, \tag{7}$$

and

$$0 \ge \Phi_n = \frac{w_n}{w} + h' u_n + \tau d_n = \frac{w_n}{w} + h' u_n + \alpha. \tag{8}$$

By a direct computation, we have

$$\begin{aligned} w_n &= v_n + u_{nn}\cos\theta + u_n(\cos\theta)_n \\ &= \frac{\sum_{\alpha=1}^{n} u_\alpha u_{\alpha n}}{v} + u_{nn}\cos\theta + u_n(\cos\theta)_n \\ &= \frac{\sum_{i=1}^{n-1} u_i u_{in}}{v} + \frac{u_n u_{nn}}{v} + u_{nn}\cos\theta + u_n(\cos\theta)_n \\ &= \frac{\sum_{i=1}^{n-1} u_i u_{ni}}{v} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{v} + u_n(\cos\theta)_n, \end{aligned} \tag{9}$$

where we denote by k_{ij} the Weingarten matrix of the boundary with respect to ν.

Differentiating u_n along $\partial\Omega$, we obtain for $i = 1, 2, \ldots, n-1$ that

$$\begin{aligned} u_{ni} &= (-v\cos\theta)_i = -v_i\cos\theta - v(\cos\theta)_i \\ &= -\left(w_i - u_{ni}\cos\theta - \sum_{l=1}^{n-1} u_l d_{li}\cos\theta - u_n(\cos\theta)_i\right)\cos\theta - v(\cos\theta)_i \\ &= -w_i\cos\theta + u_{ni}\cos^2\theta + \sum_{l=1}^{n-1} u_l d_{li}\cos^2\theta + u_n\cos\theta(\cos\theta)_i - v(\cos\theta)_i, \end{aligned}$$

furthermore, using (7), we can obtain

$$u_{ni} = \frac{h' w u_i \cos\theta + \sum_{l=1}^{n-1} u_l d_{li}\cos^2\theta - v(1+\cos^2\theta)(\cos\theta)_i}{\sin^2\theta}. \tag{10}$$

Substituting (10) into (9), we then have

$$\begin{aligned} w_n &= \frac{\sum_{i=1}^{n-1} u_i u_{ni}}{v} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{v} + u_n(\cos\theta)_n \\ &= \frac{\sum_{i=1}^{n-1} u_i\left[h' w w_i\cos\theta + \sum_{l=1}^{n-1} u_l d_{li}\cos^2\theta - v(1+\cos^2\theta)(\cos\theta)_i\right]}{v\sin^2\theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{v} + u_n(\cos\theta)_n. \end{aligned}$$

Then

$$
\begin{aligned}
0 \geq \Phi_n &= \frac{\sum_{i=1}^{n-1} u_i[h' w u_i \cos\theta + \sum_{l=1}^{n-1} u_l d_{li} \cos^2\theta - v(1+\cos^2\theta)(\cos\theta)_i]}{wv\sin^2\theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{wv} + \frac{u_n(\cos\theta)_n}{w} + h'u_n + \tau \\
&= \frac{h'\cos\theta \sum_{i=1}^{n-1} u_i^2}{v\sin^2\theta} - \frac{(1+\cos^2\theta)\sum_{i=1}^{n-1} u_i(\cos\theta)_i}{w\sin^2\theta} + \frac{\sum_{i,l=1}^{n-1} u_l d_{li} u_i \cos^2\theta}{wv\sin^2\theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{wv} + \frac{u_n(\cos\theta)_n}{w} + h'u_n + \tau \\
&= \frac{h'\cos\theta(v^2\sin^2\theta - 1)}{v\sin^2\theta} - \frac{(1+\cos^2\theta)\sum_{i=1}^{n-1} u_i(\cos\theta)_i}{w\sin^2\theta} + \frac{\sum_{i,l=1}^{n-1} u_l d_{li} u_i \cos^2\theta}{wv\sin^2\theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{wv} + \frac{u_n(\cos\theta)_n}{w} \\
&\quad + h'u_n + \tau \\
&= -\frac{h'\cos\theta}{v\sin^2\theta} - \frac{(1+\cos^2\theta)\sum_{i=1}^{n-1} u_i(\cos\theta)_i}{w\sin^2\theta} + \frac{\sum_{i,l=1}^{n-1} u_l d_{li} u_i \cos^2\theta}{wv\sin^2\theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{wv} + \frac{u_n(\cos\theta)_n}{w} + \tau.
\end{aligned}
$$

Without the loss of generality, we may assume that v is large such that if τ is chosen large enough determined by θ and the geometry of $\partial\Omega$, the right hand of the above inequality will be positive, which shows that this case will not occur at all.

Case II: $x_0 \in \Omega_\mu$.

At this point, we can assume that $|Du|$ is large enough such that $|Du|$, w and v are equivalent with each other. Remark that the Einstein summation convention will be adopted during all the calculations if no otherwise specified.

