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Research Article

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The regularity of weak solutions for certain *n*-dimensional strongly coupled parabolic systems

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Abstract: This paper is concerned with the n-dimensional strongly coupled parabolic systems with triangular form in the cylinder $\Omega \times (0, T]$. We investigate L^2 and Hölder regularity of the derivatives of weak solutions (u_1, u_2) for the systems in the following two cases: one is that the boundedness of u_1 and u_2 has not been shown in existence result of solutions; the other is that the boundedness of u_1 or u_2 has been shown in existence result of solutions. By using difference ratios and Steklov averages methods and various estimates, we prove that if (u_1, u_2) is a weak solution of the system, then for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, u_1, u_2 belong to $C^{\alpha', \alpha'/2}(\bar{\Omega}' \times [t', T])$ and $W_2^{2,1}(\Omega' \times (t', T])$ under certain conditions, and u_1, u_2 belong to $C^{2+\alpha', 1+\alpha'/2}(\bar{\Omega}' \times [t', T])$ under stronger assumptions. Applications of these results are given to two ecological models with cross-diffusion.

Keywords: strongly coupled parabolic systems, regularity, weak solutions, triangular systems

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1 Introduction

Strongly coupled elliptic and parabolic systems often appear in different fields of physics, chemistry, biology, ecology, and engineering sciences (see [5,8,11,21,25–27,29] and the references therein). In the past three decades, they have been given considerable attention in the literature in both theory and applications. The two important issues are the existence and regularity of solutions. The papers in [1–3, 5–8,10,15,18–20,25,28,30] are concerned with the existence of weak and classical solutions, and those in [14–17,22,23] are for the regularity of weak solutions.

Local existence (in time) of solutions to the strongly coupled parabolic systems with boundary conditions and initial conditions was established by Amann in a series of important papers [1–3] under the conditions that the boundary of the domain and the functions making up the system are smooth. There are extensive literature using the results of [1–3]. Hence, the solutions obtained in them have good regularity (see [6, 19, 23, 28] and the references therein). In other literature, since the functions making up the strongly coupled parabolic systems do not satisfy the conditions in [1–3], the results of [1–3] cannot be used, and the solutions obtained in these studies are weak solutions (see [6,8,10,11,26]). Then it is meaningful to investigate the regularity of the weak solutions. In previous studies [22,23], applying Gagliardo-Nirenberg type inequalities, the regularity of weak solutions for the one-dimensional cross-diffusion systems in population dynamics was studied by Shim, and the uniform W_2^1-bound of the solutions was obtained.

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In a series of papers [14–17], assuming different additional structure conditions, the regularity of weak solutions for the n-dimensional strongly coupled parabolic systems was investigated by Le. The Hölder regularity of bounded solutions was obtained in [14] by the perturbation method, and imbedding theorems of Campanato-Morrey spaces, the Hölder regularity of BMO weak solutions was proved in [16] by using the nonlinear heat approximation and BMO preserving homotopy, and the partial regularity results for bounded weak solutions were established in [17] by the method of heat approximation. Besides, some sufficient conditions on the structure of the systems to guarantee the boundedness and Hölder continuity of weak solutions were found in [15].

Strongly coupled parabolic systems are very difficult to analyze (see [4]). Some counterexamples show that we cannot expect weak solutions for general strongly coupled systems to be regular everywhere (see [14,24]). Therefore, to establish Hölder regularity for the derivatives of weak solutions, we consider strongly coupled systems of special form such as triangular systems. In addition, we note that in some model problems, the systems are of triangular form. For example, consider a two-species ecological model with cross-diffusion and self-diffusion on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, \dots)$. Let $u_1 = u_1(x, t)$ and $u_2 = u_2(x, t)$ be the population densities of the two species, respectively, where $x = (x_1, \dots, x_n)$. Assume that the cross-diffusion pressures of the first species is zero. According to [3, 21], the nonnegative vector function (u_1, u_2) is governed by parabolic system in the following form:

$$\begin{cases} u_{1t} = \Delta[(\kappa_1(x,t) + \gamma_{11}(x,t)u_1)u_1] + \operatorname{div}[e_1(x,t)u_1\nabla\varphi] + [d_{11}(x,t) + d_{12}(x,t)u_1 + d_{13}(x,t)u_2]u_1 & ((x,t) \in D_T), \\ u_{2t} = \Delta[(\kappa_2(x,t) + \gamma_{21}(x,t)u_1 + \gamma_{22}(x,t)u_2)u_2] + \operatorname{div}[e_2(x,t)u_2\nabla\varphi] + [d_{21}(x,t) + d_{22}(x,t)u_1 + d_{23}(x,t)u_2]u_2 & ((x,t) \in D_T), \end{cases} \tag{1.0}$$

where $D_T := \Omega \times (0, T]$ for $T > 0$, $u_{lt} := \partial u_l / \partial t$, ∇ is the gradient operator, and Δ is the Laplace operator, and where $\gamma_{21}(x, t)$ is the cross-diffusion rate of the second species, $\kappa_l(x, t)$, $\gamma_{ll}(x, t)$ are the diffusion rate and self-diffusion rate of the l-species, respectively, and φ is a known outer potential. This system is of triangular form because the cross-diffusion terms occur only in the second equation, and therefore, the diffusion matrix is triangular. In general, the known functions κ_l, γ_{ll}, γ_{21}, e_l, and d_{lj} $(l = 1, 2;\ j = 1, 2, 3)$ are allowed to be dependent on x and t.

Motivated by the aforementioned ecological models, in this paper, we consider a class of n-dimensional strongly coupled parabolic systems in the following form:

$$u_{lt} = \operatorname{div}\left[\sum_{j=1}^{l} a_{lj}(x, t, [\mathscr{U}]_l)\nabla u_j\right] + b_l(x, t, \mathscr{U}, \nabla[\mathscr{U}]_l) \quad ((x, t) \in D_T),\ \ l = 1, 2, \tag{1.1}$$

where $\mathscr{U} := (u_1, u_2)$, $[\mathscr{U}]_1 := u_1$, $[\mathscr{U}]_2 := (u_1, u_2)$, $\nabla[\mathscr{U}]_1 := \nabla u_1$ and $\nabla[\mathscr{U}]_2 := (\nabla u_1, \nabla u_2)$.

For the triangular systems, sufficient conditions for the global existence of solutions were established in the previous work [2], the existence of solutions for some model problems was obtained in [9,18,20,28], and the Hölder regularity of bounded solutions and the uniform boundedness of global solutions were investigated in [14,23]. In [26], to study a free boundary problem describing S-K-T competition ecological model, by using Steklov average method and L^∞ and Hölder estimates in [12], we investigated L^2 and Hölder regularity of the derivatives of weak solutions for two special one-dimensional triangular systems (see [26, Lemmas 3.5, 4.1]).

In this paper, by using difference ratios and Steklov averages methods and various estimates, we will prove that if $\mathscr{U} = (u_1, u_2)$ is a weak solution from $V_2^{1,0}(D_T)$ to (1.1), then for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, u_1, u_2 belong to $C^{\alpha', \alpha'/2}(\bar{\Omega}' \times [t', T])$ and $W_2^{2,1}(\Omega' \times (t', T])$ under certain conditions, and u_1, u_2 belong to $C^{2+\alpha', 1+\alpha'/2}(\bar{\Omega}' \times [t', T])$ under stronger assumptions. We investigate the regularity in the following two cases: one is that the boundedness of u_1 and u_2 on $\bar{D}_T$ has not been shown in existence result of solutions (see [6, Theorem 1.1], [8, Theorem 2.1], [10, Theorem 1], and [26, Proposition 3.1]); the other is that the boundedness of u_1 or u_2 has been shown in existence result of solutions (see [10, Theorem 2], [11, Theorem 1.1], and [26, Proposition 3.8]). According to the form of the equations in (1.1), to establish the regularity of (u_1, u_2), we first study the regularity of u_1 from the first equation of (1.1), and then apply the obtained result to investigate the regularity of u_2 from the second equation.

This paper is organized as follows: in Section 2, we give the definitions, hypotheses, notations, main theorems, and some preliminaries. Sections 3 and 4 are devoted to the proofs of the two main theorems, respectively. Finally, in Section 5, we give applications of the aforementioned results to two ecological models with cross-diffusion.

2 Hypotheses, main results, and preliminaries

2.1 Definitions, hypotheses, and main results

In this paper, we follow the function space notations adopted in [12]. The symbol $\Omega^* \subset\subset \Omega$ means that $\Omega^* \subset \Omega$ and $\bar{\Omega}^* \subset \Omega$, where $\bar{\Omega}^*$ denotes the closure of Ω^*.

Since we study only interior regularity of solutions, then the weak solutions of system (1.1) are defined as follows:

Definition 2.1. A vector function $\mathscr{U} = (u_1, u_2)$ is called a weak solution of system (1.1) if for each $l = 1, 2$, $u_l \in V_2^{1,0}(D_T)$, and $u_{lt} \in L^2([0, T]; (H^1(\Omega))^*)$, and if $\mathscr{U}$ satisfies

$$\int_{\tau'}^{\tau} \langle u_{lt}, \eta\rangle \mathrm{d}t + \int_{\tau'}^{\tau}\int_{\Omega'} \left\{\left[\sum_{j=1}^{l} a_{lj}(x, t, [\mathscr{U}]_l)\nabla u_j\right] \cdot \nabla\eta - b_l(x, t, \mathscr{U}, \nabla[\mathscr{U}]_l)\eta\right\} \mathrm{d}x\mathrm{d}t = 0, \quad l = 1, 2, \tag{2.1}$$

for any $\tau', \tau \in (0, T]$, $\Omega' \subset\subset \Omega$, and for any $\eta \in \overset{\circ}{V}{}_2^{1,0}(\Omega' \times (\tau', T])$, where $\langle\cdot,\cdot\rangle$ denotes the duality parring between $H^1(\Omega')$ and its dual space $(H^1(\Omega'))^*$.

In the following discussions, we always use $\mathscr{U} = (u_1, u_2)$ to denote a nonconstant weak solution of (1.1) given by existence result of solutions. In addition, for each $l = 1, 2$, we define a interval S_l corresponding to function u_l. S_l is the closure of $(\underline{a}_l, \bar{a}_l)$, where

$$\begin{cases} \underline{a}_l = \operatorname*{ess\,inf}_{D_T} u_l & \text{if it is known that } u_l \text{ is bounded in } D_T \text{ from below}, \\ \underline{a}_l = -\infty & \text{if it is not known that } u_l \text{ is bounded in } D_T \text{ from below}, \end{cases}$$

and

$$\begin{cases} \bar{a}_l = \operatorname*{ess\,sup}_{D_T} u_l & \text{if it is known that } u_l \text{ is bounded in } D_T \text{ from above}, \\ \bar{a}_l = +\infty & \text{if it is not known that } u_l \text{ is bounded in } D_T \text{ from above}. \end{cases}$$

Set

$$\begin{aligned} &\mathcal{S} := S_1 \times S_2, \quad [\mathcal{S}]_1 := S_1, \quad [\mathcal{S}]_2 := S_1 \times S_2, \\ &\mathscr{W} := (w_1, w_2), \quad [\mathscr{W}]_1 := w_1, \quad [\mathscr{W}]_2 := (w_1, w_2), \\ &\mathbf{p} := (p_1, \ldots, p_n), \quad \mathbf{p}_k := (p_{k1}, \ldots, p_{kn}), \quad \mathbb{P} := (\mathbf{p}_1, \mathbf{p}_2), \quad [\mathbb{P}]_1 := \mathbf{p}_1, \quad [\mathbb{P}]_2 := (\mathbf{p}_1, \mathbf{p}_2), \end{aligned}$$

and set

$$q_0 := \frac{2(2+n)}{n}, \quad q_1 := \frac{2+n}{2(1-\chi_1)} \tag{2.2}$$

for some $\chi_1 \in (0, 1/4)$. Then (q_0, q_0) satisfies [12, Chapter II, equality (3.3)] with $q = r = q_0$, and (q_1, q_1) satisfies [12, Chapter III, equality (7.2)] with $q = r = q_1$. Specially, $q_0 = 6$ and $q_1 = 3/[2(1-\chi_1)] \in (3/2, 2)$ for $n = 1$.

To investigate the regularity of the weak solutions, we will make the hypotheses (H_1) and (H_2):

(H1)(I) For $l = 1, 2$, $i = 1, \ldots, n$, assume that $a_{ll}(x, t, [\mathscr{W}]_l) \in C^1(D_T \times [\mathcal{S}]_l)$, $a_{llx_ix_i}(x, t, [\mathscr{W}]_l)$, $\in C(D_T \times [\mathcal{S}]_l)$, and that $a_{22x_iw_1}(x, t, \mathscr{W})$, $a_{22w_1x_i}$, $a_{22w_1w_1} \in C(D_T \times \mathcal{S})$. In addition, for $(x, t) \in D_T$, $\mathscr{W} \in \mathcal{S}$ and $[\mathbb{P}]_l \in (\mathbb{R}^n)^l$, functions $a_{21}(x,t,\mathscr{W})$ and $b_l(x,t,\mathscr{W},[\mathbb{P}]_l)$ have the first partial derivatives with respect to their variables in a pointwise sense.

(II)

(a) If S_1 and S_2 are all unbounded, we assume that for $(x,t)\in D_T$, $\mathscr{W}\in\mathcal{S}$, and $\mathbb{P}\in(\mathbb{R}^n)^2$,

$$|a_{21}(x,t,\mathscr{W})|\le\mu(0),\quad \nu(|[\mathscr{W}]_{l-1}|)\le a_{ll}(x,t,[\mathscr{W}]_l)\le\mu(|[\mathscr{W}]_{l-1}|),\quad l=1,2 \tag{2.3}$$

and

$$|b_l(x,t,\mathscr{W},[\mathbb{P}]_l)|\le\mu(|[\mathscr{W}]_{l-1}|)\left[(1+|w_l|^{\delta_l})\sum_{k=1}^{l}|\mathbf{p}_k|+\sum_{m=l}^{2}|w_m|^{\sigma_{lm}}\right]+\phi_l(x,t),\quad l=1,2, \tag{2.4}$$

where $\nu(\theta)$ is a positive nonincreasing continuous function for $\theta\ge0$ and $\mu(\theta)$ is a positive nondecreasing continuous function for $\theta\ge0$ and $|[\mathscr{W}]_{l-1}|=0$ for $l=1$, where δ_l, σ_{ll}, and σ_{12} are all nonnegative constants satisfying

$$\delta_l\le2(1-\chi_1)/n,\quad \sigma_{ll}\le4(1-\chi_1)/n,\quad l=1,2, \tag{2.5}$$

$$\sigma_{12}\le3\quad\text{for } n=1,\quad \sigma_{12}\le2(1-\chi_1)/n\quad\text{for } n=2,3,\dots, \tag{2.6}$$

and $\phi_l(x,t)$ is nonnegative function with the finite norm

$$\begin{cases}\|\phi_1\|_{L^{2q_1}(D_T)}\le\varsigma,\quad \|\phi_2\|_{L^{q_1}(D_T)}+\|\phi_2\|_{L^2(D_T)}\le\varsigma,\\ \sup\limits_{0\le t\le T}\|\phi_1,\quad \phi_2\|_{L^{q_2}(\Omega)}\le\varsigma.\end{cases} \tag{2.7}$$

Here, q_2, ς are positive constants with $q_2=1$ for $n=1$, and $q_2>n$ for $n=2,3,\dots$.

(b) For any fixed $l\in\{1,2\}$, if there exists nonempty set $E\subset\{l,2\}$ such that for each $m\in E$, S_m is bounded, we assume that the conditions in (a) are satisfied except that in (2.4) for $b_l(x,t,\mathscr{W},[\mathbb{P}]_l)$, there is no item $|w_m|^{\sigma_{lm}}$ for each $m\in E$, there is no item $|w_l|^{\delta_l}$ if $l\in E$, and $\mu(|[\mathscr{W}]_{l-1}|)$ is replaced by $\mu(|[\mathscr{W}]_{l-1}|+\sum_{m\in E}|w_m|)$.

(III)

(A) If S_2 is unbounded, assume that for $(x,t)\in D_T$, $\mathscr{W}\in\mathcal{S}$ and $\mathbb{P}\in(\mathbb{R}^n)^2$,

$$|a_{21x_i}(x,t,\mathscr{W})|+|a_{21w_k}|\le\mu(|\mathscr{W}|),\quad i=1,\dots,n,\ k=1,2, \tag{2.8}$$

and

$$\begin{cases}\left|\dfrac{\partial}{\partial x_i}b_l(x,t,\mathscr{W},[\mathbb{P}]_l)\right|\le\mu(|[\mathscr{W}]_l|)\left[\sum_{k=1}^{l}|\mathbf{p}_k|+(2-l)|w_2|^{\tilde\sigma_{11}}+\tilde\phi_{l1}(x,t)\right],\\ \left|\dfrac{\partial}{\partial w_l}b_l\right|\le\mu(|[\mathscr{W}]_l|)\left[\sum_{k=1}^{l}|\mathbf{p}_k|+(2-l)|w_2|^{\tilde\sigma_{12}}+\tilde\phi_{l2}(x,t)\right],\\ \left|\dfrac{\partial}{\partial w_r}b_l\right|\le\mu(|[\mathscr{W}]_l|)\left[\sum_{k=1}^{l}|\mathbf{p}_k|+(2-l)|w_2|^{\tilde\sigma_{13}}+\tilde\phi_{l3}(x,t)\right],\quad r=1,2,\ r\ne l,\\ \left|\dfrac{\partial}{\partial p_{ki}}b_l\right|\le\mu(|[\mathscr{W}]_l|)[(2-l)|w_2|^{\tilde\sigma_{14}}+\tilde\phi_{l4}(x,t)],\quad i=1,\dots,n,\ k=1,l;\ l=1,2,\end{cases} \tag{2.9}$$

where $\tilde\sigma_{1r}$ are nonnegative constants satisfying

$$\tilde\sigma_{11},\tilde\sigma_{12}\le4(1-\chi_1)/n,\quad \tilde\sigma_{13},\tilde\sigma_{14}\le2(1-\chi_1)/n, \tag{2.10}$$

and where $\tilde\phi_{lk}$ are nonnegative functions satisfying

$$\begin{cases} \sup\limits_{0\le t\le T} \|\tilde{\phi}_{11}^{1/2}, \tilde{\phi}_{12}^{1/2}, \tilde{\phi}_{13}, \tilde{\phi}_{14}\|_{L^{\tilde{q}_2}(\Omega)} \le \tilde{\varsigma} \quad \text{for } n = 2, 3, \ldots, \\ \sup\limits_{0\le t\le T} \|\tilde{\phi}_{21}^{1/2}, \tilde{\phi}_{22}^{1/2}, \tilde{\phi}_{23}^{1/2}, \tilde{\phi}_{24}\|_{L^{\tilde{q}_2}(\Omega)} \le \tilde{\varsigma} \quad \text{for } n = 1, 2, \ldots. \end{cases} \tag{2.11}$$

Here, $\tilde{q}_2$ and $\tilde{\varsigma}$ are also positive constants with $\tilde{q}_2 = 1$ for $n = 1$, and $\tilde{q}_2 > n$ for $n = 2, 3, \ldots$.

(B) If S_2 is bounded, we assume that the conditions in (A) are satisfied except that in (2.9) with $l = 1$, there are no items $|w_2|^{\tilde{\sigma}_{1r}}$, $r = 1, \ldots, 4$, and $\mu(|[\mathscr{W}]_1|)$ is replaced by $\mu(|\mathscr{W}|)$.

(H2) For any $\Omega' \subset\subset \Omega$, $t' \in (0, T]$ and for any bounded domains $\mathcal{S}' = S_1' \times S_2' \subset \mathcal{S}$ and $\mathbb{E}' \subset (\mathbb{R}^n)^l$, there exists $\alpha_0' \in (0, 1)$, such that $a_{ljx_i}(x, t, [\mathscr{W}]_l)$, $a_{ljw_k} \in C^{\alpha_0'}(\bar{\Omega}' \times [t', T] \times [\bar{\mathcal{S}}']_l)(i = 1, \ldots, n;\ j, k = 1, l)$, and $b_l(x, t, \mathscr{W}, [\mathbb{P}]_l) \in C^1(\bar{\Omega}' \times [t', T] \times \bar{\mathcal{S}}' \times \bar{\mathbb{E}}')$.

The main results of this paper are the following two theorems:

Theorem 2.1. *Let $\mathscr{U} = \mathscr{U}(x, t)$ be a weak solution of system* (1.1). *If hypothesis* (H$_1$) *holds, then for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, u_1 and u_2 are Hölder continuous in $\bar{\Omega}' \times [t', T]$, u_1 belongs to $W^{2,1}_{q_3}(\Omega' \times (t', T])$, and u_2 belongs to $W^{2,1}_2(\Omega' \times (t', T])$, where $q_3 = 2q_1$ if $n \ge 2$, $q_3 = \min\{2q_1, 6\}$ if $n = 1$.*

Theorem 2.2. *Let $\mathscr{U} = \mathscr{U}(x, t)$ be a weak solution of system* (1.1). *If hypotheses* (H$_1$) *and* (H$_2$) *all hold, then for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, there exists $\alpha' \in (0, 1)$, such that u_1 and u_2 belong to $C^{2+\alpha', 1+\alpha'/2}(\bar{\Omega}' \times [t', T])$.*

2.2 Some more notations and preliminaries

We introduce some more functions and notations used throughout the paper. For function $w = w(x, t)$, we denote

$$w^{(\sigma)} := \max\{w - \sigma, 0\}, \quad \mathcal{A}_{w,\sigma}(t) := \{x : w(x, t) > \sigma,\ x \in \Omega\}, \tag{2.12}$$

$$w_h(x, t) := \frac{1}{h} \int_t^{t+h} w(x, \theta) \mathrm{d}\theta, \quad w_{(t)} := \frac{w(x, t + h) - w(x, t)}{h}, \tag{2.13}$$

and

$$w_{(x_k)}(x, t) := \frac{w(x + h_k, t) - w(x, t)}{h_k}, \tag{2.14}$$

where $k \in \{1, \ldots, n\}$ and $x + h_k := (x_1, \ldots, x_{k-1}, x_k + h_k, x_{k+1}, \ldots, x_n)$. Then $w_h(x, t)$ is the Steklov average of w in t, and $w_{(t)}$, $w_{(x_k)}$ are difference ratios. The expression $[f(x, t, w(x, t), \nabla w(x, t))]_{x_i}$ means that

$$[f(x, t, w(x, t), \nabla w(x, t))]_{x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial w} w_{x_i} + \sum_{j=1}^n \frac{\partial f}{\partial w_{x_j}} w_{x_j x_i}.$$

Let

$$\begin{cases} \Lambda_1(\mathscr{U}) := |u_2|^{\tilde{\sigma}_{11}/2} + |u_2|^{\tilde{\sigma}_{12}/2} + |u_2|^{\tilde{\sigma}_{13}} + |u_2|^{\tilde{\sigma}_{14}}, \quad F_l(\mathscr{U}) := \sum\limits_{m=l}^{2} |u_m|^{\sigma_{lm}}, \quad l = 1, 2, \\ \tilde{\Phi}_1(x, t) := \tilde{\phi}_{11}^{1/2} + \tilde{\phi}_{12}^{1/2} + \tilde{\phi}_{13} + \tilde{\phi}_{14}, \quad \tilde{\Phi}_2(x, t) := \tilde{\phi}_{21}^{1/2} + \tilde{\phi}_{22}^{1/2} + \tilde{\phi}_{23}^{1/2} + \tilde{\phi}_{24}, \end{cases} \tag{2.15}$$

and let

$$\tilde{q}_0 := \frac{2q_0}{q_0 - 2} = n + 2, \quad \hat{q}_0 := \frac{q_0}{q_0 - 1} = \frac{2n + 4}{n + 4}. \tag{2.16}$$

For $\rho > 0$ and $\bar{P} := (\bar{x}, \bar{t}) \in D_T$, we denote

$$B_\rho(\bar{x}) := \{x : |x - \bar{x}| < \rho\}, \quad Q_{\rho,\tau}(\bar{P}) := B_\rho(\bar{x}) \times (\bar{t} - \tau, \bar{t}), \quad Q_\rho(\bar{P}) := Q_{\rho,\rho^2}(\bar{P}).$$

We define a function $\xi_\rho = \xi_\rho(x, t)$ corresponding to the two cylinders $Q_\rho(\bar{P})$ and $Q_{2\rho}(\bar{P})$ as follows:

$$\begin{cases} \xi_\rho \text{ is a smooth function taking values in } [0, 1], \text{ such that } \xi_\rho = 0 \text{ on the lateral and} \\ \text{base of } Q_{2\rho}(\bar{P}),\ \xi_\rho = 1 \text{ for } (x, t) \in Q_\rho(\bar{P}) \text{ and } |\xi_{\rho t}| + |\nabla \xi_\rho|^2 \le C/\rho^2 \text{ for all } (x, t) \in Q_{2\rho}(\bar{P}). \end{cases} \tag{2.17}$$

In the following discussions, let $t' \in (0, T)$ be an arbitrary fixed number, and let $\Omega' \subset\subset \Omega$ be an arbitrary fixed domain. Set $t_j = jt'/20$, $j = 1, \dots, 20$. Choose subdomains Ω_j with smooth boundaries, such that $\Omega' \subset\subset \Omega_{20} \subset\subset \dots \subset\subset \Omega_1 \subset\subset \Omega$, $\operatorname{dist}(\Omega_j, \partial\Omega_{j-1}) = d := \operatorname{dist}(\Omega', \partial\Omega)/20$. Denote

$$D_{j,T} := \Omega_j \times (t_j, T], \quad \bar{D}_{j,T} := \bar{\Omega}_j \times [t_j, T], \quad D_{j,\tau} := \Omega_j \times (t_j, \tau].$$

For each $j = 1, \dots, 20$, we define function $\lambda_j = \lambda_j(x, t)$ as follows:

$$\begin{cases} \lambda_j \text{ is a smooth function with values between 0 and 1, such that } \lambda_j = 0 \text{ for } x \notin \Omega_{j-1} \\ \text{or } t \le t_{j-1},\ \lambda_j = 1 \text{ for } (x, t) \in \bar{D}_{j,T} \text{ and } |\nabla \lambda_j| + |\lambda_{jt}| \le C(d, t') \text{ for all } (x, t) \in \bar{D}_T. \end{cases} \tag{2.18}$$

Lemma 2.3. *There exist positive constants C and $\alpha_0 \in (0, 1)$ such that for each $l = 1, 2$,*

$$\|u_l\|_{L^{q_0}(D_T)} \le C\|u_l\|_{V_2^{1,0}(D_T)}, \tag{2.19}$$

$$\||u_l|^{\delta_l}\|_{L^{\tilde{q}_0}(D_T)} + \|F_l(\mathscr{U})\|_{L^{\hat{q}_0}(D_T)} \le C, \quad \||u_l|^{2\delta_l}, F_l(\mathscr{U}), |u_2|^{\sigma_{12}}\|_{L^{q_1}(D_T)} \le C, \tag{2.20}$$

$$\||u_2|^{\sigma_{12}}\|_{L^2(D_T)} \le C, \tag{2.21}$$

and for $n \ge 2$ and any $B_\rho(\bar{x}) \subset \Omega$,

$$\int_{B_\rho(\bar{x})} [|u_2|^{2\sigma_{12}} + \Lambda_1^2(\mathscr{U}) + \phi_l^2 + \tilde{\Phi}_l^2]\mathrm{d}x \le C\rho^{n-2+2\alpha_0}. \tag{2.22}$$

Proof. It follows from [12, Chapter II, inequality (3.4)] that (2.19) holds. By a direct computation, we see from (2.2) and (2.16) that

$$\min\{q_0/\tilde{q}_0, q_0/(2q_1)\} = 2(1 - \chi_1)/n, \quad \min\{q_0/\hat{q}_0, q_0/q_1\} = 4(1 - \chi_1)/n.$$

Thus, by (2.5) and (2.6), we further obtain the relations

$$2\sigma_{12} \le q_0, \quad 2\delta_l q_1, \delta_l \tilde{q}_0 \le q_0, \quad \sigma_{lm} q_1, \sigma_{lm} \hat{q}_0 \le q_0 \quad \text{for } m = l, 2.$$

These, together with (2.15) and (2.19), give (2.20) and (2.21).

We next prove that (2.22) holds for $n \ge 2$. Conditions (2.6) and (2.7) show that $1 - n/q_2 > 0$ and $2 - n\sigma_{12} \ge 2\chi_1 > 0$. Hence,

$$\int_{B_\rho(\bar{x})} \phi_l^2 \mathrm{d}x \le \|\phi_l^2\|_{L^{q_2/2}(B_\rho(\bar{x}))} \left(\int_{B_\rho(\bar{x})} \mathrm{d}x \right)^{1-2/q_2} \le C\rho^{n-2+2(1-n/q_2)},$$

$$\int_{B_\rho(\bar{x})} |u_2|^{2\sigma_{12}} \mathrm{d}x \le \|u_2\|^{2\sigma_{12}}_{L^2(B_\rho(\bar{x}))} \left(\int_{B_\rho(\bar{x})} \mathrm{d}x \right)^{1-\sigma_{12}} \le C\rho^{n-2+(2-n\sigma_{12})}.$$

Moreover, by (2.15), (2.10), and (2.11), $\int_{B_\rho(\bar{x})} \Lambda_1^2(\mathscr{U})\mathrm{d}x$ and $\int_{B_\rho(\bar{x})} \tilde{\Phi}_l^2 \mathrm{d}x$ satisfy the similar inequalities. Setting $\alpha_0 := \min\{\chi_1, 1 - n/q_2, 1 - n/\tilde{q}_2\}$, we obtain (2.22). □

The following preliminary lemma will be used to investigate L^∞ estimates.

Lemma 2.4. *Assume that w belongs to $V_2(D_T)$ and $\hat{\sigma}$ is a nonnegative constant. Then the following statements hold true:*

(*i*) *Let $Q_{\rho,\tau}(\bar{P}) \subset D_T$ be an arbitrary cylinder, and let $\zeta = \zeta(x,t)$ be an arbitrary piecewise-smooth continuous nonnegative function that does not exceed 1 and is equal to zero on the lateral surface and the lower base of the cylinder $Q_{\rho,\tau}(\bar{P})$. Assume that for any τ_1, τ_2, and σ, $\bar{t} - \tau < \tau_1 < \tau_2 < \bar{t}$, $\sigma \geq \hat{\sigma}$,*

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau_1 \le t \le \tau_2} \|w^{(\sigma)}(x,t)\zeta(x,t)\|^2_{L^2(\Omega)} + \underline{\varsigma} \int_{\tau_1}^{\tau_2} \int_{\mathscr{A}_{w,\sigma}(t)} |\nabla w|^2 \zeta^2 \mathrm{d}x\mathrm{d}t \\ &\le \|w^{(\sigma)}(x,\tau_1)\zeta(x,\tau_1)\|^2_{L^2(\Omega)} + \int_{\tau_1}^{\tau_2} \int_{\mathscr{A}_{w,\sigma}(t)} \{\bar{\varsigma}(|\nabla\zeta|^2 + \zeta|\zeta_t|)(w^{(\sigma)})^2 + \mathscr{G}(x,t)\zeta^2[(w-\sigma)^2 + \sigma^2]\}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.23}$$

where $\underline{\varsigma}$ and $\bar{\varsigma}$ are positive constants, and function $\mathscr{G}(x,t)$ is in $L^{q,r}(D_T)$ for some q, r satisfying [12, *Chapter III, equality* (7.2)]. *Then for any $\Omega'' \subset\subset \Omega$ and $t'' \in (0,T)$, $\operatorname{ess\,sup}_{\Omega'' \times (t'',T]} w$ does not exceed a constant $\tilde{M}$ determined only by T, $\hat{\sigma}$, $\|\mathscr{G}(x,t)\|_{L^{q,r}(D_T)}$, $\|w\|_{L^2(D_T)}$, t'' and $\operatorname{dist}(\Omega'', \partial\Omega)$.*

(*ii*) *Assume that for any $\tau \in (0,T]$,*

$$\int_\Omega (w^{(\sigma)}(x,\tau))^2 \mathrm{d}x + \int_0^\tau \int_{\mathscr{A}_{w,\sigma}(t)} |\nabla w|^2 \mathrm{d}x\mathrm{d}t \le \int_0^\tau \int_{\mathscr{A}_{w,\sigma}(t)} \mathscr{G}(x,t)[(w-\sigma)^2 + \sigma^2]\mathrm{d}x\mathrm{d}t, \tag{2.24}$$

where function $\mathscr{G}(x,t)$ is in $L^{q,r}(D_T)$ for some q, r satisfying [12,*Chapter III, equality* (7.2)]. *Then $\operatorname{ess\,sup}_{D_T} w$ does not exceed a constant $\tilde{M}$ determined only by T, $\hat{\sigma}$ and $\|\mathscr{G}(x,t)\|_{L^{q,r}(D_T)}$.*

Proof. By inequality (2.23), we can conclude that w satisfies [12, Chapter III, inequality (8.2)]. The deduction is the same as that of [12, Chapter III, inequality (7.14)] from [12, Chapter III, inequality (7.8)]. Using [12, Chapter II, Theorem 6.2 and Remark 6.4], we further obtain the result of part (i) of this lemma.

Since inequality (2.24) has the same property as that of [12, Chapter III, inequality (7.8)], then by [12, Chapter II, Theorem 6.1 and Remark 6.2], the similar proof as that of [12, Chapter III, estimate (7.15)] gives the result of part (ii) of this lemma. □

3 The proof of Theorem 2.1

In this section, assume that hypothesis (H_1) holds. We will only prove Theorem 2.1 for the case that S_1 and S_2 are all unbounded, because the proofs of Theorem 2.1 for the other cases are more simple.

3.1 Hölder estimate of u_1

For simplicity, in this section, we will use M_1, M_2, C, $C(\cdots)$, C_j and $\alpha_j (j = 1, 2, \ldots)$ to denote positive constants depending only on the parameters

$$\begin{cases} T,\ \text{mes}\ \Omega,\ \operatorname{dist}(\Omega', \partial\Omega),\ t',\ \nu(0),\ \mu(0),\ \varsigma,\ \tilde{\varsigma},\ \chi_1,\ n,\ q_2,\ \tilde{q}_2, \\ \|u_l\|_{V_2^{1,0}(D_T)},\ \delta_l,\ \sigma_{ll},\ \sigma_{12},\ \tilde{\sigma}_{1r},\ r = 1,\ldots,4,\ l = 1,2, \end{cases} \tag{3.1}$$

and the quantities appearing in parentheses. In the same lemma, the same letter C will be used to denote different constants depending on the same set of arguments. We first give the Hölder estimate of u_1.

Lemma 3.1. *The integrals in the first equality of* (2.1) *exist and are finite. There exist positive constants C and* α_1 *such that*

$$\|u_1\|_{C^{\alpha_1,\alpha_1/2}(\bar{D}_{2,T})} \le C, \quad \alpha_1 \in (0,1). \tag{3.2}$$

Proof. Step 1. We show that the integrals in the first equality of (2.1) exist and are finite. For any $\tau \in (t_1, T]$ and $\eta \in \overset{\circ}{V}{}_2^{1,0}(D_{1,\tau})$, using condition (2.4) and [12, Chapter II, inequalities (1.5) and (3.4)], we have

$$\begin{aligned}
&\iint_{D_{1,\tau}} |b_1(x,t,\mathscr{U},\nabla[\mathscr{U}]_1)\eta| \mathrm{d}x\mathrm{d}t \\
&\le \iint_{D_{1,\tau}} \{\mu(0)[(1+|u_1|^{\delta_1})|\nabla u_1| + F_1(\mathscr{U})] + \phi_1(x,t)\}|\eta|\mathrm{d}x\mathrm{d}t \\
&\le C\{[\||u_1|^{\delta_1}\|_{L^{\tilde{q}_0}(D_{1,T})}\|\nabla u_1\|_{L^2(D_{1,T})} + \|F_1(\mathscr{U})\|_{L^{\hat{q}_0}(D_{1,T})}]\|\eta\|_{L^{q_0}(D_{1,T})} + [\|\nabla u_1\|_{L^2(D_{1,T})} + \|\phi_1\|_{L^2(D_{1,T})}]\|\eta\|_{L^2(D_{1,T})}\},
\end{aligned}$$

where $F_1(\mathscr{U})$, $\tilde{q}_0$, and $\hat{q}_0$ are defined by (2.15) and (2.16). Therefore, by (2.20) and (2.7), $\iint_{D_{1,\tau}} |b_1\eta|\mathrm{d}x\mathrm{d}t$ is finite. Besides, it follows from condition (2.3) that $\int_{t_1}^{\tau} \langle u_{1t}, \eta\rangle \mathrm{d}t + \iint_{D_{1,\tau}} a_{11}(x,t,[\mathscr{U}]_1)\nabla u_1 \cdot \nabla\eta \mathrm{d}x\mathrm{d}t$ is also finite. Since $t_1 = t'/20$ is an arbitrary fixed number in $(0,T)$ and Ω_1 is an arbitrary subdomain of Ω, then the integrals in the first equality of (2.1) exist and are finite.