Since x_0 is the maximum point, we then have

$$0 = \Phi_i = \frac{w_i}{w} + h'u_i + \tau d_i,$$

and it follows that

$$w_i = -w(h'u_i + \tau d_i). \tag{11}$$

By the definition of w, we have

$$w_i = \frac{u_l u_{li}}{v} + u_{li} d_l \cos\theta + u_l d_{li} \cos\theta + u_l d_l (\cos\theta)_i = \left(\frac{u_l}{v} + d_l\cos\theta\right) u_{li} + u_l d_{li}\cos\theta + u_l d_l(\cos\theta)_i.$$

Therefore,

$$-w(h'u_i + \tau d_i) = \left(\frac{u_l}{v} + \cos\theta d_l\right) u_{li} + u_l d_{li}\cos\theta + u_l d_l(\cos\theta)_i. \tag{12}$$

We now come to deal with Φ_{ij}. By (11), we derive that

$$
\begin{aligned}
\Phi_{ij} &= \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + h'u_{ij} + h''u_i u_j + \tau d_{ij} \\
&= \frac{w_{ij}}{w} - (h'u_i + \tau d_i)(h'u_j + \tau d_j) + h'u_{ij} + h''u_i u_j + \tau d_{ij} \\
&= \frac{w_{ij}}{w} - \tau h' u_i d_j - \tau h' u_j d_i - \tau^2 d_i d_j + h'u_{ij} + [h'' - (h')^2] u_i u_j + \tau d_{ij}.
\end{aligned}
$$

Following [25], we take the coordinate around x_0 such that (u_{ij}) is diagonal at this point, and all the following calculation will be done at this point. Denoted by F^{ij} the derivative $\frac{\partial\sigma_k(u_{ij})}{\partial u_{ij}}$ and F the sum $\sum_{i=1}^n F^{ii}$. We then have

$$0 \geq F^{ij}\Phi_{ij} = \frac{F^{ij} w_{ij}}{w} + [h'' - (h')^2] F^{ij} u_i u_j + h' F^{ij} u_{ij} + \tau F^{ij} d_{ij} - 2\tau h' F^{ij} u_i d_j - \tau^2 F^{ij} d_i d_j = I + II + III, \tag{13}$$

where

$$I = \frac{F^{ij} w_{ij}}{w},$$
$$II = [h'' - (h')^2] F^{ij} u_i u_j,$$
$$III = h' F^{ij} u_{ij} + \tau F^{ij} d_{ij} - 2\tau h' F^{ij} u_i d_j - \tau^2 F^{ij} d_i d_j.$$

For the last term, we can easily have

$$III = h' F^{ij} u_{ij} + \tau F^{ij} d_{ij} - 2\tau h' F^{ij} u_i d_j - \tau^2 F^{ij} d_i d_j \geq -C|Du|F. \tag{14}$$

In the following, we come to deal with the first term I. The key point is to calculate $F^{ij} w_{ij}$. By a direct calculation, we can deduce that

$$\begin{aligned} w_{ij} &= \left(\frac{u_l}{v} + d_l \cos\theta\right) u_{lij} + \left(\frac{u_l}{v} + d_l \cos\theta\right)_j u_{li} + u_{lj} d_{li} \cos\theta + u_l (d_{li} \cos\theta)_j + u_{lj} d_l (\cos\theta)_i + u_l (d_l (\cos\theta)_i)_j \\ &= \left(\frac{u_l}{v} + d_l \cos\theta\right) u_{lij} + \left(\frac{u_{lj}}{v} - \frac{u_l u_k u_{kj}}{v^3}\right) u_{li} + (d_l \cos\theta)_j u_{li} + u_{lj} d_{li} \cos\theta + u_l (d_{li} \cos\theta)_j + u_{lj} d_l (\cos\theta)_i \\ &\quad + u_l (d_l (\cos\theta)_i)_j. \end{aligned}$$

Hence,

$$\begin{aligned} F^{ij} w_{ij} &= \left(\frac{u_l}{v} + d_l \cos\theta\right) D_l f + \left(\frac{1}{v} - \frac{u_i^2}{v^3}\right) F^{ii} u_{ii}^2 + (d_i \cos\theta)_i F^{ii} u_{ii} \\ &\quad + F^{ii} u_{ii} d_{ii} \cos\theta + F^{ij} u_l (d_{li} \cos\theta)_j + F^{ii} u_{ii} d_i (\cos\theta)_i + F^{ij} u_l (d_l (\cos\theta)_i)_j \\ &\geq \left(\frac{1}{v} - \frac{u_i^2}{v^3}\right) F^{ii} u_{ii}^2 + 2(d_i \cos\theta)_i F^{ii} u_{ii} - C|Du|F - C|Du|. \end{aligned} \tag{15}$$

For the choice of the coordinate and (12), we have at x_0 that

$$-w(h' u_i + \tau d_i) = \left(\frac{u_i}{v} + d_i \cos\theta\right) u_{ii} + u_l (d_l \cos\theta)_i, \quad i = 1, 2, \ldots, n. \tag{16}$$

Setting

$$K = \left\{ i \in I \mid |d_i \cos\theta| + \frac{b}{8n} \leq \left|\frac{u_i}{v}\right| \right\},$$

where $I = \{1, 2, \ldots, n\}$. It is obvious that the index set K is not empty and if we further assume that v is large enough, we can assume that

$$|\tau d_i| \leq \frac{1}{2} h' |u_i|, \ |u_l (d_l \cos\theta)_i| \leq \frac{1}{4} |h' w u_i| \quad \text{for } i \in K.$$

Note that we here need h' have a positive bound, which will be satisfied later. Under these assumptions, we have

$$-C h' w |u_i| \leq u_{ii} \leq 0 \quad \text{for } i \in K. \tag{17}$$

Then for $i \in K$, we have by (3) that

$$F^{ii} \geq F^{kk} \geq CF.$$

Hence,

$$\begin{aligned} F^{ij} w_{ij} &\geq \sum_{i=1}^{n} \left(\left(\frac{1}{v} - \frac{u_i^2}{v^3}\right) F^{ii} u_{ii}^2 - 2(d_i \cos\theta)_i F^{ii} u_{ii} \right) - C|Du|F - C|Du| \\ &= \sum_{i \in K} \left(\left(\frac{1}{v} - \frac{u_i^2}{v^3}\right) F^{ii} u_{ii}^2 - 2(d_i \cos\theta)_i F^{ii} u_{ii} \right) + \sum_{i \notin K} \left(\left(\frac{1}{v} - \frac{u_i^2}{v^3}\right) F^{ii} u_{ii}^2 - 2(d_i \cos\theta)_i F^{ii} u_{ii} \right) \\ &\quad - (C|Du|F + C|Du|) \\ &= T_1 + T_2 + T_3. \end{aligned} \tag{18}$$

For the term T_1, according to (17), we have

$$T_1 = \sum_{i\in K}\left(\left(\frac{1}{v} - \frac{u_i^2}{v^3}\right)F^{ii}u_{ii}^2 - 2(d_i\cos\theta)_iF^{ii}u_{ii}\right) \geq \sum_{i\in K}(-2(d_i\cos\theta)_iF^{ii}u_{ii}) \geq -Cv^2F \tag{19}$$

and for the term T_2, because of the definition of K and the fact $ax^2 + bx \geq -\frac{b^2}{4a}$ for $a > 0$, we have

$$T_2 \geq \sum_{i\notin K}\left(\left(\frac{1}{v} - \frac{u_i^2}{v^3}\right)F^{ii}u_{ii}^2 - 2(d_i\cos\theta)_iF^{ii}u_{ii}\right) \geq \sum_{i\notin K}\left(\frac{C}{vF^{ii}}(F^{ii}u_{ii})^2 - 2(d_i\cos\theta)_iF^{ii}u_{ii}\right) \geq -CvF. \tag{20}$$

It follows that

$$I = \frac{F^{ij}w_{ij}}{w} \geq -CvF - CF - C. \tag{21}$$

For the term II,

$$II = [h'' - (h')^2]F^{ij}u_iu_j = [h'' - (h')^2]\sum_{i=1}^{n}F^{ii}u_i^2 \geq [h'' - (h')^2]\sum_{i\in K}F^{ii}u_i^2 \geq C[h'' - (h')^2]v^2F. \tag{22}$$

By the Newton-MacLaurin inequality stated in Proposition 2.3, we have

$$F \geq C > 0, \tag{23}$$

and therefore,

$$0 \geq \frac{F^{ij}\Phi_{ij}}{F} = I + II + III \geq C[h'' - (h')^2]v^2 - Cv - C - \frac{C}{F} \geq C[h'' - (h')^2]v^2 - Cv - C. \tag{24}$$

If we take $h(t) = \frac{1}{2}\ln\frac{1}{(3M-t)}$, then $h'' - (h')^2 = (h')^2$ and $h(t)$ satisfies all the assumptions we have set in advance. Thus, we bound the gradient at this point such that $v \leq C$, then we derive the gradient estimate near the boundary by a standard discussion. Thus, we complete the proof of Theorem 3.1. □

4 Oblique derivative boundary value

In this section, we will obtain the a priori gradient estimate of the solution to Hessian equations with the oblique derivative boundary value. Specifically, we will show the following result.

Theorem 4.1. *Let Ω be a smooth bounded domain in $\mathbf{R}^n (n \geq 2)$ and u be the admissible solution to the following Hessian equations with the oblique derivative boundary value,*

$$\begin{cases} \sigma_k(u_{ij}) = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \beta} = \varphi(x,u) & \text{on } \partial\Omega, \end{cases} \tag{25}$$

where $f(x,t)$ is a positive smooth function defined on $\Omega \times \mathbf{R}$ with $f_t \geq 0$, $\varphi(x,t)$ is a smooth function defined on $\overline{\Omega} \times \mathbf{R}$ and β is a smooth unit vector field along $\partial\Omega$ with $\langle \beta, \nu\rangle \geq c_0 > 0$ for some positive constant c_0, and ν is denoted to be the inward unit normal along $\partial\Omega$. Also we assume that we have already obtained the C^0 estimate as $|u| \leq M$. Then, there exists a positive constant $C = C(M, n, \Omega, c_0, |\beta|_{C^3(\partial\Omega)}, |f|_{C^1(\Omega\times[-M,M])}, |\varphi|_{C^3(\Omega\times[-M,M])})$ such that

$$|Du| \leq C. \tag{26}$$

Proof. First, we say some words about the boundary value.