Step 2. We prove that

$$\operatorname*{ess\,sup}_{D_{1,T}} |u_1| \le M_1. \tag{3.3}$$

Let cylinder $Q_{\rho,\tau}(\bar{P})$ and function $\zeta = \zeta(x,t)$ be same as those in Lemma 2.4, and let $\hat{\sigma} \ge 1$. Setting $\eta = u_1^{(\sigma)}\zeta^2(x,t)$ for $\sigma \ge \hat{\sigma}$ in the first equality of (2.1) yields, for any τ_1 and τ^*, $\bar{t} - \tau < \tau_1 \le \tau^* < \bar{t}$,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega} (u_1^{(\sigma)}\zeta)^2 \mathrm{d}x \Big|_{t=\tau_1}^{t=\tau^*} + \int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} a_{11}(x,t,u_1)|\nabla u_1|^2\zeta^2 \mathrm{d}x\mathrm{d}t \\
&= \int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} \{-a_{11}(x,t,u_1)u_1^{(\sigma)}\nabla u_1 \cdot (2\zeta\nabla\zeta) + (u_1^{(\sigma)})^2\zeta\zeta_t + b_1(x,t,\mathscr{U},\nabla u_1)u_1^{(\sigma)}\zeta^2\}\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.4}$$

where notations $u_1^{(\sigma)}$ and $\mathcal{A}_{u_1,\sigma}$ are defined by (2.12). By using conditions (2.3), (2.4), and Cauchy's inequality with ε, we find from (3.4) that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega} (u_1^{(\sigma)}\zeta)^2 \mathrm{d}x \Big|_{t=\tau_1}^{t=\tau^*} + \int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} a_{11}(x,t,u_1)|\nabla u_1|^2\zeta^2 \mathrm{d}x\mathrm{d}t \\
&\le C\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} \{(u_1^{(\sigma)})^2\zeta|\zeta_t| + |\nabla u_1|u_1^{(\sigma)}\zeta|\nabla\zeta| + [(1+|u_1|^{\delta_1})|\nabla u_1| + F_1(\mathscr{U}) + \phi_1]u_1^{(\sigma)}\zeta^2\}\mathrm{d}x\mathrm{d}t \\
&\le \varepsilon\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} |\nabla u_1|^2\zeta^2 \mathrm{d}x\mathrm{d}t + \frac{C}{\varepsilon}\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_1,\sigma}(t)} \{(u_1^{(\sigma)})^2(\zeta|\zeta_t| + |\nabla\zeta|^2) + [(1+|u_1|^{2\delta_1})(u_1^{(\sigma)})^2 + (F_1(\mathscr{U}) \\
&\quad + \phi_1)u_1^{(\sigma)}]\zeta^2\}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Note that $|u_1 - \sigma| \le \max\{1, (u_1-\sigma)^2\}$ and $\sigma \ge 1$. Then

$$u_1^{(\sigma)} \le |u_1 - \sigma| \le (u_1 - \sigma)^2 + \sigma^2.$$

In view of $a_{11}(x, t, u_1) \ge \nu(0)$, taking $\varepsilon = \min\{1/2, \nu(0)/2\}$, we further obtain, for any $\bar{t} - \tau < \tau_1 < \tau_2 < \bar{t}$,

$$\sup_{\tau_1 \le t \le \tau_2} \|u_1^{(\sigma)}(x, t)\zeta(x, t)\|^2_{L^2(\Omega)} + \frac{\nu(0)}{2} \int_{\tau_1}^{\tau_2} \int_{\mathcal{A}_{u_1,\sigma}(t)} |\nabla u_1|^2 \zeta^2 dx dt$$
$$\le \|u_1^{(\sigma)}(x, \tau_1)\zeta(x, \tau_1)\|^2_{L^2(\Omega)} + \int_{\tau_1}^{\tau_2} \int_{\mathcal{A}_{u_1,\sigma}(t)} \{\mathcal{G}_1(x, t)\zeta^2[(u_1 - \sigma)^2 + \sigma^2] + C(|\nabla\zeta|^2 + \zeta|\zeta_t|)(u_1^{(\sigma)})^2\} dx dt,$$

where $\mathcal{G}_1(x, t) = C[1 + |u_1|^{2\delta_1} + F_1(\mathcal{U}) + \phi_1]$. It follows from (2.7) and (2.20) that $\|\mathcal{G}_1(x, t)\|_{L^{q_1}(D_T)}$ is bounded from above by C. Then u_1 satisfies inequality (2.23) with w replaced by u_1. Lemma 2.4 shows that $\operatorname{ess\,sup}_{D_{1,T}} u_1$ does not exceed a constant C. The similar argument implies that $\operatorname{ess\,sup}_{D_{1,T}}(-u_1)$ also does not exceed C. Hence, (3.3) holds.

Step 3. We show Hölder estimate (3.2). Set

$$B_{1,i}(x, t, w, \mathbf{p}) := a_{11}(x, t, w)p_i, \quad B_1(x, t, w, \mathbf{p}) := -b_1(x, t, w, u_2, \mathbf{p}). \tag{3.5}$$

It follows from the first integral equality of (2.1) that for any $\tau_0, \tau \in (t_1, T]$ and for any $\eta \in \overset{\circ}{W}{}^{1,1}_2(D_{1,T})$,

$$\int_{\Omega_1} u_1(x, t)\eta(x, t) dx \Bigg|_{\tau_0}^{\tau} + \int_{\tau_0}^{\tau} \int_{\Omega_1} \left[-u_1\eta_t + \sum_{i=1}^{n} B_{1,i}(x, t, u_1, \nabla u_1)\eta_{x_i} + B_1(x, t, u_1, \nabla u_1)\eta \right] dx dt = 0. \tag{3.6}$$

From estimate (3.3) and conditions (2.3) and (2.4), we obtain, for $(x, t) \in D_{1,T}$, $w \in [-M_1, M_1] \cap S_1$ and $\mathbf{p} \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} B_{1,i}(x, t, w, \mathbf{p})p_i \ge \nu(0)|\mathbf{p}|^2, \quad |B_{1,i}(x, t, w, \mathbf{p})| \le \mu(0)|\mathbf{p}|, \tag{3.7}$$

and

$$|B_1(x, t, w, \mathbf{p})| \le C(|\mathbf{p}| + |u_2|^{\sigma_{12}} + \phi_1 + 1). \tag{3.8}$$

In view of (2.7) and (2.20), this equality has the same form as [12, Chapter V, equality (1.6)]. Then by using (3.3), (3.6)–(3.8), and [12, Chapter V, Theorem 1.1], we obtain estimate (3.2). □

3.2 Estimate of $\|u_1\|_{W_2^{2,1}(\Omega' \times (t', T])}$

Based on Hölder estimate of u_1, we will prove that u_1 has weak derivative u_{1t} by using the Steklov average method. In view of hypothesis (H$_1$)(I), we obtain, for some positive constant Θ_1,

$$\|a_{11}(x, t, w)\|_{C^1(\Xi_1)}, \quad \|a_{11x_ix_i}(x, t, w)\|_{C(\Xi_1)} \le \Theta_1, \quad i = 1, \dots, n, \tag{3.9}$$

where $\Xi_1 := \bar{D}_{1,T} \times (S_1 \cap [-M_1, M_1])$.

Lemma 3.2. *Function u_1 has weak derivative u_{1t} in $D_{3,T}$, and satisfies the inequality*

$$\operatorname*{ess\,sup}_{t_3 \le t \le T} \int_{\Omega_3} |\nabla u_1|^2 dx + \iint_{D_{3,T}} u_{1t}^2 dx dt \le C(\Theta_1). \tag{3.10}$$

Proof. Let $x_2 \in \Omega_2$ be fixed, and let $u_1^* = u_1(x_2, t_2)$. Define

$$\hat{w}_1 = \hat{w}_1(x,t) = \int_{u_1^*}^{u_1} a_{11}(x,t,\omega)\mathrm{d}\omega. \tag{3.11}$$

Then

$$\hat{w}_{1x_i} = u_{1x_i} a_{11}(x,t,u_1) + \int_{u_1^*}^{u_1} a_{11x_i}(x,t,\omega)\mathrm{d}\omega, \quad i = 1,\dots,n. \tag{3.12}$$

By using integration by parts, we see from the first equality of (2.1) that for any $\eta \in \overset{\circ}{W}{}_2^{1,1}(D_{2,T})$ and $\tau \in (t_2, T]$,

$$\int_{\Omega_2} u_1 \eta \mathrm{d}x \Bigg|_{t_2}^{\tau} - \iint_{D_{2,\tau}} u_1 \eta_t \mathrm{d}x\mathrm{d}t = \iint_{D_{2,\tau}} [-\nabla \hat{w}_1 \cdot \nabla \eta + \hat{f}_1(x,t)\eta]\mathrm{d}x\mathrm{d}t, \tag{3.13}$$

where

$$\hat{f}_1(x,t) = -\sum_{i=1}^{n} \left[\int_{u_1^*}^{u_1} a_{11x_i}(x,t,\omega)\mathrm{d}\omega \right]_{x_i} + b_1(x,t,\mathscr{U},\nabla u_1). \tag{3.14}$$

From the similar arguments as those of [12, Chapter III, Section 2], it follows that for any given $h \in (0, T - t_3)$ and $\tau \in (t_2, T - h]$ and for any $\eta \in \overset{\circ}{V}{}_2^{1,0}(D_{2,T})$,

$$\iint_{D_{2,\tau}} u_{1(t)} \eta \mathrm{d}x\mathrm{d}t = \iint_{D_{2,\tau}} \{-\nabla \hat{w}_{1h} \cdot \nabla \eta + [\hat{f}_1(x,t)]_h \eta\}\mathrm{d}x\mathrm{d}t. \tag{3.15}$$

Here and below, notations w_{1h} and $w_{1(t)}$ are defined by (2.13). Note that by (3.11),

$$\hat{w}_{1(t)} = u_{1(t)} \int_0^1 a_{11}(x,t^\vartheta,u_1^\vartheta)\mathrm{d}\vartheta + \int_0^1 \left[\int_{u_1^*}^{u_1^\vartheta} a_{11t^\vartheta}(x,t^\vartheta,\omega)\mathrm{d}\omega \right]\mathrm{d}\vartheta,$$

where $t^\vartheta = t + \vartheta h$ and $\mathscr{U}^\vartheta = \vartheta \mathscr{U}(x,t+h) + (1-\vartheta)\mathscr{U}(x,t)$. Thus,

$$u_{1(t)} = \left\{ \hat{w}_{1(t)} - \int_0^1 \left[\int_{u_1^*}^{u_1^\vartheta} a_{11t^\vartheta}(x,t^\vartheta,\omega)\mathrm{d}\omega \right]\mathrm{d}\vartheta \right\} \Bigg/ \int_0^1 a_{11}(x,t^\vartheta,u_1^\vartheta)\mathrm{d}\vartheta. \tag{3.16}$$

Let the vertex $\bar{P} = (\bar{x}, \bar{t})$ of cylinders $Q_\rho(\bar{P})$ and $Q_{2\rho}(\bar{P})$ be in $D_{3,T-h}$ for $h \in (0, T - t_3)$, and let $\rho \le \rho_1 := \min\{d/4, \sqrt{I}/4, 1\}$, where $I := t'/20$. Thus, $Q_{2\rho}(\bar{P}) \subset D_{2,T}$. Function ξ_ρ is defined by (2.17). Setting $\eta = \hat{w}_{1(t)}\xi_\rho^2$ in (3.15), noting that $\hat{w}_{1(t)} = (\hat{w}_{1h})_t$, and using (3.16), we deduce that

$$\begin{aligned} &\iint_{Q_{2\rho}(\bar{P})} \left[\int_0^1 a_{11}(x,t^\vartheta,u_1^\vartheta)\mathrm{d}\vartheta \right]^{-1} \hat{w}_{1(t)}^2 \xi_\rho^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_{B_{2\rho}(\bar{x})} |\nabla \hat{w}_{1h}(x,\bar{t})|^2 \xi_\rho^2(x,\bar{t})\mathrm{d}x \\ &= \iint_{Q_{2\rho}(\bar{P})} \int_0^1 \int_{u_1^*}^{u_1^\vartheta} a_{11t^\vartheta}(x,t^\vartheta,\omega)\mathrm{d}\omega\mathrm{d}\vartheta \cdot \left[\int_0^1 a_{11}(x,t^\vartheta,u_1^\vartheta)\mathrm{d}\vartheta \right]^{-1} \hat{w}_{1(t)}\xi_\rho^2 \mathrm{d}x\mathrm{d}t \\ &\quad + \iint_{Q_{2\rho}(\bar{P})} \{-2\hat{w}_{1(t)}\xi_\rho \nabla \hat{w}_{1h} \cdot \nabla \xi_\rho + |\nabla \hat{w}_{1h}|^2 \xi_\rho \xi_{\rho t} + [\hat{f}_1(x,t)]_h \hat{w}_{1(t)}\xi_\rho^2\}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.17}$$

By (3.2), (3.9), and Cauchy's inequality with ε, from (3.17), we obtain, for any $\varepsilon \in (0,1)$,

$$\iint\limits_{Q_{2\rho}(\bar{P})}\left[\int_0^1 a_{11}(x,t^\vartheta,u_1^\vartheta)\mathrm{d}\vartheta\right]^{-1}\hat{w}_{1(t)}^2\xi_\rho^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int\limits_{B_{2\rho}(\bar{x})}|\nabla\hat{w}_{1h}(x,\bar{t})|^2\xi_\rho^2(x,\bar{t})\mathrm{d}x$$
$$\leq \varepsilon\iint\limits_{Q_{2\rho}(\bar{P})}\hat{w}_{1(t)}^2\xi_\rho^2\mathrm{d}x\mathrm{d}t+\frac{C}{\varepsilon}\iint\limits_{Q_{2\rho}(\bar{P})}\{|\nabla\hat{w}_{1h}|^2(|\nabla\xi_\rho|^2+|\xi_\rho\xi_{\rho t}|)+[\hat{f}_1(x,t)]_h^2\xi_\rho^2+\xi_\rho^2\}\mathrm{d}x\mathrm{d}t.$$

Using (2.3) and choosing $\varepsilon=\min\{1/2,1/(2\mu(0))\}$, we further have

$$\iint\limits_{Q_{2\rho}(\bar{P})}\hat{w}_{1(t)}^2\xi_\rho^2\mathrm{d}x\mathrm{d}t+\int\limits_{B_{2\rho}(\bar{x})}|\nabla\hat{w}_{1h}(x,\bar{t})|^2\xi_\rho^2(x,\bar{t})\mathrm{d}x\leq C\iint\limits_{Q_{2\rho}(\bar{P})}\{[|\nabla\hat{w}_{1h}|^2(|\nabla\xi_\rho|^2+|\xi_\rho\xi_{\rho t}|)+[\hat{f}_1(x,t)]_h^2\xi_\rho^2+\xi_\rho^2\}\mathrm{d}x\mathrm{d}t. \tag{3.18}$$

In addition, it follows from (3.12), (3.14), (2.4), (3.2), and (3.9) that

$$|\nabla\hat{w}_1|^2\leq C(1+|\nabla u_1|^2),\quad \hat{f}_1^2(x,t)\leq C\Big(|\nabla u_1|^2+|u_2|^{2\sigma_{12}}+\phi_1^2+1\Big)\quad ((x,t)\in D_{2,T}). \tag{3.19}$$

Hence, using (2.21) and (2.7) yields that $|\nabla\hat{w}_1|$, $\hat{f}_1(x,t)\in L^2(D_{2,T})$. Moreover, by [12, Chapter II, Lemma 4.7],

$$\|\nabla\hat{w}_{1h}\|_{L^2(D_{2,T})}+\|[\hat{f}_1(x,t)]_h\|_{L^2(D_{2,T})}\leq C.$$

Therefore, inequality (3.18) and the definition of function ξ_ρ yield

$$\iint\limits_{Q_\rho(\bar{P})}\hat{w}_{1(t)}^2\mathrm{d}x\mathrm{d}t+\int\limits_{B_\rho(\bar{x})}|\nabla\hat{w}_{1h}|^2(x,\bar{t})\mathrm{d}x\leq C/\rho^2.$$

Hence,

$$\iint\limits_{D_{3,T-h}}\hat{w}_{1(t)}^2\mathrm{d}x\mathrm{d}t+\sup_{t_3\leq t\leq T-h}\int\limits_{\Omega_3}|\nabla\hat{w}_{1h}|^2\mathrm{d}x\leq C,$$

where C is independent of h. [12, Chapter II, Lemma 4.11] further shows that $\hat{w}_1$ has weak derivative $\hat{w}_{1t}$ in $D_{3,T}$ and satisfies

$$\operatorname*{ess\,sup}_{t_3\leq t\leq T}\int\limits_{\Omega_3}|\nabla\hat{w}_1|^2(x,t)\mathrm{d}x+\iint\limits_{D_{3,T}}\hat{w}_{1t}^2\mathrm{d}x\mathrm{d}t\leq C. \tag{3.20}$$

Then u_1 has weak derivative $u_{1t}=[\hat{w}_{1t}-\int_{u_1^*}^{u_1}a_{11t}(x,t,\omega)\mathrm{d}\omega]/a_{11}(x,t,u_1)$, and estimate (3.20), together with (3.2) and (3.9), leads to estimate (3.10). □

To prove that u_1 has the second derivatives with respect to x, we need the following lemma:

Lemma 3.3. *Let $n\geq 2$, and let $\bar{P}\in D_{5,T}$, $\rho\leq\rho_2:=\rho_1/2$. If $\zeta=\zeta(x,t)$ is an arbitrary bounded function from $\overset{\circ}{V}_2(Q_\rho(\bar{P}))$, then there exist positive constants $C=C(\Theta_1)$ and $\alpha_2=\alpha_2(\Theta_1)$, $\alpha_2\in(0,1)$, such that*

$$\iint\limits_{Q_\rho(\bar{P})}[|\nabla u_1|^2+|u_2|^{2\sigma_{12}}+\Lambda_1^2(\mathscr{U})+\phi_1^2+\tilde{\Phi}_1^2]\zeta^2\mathrm{d}x\mathrm{d}t\leq C\rho^{2\alpha_2}\iint\limits_{Q_\rho(\bar{P})}|\nabla\zeta|^2\mathrm{d}x\mathrm{d}t. \tag{3.21}$$

Proof. We divide the proof into three steps.

Step 1. We prove that if $\bar{P}\in D_{4,T}$ and $\rho\leq\rho_1$, then for some $\tilde{\alpha}\in(0,1)$,

$$\iint\limits_{Q_\rho(\bar{P})}|\nabla u_1|^2\mathrm{d}x\mathrm{d}t\leq C\rho^{n+2\tilde{\alpha}}. \tag{3.22}$$

We see that $Q_{2\rho}(\bar{P}) \subset D_{3,T}$. Let (x^*, t^*) be a given point in $Q_\rho(\bar{P})$. Choosing $\eta = (u_1(x,t) - u_1(x^*, t^*))\xi_\rho^2$ and $\tau = \bar{t}$ in the first equality of (2.1) yields

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}(\bar{x})} (u_1(x,\bar{t}) - u_1(x^*, t^*))^2\xi_\rho^2 dx + \iint_{Q_{2\rho}(\bar{P})} a_{11}(x,t,u_1)|\nabla u_1|^2\xi_\rho^2 dxdt \\
&= \iint_{Q_{2\rho}(\bar{P})} \{(u_1(x,t) - u_1(x^*,t^*))^2\xi_\rho\xi_{\rho t} - 2(u_1(x,t) - u_1(x^*,t^*))\xi_\rho a_{11}(x,t,u_1)\nabla u_1\cdot\nabla\xi_\rho \\
&\quad + b_1(x,t,\mathscr{U},\nabla u_1)(u_1(x,t) - u_1(x^*,t^*))\xi_\rho^2\}dxdt.
\end{aligned} \tag{3.23}$$

By using (2.3), (2.4), (3.2), Cauchy's inequality with ε, and [12, Chapter II, formula (1.5)], we deduce from (3.23) that for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}(\bar{x})} (u_1(x,\bar{t}) - u_1(x^*, t^*))^2\xi_\rho^2 dx + \iint_{Q_{2\rho}(\bar{P})} \nu(0)|\nabla u_1|^2\xi_\rho^2 dxdt \\
&\le \varepsilon\iint_{Q_{2\rho}(\bar{P})} |\nabla u_1|^2\xi_\rho^2 dxdt + \frac{C}{\varepsilon}\iint_{Q_{2\rho}(\bar{P})} (u_1(x,t) - u_1(x^*,t^*))^2(|\xi_\rho\xi_{\rho t}| + |\nabla\xi_\rho|^2 + \xi_\rho^2)dxdt \\
&\quad + C\iint_{Q_{2\rho}(\bar{P})} \big(|u_2|^{\sigma_{12}} + \phi_1\big)|u_1(x,t) - u_1(x^*,t^*)|\xi_\rho^2 dxdt \\
&\le \varepsilon\iint_{Q_{2\rho}(\bar{P})} |\nabla u_1|^2\xi_\rho^2 dxdt + \frac{C}{\varepsilon}\rho^{2\alpha_1}\rho^{2+n}\max_{Q_{2\rho}(\bar{P})}[|\xi_\rho\xi_{\rho t}| + |\nabla\xi_\rho|^2 + \xi_\rho^2] + C\rho^{\alpha_1}[\||u_2|^{\sigma_{12}}\|_{L^{q_1}(Q_{2\rho}(\bar{P}))} \\
&\quad + \|\phi_1\|_{L^{q_1}(D_T)}](\rho^{n+2})^{1-1/q_1}.
\end{aligned} \tag{3.24}$$

Note that

$$\max_{Q_{2\rho}(\bar{P})}[|\xi_\rho\xi_{\rho t}| + |\nabla\xi_\rho|^2 + \xi_\rho^2] \le C/\rho^2, \quad (\rho^{n+2})^{1-1/q_1} = \rho^{n+2\chi_1}.$$

Setting $\varepsilon = \nu(0)/2$ and $\tilde{\alpha} = \min\{\chi_1, \alpha_1\}$, and using (2.20) and (2.7), we further obtain inequality (3.22).