Taking a unit normal moving frame along $\partial\Omega$, denoted by $\{e_1, e_2, \ldots, e_{n-1}, \nu\}$, then β can be represented as

$$\beta = \beta_n \nu + \sum_{l=1}^{n-1} \beta_l e_l, \tag{27}$$

where $\beta_n = \langle \beta, \nu \rangle = \cos\theta$, which is bounded from below by the positive constant c_0 according to the conditions of Theorem 4.1.

By the boundary data, we have

$$\varphi(x, u) = \frac{\partial u}{\partial \beta} = \langle Du, \beta \rangle = \frac{\partial u}{\partial \nu}\beta_n + \sum_{l=1}^{n-1} \beta_l u_l. \tag{28}$$

Setting $w = u - \frac{\varphi d}{\cos\theta}$, we then have

$$\varphi(x, u) = \frac{\partial\left(w + \frac{\varphi d}{\cos\theta}\right)}{\partial \nu}\cos\theta + \sum_{l=1}^{n-1} \beta_l \left(w + \frac{\varphi d}{\cos\theta}\right)_l, \tag{29}$$

which indicates that

$$0 = \frac{\partial w}{\partial \nu}\beta_n + \sum_{l=1}^{n-1} \beta_l w_l. \tag{30}$$

Therefore, we have

$$\frac{\partial w}{\partial \nu} = -\sum_{l=1}^{n-1} \frac{\beta_l}{\beta_n} w_l, \tag{31}$$

and it follows by Cauchy inequality and the fact $\sum_{i=1}^{n} \beta_i^2 = 1$ that

$$\left(\frac{\partial w}{\partial \nu}\right)^2 \leq |Dw|^2 \cdot \sin^2\theta. \tag{32}$$

As before, we only need to obtain the gradient estimate near boundary, denoted by Ω_μ, where $\mu \leq \mu_1$ is a positive constant to be determined later. We extend β smoothly to Ω_μ, also denoted by β, such that $\langle \beta, Dd \rangle = \cos\theta \geq c_0$ is also assumed to be still valid. Denote by

$$\phi = |Dw|^2 - \left(\sum_{\alpha=1}^{n} w_\alpha d_\alpha\right)^2 = \sum_{\alpha,\delta=1}^{n} (\delta_{\alpha\delta} - d_\alpha d_\delta) w_\alpha w_\delta = \sum_{\alpha,\delta=1}^{n} C^{\alpha\delta} w_\alpha w_\delta$$

and take the auxiliary function

$$\Phi = \log\phi + h(u) + \tau d,$$

where $h(t)$ is a smooth function and τ is a positive constant. Both of them will be determined later.

Assume the maximum of Φ on Ω_μ is achieved at x_0. Also by the interior gradient estimate, which has been derived in [1], we only need to consider the two following cases.

Case I: $x_0 \in \partial\Omega$.

As in Section 3, we choose a coordinate around x_0 such that $\nu = \frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial x_i} (i = 1, 2, \ldots, n-1)$ are tangent to $\partial\Omega$. We also have that

$$\frac{\partial d}{\partial x_i} = 0, \quad \frac{\partial d}{\partial x_n} = 1, \quad \frac{\partial^2 d}{\partial x_n \partial x_\alpha} = 0, \quad \frac{\partial^2 d}{\partial x_i \partial x_j} = -\kappa_i \delta_{ij},$$

where $1 \leq i, j < n-1$, $1 \leq \alpha \leq n$ and κ_i $(i = 1, 2, \ldots, n-1)$ are the principal curvatures of $\partial\Omega$ at $x_0 \in \partial\Omega$.

By the fact that x_0 is the maximum point of Φ on the boundary, it follows that

$$0 = \Phi_i = \frac{\phi_i}{\phi} + h' u_i, \quad i = 1, 2, \ldots, n-1 \tag{33}$$

and

$$0 \geq \Phi_n = \frac{\phi_n}{\phi} + h' u_n + \alpha d_n = \frac{w_n}{w} + h' u_n + \tau. \tag{34}$$

From (33), we obtain

$$-\phi h' u_i = (|Dw|^2)_i - \left[\left(\sum_{\alpha=1}^{n} w_\alpha d_\alpha\right)^2\right]_i = 2\sum_{j=1}^{n-1} w_{ij} w_j - 2w_n \sum_{j=1}^{n-1} d_{ij} w_j, \quad i = 1, 2, \ldots, n-1. \tag{35}$$

We then deal with the term ϕ_n as follows:

$$\begin{aligned}
\phi_n &= 2\sum_{\alpha=1}^{n} w_\alpha w_{\alpha n} - 2w_n w_{nn} = 2\sum_{i=1}^{n-1} w_i w_{in} = 2\sum_{i=1}^{n-1} w_i w_{ni} + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= -2\sum_{i=1}^{n-1} w_i \left(\frac{\beta_l}{\beta_n} w_l\right)_i + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= -\frac{2\sum_{i,l=1}^{n-1} w_i w_{li} \beta_l}{\beta_n} - 2\sum_{i,l=1}^{n-1} w_i w_l \left(\frac{\beta_l}{\beta_n}\right)_i + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= \frac{\phi h' \sum_{l=1}^{n-1} u_l \beta_l}{\beta_n} - \frac{2w_n \sum_{l,j=1}^{n-1} d_{lj} w_j \beta_l}{\beta_n} - 2\sum_{i,l=1}^{n-1} w_i w_l \left(\frac{\beta_l}{\beta_n}\right)_i + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j.
\end{aligned} \tag{36}$$

Note that the last equality comes from (35), and we denote by κ_{ij} the Weingarten matrix of the boundary with respect to ν.

Therefore, it follows that

$$\begin{aligned}
0 \geq \Phi_n &= \frac{\frac{\phi h' \sum_{l=1}^{n-1} u_l \beta_l}{\beta_n} - \frac{2w_n \sum_{j,l=1}^{n-1} d_{lj} w_j \beta_l}{\beta_n} - 2\sum_{i,l=1}^{n-1} w_i w_l \left(\frac{\beta_l}{\beta_n}\right)_i + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j}{\phi} + h' u_n + \tau \\
&= \frac{-\frac{2w_n \sum_{l,j=1}^{n-1} d_{lj} w_j \beta_l}{\beta_n} - 2\sum_{i,j=1}^{n-1} w_i w_l \left(\frac{\beta_l}{\beta_n}\right)_i + 2\sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j}{\phi} + \frac{h' \varphi}{\cos\theta} + \tau.
\end{aligned} \tag{37}$$

We may assume in advance that

$$0 < h'(t) < 1, \quad \forall t \in [-M, M]. \tag{38}$$

Thus, if we set τ large enough, depending on c_0, $|\beta|_{C^1(\partial\Omega)}$, n and the geometry of $\partial\Omega$, we can conclude that this case does not occur at all.

Case II: $x_0 \in \Omega_\mu$.

All the calculations will proceed at this point, and the Einstein summation convention will be adopted during all the calculations if no otherwise specified. Also, we denoted by F^{ij} the derivative $\frac{\partial \sigma_k(u_{ij})}{\partial u_{ij}}$ and F the sum $\sum_{i=1}^{n} F^{ii}$.

According to [1], we know that

$$\sup_\Omega |Du| \leq C_1 \left(1 + \sup_{\partial\Omega} |Du|\right), \tag{39}$$

where C_1 is a positive constant depending only on Ω, n, k, $|D_x f|_{C^0(\Omega\times[-M,M])}$. One can verify this point by setting a auxiliary function $\chi = \log|Du|^2 + \alpha|x|^2$ and checking that $F^{ij}\chi_{ij} \geq 0$ once we set α to be small and $|Du|$ to be large enough. Remark that we have supposed with out loss of generality that the point 0 is located out of $\overline{\Omega}$.

Now we assume that the maximum value of $|Du|$ on $\partial\Omega$ is achieved at the point x_1, without loss of generality, we can suppose that

$$|Du|^2(x_1) \geq 4\sup_{\partial\Omega}\left(\left|\frac{\varphi}{\cos\theta}\right|\right)^2, \tag{40}$$

otherwise we have finish the estimate of the gradient of the solutions.

By the fact that $\Phi(x_0) \geq \Phi(x_1)$, it follows that

$$\begin{aligned}\phi(x_0) &\geq C(\tau,\mu)e^{-2Mh'}\phi(x_1) = C(\tau,\mu)e^{-2Mh'}\left[|Dw|^2 - \left(\frac{\partial w}{\partial\nu}\right)^2\right](x_1)\\ &\geq C(\tau,\mu)e^{-2Mh'}[|Dw|^2\cos^2\theta](x_1)\\ &\geq {c_0}^2C(\tau,\mu)e^{-2Mh'}|Dw|^2(x_1)\\ &= {c_0}^2C(\tau,\mu)e^{-2Mh'}|Du - \frac{\phi}{\cos\theta}\nu|^2(x_1)\\ &\geq \frac{{c_0}^2C(\tau,\mu)e^{-2Mh'}}{4}|Du|^2(x_1),\end{aligned} \tag{41}$$

remark that the last inequality above comes from (40) and the fact that $(x-y)^2 \geq \frac{x^2}{2} - y^2$.

Joining with (39) and assuming once again that

$$0 < h'(t) < \frac{1}{2M}, \quad \forall t \in [-M, M], \tag{42}$$

we then derive

$$\begin{aligned}\phi(x_0) &\geq \frac{c_0C(\tau,\mu)e^{-2Mh'}}{4C_1}\left(\sup_\Omega|Du|^2 - C_1\right)\\ &\geq \frac{c_0C(\tau,\mu)e^{-2Mh'}}{8C_1}\sup_\Omega|Du|^2\\ &\geq \frac{c_0C(\tau,\mu)e^{-2Mh'}}{8C_1}|Du|^2(x_0)\\ &\geq \frac{c_0C(\tau,\mu)}{9C_1e}|Dw|^2(x_0) \triangleq C_0|Dw|^2(x_0).\end{aligned} \tag{43}$$

Without the loss of generality, we can assume that $C_0 \in (0,1)$.