Step 2. We show that if $\bar{P} \in D_{5,T}$ and $\rho \le \rho_2$, then

$$\operatorname*{ess\,sup}_{t_5\le t\le T}\int_{B_\rho(\bar{x})} |\nabla u_1(x,t)|^2 dx \le C\rho^{n-2+2\tilde{\alpha}}. \tag{3.25}$$

We find that $Q_{2\rho}(\bar{P}) \subset D_{4,T}$. Letting $h \to 0$ in (3.18) and using (3.19) and (2.7), we have

$$\begin{aligned}
&\iint_{Q_{2\rho}(\bar{P})} \hat{w}_{1t}^2\xi_\rho^2 dxdt + \int_{B_{2\rho}(\bar{x})} |\nabla\hat{w}_1(x,\bar{t})|^2\xi_\rho^2(x,\bar{t})dx \\
&\le C\iint_{Q_{2\rho}(\bar{P})} \{|\nabla\hat{w}_1|^2(|\nabla\xi_\rho^2| + |\xi_\rho\xi_{\rho t}|) + \hat{f}_1^2\xi_\rho^2 + \xi_\rho^2\}dxdt \\
&\le \frac{C}{\rho^2}\iint_{Q_{2\rho}(\bar{P})} (1 + |\nabla u_1|^2)dxdt + C\iint_{Q_{2\rho}(\bar{P})} [|\nabla u_1|^2 + |u_2|^{2\sigma_{12}} + \phi_1^2 + 1]\xi_\rho^2 dxdt.
\end{aligned}$$

Thus, (3.22) and (2.22) further imply that

$$\iint_{Q_\rho(\bar{P})} \hat{w}_{1t}^2 dxdt + \int_{B_\rho(\bar{x})} |\nabla\hat{w}_1(x,\bar{t})|^2 dx \le C\rho^{n-2+2\tilde{\alpha}},$$

which, together with (3.12), leads to inequality (3.25).

Step 3. Employing [12, Chapter II, Lemma 5.2], we find from (3.25) and (2.22) that (3.21) holds. □

We next investigate the second partial derivatives of u_1 with respect to x by dealing with difference ratios of u_{1x_i}.

Lemma 3.4. *Function u_1 has weak derivatives $u_{1x_ix_k}$ in $D_{6,T}$ for $i, k = 1, \dots, n$, and satisfies*

$$\|u_1\|_{W_2^{2,1}(D_{6,T})} \le C(\Theta_1, \mu(M_1)). \tag{3.26}$$

Proof. Step 1. We prove that (3.26) holds for $n = 1$. Let $\hat{w}_1 = \hat{w}_1(x, t)$ be defined by (3.11). From equality (3.13) and estimate (3.10), we find that $\hat{w}_1$ has the second partial derivative $\hat{w}_{1xx}$ and satisfies

$$\hat{w}_{1xx} = u_{1t} - \hat{\hat{f}}_1(x, t) \quad ((x, t) \in D_{5,T}),$$

which, together with inequalities (3.19) and (3.20), implies that $\hat{w}_{1xx} \in L^2(D_{5,T})$ and $\hat{w}_{1x} \in V_2(D_{5,T})$. It follows from [12, Chapter II, inequality (3.4)] with $n = 1$ that $\|\hat{w}_{1x}\|_{L^6(D_{5,T})}$ and $\|\hat{w}_{1x}\|_{L^{\infty,4}(D_{5,T})}$ are estimated from above by C. Thus, by (3.12), (3.2), and (3.9), we have

$$\|u_{1x}\|_{L^6(D_{5,T})} + \|u_{1x}\|_{L^{\infty,4}(D_{5,T})} \le C \tag{3.27}$$

and

$$u_{1xx} = [a_{11}(x, t, u_1)]^{-1}\left[\hat{w}_{1xx} - u_{1x}a_{11x}(x, t, u_1) - \int_{u_1^*}^{u_1} a_{11xx}(x, t, \omega)\mathrm{d}\omega\right]$$
$$- [a_{11}(x, t, u_1)]^{-2}\left[\hat{w}_{1x} - \int_{u_1^*}^{u_1} a_{11x}(x, t, \omega)\mathrm{d}\omega\right][u_{1x}a_{11u_1}(x, t, u_1) + a_{11x}].$$

Hence, (2.3) and (3.9) lead to the inequality $|u_{1xx}| \le C[|\hat{w}_{1xx}| + \hat{w}_{1x}^2 + u_{1x}^2 + 1]$. Estimate (3.27) further shows that $\|u_{1xx}\|_{L^2(D_{5,T})}$ is estimated from above by C. In view of (3.10), we find that (3.26) holds for $n = 1$.

Step 2. We prove that (3.26) holds for $n \ge 2$. Let $\bar{P} \in D_{6,T}$ and $\rho \le \rho_3 := \rho_2/2$. Then $Q_{2\rho}(\bar{P}) \subset D_{5,T}$. For any fixed $k \in \{1, \dots, n\}$ and for $|h_k| \le \min\{d/4, \rho_3/4\}$, taking $\eta = [(u_{1(x_k)}\xi_\rho^2)(x - h_k, t)]_{(x_k)}$ and $\tau = \bar{t}$ in the first equality of (2.1). Here and below, notation $w_{(x_k)}(x, t)$ is defined by (2.14). Then employing [12, Chapter 2, formula (4.9)] and integrating by parts leads to the equality:

$$\begin{aligned}\frac{1}{2}\int_{B_{2\rho}(\bar{x})} u_{1(x_k)}^2(x, \bar{t})\xi_\rho^2(x, \bar{t})\mathrm{d}x + \iint_{Q_{2\rho}(\bar{P})}\Big\{&-u_{1(x_k)}^2\xi_\rho\xi_{\rho t} + \sum_{i=1}^n [a_{11}(x, t, u_1)u_{1x_i}]_{(x_k)}\big(u_{1(x_k)}\xi_\rho^2\big)_{x_i}\\ &- [b_1(x, t, \mathscr{U}, \nabla u_1)]_{(x_k)}u_{1(x_k)}\xi_\rho^2\Big\}\mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{3.28}$$

Note that

$$[a_{11}(x, t, u_1)u_{1x_i}]_{(x_k)} = u_{1(x_k)x_i}\int_0^1 a_{11}(x^\theta, t, u_1^\theta)\mathrm{d}\theta + u_{1(x_k)}\int_0^1 \frac{\partial a_{11}}{\partial u_1^\theta}u_{1x_i}^\theta\mathrm{d}\theta + \int_0^1 \frac{\partial a_{11}}{\partial x_k^\theta}u_{1x_i}^\theta\mathrm{d}\theta \tag{3.29}$$

and

$$[b_1(x, t, \mathscr{U}, \nabla u_1)]_{(x_k)} = \sum_{i=1}^n u_{1(x_k)x_i}\int_0^1 \frac{\partial b_1(x^\theta, t, \mathscr{U}^\theta, \nabla u_1^\theta)}{\partial u_{1x_i}^\theta}\mathrm{d}\theta + \sum_{m=1}^2 u_{m(x_k)}\int_0^1 \frac{\partial b_1}{\partial u_m^\theta}\mathrm{d}\theta + \int_0^1 \frac{\partial b_1}{\partial x_k^\theta}\mathrm{d}\theta, \tag{3.30}$$

where $x^\theta = (x_1^\theta, \dots, x_n^\theta) := x + \theta h_k$ and $\mathscr{U}^\theta(x, t) := (1 - \theta)\mathscr{U}(x, t) + \theta\mathscr{U}(x + h_k, t)$ and $u_{1(x_k)x_i} = (u_{1(x_k)})_{x_i}$. Using (3.28)–(3.30), (2.9), (3.2), (3.9), and Cauchy's inequality, we deduce that for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2\rho}(\bar{x})} u_{1(x_k)}^2(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})} |\nabla u_{1(x_k)}|^2\xi_\rho^2\int\limits_0^1 a_{11}(x^\theta,t,u_1^\theta)\mathrm{d}\theta\mathrm{d}x\mathrm{d}t\\
&\le \iint\limits_{Q_{2\rho}(\bar{P})} u_{1(x_k)}^2|\xi_\rho\xi_{\rho t}|\mathrm{d}x\mathrm{d}t + \iint\limits_{Q_{2\rho}(\bar{P})}\Bigg\{\Bigg[|u_{1(x_k)}|\int\limits_0^1|\nabla u_1^\theta|\mathrm{d}\theta + \int\limits_0^1|\nabla u_1^\theta|\mathrm{d}\theta\Bigg]|\nabla u_{1(x_k)}|\xi_\rho^2\\
&\quad + \Bigg[|\nabla u_{1(x_k)}| + |u_{1(x_k)}|\int\limits_0^1|\nabla u_1^\theta|\mathrm{d}\theta + \int\limits_0^1|\nabla u_1^\theta|\mathrm{d}\theta\Bigg]2|u_{1(x_k)}\xi_\rho||\nabla\xi_\rho|\Bigg\}\mathrm{d}x\mathrm{d}t\\
&\quad + \iint\limits_{Q_{2\rho}(\bar{P})}\Bigg[|\nabla u_{1(x_k)}|\int\limits_0^1(|u_2^\theta|^{\tilde{\sigma}_{14}} + \tilde{\phi}_{14}(x^\theta,t))\mathrm{d}\theta + |u_{2(x_k)}|\int\limits_0^1(|\nabla u_1^\theta| + |u_2^\theta|^{\tilde{\sigma}_{13}} + \tilde{\phi}_{13}(x^\theta,t))\mathrm{d}\theta\\
&\quad + |u_{1(x_k)}|\int\limits_0^1(|\nabla u_1^\theta| + |u_2^\theta|^{\tilde{\sigma}_{12}} + \tilde{\phi}_{12}(x^\theta,t))\mathrm{d}\theta + \int\limits_0^1(|\nabla u_1^\theta| + |u_2^\theta|^{\tilde{\sigma}_{11}} + \tilde{\phi}_{11}(x^\theta,t))\mathrm{d}\theta\Bigg]|u_{1(x_k)}|\xi_\rho^2\mathrm{d}x\mathrm{d}t\\
&\le \varepsilon\iint\limits_{Q_{2\rho}(\bar{P})}|\nabla u_{1(x_k)}|^2\xi_\rho^2\mathrm{d}x\mathrm{d}t + \frac{C}{\varepsilon}(J_1+J_2),
\end{aligned} \tag{3.31}$$

where

$$J_1 := \iint\limits_{Q_{2\rho}(\bar{P})}\{u_{1(x_k)}^2[|\xi_\rho\xi_{\rho t}| + |\nabla\xi_\rho|^2 + \xi_\rho^2] + u_{2(x_k)}^2\xi_\rho^2\}\mathrm{d}x\mathrm{d}t,$$

$$J_2 := \int\limits_0^1\Bigg\{\iint\limits_{Q_{2\rho}(\bar{P})}[|\nabla u_1^\theta|^2 + \Lambda_1^2(\mathscr{U}^\theta) + \tilde{\Phi}_1^2(x^\theta,t)](u_{1(x_k)}^2+1)\xi_\rho^2\mathrm{d}x\mathrm{d}t\Bigg\}\mathrm{d}\theta.$$

Setting $\varepsilon = \min\{1/2, \nu(0)/2\}$ in (3.31) leads to the inequality

$$\frac{1}{2}\int\limits_{B_{2\rho}(\bar{x})} u_{1(x_k)}^2(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \frac{\nu(0)}{2}\iint\limits_{Q_{2\rho}(\bar{P})}|\nabla u_{1(x_k)}|^2\xi_\rho^2\mathrm{d}x\mathrm{d}t \le \bar{C}_1(J_1+J_2). \tag{3.32}$$

We next estimate J_1, J_2. Since $u_{jx_k}\in L^2(D_T)$ for $j=1,2$, then by (2.17), we obtain the inequality $J_1\le C/\rho^2$. Taking $\zeta = u_{1(x_k)}\xi_\rho$ and $\zeta=\xi_\rho$ in (3.21), respectively, we obtain

$$\begin{aligned}
J_2 &\le C_1^*\rho^{2\alpha_2}\iint\limits_{Q_{2\rho}(\bar{P})}[|\nabla u_{1(x_k)}|^2\xi_\rho^2 + \big(u_{1(x_k)}^2+1\big)|\nabla\xi_\rho|^2]\mathrm{d}x\mathrm{d}t\\
&\le C_1^*\rho^{2\alpha_2}\iint\limits_{Q_{2\rho}(\bar{P})}|\nabla u_{1(x_k)}|^2\xi_\rho^2\mathrm{d}x\mathrm{d}t + C\rho^{2\alpha_2-2}.
\end{aligned}$$

Let $\rho\le\rho_4 := \min\{\rho_3, [\nu(0)/(4\bar{C}_1C_1^*)]^{1/(2\alpha_2)}\}$. Then $\bar{C}_1C_1^*\rho^{2\alpha_2}\le\nu(0)/4$. Substituting the estimates of J_1 and J_2 into inequality (3.32) yields

$$\int\limits_{B_{2\rho}(\bar{x})} u_{1(x_k)}^2(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})}|\nabla u_{1(x_k)}|^2\xi_\rho^2\mathrm{d}x\mathrm{d}t \le C/\rho^2,$$

which implies

$$\int\limits_{B_\rho(\bar{x})} u_{1(x_k)}^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_\rho(\bar{P})}|\nabla u_{1(x_k)}|^2\mathrm{d}x\mathrm{d}t \le C/\rho^2.$$

Thus, $\iint_{D_{6,T}} |\nabla u_{1(x_k)}|^2 \mathrm{d}x\mathrm{d}t$ is estimated from above by constant C. [13, Chapter 2, Lemma 4.6] further shows that u_1 has weak derivatives $u_{1x_ix_k}$ in $D_{6,T}(i = 1, \ldots, n)$, and $\|u_{1x_ix_k}\|_{L^2(D_{6,T})}$ is estimated from above by C. In view of (3.10), u_1 belongs to $W_2^{2,1}(D_{6,T})$ and estimate (3.26) holds for $n \geq 2$. Hence, we complete the proof of the lemma. □

3.3 Estimate of $\|u_2\|_{C^{\alpha,\alpha/2}(\bar{\Omega}'\times[t',T])}$

To investigate the boundedness of $|u_2|$, we need the estimate of $\||\nabla u_1|\|_{L^p}$ for any $p > 2$ when $n \geq 2$.

Lemma 3.5. *Let $n \geq 2$. For any given positive integer K, there exists positive constant $C = C(\Theta_1, \mu(M_1), K)$ such that*

$$\iint_{D_{7,T}} [|\nabla u_1|^{4+2K} + (1 + |\nabla u_1|)^{2K} |\nabla^2 u_1|^2]\mathrm{d}x\mathrm{d}t \leq C, \tag{3.33}$$

where $|\nabla^2 u_1| := \left(\sum_{i,j=1}^n u_{1x_ix_j}^2\right)^{1/2}$.

Proof. Step 1. Let $B_{1,i}(x, t, w, \mathbf{q})$ and $B_1(x, t, w, \mathbf{q})$ also be defined by (3.5), and let $\psi(x, t)$ be an arbitrary sufficiently smooth function in D_T such that $\psi(x, t) = 0$ for $x \notin \Omega_6$. For any given $r \in \{1, \ldots, n\}$, choosing $\eta = \psi_{x_r}$ in the first equality of (2.1) and integrating by parts yields

$$\iint_{D_{6,T}} \left[-u_{1t}\psi_{x_r} + \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial B_{1,i}(x, t, u_1, \nabla u_1)}{\partial u_{1x_j}} u_{1x_jx_r} + \frac{\partial B_{1,i}}{\partial u_1} u_{1x_r} + \frac{\partial B_{1,i}}{\partial x_r} \right) \psi_{x_i} - B_1(x, t, u_1, \nabla u_1)\psi_{x_r} \right] \mathrm{d}x\mathrm{d}t = 0. \tag{3.34}$$

This equality also holds for $\psi \in \overset{\circ}{W}{}_2^{1,0}(D_{6,T})$. Then from (3.9), we obtain, for $(x, t) \in D_{6,T}$, $w \in [-M_1, M_1] \cap S_1$ and $\mathbf{p} \in \mathbb{R}^n$,

$$\sum_{i=1}^n \left(\left| \frac{\partial B_{1,i}(x, t, w, \mathbf{p})}{\partial w} \right| + |B_{1,i}| \right) + \sum_{i,j=1}^n \left| \frac{\partial B_{1,i}}{\partial x_j} \right| \leq C(1 + |\mathbf{p}|). \tag{3.35}$$

Let $\bar{P} = (\bar{x}, \bar{t}) \in D_{7,T}$ and $\rho \leq \rho_4/2$. Set $g = \min\{|\nabla u_1|^2, L\}$, where L is a large positive number. For any given $s \in \{0, 1, \ldots, K\}$, choosing $\psi = g^s u_{1x_r}\xi_\rho^2$ in (3.34) and summing this equality with respect to r from 1 to n, we obtain

$$\begin{aligned} &\sum_{r=1}^n \iint_{Q_{2\rho}(\bar{P})} \Bigg[-u_{1t}(g^s u_{1x_r}\xi_\rho^2)_{x_r} + \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial B_{1,i}}{\partial u_{1x_j}} u_{1x_jx_r} + \frac{\partial B_{1,i}}{\partial u_1} u_{1x_r} + \frac{\partial B_{1,i}}{\partial x_r} \right) \\ &\quad \times (g^s u_{1x_rx_i}\xi_\rho^2 + s g^{s-1} g_{x_i} u_{1x_r}\xi_\rho^2 + g^s u_{1x_r} 2\xi_\rho \xi_{\rho x_i}) \Bigg] \mathrm{d}x\mathrm{d}t \\ &= \sum_{r=1}^n \iint_{Q_{2\rho}(\bar{P})} B_1(x, t, u_1, \nabla u_1)(g^s u_{1x_rx_r}\xi_\rho^2 + s g^{s-1} g_{x_r} u_{1x_r}\xi_\rho^2 + g^s u_{1x_r} 2\xi_\rho \xi_{\rho x_r})\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.36}$$

Note that by (3.5),

$$\sum_{r=1}^n \iint_{Q_{2\rho}(\bar{P})} \sum_{i,j=1}^n \frac{\partial B_{1,i}}{\partial u_{1x_j}} u_{1x_jx_r} \left(g^s u_{1x_rx_i}\xi_\rho^2 + s g^{s-1} g_{x_i} u_{1x_r}\xi_\rho^2 \right) \mathrm{d}x\mathrm{d}t = \iint_{Q_{2\rho}(\bar{P})} a_{11}(x, t, u_1) \left[g^s \xi_\rho^2 |\nabla^2 u_1|^2 + \frac{s}{2} g^{s-1} |\nabla g|^2 \xi_\rho^2 \right] \mathrm{d}x\mathrm{d}t,$$

and by integration by parts,

$$
\begin{aligned}
&\sum_{r=1}^{n} \iint\limits_{Q_{2\rho}(\bar{P})} -u_{1t}(g^s u_{1x_r}\xi_\rho^2)_{x_r}\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\iint\limits_{Q_{2\rho}(\bar{P})} g^s(|\nabla u_1|^2)_t\xi_\rho^2\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\iint\limits_{Q_{2\rho}(\bar{P})} \{[g^s|\nabla u_1|^2\xi_\rho^2]_t - (g^s)_t g\xi_\rho^2 - g^s|\nabla u_1|^2 2\xi_\rho\xi_{\rho t}\}\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int\limits_{B_{2\rho}(\bar{x})} \left(g^s|\nabla u_1|^2\xi_\rho^2 - \frac{s}{s+1}g^{s+1}\xi_\rho^2\right)(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})}\left[\frac{s}{s+1}g^{s+1} - g^s|\nabla u_1|^2\right]\xi_\rho\xi_{\rho t}\mathrm{d}x\mathrm{d}t \\
&\geq \frac{1}{2(s+1)}\int\limits_{B_{2\rho}(\bar{x})} g^{s+1}(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x - \iint\limits_{Q_{2\rho}(\bar{P})}(1+|\nabla u_1|^2)g^s|\xi_\rho\xi_{\rho t}|\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.37}
$$

Inequality (3.37) is correct, although in deriving it we made use the partial derivatives u_{1x_it}. Indeed, we can first take the sequence infinitely differentiable functions v_m converging to u_1 in the norm of $W_2^{2,1}(D_{6,T})$ as $m \to +\infty$ and obtain (3.37) for v_m. Then letting $m \to +\infty$ we obtain (3.37) for u_1.