At x_0, we also follow [25] to choose the coordinate such that (u_{ij}) is diagonal.

For $k = 1,2,\ldots,n$, denote by $T_k = \sum_{l=1}^n C^{kl}w_l$ and $\overrightarrow{T} = (T_1, T_2, \ldots, T_n)$, it is obvious to observe that $|\overrightarrow{T}| \leq |Dw|$ and

$$\phi = \sum_{i,j=1}^n C^{ij}w_iw_j = \sum_{j=1}^n T_jw_j = \langle\overrightarrow{T}, Dw\rangle. \tag{44}$$

Considering the lower bound we just derived in (43), we obtain

$$C_0|Dw| \leq |T| \leq |Dw|. \tag{45}$$

Without the loss of generality, we further assume by the Pigeon-Hole Principle that

$$T_1w_1 \geq \frac{C_0}{n}|Dw|^2, \tag{46}$$

and therefore,

$$\frac{w_1}{T_1} \geq \frac{C_0}{n}, \tag{47}$$

and we can set μ is small such that

$$\frac{u_1}{T_1} \geq \frac{C_0}{3n}. \tag{48}$$

By a direct calculation, we have

$$\begin{aligned} w_i &= u_i\left(1 - \frac{\varphi_z d}{\cos\theta}\right) + \varphi_i\left(\frac{d}{\cos\theta}\right) + \varphi\left(\frac{d}{\cos\theta}\right)_i; \\ w_{ij} &= u_{ij}\left(1 - \frac{\varphi_z d}{\cos\theta}\right) - \frac{\varphi_{zz} d}{\cos\theta}u_i u_j - \varphi_{zj}u_i\left(\frac{d}{\cos\theta}\right) - \varphi_z u_i\left(\frac{d}{\cos\theta}\right)_j + \varphi_z u_j\left(\frac{d}{\cos\theta}\right)_i \\ &\quad + \frac{d}{\cos\theta}\varphi_{ij} + \varphi_{zi}u_j\left(\frac{d}{\cos\theta}\right) + \left(\frac{d}{\cos\theta}\right)_i \varphi_j + \left(\frac{d}{\cos\theta}\right)_j \varphi_i + \left(\frac{d}{\cos\theta}\right)_{ij}\varphi. \end{aligned} \tag{49}$$

By the assumption that x_0 is the maximum point, we then have $\Phi_i = 0$ for $i = 1, 2, \ldots, n$, and it follows that

$$\frac{\phi_i}{\phi} + h'u_i + \tau d_i = 0, \tag{50}$$

especially for $i = 1$,

$$\sum_{l=1}^{n} T_l w_{l1} = -\frac{\phi}{2}(h'u_1 + \tau d_1) - \sum_{k,l=1}^{n} \frac{C^{kl}{}_{,1}}{2} w_k w_l, \tag{51}$$

then by (45)–(48), we have

$$u_{11}\left(1 - \frac{\varphi_z d}{\cos\theta}\right) \leq -\frac{u_1}{2T_1}h'\phi + Cd|Dw|^2 + C|Dw|. \tag{52}$$

If we assume that h' has a positive lower bound and $|Dw|$ is large enough, and μ is small enough, then we can obtain

$$u_{11} < 0, \tag{53}$$

and thus,

$$F^{11} \geq F^{kk} \geq C(n,k)F. \tag{54}$$

Now, it is turn for us to deal with the second order derivatives of Φ. With the help of the first-order condition (50), it follows that

$$\begin{aligned} \Phi_{ij} &= \frac{\left(\sum_{k,l=1}^{n} C^{kl} w_k w_l\right)_{ij}}{\phi} - (h'u_i + \tau d_i)(h'u_j + \tau d_j) + h'u_{ij} + h''u_i u_j + \tau d_{ij} \\ &= \frac{\left(\sum_{k,l=1}^{n} C^{kl} w_k w_l\right)_{ij}}{\phi} - h'u_i d_j - h'u_j d_i - \tau^2 d_i d_j + h'u_{ij} + [h'' - (h')^2]u_i u_j + \tau d_{ij}. \end{aligned} \tag{55}$$

Hence, we have at x_0 that

$$
\begin{aligned}
0 \ge F^{ij}\Phi_{ij} &= \frac{F^{ij}\Big(\sum_{k,l=1}^{n} C^{kl} w_k w_l\Big)_{ij}}{\phi} - 2h' \sum_{i,j=1}^{n} F^{ij} u_i d_j - \tau^2 \sum_{i,j=1}^{n} F^{ij} d_i d_j + h'kf \\
&\quad + [h'' - (h')^2] \sum_{i,j=1}^{n} F^{ij} u_i u_j + \tau \sum_{i,j=1}^{n} F^{ij} d_{ij} \\
&\ge \sum_{i,j=1}^{n} \frac{F^{ij}(C^{kl} w_k w_l)_{ij}}{\phi} - (\tau^2 + 1) \sum_{i,j=1}^{n} F^{ij} d_i d_j + h'kf + [h'' - 2(h')^2] \sum_{i,j=1}^{n} F^{ij} u_i u_j + \tau \sum_{i,j=1}^{n} F^{ij} d_{ij} \\
&= I + II + III + IV + V.
\end{aligned} \tag{56}
$$

It is a simple and direct calculation to deal with the last four terms. According to (45)–(48) and (54), we have

$$
\begin{aligned}
II &= -(\tau^2 + 1) F^{ij} d_i d_j \ge -(\tau^2 + 1) F, \\
III &= h'kf \ge 0, \\
IV &= [h'' - 2(h')^2] \sum_{i,j=1}^{n} F^{ij} u_i u_j \ge [h'' - 2(h')^2] F^{11} u_1^2 \ge C_2 [h'' - 2(h')^2] |Dw|^2 F, \\
V &= \tau F^{ij} d_{ij} \ge -k_0 \tau \sum_{i=1}^{n} F^{ii} = -k_0 \tau F,
\end{aligned} \tag{57}
$$

where k_0 is a positive constant related to the geometry of $\partial\Omega$.

To deal with the term I, we have

$$
\begin{aligned}
I &= \frac{\sum_{i,j,k,l=1}^{n} F^{ij} C^{kl}{}_{,ij} w_k w_l}{\phi} + \frac{2\sum_{i,j,k,l=1}^{n} F^{ij} C^{kl} w_{ijk} w_l}{\phi} + \frac{4\sum_{i,j,k,l=1}^{n} F^{ij} C^{kl}{}_{,j} w_{ik} w_l}{\phi} + \frac{2\sum_{i,j,k,l=1}^{n} F^{ij} C^{kl} w_{ik} w_{jl}}{\phi} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{58}
$$

We consider these four terms one by one in the following text.

For the term I_1, it is easy to deduce that

$$
I_1 = \frac{\sum_{i,j,k,l=1}^{n} F^{ij} C^{kl}{}_{,ij} w_k w_l}{\phi} \ge -CF. \tag{59}
$$

For the term I_2, we need a subtle operation as follows:

$$
\begin{aligned}
\phi I_2 &= 2 \sum_{i,j,k,l=1}^{n} F^{ij} C^{kl} w_{ijl} w_k \\
&= 2 \sum_{i,j,l=1}^{n} F^{ij} T_l \left(u - \frac{\varphi(x,u) d}{\cos\theta} \right)_{ijl} \\
&= 2 \sum_{i,j,l=1}^{n} T_l \left[F^{ij} u_{ijl} + F^{ij} \left(\frac{\varphi(x,u) d}{\cos\theta} \right)_{ijl} \right] \\
&= 2 \sum_{l=1}^{n} T_l \left[f_l + \sum_{i,j=1}^{n} F^{ij} \left(\frac{\varphi(x,u) d}{\cos\theta} \right)_{ijl} \right].
\end{aligned} \tag{60}
$$

To proceed, we should compute $\left(\frac{\varphi(x,u)d}{\cos\theta}\right)_{ijl}$. By a direct calculation,

$$
\begin{aligned}
\left(\frac{\varphi(x,u) d}{\cos\theta} \right)_{ijl} &= (\varphi)_{ijl} \left(\frac{d}{\cos\theta} \right) + (\varphi)_{ij} \left(\frac{d}{\cos\theta} \right)_l + (\varphi)_{il} \left(\frac{d}{\cos\theta} \right)_j + (\varphi)_{jl} \left(\frac{d}{\cos\theta} \right)_i \\
&\quad + (\varphi)_i \left(\frac{d}{\cos\theta} \right)_{jl} + (\varphi)_j \left(\frac{d}{\cos\theta} \right)_{il} + (\varphi)_l \left(\frac{d}{\cos\theta} \right)_{ij} + \varphi \left(\frac{d}{\cos\theta} \right)_{ijl},
\end{aligned} \tag{61}
$$

where

$$\begin{aligned}(\varphi)_i &= \varphi_i + \varphi_z u_i,\\ (\varphi)_{ij} &= (\varphi_i + \varphi_z u_i)_j = \varphi_{ij} + \varphi_{iz}u_j + \varphi_{zj}u_i + \varphi_{zz}u_iu_j + \varphi_z u_{ij},\\ (\varphi)_{ijl} &= (\varphi_{ij} + \varphi_{iz}u_j + \varphi_{zj}u_i + \varphi_{zz}u_iu_j + \varphi_z u_{ij})_l\\ &= \varphi_{ijl} + \varphi_{ijz}u_l + \varphi_{izl}u_j + \varphi_{izz}u_ju_l + \varphi_{iz}u_{lj} + \varphi_{zjl}u_i + \varphi_{zjz}u_iu_l + \varphi_{zj}u_{il}\\ &\quad + \varphi_{zzl}u_iu_j + \varphi_{zzz}u_iu_ju_l + \varphi_{zz}u_{li}u_j + \varphi_{zz}u_iu_{lj} + \varphi_{zl}u_{ij} + \varphi_{zz}u_lu_{ij} + \varphi_z u_{ijl}.\end{aligned} \tag{62}$$