Thus, substituting the aforementioned two relations into (3.36) and using (3.8), (3.35) leads to

$$
\begin{aligned}
&\frac{1}{2(s+1)}\int\limits_{B_{2\rho}(\bar{x})} g^{s+1}(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})} a_{11}(x,t,u_1)[g^s|\nabla^2 u_1|^2 + \frac{s}{2}g^{s-1}|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t \\
&\leq C\iint\limits_{Q_{2\rho}(\bar{P})}(1+|\nabla u_1|^2)g^s|\xi_\rho\xi_{\rho t}|\mathrm{d}x\mathrm{d}t + C\iint\limits_{Q_{2\rho}(\bar{P})}[g^s|\nabla^2 u_1||\nabla u_1|\xi_\rho|\nabla\xi_\rho| + (|\nabla u_1|^2 + |u_2|^{\sigma_{12}} + \phi_1 \\
&\quad + 1)(sg^{s-1}|\nabla g||\nabla u_1|\xi_\rho^2 + g^s|\nabla^2 u_1|\xi_\rho^2 + g^s|\nabla u_1|\xi_\rho|\nabla\xi_\rho|)]\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Note that $sg^{s-1}|\nabla g||\nabla u_1| = sg^{s-\frac{1}{2}}|\nabla g|$. Then Cauchy's inequality with ε further shows that for any $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&\int\limits_{B_{2\rho}(\bar{x})} g^{s+1}(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})}[g^s|\nabla^2 u_1|^2 + sg^{s-1}|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t \\
&\quad\leq \varepsilon\iint\limits_{Q_{2\rho}(\bar{P})}[g^s|\nabla^2 u_1|^2\xi_\rho^2 + sg^{s-1}|\nabla g|^2\xi_\rho^2]\mathrm{d}x\mathrm{d}t + \frac{C}{\varepsilon}[J_{3,s} + J_{4,s} + J_{5,s}],
\end{aligned}
\tag{3.38}
$$

where

$$
\begin{aligned}
J_{3,s} &= \iint\limits_{Q_{2\rho}(\bar{P})}|\nabla u_1|^2(1+|\nabla u_1|^2)(1+g)^s\xi_\rho^2\mathrm{d}x\mathrm{d}t, \\
J_{4,s} &= \iint\limits_{Q_{2\rho}(\bar{P})}\left(|u_2|^{2\sigma_{12}} + \phi_1^2\right)(1+g)^s\xi_\rho^2\mathrm{d}x\mathrm{d}t, \\
J_{5,s} &= \iint\limits_{Q_{2\rho}(\bar{P})}(1+|\nabla u_1|^2)(1+g)^s(|\nabla\xi_\rho|^2 + |\xi_\rho\xi_{\rho t}|)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Here and below, for $s = 0$ and $g \geq 0$, the expression g^s is taken equal to unity and the expression sg^{s-1} is taken equal to 0.

We next estimate $J_{3,s}$, $J_{4,s}$. By taking $\zeta = (1+|\nabla u_1|^2)^{1/2}(1+g)^{s/2}\xi_\rho$ and $\zeta = (1+g)^{s/2}\xi_\rho$ in (3.21), respectively, we have

$$
J_{3,s} \leq C\rho^{2\alpha_2}\iint\limits_{Q_{2\rho}(\bar{P})}\{[|\nabla^2 u_1|^2(1+g)^s + s(1+g)^{s-1}|\nabla g|^2]\xi_\rho^2 + (1+|\nabla u_1|^2)(1+g)^s|\nabla\xi_\rho|^2\}\mathrm{d}x\mathrm{d}t
$$

and

$$J_{4,s} \le C\rho^{2\alpha_2} \iint\limits_{Q_{2\rho}(\bar{P})} [s(1+g)^{s-2}|\nabla g|^2 \xi_\rho^2 + (1+g)^s|\nabla \xi_\rho|^2]\mathrm{d}x\mathrm{d}t$$
$$\le C\rho^{2\alpha_2} \iint\limits_{Q_{2\rho}(\bar{P})} [s(1+g)^{s-1}|\nabla g|^2 \xi_\rho^2 + (1+g)^s|\nabla \xi_\rho|^2]\mathrm{d}x\mathrm{d}t.$$

Substituting the estimates of $J_{3,s}$, $J_{4,s}$ into (3.38) and setting $\varepsilon = 1/2$ further yields

$$\int\limits_{B_{2\rho}(\bar{x})} g^{s+1}(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})} [g^s|\nabla^2 u_1|^2 + sg^{s-1}|\nabla g|^2]\xi_\rho^2 \mathrm{d}x\mathrm{d}t \le C_2^*\rho^{2\alpha_2} \iint\limits_{Q_{2\rho}(\bar{P})} [|\nabla^2 u_1|^2(1+g)^s + s(1+g)^{s-1}|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t + CJ_{5,s}. \tag{3.39}$$

Specially, letting $s = 1$ in (3.39) and $\rho \le \min\{\rho_4/2, [1/(8C_2^*)]^{1/(2\alpha_2)}\}$ leads to the inequality

$$\iint\limits_{Q_{2\rho}(\bar{P})} [g|\nabla^2 u_1|^2 + |\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t \le C \iint\limits_{Q_{2\rho}(\bar{P})} |\nabla^2 u_1|^2 \xi_\rho^2 \mathrm{d}x\mathrm{d}t + CJ_{5,1}. \tag{3.40}$$

We find that there exists a constant C_3^* such that $(1+g)^s \le C_3^*(1+g^s)$ and $(1+g)^{s-1} \le C_3^*(1+sg^{s-1})$ for $s \in \{0,1,\dots,K\}$. If $\rho \le \rho_5 := \min\{\rho_4/2, [1/(8C_2^*)]^{1/(2\alpha_2)}, [1/(8KC_2^*C_3^*)]^{1/(2\alpha_2)}\}$, then $KC_2^*C_3^*\rho^{2\alpha_2} \le 1/8$ and $C_2^*\rho^{2\alpha_2} \le 1/8$. Inequality (3.39) implies that

$$\iint\limits_{Q_{2\rho}(\bar{P})} [g^s|\nabla^2 u_1|^2 + sg^{s-1}|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t \le C \iint\limits_{Q_{2\rho}(\bar{P})} [|\nabla^2 u_1|^2 + s|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t + CJ_{5,s},$$

which, together with (3.40), yields

$$\iint\limits_{Q_{2\rho}(\bar{P})} [g^s|\nabla^2 u_1|^2 + sg^{s-1}|\nabla g|^2]\xi_\rho^2\mathrm{d}x\mathrm{d}t \le C \iint\limits_{Q_{2\rho}(\bar{P})} |\nabla^2 u_1|^2 \xi_\rho^2\mathrm{d}x\mathrm{d}t + CJ_{5,s}. \tag{3.41}$$

Step 2. In another aspect, taking $\zeta = (1+g)^{(s+1)/2}\xi_\rho$ in (3.21) we obtain, for $s \in \{0,1,\dots,K\}$,

$$\iint\limits_{Q_{2\rho}(\bar{P})} |\nabla u_1|^2(1+g)^{s+1}\xi_\rho^2\mathrm{d}x\mathrm{d}t \le C\rho^{2\alpha_2} \iint\limits_{Q_{2\rho}(\bar{P})} [(1+g)^{s-1}|\nabla g|^2\xi_\rho^2 + (1+g)^{s+1}|\nabla \xi_\rho|^2]\mathrm{d}x\mathrm{d}t \le C\rho^{2\alpha_2} \iint\limits_{Q_{2\rho}(\bar{P})} (1+g)^{s-1}|\nabla g|^2\xi_\rho^2\mathrm{d}x\mathrm{d}t + CJ_{5,s}. \tag{3.42}$$

Step 3. Set $\bar{\rho}_s = \rho/2^s$. Let $\xi_{\bar{\rho}_s} = \xi_{\bar{\rho}_s}(x,t)$ be defined by (2.17) with ρ replaced by $\bar{\rho}_s$, and let $Q_{2\rho}(\bar{P})$ and ξ_ρ in (3.41) and (3.42) be replaced by $Q_{2\bar{\rho}_s}(\bar{P})$ and $\xi_{\bar{\rho}_s}$, respectively.

Specially, by setting $s = 0$ in (3.42) and using (3.26), (2.17) leads to

$$\iint\limits_{Q_{2\bar{\rho}_0}(\bar{P})} |\nabla u_1|^2(1+g)\xi_{\bar{\rho}_0}^2\mathrm{d}x\mathrm{d}t$$
$$\le C \iint\limits_{Q_{2\bar{\rho}_0}(\bar{P})\cap\{|\nabla u_1|^2\le L\}} \frac{|\nabla u_1|^2|\nabla^2 u_1|^2\xi_{\bar{\rho}_0}^2}{(1+|\nabla u_1|^2)}\mathrm{d}x\mathrm{d}t + C\iint\limits_{Q_{2\bar{\rho}_0}(\bar{P})} (1+|\nabla u_1|^2)[|\nabla\xi_{\bar{\rho}_0}|^2 + |\xi_{\bar{\rho}_0}\xi_{\bar{\rho}_0 t}|]\mathrm{d}x\mathrm{d}t$$
$$\le C/\rho^2.$$

Thus,

$$\iint\limits_{Q_{2\bar{\rho}_1}(\bar{P})} |\nabla u_1|^2(1+g)\mathrm{d}x\mathrm{d}t = \iint\limits_{Q_{\bar{\rho}_0}(\bar{P})} |\nabla u_1|^2(1+g)\mathrm{d}x\mathrm{d}t \le C/\rho^2. \tag{3.43}$$

Inequality (3.41) with $s = 1$, together with (3.43), implies that

$$\iint\limits_{Q_{2\bar\rho_1}(\bar P)} |\nabla g|^2 \xi_{\bar\rho_1}^2 \mathrm{d}x\mathrm{d}t \le C \iint\limits_{Q_{2\bar\rho_1}(\bar P)} |\nabla^2 u_1|^2 \mathrm{d}x\mathrm{d}t + \frac{C}{\rho^2} \iint\limits_{Q_{2\bar\rho_1}(\bar P)} (1 + |\nabla u_1|^2)(1 + g)\mathrm{d}x\mathrm{d}t \le C(1/\rho^2). \tag{3.44}$$

Moreover, for $s = 1, \dots, K$, it follows from (3.41) that

$$\begin{aligned}\iint\limits_{Q_{2\bar\rho_s}(\bar P)} [g^s|\nabla^2 u_1|^2 + g^{s-1}|\nabla g|^2]\xi_{\bar\rho_s}^2 \mathrm{d}x\mathrm{d}t \le C(1/\rho^2) + C \iint\limits_{Q_{2\bar\rho_s}(\bar P)} (1 + |\nabla u_1|)^2(1 + g)^s\Big(|\nabla \xi_{\bar\rho_s}|^2 \\ + |\xi_{\bar\rho_s}\big(\xi_{\bar\rho_s}\big)_t|\Big)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.45}$$

and from (3.42), (3.44), and (3.45) that

$$\begin{aligned}&\iint\limits_{Q_{2\bar\rho_s}(\bar P)} |\nabla u_1|^2 (1 + g)^{s+1}\xi_{\bar\rho_s}^2 \mathrm{d}x\mathrm{d}t \\ &\le C \iint\limits_{Q_{2\bar\rho_s}(\bar P)} |\nabla g|^2 \xi_{\bar\rho_s}^2 \mathrm{d}x\mathrm{d}t + C \iint\limits_{Q_{2\bar\rho_s}(\bar P)} g^{s-1}|\nabla g|^2 \xi_{\bar\rho_s}^2 \mathrm{d}x\mathrm{d}t \\ &\quad + C \iint\limits_{Q_{2\bar\rho_s}(\bar P)} (1 + |\nabla u_1|)^2(1 + g)^s\Big(|\nabla \xi_{\bar\rho_s}|^2 + |\xi_{\bar\rho_s}\big(\xi_{\bar\rho_s}\big)_t|\Big)\mathrm{d}x\mathrm{d}t \\ &\le C(1/\rho^2) + C \iint\limits_{Q_{2\bar\rho_s}(\bar P)} (1 + |\nabla u_1|)^2(1 + g)^s\Big(|\nabla \xi_{\bar\rho_s}|^2 + |\xi_{\bar\rho_s}\big(\xi_{\bar\rho_s}\big)_t|\Big)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.46}$$

In view of (3.43), by considering in succession (3.46) for $s = 1, \dots, K$, we can find from inequalities (3.45) and (3.46) that

$$\iint\limits_{Q_{\bar\rho_K}} [g^K|\nabla^2 u_1|^2 + g^{K-1}|\nabla g|^2 + |\nabla u_1|^2(1 + g)^{K+1}]\mathrm{d}x\mathrm{d}t \le C(1/\rho^2).$$

Letting $L \to +\infty$ leads to the estimate

$$\iint\limits_{Q_{\bar\rho_K}} [|\nabla u_1|^{2K}|\nabla^2 u_1|^2 + |\nabla u_1|^{4+2K}]\mathrm{d}x\mathrm{d}t \le C(1/\rho^2).$$

This gives estimate (3.33). □

We next use (3.33) to estimate $\operatorname{ess\,sup}_{D_{8,T}}|u_2|$ from the second equality of (2.1).

Lemma 3.6. *There exists positive constant $M_2 = M_2(\Theta_1, \nu(M_1), \mu(M_1))$ such that*

$$\operatorname*{ess\,sup}_{D_{8,T}}|u_2| \le M_2. \tag{3.47}$$

Proof. Step 1. From (2.3) and (3.3), we see that for $(x, t) \in \bar D_{1,T}$ and $\mathscr{W} \in (S_1 \cap [-M_1, M_1]) \times S_2$,

$$\nu(M_1) \le a_{22}(x, t, \mathscr{W}) \le \mu(M_1), \quad |a_{21}(x, t, \mathscr{W})| \le \mu(M_1). \tag{3.48}$$

Using (3.48), (2.4), (2.7), (2.20), and the similar argument as that of Step 1 in Lemma 3.1, we can prove that the integrals in the second equality of (2.1) exist and are finite.

Step 2. We show that u_2 has estimate (3.47). For any given $\bar P \in D_{8,T}$ and $Q_{\rho,\tau}(\bar P) \subset D_{7,T}$, let $\zeta = \zeta(x, t)$ be same as that in Lemma 2.4. Choosing $\hat\sigma \ge 1$ and taking $\eta = u^{(\sigma)}\zeta^2$ in the second equality of (2.1) for $\sigma \ge \hat\sigma$, we obtain, for any τ_1 and τ^*, $\bar t - \tau < \tau_1 \le \tau^* < \bar t$,

$$\frac{1}{2}\int_{\Omega}(u_2^{(\sigma)}\zeta)^2\mathrm{d}x\Bigg|_{t=\tau_1}^{t=\tau^*}+\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}a_{22}(x,t,\mathcal{U})|\nabla u_2|^2\zeta^2\mathrm{d}x\mathrm{d}t$$
$$=\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}\{-[\zeta^2\nabla u_2+2u_2^{(\sigma)}\zeta\nabla\zeta]\cdot a_{21}(x,t,\mathcal{U})\nabla u_1-a_{22}(x,t,\mathcal{U})u_2^{(\sigma)}\nabla u_2\cdot(2\zeta\nabla\zeta)+(u_2^{(\sigma)})^2\zeta\zeta_t$$
$$+b_2(x,t,\mathcal{U},\nabla\mathcal{U})u_2^{(\sigma)}\zeta^2\}\mathrm{d}x\mathrm{d}t.$$

By (2.4), (3.2), (3.48) and Cauchy's inequality with ε, we find that

$$\frac{1}{2}\int_{\Omega}[u_2^{(\sigma)}\zeta(x,t)]^2\mathrm{d}x\Bigg|_{\tau_1}^{\tau^*}+\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}a_{22}(x,t,\mathcal{U})|\nabla u_2|^2\zeta^2\mathrm{d}x\mathrm{d}t$$
$$\le C\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}\{[|\nabla u_2|\zeta^2+u_2^{(\sigma)}\zeta|\nabla\zeta|]|\nabla u_1|+|\nabla u_2|u_2^{(\sigma)}\zeta|\nabla\zeta|+[u_2^{(\sigma)}]^2\zeta|\zeta_t|+[\big(1+|u_2|^{\delta_2}\big)(|\nabla u_2|+|\nabla u_1|)+F_2(\mathcal{U})$$
$$+\phi_2]u_2^{(\sigma)}\zeta^2\}\mathrm{d}x\mathrm{d}t$$
$$\le\varepsilon\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}|\nabla u_2|^2\zeta^2\mathrm{d}x\mathrm{d}t+\frac{C}{\varepsilon}\int_{\tau_1}^{\tau^*}\int_{\mathcal{A}_{u_2,\sigma}(t)}\{(\zeta|\zeta_t|+|\nabla\zeta|^2)[u_2^{(\sigma)}]^2+\big(1+|u_2|^{2\delta_2}\big)[u_2^{(\sigma)}]^2\zeta^2+|\nabla u_1|^2\zeta^2+[F_2(\mathcal{U})$$
$$+\phi_2]u_2^{(\sigma)}\zeta^2\}\mathrm{d}x\mathrm{d}t.$$

Setting $\varepsilon=\min\{1/2,\nu(M_1)/2\}$ further yields, for any τ_1 and τ_2, $\bar{t}-\tau<\tau_1<\tau_2<\bar{t}$,

$$\sup_{\tau_1\le t\le\tau_2}\|u_2^{(\sigma)}(x,t)\zeta(x,t)\|^2_{L^2(\Omega)}+\nu(M_1)\int_{\tau_1}^{\tau_2}\int_{\mathcal{A}_{u_2,\sigma}(t)}|\nabla u_2|^2\zeta^2\mathrm{d}x\mathrm{d}t$$
$$\le\|u_2^{(\sigma)}(x,\tau_1)\zeta(x,\tau_1)\|^2_{L^2(\Omega)}+\int_{\tau_1}^{\tau_2}\int_{\mathcal{A}_{u_2,\sigma}(t)}\{\mathcal{G}_2(x,t)\zeta^2[(u_2-\sigma)^2+\sigma^2]+C(|\nabla\zeta|^2+\zeta|\zeta_t|)[u_2^{(\sigma)}]^2\}\mathrm{d}x\mathrm{d}t,$$

where $\mathcal{G}_2(x,t)=C\big(|\nabla u_1|^2+|u_2|^{2\delta_2}+F_2(\mathcal{U})+\phi_2+1\big)$. It follows from (2.7), (2.20), (3.27), and (3.33) that $\mathcal{G}_2(x,t)\in L^{q_1}(D_{7,T})$. Then Lemma 2.4 shows that $\operatorname{ess\,sup}_{D_{8,T}}u_2$ does not exceed constant C. The similar argument implies that $\operatorname{ess\,sup}_{D_{8,T}}(-u_2)$ also does not exceed C. Thus, (3.47) holds. □

To obtain Hölder estimate for u_1, we need to estimate $\|u_1\|_{W^{2,1}_{q_3}(D_{10,T})}$.

Lemma 3.7. *We have*

$$\|u_{1x_i}\|_{C^{\alpha_3,\alpha_3/2}(\bar{D}_{10,T})}\le\Upsilon_1,\quad\|u_1\|_{W^{2,1}_{q_3}(D_{10,T})}\le C,\quad\alpha_3\in(0,1),\quad i=1,\dots,n,\tag{3.49}$$

where constants Υ_1 and C *depend on* Θ_1, $\nu(M_1)$ *and* $\mu(|(M_1,M_2)|)$, *and* q_3 *is same as that in* Theorem 2.1.

Proof. Step 1. We show that when $n\ge2$,

$$\operatorname*{ess\,sup}_{D_{9,T}}|\nabla u_1|\le C.\tag{3.50}$$

Let $\tilde{z}=|\nabla u_1|^2\lambda_9^2$, where function λ_9 is defined by (2.18), and let $\sigma\ge1$. Set $\psi=u_{1x_r}\lambda_9^2\max\{\tilde{z}-\sigma,0\}=u_{1x_r}\lambda_9^2\tilde{z}^{(\sigma)}$ in (3.34), and sum the resulting equations with respect to r from 1 to n. Then

$$\sum_{r=1}^{n}\int_0^\tau\int_\Omega -u_{1t}[u_{1x_r}\lambda_9^2\tilde{z}^{(\sigma)}]_{x_r}\mathrm{d}x\mathrm{d}t+\sum_{r=1}^{n}\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}\Bigg\{\sum_{i=1}^{n}\Bigg[\Bigg(\sum_{j=1}^{n}\frac{\partial B_{1,i}(x,t,u_1,\nabla u_1)}{\partial u_{1x_j}}u_{1x_jx_r}+\frac{\partial B_{1,i}}{\partial u_1}u_{1x_r}+\frac{\partial B_{1,i}}{\partial x_r}\Bigg)(u_{1x_rx_i}\lambda_9^2(\tilde{z}-\sigma)+u_{1x_r}\lambda_9^2\tilde{z}_{x_i}+2u_{1x_r}(\tilde{z}-\sigma)\lambda_9\lambda_{9x_i})\Bigg]-B_1(x,t,u_1,\nabla u_1)[u_{1x_rx_r}\lambda_9^2(\tilde{z}-\sigma)+u_{1x_r}\lambda_9^2\tilde{z}_{x_r}+2u_{1x_r}(\tilde{z}-\sigma)\lambda_9\lambda_{9x_r}]\Bigg\}\mathrm{d}x\mathrm{d}t=0. \tag{3.51}$$

Note that by integration by parts,

$$\begin{aligned}&\sum_{r=1}^{n}\int_0^\tau\int_\Omega -u_{1t}[u_{1x_r}\lambda_9^2\tilde{z}^{(\sigma)}]_{x_r}\mathrm{d}x\mathrm{d}t\\&=\frac{1}{4}\int_\Omega(\tilde{z}^{(\sigma)}(x,\tau))^2\mathrm{d}x-\int_0^\tau\int_\Omega|\nabla u_1|^2\tilde{z}^{(\sigma)}\lambda_9\lambda_{9t}\mathrm{d}x\mathrm{d}t\\&\geq\frac{1}{4}\int_\Omega(\tilde{z}^{(\sigma)}(x,\tau))^2\mathrm{d}x-\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}|\nabla u_1|^2(\tilde{z}-\sigma)|\lambda_9\lambda_{9t}|\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.52}$$

and by (3.5),

$$\begin{aligned}&\sum_{r=1}^{n}\sum_{i,j=1}^{n}\frac{\partial B_{1,i}(x,t,u_1,\nabla u_1)}{\partial u_{1x_j}}u_{1x_jx_r}[u_{1x_rx_i}\lambda_9^2(\tilde{z}-\sigma)+u_{1x_r}\lambda_9^2\tilde{z}_{x_i}]\\&=a_{11}(x,t,u_1)\Big[|\nabla^2u_1|^2(\tilde{z}-\sigma)\lambda_9^2+\frac{1}{2}|\nabla\tilde{z}|^2-|\nabla u_1|^2\lambda_9\nabla\lambda_9\cdot\nabla\tilde{z}\Big].\end{aligned} \tag{3.53}$$

Substituting (3.52) and (3.53) into (3.51) and using (3.8), (3.35), (3.47), and Cauchy's inequality with ε, we can obtain

$$\begin{aligned}&\frac{1}{4}\int_\Omega(\tilde{z}^{(\sigma)})^2(x,\tau)\mathrm{d}x+\frac{\nu(0)}{2}\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}\Big\{|\nabla^2u_1|^2(\tilde{z}-\sigma)\lambda_9^2+\frac{1}{2}|\nabla\tilde{z}|^2\Big\}\mathrm{d}x\mathrm{d}t\\&\leq C\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}\{(1+|\nabla u_1|)^6(\lambda_9^2+|\nabla\lambda_9|^2)+|\nabla u_1|^2(\tilde{z}-\sigma)(\lambda_9^2+|\nabla\lambda_9|^2+|\lambda_9\lambda_{9t}|)+(\phi_1^2+1)[(\tilde{z}-\sigma)\lambda_9^2\\&\quad+\tilde{z}]\}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Note that $\lambda_8=1$ for $(x,t)\in\Omega_8\times(t_8,T]$ and $\lambda_9=0$ for $x\notin\Omega_8$ or $t\leq t_8$. Then we further obtain $\{(x,t):x\in\mathcal{A}_{\tilde{z},\sigma}(t),t\in(0,\tau]\}\subseteq D_{8,\tau}$, and

$$\begin{aligned}&\int_\Omega(\tilde{z}^{(\sigma)}(x,\tau))^2\mathrm{d}x+\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}|\nabla\tilde{z}|^2\mathrm{d}x\mathrm{d}t\\&\leq C\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}\{(1+|\nabla u_1||\lambda_8|)^6(\lambda_9^2+|\nabla\lambda_9|^2)+|\nabla u_1|^2\lambda_8^2(\tilde{z}-\sigma)(\lambda_9^2+|\nabla\lambda_9|^2+|\lambda_9\lambda_{9t}|)+(\phi_1^2+1)[(\tilde{z}-\sigma)\lambda_9^2\\&\quad+\tilde{z}]\}\mathrm{d}x\mathrm{d}t\\&\leq C\int_0^\tau\int_{\mathcal{A}_{\tilde{z},\sigma}(t)}\mathcal{G}_3(x,t)[(\tilde{z}-\sigma)^2+\sigma^2]\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\mathcal{G}_3(x,t)=C(|\nabla u_1|^6\lambda_8^2+\phi_1^2+1)$. From (2.7) and (3.33) we find that $\mathcal{G}_3(x,t)\in L^{q_1}(D_T)$. Result (ii) of Lemma 2.4 shows that $\sup_{D_T}\tilde{z}$ does not exceed constant C. Therefore, estimate (3.50) holds for $n\geq 2$.

Step 2. We prove that estimate (3.49) holds for $n \geq 1$. By estimates (3.2) and (3.26), the first equality of (2.1) yields

$$u_{1t} = \operatorname{div}[a_{11}(x,t,u_1)\nabla u_1] + b_1(x,t,\mathscr{U},\nabla u_1) \quad ((x,t) \in D_{9,T}). \tag{3.54}$$

For fixed $x_9 \in \Omega_9$, set $u_1^{**} = u_1(x_9, t_9)$. Let

$$\check{z}_1 = \lambda_{10}^2 \int_{u_1^{**}}^{u_1} a_{11}(x,t,\omega)\mathrm{d}\omega.$$

By a direct computation we find from (3.54) that $\check{z}_1$ is a solution in $W_2^{2,1}(D_{9,T})$ for a linear problem in the form

$$\begin{cases} z_t = \bar{a}_1(x,t)\Delta z + \bar{f}_1(x,t) \quad ((x,t) \in D_{9,T}), \\ z(x,t) = 0 \quad (x \in \partial\Omega_9, \quad t \in [t_9, T]), \quad z(x,t_9) = 0 \quad (x \in \Omega_9), \end{cases} \tag{3.55}$$

where

$$\begin{cases} \bar{a}_1(x,t) = a_{11}(x,t,u_1), \\ \bar{f}_1(x,t) = \lambda_{10}^2 a_{11}(x,t,u_1) b_1(x,t,\mathscr{U},\nabla u_1) + (\lambda_{10}^2)_t \displaystyle\int_{u_1^{**}}^{u_1} a_{11}(x,t,\omega)\mathrm{d}\omega \\ \qquad - a_{11}(x,t,u_1)\left\{\Delta(\lambda_{10}^2)\displaystyle\int_{u_1^{**}}^{u_1} a_{11}(x,t,\omega)\mathrm{d}\omega + 2a_{11}(x,t,u_1)\nabla(\lambda_{10}^2)\cdot\nabla u_1 \right. \\ \qquad \left. + 2\nabla(\lambda_{10}^2)\cdot \displaystyle\int_{u_1^{**}}^{u_1} \nabla a_{11}(x,t,\omega)\mathrm{d}\omega + \lambda_{10}^2 \operatorname{div}\left[\int_{u_1^{**}}^{u_1} \nabla a_{11}(x,t,\omega)\mathrm{d}\omega\right]\right\}. \end{cases} \tag{3.56}$$

Hence,

$$\begin{cases} \nabla u_1 = \left[\nabla \check{z}_1 - \displaystyle\int_{u_1^{**}}^{u_1} \nabla a_{11}(x,t,\omega)\mathrm{d}\omega\right] \Big/ a_{11}(x,t,u_1) \quad ((x,t) \in D_{10,T}), \\ (\nabla u_1)_{x_k} = \left\{(\nabla\check{z}_1)_{x_k} - [a_{11}(x,t,u_1)]_{x_k}\nabla u_1 - \left[\displaystyle\int_{u_1^{**}}^{u_1} \nabla a_{11}(x,t,\omega)\mathrm{d}\omega\right]_{x_k}\right\} \Big/ a_{11}(x,t,u_1) \quad ((x,t) \in D_{10,T}). \end{cases} \tag{3.57}$$

Estimate (3.2) shows that $\bar{a}_1(x,t) \in C(\bar{D}_{9,T})$, and (2.4), (3.56), (3.2), (3.9), and (3.47) yield

$$|\nabla \bar{a}_1(x,t)| \leq C(|\nabla u_1| + 1), \quad |\bar{f}_1(x,t)| \leq C(|\nabla u_1| + \phi_1 + 1).$$

Thus, it follows from (2.7) and (3.27) that $|\nabla\bar{a}_1(x,t)| \in L^{\infty,4}(D_{9,T}) \cap L^6(D_{9,T})$ and $\bar{f}_1(x,t) \in L^{q_3}(D_{9,T})$ for $n = 1$, and from (2.7) and (3.50) that $|\nabla\bar{a}_1(x,t)| \in L^\infty(D_{9,T})$ and $\bar{f}_1(x,t) \in L^{q_3}(D_{9,T})$ for $n \geq 2$. Then applying [12, Chapter IV, Section 9, Theorem 9.1], we see that problem (3.55) has a unique solution z in $W_{q_3}^{2,1}(D_{9,T})$, and

$$\|z\|_{W_{q_3}^{2,1}(D_{9,T})} \leq C. \tag{3.58}$$

Since $\check{z}_1$ is also a solution in $W_2^{2,1}(D_{9,T})$ for problem (3.55), then

$$\begin{cases} (z - \check{z}_1)_t = \bar{a}_1(x,t)\Delta(z - \check{z}_1) & ((x,t) \in D_{9,T}), \\ (z - \check{z}_1)(x,t) = 0 & (x \in \partial\Omega_9, \ t \in [t_9, T]), \\ (z - \check{z}_1)(x,t_9) = 0 & (x \in \Omega_9). \end{cases} \tag{3.59}$$

We multiply the equation in (3.59) by $z - \check{z}_1$, integrate it over $D_{9,\tau}$ and use Cauchy's inequality to find that for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}&\int_{\Omega_9} (z - \check{z}_1)^2(x, \tau)\mathrm{d}x + \int_{t_9}^{\tau}\int_{\Omega_9} \bar{a}_1(x, t)|\nabla(z - \check{z}_1)|^2\mathrm{d}x\mathrm{d}t\\ &= \int_{t_9}^{\tau}\int_{\Omega_9} -[\nabla\bar{a}_1(x, t)\cdot\nabla(z - \check{z}_1)](z - \check{z}_1)\mathrm{d}x\mathrm{d}t\\ &\leq \varepsilon\int_{t_9}^{\tau}\int_{\Omega_9} |\nabla(z - \check{z}_1)|^2\mathrm{d}x\mathrm{d}t + \frac{C}{\varepsilon}\int_{t_9}^{\tau}\left\{\|\nabla\bar{a}_1(x, t)\|^2_{L^\infty(\Omega_9)}\int_{\Omega_9}(z - \check{z}_1)^2\mathrm{d}x\right\}\mathrm{d}t.\end{aligned}$$

Setting $\varepsilon = \nu(0)/2$ leads to the inequality

$$\int_{\Omega_9} (z - \check{z}_1)^2(x, \tau)\mathrm{d}x + \int_{t_9}^{\tau}\int_{\Omega_9} |\nabla(z - \check{z}_1)|^2\mathrm{d}x\mathrm{d}t \leq C\int_{t_9}^{\tau}\left\{\|\nabla\bar{a}_1(x, t)\|^2_{L^\infty(\Omega_9)}\int_{\Omega_9}(z - \check{z}_1)^2\mathrm{d}x\right\}\mathrm{d}t.$$

Gronwall's inequality further implies that $z = \check{z}_1$ in $\bar{D}_{9,T}$. Since (2.2) leads to the relation $q_3 > n + 2$, then it follows from (3.58) and Sobolev embedding theorem that for some $\alpha_3 \in (0, 1)$,

$$\|\check{z}_{1x_i}\|_{C^{\alpha_3,\alpha_3/2}(\bar{D}_{9,T})} \leq C'\|\check{z}_1\|_{W^{2,1}_{q_3}(D_{9,T})} \leq C, \quad i = 1, \dots, n.$$

This, together with (3.57), shows that (3.49) holds. □

Lemma 3.8. *We have*

$$\|u_2\|_{C^{\alpha_4,\alpha_4/2}(\bar{D}_{11,T})} \leq C, \quad \alpha_4 \in (0, 1), \tag{3.60}$$

where $\alpha_4 = \alpha_4(\Theta_1, \nu(M_1), \mu(|(M_1, M_2)|))$ *and* $C = C(\Theta_1, \nu(M_1), \mu(|(M_1, M_2)|))$.

Proof. Let

$$\begin{cases}B_{2,i}(x, t, w, \mathbf{p}) := a_{22}(x, t, u_1, w)p_i + a_{21}(x, t, u_1, w)u_{1x_i},\\ B_2(x, t, w, \mathbf{p}) := -b_2(x, t, u_1, w, \nabla u_1, \mathbf{p}).\end{cases} \tag{3.61}$$

From (3.47) and (3.49), it follows that for $(x, t) \in D_{10,T}$, $w \in [-M_2, M_2] \cap S_2$ and $\mathbf{p} \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} B_{2,i}(x, t, w, \mathbf{p})p_i \geq \frac{\nu(M_1)}{2}|\mathbf{p}|^2 - C, \quad |B_{2,i}(x, t, w, \mathbf{p})| \leq C(|\mathbf{p}| + 1), \tag{3.62}$$

$$|B_2(x, t, w, \mathbf{p})| \leq C(|\mathbf{p}| + \phi_2 + 1). \tag{3.63}$$

Moreover, it follows from the second integral equality of (2.1) that for any $\tau_0, \tau \in (t_{10}, T]$ and for any $\eta \in \overset{\circ}{W}{}^{1,1}_2(D_{10,T})$,

$$\int_{\Omega_{10}} u_2(x, t)\eta(x, t)\mathrm{d}x\bigg|_{\tau_0}^{\tau} + \int_{\tau_0}^{\tau}\int_{\Omega_{10}}\left[-u_2\eta_t + \sum_{i=1}^{n} B_{2,i}(x, t, u_2, \nabla u_2)\eta_{x_i} + B_2(x, t, u_2, \nabla u_2)\eta\right]\mathrm{d}x\mathrm{d}t = 0. \tag{3.64}$$

Then employing [12, Chapter V, Section 1, Theorem 1.1] again, we see from (2.7), (2.20), (3.47), and (3.62)–(3.64) that (3.60) holds. □

3.4 Estimate of $\|u_2\|_{W_2^{2,1}(\Omega'\times(t',T])}$

In this subsection, by using the similar methods as those in Section 3.2, we will establish the estimate of $\|u_2\|_{W_2^{2,1}(\Omega'\times(t',T])}$ from the second equation of (1.1). Based on estimates (3.47) and (3.49), we first use Steklov average method to prove that u_2 has weak derivative u_{2t}.

By hypothesis (H_1)(I), for some positive constant Θ_2,

$$\|a_{22}(x,t,\mathscr{W})\|_{C^1(\Xi_2)},\ \|a_{22x_ix_i},\ a_{22x_iw_j},\ a_{22w_1w_1}\|_{C(\Xi_2)}\le\Theta_2, \tag{3.65}$$

where $i=1,\dots,n$, $j=1,2$ and $\Xi_2:=\bar D_{8,T}\times(S_1\cap[-M_1,M_1])\times(S_2\cap[-M_2,M_2])$.