Note that

$$\sum_{i,j=1}^{n} F^{ij}u_{ij} = kf, \quad \sum_{i,j=1}^{n} F^{ij}u_{ijl} = D_l f, \quad \sum_{j=1}^{n} F^{ij}u_{lj} = F^{ii}u_{li} \quad (\text{fixed } i), \quad 0 < \sum_{j=1}^{n} F^{ij}u_iu_j \le |Du|^2 F,$$

and therefore, we have

$$\phi I_2 \ge -Cd|Dw|^4F - C|Dw|^3F - Cd|Dw|^2\sum_{i=1}^{n}|F^{ii}u_{ii}| - C|Dw|\sum_{i=1}^{n}|F^{ii}u_{ii}| - C|Dw|^2. \tag{63}$$

Almost the same procedure, we can settle the remained two terms.

$$\phi I_3 = 4\sum_{i,j,p,l=1}^{n} F^{ij}C^{pl}{}_{,j}w_{ip}w_l \ge 2\sum_{i,l=1}^{n} w_lC^{il}{}_{i}F^{ii}u_{ii} - C|Dw|^3F \ge -C|Dw|^3F - C|Dw|\sum_{i=1}^{n}|F^{ii}u_{ii}|, \tag{64}$$

and

$$\phi I_4 = 2\sum_{i,j,p,l=1}^{n} F^{ij}C^{pl}w_{ip}w_{jl} \ge 2\sum_{i=1}^{n}F^{ii}C^{ii}u_{ii}{}^2 - Cd|Dw|^2\sum_{i=1}^{n}|F^{ii}u_{ii}| - Cd|Dw|^4F - C|Dw|^3F. \tag{65}$$

Taking into account (59), (63), (64), and (65), we can obtain

$$\phi I \ge 2\sum_{i=1}^{n}F^{ii}C^{ii}u_{ii}{}^2 - Cd|Dw|^2\sum_{i=1}^{n}|F^{ii}u_{ii}| - C|Dw|\sum_{i=1}^{n}|F^{ii}u_{ii}| - Cd|Dw|^4F - C|Dw|^3F - C|Dw|^2. \tag{66}$$

Denoting by

$$H = 2\sum_{i=1}^{n}F^{ii}C^{ii}u_{ii}{}^2 - Cd|Dw|^2\sum_{i=1}^{n}|F^{ii}u_{ii}| - C|Dw|\sum_{i=1}^{n}|F^{ii}u_{ii}|, \tag{67}$$

and we will bound H from below in the following.

Let $C^{i_0i_0}$ be the smallest of $\{C^{ii}\}_{i=1}^{n}$, without the loss of generality, we can assume $i_0 = 1$. Then, we have $C^{ii} \ge \frac{1}{2}$ for any $i \ge 2$, otherwise it follows that $\sum_{i=1}^{n} C^{ii} < 1 + (n-2) = n-1$, which contradicts with $\sum_{i=1}^{n} C^{ii} = n-1$. Then by the equation, we can obtain

$$F^{11}u_{11} = kf - \sum_{\alpha=2}^{n} F^{\alpha\alpha}u_{\alpha\alpha}, \tag{68}$$

and therefore, we have by the simple fact $ax^2 + bx \ge -\frac{b^2}{4a}$ if $a > 0$ that

$$H \ge \sum_{i=2}^{n}F^{ii}u_{ii}^2 - C(d|Dw|^2 + |Dw|)\sum_{i=2}^{n}F^{ii}|u_{ii}| - kf \ge -C\sum_{i=1}^{n}F^{ii}(d|Dw|^2 + |Dw|)^2 - C. \tag{69}$$

Plugging this into (66) and joining with (43), we can derive

$$I \ge -Cd|Dw|^2F - C|Dw|F. \tag{70}$$

Therefore, combining (57) and (70), we can obtain

$$0 \ge \frac{F^{ij}\Phi_{ij}}{F} \ge C_2[h'' - 2(h')^2]|Dw|^2 - C_3d|Dw|^2 - C|Dw|, \tag{71}$$

where we use once again the fact $F \ge C > 0$.

Now, we set

$$h(t) = \frac{1}{4}\ln\frac{1}{(3M - t)}, \tag{72}$$

and it satisfies all the assumptions we have made in advance. Let μ be small enough so that $C_3\mu \leq C_2(h')^2$, we then obtain

$$C_2\left(\frac{1}{16M}\right)^2|Dw|^2 - C|Dw| \leq 0, \tag{73}$$

and this will lead to the universal bound of $|Dw|$ at x_0 and we then obtain the global gradient estimate of u on $\overline{\Omega}$ by a standard discussion, and this finishes the whole proof of Theorem 4.1. □

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