Lemma 3.9. *Function u_2 has weak derivative u_{2t} in $D_{12,T}$, and satisfies*

$$\operatorname*{ess\,sup}_{t_{12}\le t\le T}\|\nabla u_2(\cdot,t)\|_{L^2(\Omega_{12})}+\|u_{2t}\|_{L^2(D_{12,T})}\le C. \tag{3.66}$$

Moreover, if the vertex $\bar P=(\bar x,\bar t)$ of cylinders $Q_\rho(\bar P)$ and $Q_{2\rho}(\bar P)$ is in $\bar\Omega_{13}\times[t_{13},T]$ and if $\rho\le\rho_5$, then

$$\begin{aligned}&\iint\limits_{Q_{2\rho}(\bar P)}u_{2t}^2\xi_\rho^2\,\mathrm{d}x\mathrm{d}t+\int\limits_{B_{2\rho}(\bar x)}|\nabla u_2(x,\bar t)|^2\xi_\rho^2(x,\bar t)\,\mathrm{d}x\\ &\le C\int\limits_{B_{2\rho}(\bar x)}\xi_\rho^2(x,\bar t)\,\mathrm{d}x+C\iint\limits_{Q_{2\rho}(\bar P)}[(1+|\nabla u_2|^2)(|\nabla\xi_\rho|^2+|\xi_\rho\xi_{\rho t}|+\xi_\rho^2)+(u_{1t}^2+|\nabla^2u_1|^2+\phi_2^2)\xi_\rho^2]\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.67}$$

Here, $C=C(\Theta_1,\Theta_2,\nu(M_1),\mu(|(M_1,M_2)|))$.

Proof. Fix $x_{11}\in\Omega_{11}$ and set $u_2^*=u_2(x_{11},t_{11})$. Let

$$\hat w_2=\hat w_2(x,t)=\int\limits_{u_2^*}^{u_2}a_{22}(x,t,u_1,\omega)\,\mathrm{d}\omega. \tag{3.68}$$

Hence, for each $i=1,\dots,n$,

$$\hat w_{2x_i}=u_{2x_i}a_{22}(x,t,\mathscr{U})+u_{1x_i}\int\limits_{u_2^*}^{u_2}a_{22u_1}(x,t,u_1,\omega)\,\mathrm{d}\omega+\int\limits_{u_2^*}^{u_2}a_{22x_i}(x,t,u_1,\omega)\,\mathrm{d}\omega. \tag{3.69}$$

It follows from the second equality of (2.1) and (3.69) that for any $\eta\in\overset{\circ}{W}{}_2^{1,1}(D_{11,T})$ and $\tau\in(t_{11},T]$,

$$\int\limits_{\Omega_{11}}u_2\eta\,\mathrm{d}x\,\Bigg|_{t_{11}}^{\tau}-\iint\limits_{D_{11,\tau}}u_2\eta_t\,\mathrm{d}x\mathrm{d}t=\iint\limits_{D_{11,\tau}}[-\nabla\hat w_2\cdot\nabla\eta+\hat f_2(x,t)\eta]\,\mathrm{d}x\mathrm{d}t,$$

where

$$\hat f_2(x,t)=-\sum_{i=1}^n\left[u_{1x_i}\int\limits_{u_2^*}^{u_2}a_{22u_1}(x,t,u_1,\omega)\,\mathrm{d}\omega+\int\limits_{u_2^*}^{u_2}a_{22x_i}(x,t,u_1,\omega)\,\mathrm{d}\omega\right]_{x_i}+\sum_{i=1}^n[a_{21}(x,t,\mathscr{U})u_{1x_i}]_{x_i}+b_2(x,t,\mathscr{U},\nabla\mathscr{U}).$$

Inequalities (2.4), (2.8), and (3.65), and estimates (3.2), (3.47), and (3.49) yield

$$|\nabla\hat w_2|\le C(1+|\nabla u_2|),\quad |\nabla u_2|\le C(|\nabla\hat w_2|+1),\quad |\hat f_2|\le C(|\nabla u_2|+|\nabla^2u_1|+\phi_2). \tag{3.70}$$

Furthermore, by (3.49) and (2.7),

$$\|\hat w_2\|_{L^\infty(D_{11,T})}\le C,\quad \||\nabla\hat w_2|,\ u_{1t},\ \hat f_2\|_{L^2(D_{11,T})}\le C. \tag{3.71}$$

Again from the similar arguments as those of [12, Chapter III, Section 2], it follows that for any given $h \in (0, T - t_{12})$ and $\tau \in (t_{11}, T - h]$ and for any $\eta \in \overset{\circ}{V}{}_2^{1,0}(D_{11,T})$,

$$\iint_{D_{11,\tau}} u_{2(t)}\eta \mathrm{d}x\mathrm{d}t = \iint_{D_{11,\tau}} \{-\nabla \hat{w}_{2h} \cdot \nabla \eta + [\hat{f}_2(x,t)]_h \eta\}\mathrm{d}x\mathrm{d}t. \tag{3.72}$$

(3.68) yields

$$\hat{w}_{2(t)} = u_{2(t)} \int_0^1 a_{22}(x, t^\vartheta, \mathscr{U}^\vartheta)\mathrm{d}\vartheta + u_{1(t)} \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22u_1^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta + \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22t^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta,$$

where $t^\vartheta = t + \vartheta h$ and $\mathscr{U}^\vartheta = \vartheta \mathscr{U}(x, t+h) + (1-\vartheta)\mathscr{U}(x,t)$. Hence,

$$u_{2(t)} = \left\{ \hat{w}_{2(t)} - u_{1(t)} \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22u_1^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta - \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22t^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta \right\} \Bigg/ \int_0^1 a_{22}(x, t^\vartheta, \mathscr{U}^\vartheta)\mathrm{d}\vartheta. \tag{3.73}$$

For any given $\bar{P} \in \bar{D}_{12,T-h}$ and $\rho \le \rho_5$, set $\eta = \hat{w}_{2(t)}\xi_\rho^2$ in (3.72) and use (3.73) to find that

$$\begin{aligned}
&\iint_{Q_{2\rho}(\bar{P})} \left[\int_0^1 a_{22}(x, t^\vartheta, \mathscr{U}^\vartheta)\mathrm{d}\vartheta \right]^{-1} \hat{w}_{2(t)}^2 \xi_\rho^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_{B_{2\rho}(\bar{x})} |\nabla \hat{w}_{2h}(x, \bar{t})|^2 \xi_\rho^2(x, \bar{t})\mathrm{d}x \\
&= \iint_{Q_{2\rho}(\bar{P})} \left\{ u_{1(t)} \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22u_1^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta + \int_0^1 \int_{u_2^*}^{u_2^\vartheta} a_{22t^\vartheta}(x, t^\vartheta, u_1^\vartheta, \omega)\mathrm{d}\omega \mathrm{d}\vartheta \right\} \\
&\quad \times \left[\int_0^1 a_{22}(x, t^\vartheta, \mathscr{U}^\vartheta)\mathrm{d}\vartheta \right]^{-1} \hat{w}_{2(t)}\xi_\rho^2 \mathrm{d}x\mathrm{d}t + \iint_{Q_{2\rho}(\bar{P})} \{-2\hat{w}_{2(t)}\xi_\rho \nabla \hat{w}_{2h} \cdot \nabla \xi_\rho + |\nabla \hat{w}_{2h}|^2 \xi_\rho \xi_{\rho t} + [\hat{f}_2(x,t)]_h \hat{w}_{2(t)}\xi_\rho^2\}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By (3.48), (3.65), and cauchy's inequality with ε, we can further obtain

$$\begin{aligned}
&\iint_{Q_{2\rho}(\bar{P})} \hat{w}_{2(t)}^2 \xi_\rho^2 \mathrm{d}x\mathrm{d}t + \int_{B_{2\rho}(\bar{x})} |\nabla \hat{w}_{2h}(x, \bar{t})|^2 \xi_\rho^2(x, \bar{t})\mathrm{d}x \\
&\le C \iint_{Q_{2\rho}(\bar{P})} \{|\nabla \hat{w}_{2h}|^2 (|\nabla \xi_\rho|^2 + |\xi_\rho \xi_{\rho t}|) + (u_{1(t)}^2 + 1)\xi_\rho^2 + [\hat{f}_2(x,t)]_h^2 \xi_\rho^2\}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.74}$$

This, together with (3.10) and (3.71), implies that u_2 has weak derivative u_{2t} in $D_{12,T}$ and estimate (3.66) holds. The deduction is the same as that of (3.10) from (3.18).

If $\bar{P} \in \bar{D}_{13,T}$ and $\rho \le \rho_5$, then $Q_{2\rho}(\bar{P}) \subset D_{12,T}$. Letting $h \to 0$ in (3.74) leads to the inequality

$$\iint_{Q_{2\rho}(\bar{P})} \hat{w}_{2t}^2 \xi_\rho^2 \mathrm{d}x\mathrm{d}t + \int_{B_{2\rho}(\bar{x})} |\nabla \hat{w}_2(x, \bar{t})|^2 \xi_\rho^2(x, \bar{t})\mathrm{d}x \le C \iint_{Q_{2\rho}(\bar{P})} [|\nabla \hat{w}_2|^2 (|\nabla \xi_\rho|^2 + |\xi_\rho \xi_{\rho t}|) + (u_{1t}^2 + 1)\xi_\rho^2 + \hat{f}_2^2 \xi_\rho^2]\mathrm{d}x\mathrm{d}t,$$

which, together with (3.70), gives (3.67). □

As we have done in the derivation of (3.26), to show that u_2 has the second partial derivatives $u_{x_i x_k}$, we need the following lemma:

Lemma 3.10. *For any given $\bar{P} \in D_{13,T}$ and $\rho \le \rho_6 := \rho_5/2$, if $\zeta(x,t)$ be an arbitrary bounded function from $\overset{\circ}{V}_2(Q_\rho(\bar{P}))$, then there exist constants C, α_5 depending on Θ_1, Θ_2, $\nu(M_1)$, and $\mu(|(M_1, M_2)|)$, such that*

$$\iint\limits_{Q_\rho(\bar{P})} (|\nabla u_2|^2 + \phi_2^2 + \tilde{\Phi}_2^2)\zeta^2 \mathrm{d}x\mathrm{d}t \le C\rho^{2\alpha_5} \iint\limits_{Q_\rho(\bar{P})} |\nabla\zeta|^2 \mathrm{d}x\mathrm{d}t, \quad \alpha_5 \in (0,1). \tag{3.75}$$

Proof. Step 1. We prove that there exists constant $\hat{\alpha} \in (0,1)$ such that

$$\iint\limits_{Q_\rho} |\nabla u_2|^2 \mathrm{d}x\mathrm{d}t \le C\rho^{n+2\hat{\alpha}} \tag{3.76}$$

for $\rho \le \rho_5$. Let (x^*, t^*) be a given point in $Q_\rho(\bar{P})$. Set $\eta = (u_2(x,t) - u_2(x^*, t^*))\xi_\rho^2$ and $\tau = \bar{t}$ in the second equality of (2.1) and use (2.4), (3.48), (3.49), (3.60), and Cauchy's inequality to obtain, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2\rho}(\bar{x})} (u_2(x,\bar{t}) - u_2(x^*,t^*))^2\xi_\rho^2 \mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})} a_{22}(x,t,\mathscr{U})|\nabla u_2|^2\xi_\rho^2 \mathrm{d}x\mathrm{d}t \\
&= \iint\limits_{Q_{2\rho}(\bar{P})} \Big\{(u_2(x,t) - u_2(x^*,t^*))^2\xi_\rho\xi_{\rho t} - a_{21}(x,t,\mathscr{U})\xi_\rho^2\nabla u_1\cdot\nabla u_2 - 2(u_2(x,t) - u_2(x^*,t^*))\xi_\rho\Big[\sum_{j=1}^{2} a_{2j}(x,t,\mathscr{U})\nabla u_j\Big] \\
&\quad \cdot\nabla\xi_\rho + b_2(x,t,\mathscr{U},\nabla\mathscr{U})(u_2(x,t) - u_2(x^*,t^*))\xi_\rho^2\Big\}\mathrm{d}x\mathrm{d}t \\
&\le \varepsilon \iint\limits_{Q_{2\rho}(\bar{P})} |\nabla u_2|^2\xi_\rho^2 \mathrm{d}x\mathrm{d}t + \frac{C}{\varepsilon}\iint\limits_{Q_{2\rho}(\bar{P})} \{(u_2(x,t) - u_2(x^*,t^*))^2[\xi_\rho^2 + |\nabla\xi_\rho|^2 + |\xi_\rho\xi_{\rho t}|] + \xi_\rho^2 + |u_2(x,t) \\
&\quad - u_2(x^*,t^*)|[|\nabla\xi_\rho|^2 + (1+\phi_2)\xi_\rho^2]\}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This inequality has the similar property as (3.24) for $\hat{\alpha} = \min\{\alpha_4, 2\chi_1\}$. Then we can obtain (3.76). The deduction is the same as that of (3.22) from (3.24).

Step 2. We show that (3.75) holds for $\rho \le \rho_6$.

By (3.66), (2.7), and (2.11), we find from [12, Chapter II, Lemma 5.3'] that (3.75) holds for $n = 1$.

We next consider the case $n \ge 2$. In view of condition (2.7) and estimates (3.49) and (3.76), it follows from (3.67) and (2.22) that

$$\begin{aligned}
&\iint\limits_{Q_{2\rho}(\bar{P})} u_{2t}^2\xi_\rho^2 \mathrm{d}x\mathrm{d}t + \int\limits_{B_{2\rho}(\bar{x})} |\nabla u_2(x,\bar{t})|^2\xi_\rho^2(x,\bar{t})\mathrm{d}x \\
&\le C\rho^n + \frac{C}{\rho^2}\iint\limits_{Q_{2\rho}(\bar{P})} (1 + |\nabla u_2|^2)\mathrm{d}x\mathrm{d}t + C\|u_1\|^2_{W_{q_3}^{2,1}(Q_{2\rho}(\bar{P}))}(\rho^{2+n})^{1-2/q_3} + \iint\limits_{Q_{2\rho}(\bar{P})} \phi_2^2 \mathrm{d}x\mathrm{d}t \\
&\le C\big[\rho^n + \rho^{n-2+2\hat{\alpha}} + (\rho^{2+n})^{1-2/q_3} + \rho^{n+2\alpha_0}\big] \\
&\le C\rho^{n-2+2\alpha_5},
\end{aligned}$$

where $\alpha_5 = \min\{\hat{\alpha}, 2 - (n+2)/q_3\} = \min\{\hat{\alpha}, 1 + \chi_1\} = \hat{\alpha}$. Thus,

$$\operatorname*{ess\,sup}_{t_{13}\le t\le T} \int\limits_{B_\rho} |\nabla u_2|^2 \mathrm{d}x \le C\rho^{n-2+2\alpha_5}.$$

Using [12, Chapter II, Lemma 5.2] and (2.22) again leads to inequality (3.75) for the case $n \ge 2$. Hence, we complete the proof of the lemma. □

Lemma 3.11. *Function u_2 has weak derivatives $u_{2x_ix_k}$ in $D_{14,T}$ for $i,k = 1,\dots,n$, and satisfies*

$$\|u_2\|_{W_2^{2,1}(D_{14,T})} \le C, \tag{3.77}$$

where $C = C(\Theta_1, \Theta_2, \nu(M_1), \mu(|(M_1, M_2)|))$.

Proof. Let $\bar{P} \in D_{14,T}$ and $\rho \le \rho_6$. Thus $Q_{2\rho}(\bar{P}) \subset D_{13,T}$. By using the similar proof as that of (3.28), we see from the second equality of (2.1) that for any given $k \in \{1, \dots, n\}$,

$$\begin{aligned}&\frac{1}{2}\int_{B_{2\rho}(\bar{x})} u_{2(x_k)}^2(x,\bar{t})\xi_\rho^2(x,\bar{t})\mathrm{d}x + \iint_{Q_{2\rho}(\bar{P})}\left\{-u_{2(x_k)}^2\xi_\rho\xi_{\rho t} + \sum_{i=1}^n [a_{22}(x,t,\mathscr{U})u_{2x_i}]_{(x_k)}\left(u_{2(x_k)}\xi_\rho^2\right)_{x_i}\right\}\mathrm{d}x\mathrm{d}t\\ &= \iint_{Q_{2\rho}(\bar{P})}\left\{-\sum_{i=1}^n [a_{21}(x,t,\mathscr{U})u_{1x_i}]_{(x_k)}\left(u_{2(x_k)}\xi_\rho^2\right)_{x_i} + [b_2(x,t,\mathscr{U},\nabla\mathscr{U})]_{(x_k)}u_{2(x_k)}\xi_\rho^2\right\}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.78}$$

Since

$$[a_{2j}(x,t,\mathscr{U})u_{jx_i}]_{(x_k)} = u_{j(x_k)x_i}\int_0^1 a_{2j}(x^\theta,t,\mathscr{U}^\theta)\mathrm{d}\theta + \sum_{r=1}^2 u_{r(x_k)}\int_0^1 \frac{\partial a_{2j}}{\partial u_r^\theta}u_{jx_i}^\theta\mathrm{d}\theta + \int_0^1\frac{\partial a_{2j}}{\partial x_k^\theta}u_{jx_i}^\theta\mathrm{d}\theta,$$

and

$$[b_2(x,t,\mathscr{U},\nabla\mathscr{U})]_{(x_k)} = \sum_{m=1}^2\sum_{i=1}^n u_{m(x_k)x_i}\int_0^1\frac{\partial b_2(x^\theta,t,\mathscr{U}^\theta,\nabla\mathscr{U}^\theta)}{\partial u_{mx_i}^\theta}\mathrm{d}\theta + \sum_{m=1}^2 u_{m(x_k)}\int_0^1\frac{\partial b_2}{\partial u_m^\theta}\mathrm{d}\theta + \int_0^1\frac{\partial b_2}{\partial x_k^\theta}\mathrm{d}\theta,$$

then by (2.8), (2.9), (3.47)–(3.49), (3.65), and Cauchy's inequality, we further obtain, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}&\sum_{i=1}^n[a_{22}(x,t,\mathscr{U})u_{2x_i}]_{(x_k)}\left(u_{2(x_k)x_i}\xi_\rho^2 + 2u_{2(x_k)}\xi_\rho\xi_{\rho x_i}\right)\\ &\quad\ge |\nabla u_{2(x_k)}|^2\xi_\rho^2\int_0^1 a_{22}(x^\theta,t,\mathscr{U}^\theta)\mathrm{d}\theta - C|u_{2(x_k)}||\nabla u_{2(x_k)}|\xi_\rho^2\int_0^1|\nabla u_2^\theta|\mathrm{d}\theta\\ &\qquad - C|u_{2(x_k)}\xi_\rho||\nabla\xi_\rho|\left[|\nabla u_{2(x_k)}| + \left(|u_{2(x_k)}|+1\right)\int_0^1|\nabla u_2^\theta|\mathrm{d}\theta\right]\\ &\quad\ge \nu(M_1)|\nabla u_{2(x_k)}|^2\xi_\rho^2 - \varepsilon|\nabla u_{2(x_k)}|^2\xi_\rho^2 - \frac{C}{\varepsilon}\left[\left(|u_{2(x_k)}|+1\right)^2\xi_\rho^2\int_0^1|\nabla u_2^\theta|^2\mathrm{d}\theta + |u_{2(x_k)}|^2|\nabla\xi_\rho|^2\right],\\ &\sum_{i=1}^n|[a_{21}(x,t,\mathscr{U})u_{1x_i}]_{(x_k)}\left(u_{2(x_k)x_i}\xi_\rho^2 + 2u_{2(x_k)}\xi_\rho\xi_{\rho x_i}\right)|\\ &\quad\le C(|\nabla u_{1(x_k)}| + |u_{2(x_k)}| + 1)\left(|\nabla u_{2(x_k)}|\xi_\rho^2 + |u_{2(x_k)}|\xi_\rho|\nabla\xi_\rho|\right)\\ &\quad\le \varepsilon|\nabla u_{2(x_k)}|^2\xi_\rho^2 + \frac{C}{\varepsilon}|\nabla u_{1(x_k)}|^2\xi_\rho^2 + \frac{C}{\varepsilon}\left(u_{2(x_k)}^2+1\right)(\xi_\rho^2 + |\nabla\xi_\rho|^2),\end{aligned}$$

and

$$\begin{aligned}&|[b_2(x,t,\mathscr{U},\nabla\mathscr{U})]_{(x_k)}u_{2(x_k)}\xi_\rho^2|\\ &\le C\Bigg[\left(|\nabla u_{2(x_k)}| + |\nabla u_{1(x_k)}|\right)\int_0^1(\tilde{\phi}_{24}(x^\theta,t)+1)\mathrm{d}\theta + |u_{2(x_k)}|\int_0^1(|\nabla u_2^\theta| + \tilde{\phi}_{22}(x^\theta,t)+1)\mathrm{d}\theta + \int_0^1(|\nabla u_2^\theta| + \tilde{\phi}_{23}(x^\theta,t)\\ &\quad + \tilde{\phi}_{21}(x^\theta,t)+1)\mathrm{d}\theta\Bigg]|u_{2(x_k)}|\xi_\rho^2\\ &\le \varepsilon|\nabla u_{2(x_k)}|^2\xi_\rho^2 + \frac{C}{\varepsilon}\left[|\nabla u_{1(x_k)}|^2\xi_\rho^2 + \left(|u_{2(x_k)}|+1\right)^2\xi_\rho^2\int_0^1(|\nabla u_2^\theta|^2 + \tilde{\Phi}_2^2(x^\theta,t)+1)\mathrm{d}\theta\right].\end{aligned}$$

Choosing $\varepsilon = \min\{1/2, \nu(M_1)/6\}$ and substituting the aforementioned three inequalities into (3.78) yields

$$\begin{aligned}&\int\limits_{B_{2\rho}(\bar{x})} u^2_{2(x_k)}(x,\bar{t})\xi^2_\rho(x,\bar{t})\mathrm{d}x + \iint\limits_{Q_{2\rho}(\bar{P})} |\nabla u_{2(x_k)}|^2\xi^2_\rho \mathrm{d}x\mathrm{d}t \\ &\leq C\iint\limits_{Q_{2\rho}(\bar{P})} u^2_{2(x_k)}[\xi^2_\rho + |\nabla\xi_\rho|^2 + |\xi_\rho\xi_{\rho t}|]\mathrm{d}x\mathrm{d}t + C\iint\limits_{Q_{2\rho}(\bar{P})} |\nabla u_{1(x_k)}|^2\xi^2_\rho \mathrm{d}x\mathrm{d}t + C\int\limits_0^1\iint\limits_{Q_{2\rho}(\bar{P})} (|\nabla u_2^\theta|^2 \\ &\quad + \tilde{\Phi}^2_2(x^\theta, t))\left(u^2_{2(x_k)} + 1\right)\xi^2_\rho \mathrm{d}x\mathrm{d}t\mathrm{d}\theta.\end{aligned} \tag{3.79}$$

In view of estimate (3.49), inequality (3.79) has the similar property as (3.31). In addition, note that (3.75) holds for all $n \geq 1$. Then by using (3.75) and the deduction similar to that of (3.26) from (3.31), we can conclude that u_2 has weak derivatives $u_{2x_ix_k}$ in $D_{14,T}$ for $i, k = 1, \ldots, n$, and estimate (3.77) holds. □

Proof of Theorem 2.1. It follows from estimates (3.2) and (3.60) that for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, $u_1, u_2 \in C^{\alpha',\alpha'/2}(\bar{\Omega}' \times [t', T])$ for some $\alpha' \in (0, 1)$, and from (3.49), (3.77) that $u_1 \in W^{2,1}_{q_3}(\Omega' \times (t', T])$ and $u_2 \in W^{2,1}_2(\Omega' \times (t', T])$. □

4 The proof of Theorem 2.2

In this section, assume that hypotheses (H_1) and (H_2) all hold. We will use L^p estimates for parabolic equations, Sobolev embedding theorem and [12, Chapter III, Theorem 12.2] to complete the proof of Theorem 2.2. We use C and $\alpha_j (j = 6, 7, \ldots)$ to denote positive constants depending only on Θ_1, Θ_2, $\nu(M_1)$, $\mu(|(M_1, M_2)|)$, $\|b_1(x, t, \mathscr{W}, \mathbf{p})\|_{C^1(\Xi_3\times[-\Upsilon_1,\Upsilon_1]^n)}$, $\|a_{11x_i}(x, t, w_1), a_{11w_1}\|_{C^{\alpha_0'}(\Xi_4)}$, $\|a_{2jx_i}(x, t, \mathscr{W}), a_{2jw_k}\|_{C^{\alpha_0'}(\Xi_3)}$ $(i = 1, \ldots, n;$ $j, k = 1, 2)$, and the quantities appearing (3.1) and in parentheses, where $\Xi_3 := \bar{D}_{14,T} \times (S_1 \cap [-M_1, M_1]) \times (S_2 \cap [-M_2, M_2])$ and $\Xi_4 := \bar{D}_{14,T} \times (S_1 \cap [-M_1, M_1])$.

Lemma 4.1. *There exist positive constants C and α_6 such that*

$$\|u_1\|_{C^{2+\alpha_6,1+\alpha_6/2}(\bar{D}_{15,T})} \leq C, \quad \alpha_6 \in (0, 1). \tag{4.1}$$

Proof. Consider the following linear problem

$$\begin{cases} w_t = \check{a}_1(x, t)\Delta w + \check{f}_1(x, t) \quad ((x, t) \in D_{14,T}), \\ w(x, t) = 0 \quad (x \in \partial\Omega_{14},\ t \in [t_{14}, T]),\ w(x, t_{14}) = 0\ (x \in \Omega_{14}), \end{cases} \tag{4.2}$$

where

$$\begin{aligned}&\check{a}_1(x, t) = a_{11}(x, t, u_1), \\ &\check{f}_1(x, t) = \{\nabla[a_{11}(x, t, u_1)]\cdot\nabla u_1 + b_1(x, t, \mathscr{U}, \nabla u_1)\}\lambda^2_{15} - a_{11}(x, t, u_1)[2\nabla u_1\cdot\nabla(\lambda^2_{15}) + u_1\Delta(\lambda^2_{15})] + u_1(\lambda^2_{15})_t.\end{aligned}$$

By a direct computation, we find that $\check{w}_1 = u_1\lambda^2_{15}$ is the weak solution of (4.2) in $W^{2,1}_{q_3}(D_{14,T})$. Moreover, hypothesis (H_2) and estimates (3.49) and (3.60) imply that $\check{a}_1(x, t)$ and $\check{f}_1(x, t)$ are Hölder continuous in $\bar{\Omega}_{14} \times [t_{14}, T]$. Then [12, Chapter III, Theorem 12.2] shows that $\check{w}_1$ belongs to $C^{2+\alpha_6,1+\alpha_6/2}(\bar{D}_{14,T})$ and $\|\check{w}_1\|_{C^{2+\alpha_6,1+\alpha_6/2}(\bar{D}_{14,T})}$ is bounded from above by a constant C. This further leads to estimate (4.1). □

To show that $u_2 \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}' \times [t', T])$, we need the estimate of $\|u_2\|_{W^{2,1}_q(D_{18,T})}$.

Lemma 4.2. *For any $q > n + 2$, we have*

$$\|u_2\|_{W^{2,1}_q(D_{18,T})} \leq C. \tag{4.3}$$

Proof. Step 1. We prove that for any positive integer K,

$$\iint_{D_{16,T}} [|\nabla u_2|^{4+2K} + (1+|\nabla u_2|)^{2K}|\nabla^2 u_2|^2]\mathrm{d}x\mathrm{d}t \le C(K). \tag{4.4}$$

Let $B_{2,i}(x,t,w,\mathbf{p})$ and $B_2(x,t,w,\mathbf{p})$ be defined by (3.61). It follows from (3.48), (3.60), and (4.1) that for $(x,t)\in\bar{D}_{15,T}$, $w\in[-M_2,M_2]\cap S_2$ and $\mathbf{p}\in\mathbb{R}^n$, $B_2(x,t,w,\mathbf{p})$ satisfies (3.63), and $B_{2,i}(x,t,w,\mathbf{p})(i=1,\dots,n)$ satisfy

$$\nu(M_1)\sum_{j=1}^n \Psi_j^2 \le \sum_{i,j=1}^n \frac{\partial B_{2,i}(x,t,w,\mathbf{p})}{\partial p_j}\Psi_i\Psi_j \le C\sum_{j=1}^n \Psi_j^2, \tag{4.5}$$

$$\sum_{i=1}^n\left(\left|\frac{\partial B_{2,i}(x,t,w,\mathbf{p})}{\partial w}\right| + |B_{2,i}|\right) + \sum_{i,j=1}^n\left|\frac{\partial B_{2,i}}{\partial x_j}\right| \le C(1+|\mathbf{p}|). \tag{4.6}$$

Let $\psi=\psi(x,t)$ be smooth function satisfying $\psi=0$ for $x\notin\Omega_{15}$. As we have done in Lemma 3.5, for any given $r\in\{1,\dots,n\}$ setting $\eta=\psi_{x_r}$ in the second equality of (2.1), we have

$$\iint_{D_{14,T}}\left[-u_{2t}\psi_{x_r} + \sum_{i=1}^n\left(\sum_{j=1}^n \frac{\partial B_{2,i}(x,t,u_2,\nabla u_2)}{\partial u_{2x_j}}u_{2x_jx_r} + \frac{\partial B_{2,i}}{\partial u_2}u_{2x_r} + \frac{\partial B_{2,i}}{\partial x_r}\right)\psi_{x_i} - B_2(x,t,u_2,\nabla u_2)\psi_{x_r}\right]\mathrm{d}x\mathrm{d}t = 0.$$

Note that (3.75) holds for $n\ge 1$. Then by inequalities (3.63), (3.75), (4.5), and (4.6), and by the proof similar to that of (3.33), we can obtain estimate (4.4).

Step 2. Based on (4.4), the proof similar to that of (3.50) further yields

$$\operatorname*{ess\,sup}_{D_{17,T}}|\nabla u_2| \le \Upsilon_2 \tag{4.7}$$

for some positive constant Υ_2.

Step 3. Consider the following linear problem:

$$\begin{cases} z_t = \bar{a}_2(x,t)\Delta z + \bar{f}_2(x,t) \quad ((x,t)\in D_{17,T}),\\ z(x,t)=0 \quad (x\in\partial\Omega_{17},\quad t\in[t_{17},T]),\quad z(x,t_{17})=0 \quad (x\in\Omega_{17}), \end{cases} \tag{4.8}$$

where

$$\bar{a}_2(x,t) = a_{22}(x,t,\mathscr{U}),$$

$$\begin{aligned}\bar{f}_2(x,t) = {}& \lambda_{18}^2 a_{22}(x,t,\mathscr{U})[b_2(x,t,\mathscr{U},\nabla\mathscr{U}) + \operatorname{div}(a_{21}(x,t,\mathscr{U})\nabla u_1)] + (\lambda_{18}^2)_t\int_{u_2^{**}}^{u_2} a_{22}(x,t,u_1,\omega)\mathrm{d}\omega - a_{22}(x,t,\mathscr{U})\\ &\times\Bigg\{\Delta(\lambda_{18}^2)\int_{u_2^{**}}^{u_2} a_{22}(x,t,u_1,\omega)\mathrm{d}\omega + 2a_{22}(x,t,\mathscr{U})\nabla(\lambda_{18}^2)\cdot\nabla u_2 + 2\nabla(\lambda_{18}^2)\cdot\int_{u_2^{**}}^{u_2}\nabla a_{22}(x,t,u_1,\omega)\mathrm{d}\omega\\ &+\lambda_{18}^2\operatorname{div}\left[\int_{u_2^{**}}^{u_2}\nabla a_{22}(x,t,u_1,\omega)\mathrm{d}\omega\right]\Bigg\},\end{aligned}$$

where $u_2^{**}=u_2(x_{17},t_{17})$ for fixed $x_{17}\in\Omega_{17}$. Hence, $\bar{a}_2(x,t)$ is Hölder continuous in $\bar{\Omega}_{17}\times[t_{17},T]$. It follows from estimates (4.1) and (4.7) that $|\nabla\bar{a}_2|, \bar{f}_2\in L^\infty(D_{17,T})$. Then problem (4.8) has a unique solution z in $W_q^{2,1}(D_{17,T})$ for any $q>n+2$, and

$$\|z\|_{W_q^{2,1}(D_{17,T})} \le C(q). \tag{4.9}$$

Let $\check{z}_2 = \lambda_{18}^2 \int_{u_2^{**}}^{u_2} a_{22}(x, t, u_1, \omega)\mathrm{d}\omega$. A direct computation shows that $\check{z}_2$ is also a solution in $W_2^{2,1}(D_{17,T})$ of (4.8). The similar argument as that of Lemma 3.7 shows that $\check{z}_2 = z$ in $\bar{D}_{17,T}$. Note that $\check{z}_2 = \int_{u_2^{**}}^{u_2} a_{22}(x, t, u_1, \omega)\mathrm{d}\omega$ for $(x, t) \in D_{18,T}$. Thus,

$$\nabla u_2 = \left[\nabla \check{z}_2 - \int_{u_2^{**}}^{u_2} \nabla a_{22}(x, t, u_1, \omega)\mathrm{d}\omega\right] \Big/ a_{22}(x, t, \mathscr{U}) \quad ((x, t) \in D_{18,T}),$$

$$(\nabla u_2)_{x_k} = \left\{(\nabla \check{z}_2)_{x_k} - [a_{22}(x, t, \mathscr{U})]_{x_k} \nabla u_2 - \left[\int_{u_2^{**}}^{u_2} \nabla a_{22}(x, t, u_1, \omega)\mathrm{d}\omega\right]_{x_k}\right\} \Big/ a_{22}(x, t, \mathscr{U}) \quad ((x, t) \in D_{18,T}).$$

These, together with (4.9), leads to estimate (4.3). □

Furthermore, estimate (4.3) and Sobolev embedding theorem yield

$$\|u_{2x_i}\|_{C^{\alpha_7,\alpha_7/2}(\bar{D}_{18,T})} \leq C, \quad \alpha_7 \in (0, 1), \quad i = 1, \dots, n. \tag{4.10}$$

Lemma 4.3. *There exist positive constants C, α_8 such that*

$$\|u_2\|_{C^{2+\alpha_8,1+\alpha_8/2}(\bar{D}_{19,T})} \leq C, \tag{4.11}$$

where C, α_8 depend on $\|b_1(x, t, \mathscr{W}, \mathbf{p})\|_{C^1(\Xi_5 \times [-\Upsilon_1,\Upsilon_1]^n)}$ and $\|b_2(x, t, \mathscr{W}, \mathbb{P})\|_{C^1(\Xi_5 \times [-\Upsilon_1,\Upsilon_1]^n \times [-\Upsilon_2,\Upsilon_2]^n)}$, $\Xi_5 := \bar{D}_{18,T} \times (S_1 \cap [-M_1, M_1]) \times (S_2 \cap [-M_2, M_2])$.

Proof. Let $\check{w}_2 = u_2 \lambda_{19}^2$. Then a direct computation shows that $\check{w}_2 \in W_q^{2,1}(D_{18,T})$ satisfies

$$\check{w}_{2t} = \check{a}_2(x, t)\Delta \check{w}_2 + \check{f}_2(x, t) \quad ((x, t) \in D_{18,T}),$$
$$\check{w}_2(x, t) = 0 \quad (x \in \partial\Omega_{18},\ t \in [t_{18}, T]), \quad \check{w}_2(x, t_{18}) = 0 \quad (x \in \Omega_{18}),$$

where

$$\check{a}_2(x, t) = a_{22}(x, t, \mathscr{U}),$$
$$\check{f}_2(x, t) = \{\nabla a_{22}(x, t, \mathscr{U}) \cdot \nabla u_2 + b_2(x, t, \mathscr{U}, \nabla \mathscr{U}) + \operatorname{div}[a_{21}(x, t, \mathscr{U})\nabla u_1]\}\lambda_{19}^2$$
$$- a_{22}(x, t, \mathscr{U})[2\nabla u_2 \cdot \nabla(\lambda_{19}^2) + u_2 \Delta(\lambda_{19}^2)] + u_2(\lambda_{19}^2)_t,$$

By using (4.1), (4.10), and hypothesis (H$_2$), we find that $\check{a}_2(x, t)$ and $\check{f}_2(x, t)$ are Hölder continuous in $\bar{D}_{18,T}$. Again by [12, Chapter III, Theorem 12.2], we conclude that $\check{w}_2$ is in $C^{2+\tilde{\alpha},1+\tilde{\alpha}/2}(\bar{D}_{18,T})$ and u_2 satisfies estimate (4.11). □

Proof of Theorem 2.2. Lemmas 4.1 and 4.3 show that for any $\Omega' \subset\subset \Omega$ and $t' \in (0, T)$, $u_1, u_2 \in C^{2+\alpha',1+\alpha'/2}(\bar{\Omega}' \times [t', T])$ for some $\alpha' \in (0, 1)$. □

5 Applications

In this section, we give some applications to the regularity of nonnegative weak solutions for two ecological models with cross-diffusion.

5.1 Regularity of bounded nonnegative weak solutions for (1.0)

Consider system (1.0). It is a special case of (1.1) with

$$\begin{cases} a_{11}(x,t,[\mathscr{U}]_1) = \kappa_1(x,t) + 2\gamma_{11}(x,t)u_1, \quad a_{21}(x,t,[\mathscr{U}]_2) = \gamma_{21}(x,t)u_2, \\ a_{22}(x,t,[\mathscr{U}]_2) = \kappa_2(x,t) + \gamma_{21}(x,t)u_1 + 2\gamma_{22}(x,t)u_2, \end{cases}$$

and

$$\begin{cases} b_1(x,t,\mathscr{U},\nabla u_1) = (\nabla\kappa_1 + 2u_1\nabla\gamma_{11})\cdot\nabla u_1 + e_1\nabla\varphi\cdot\nabla u_1 + u_1\Delta\kappa_1 + u_1^2\Delta\gamma_{11} \\ \qquad + u_1 \operatorname{div}(e_1\nabla\varphi) + [d_{11}(x,t) + d_{12}(x,t)u_1 + d_{13}(x,t)u_2]u_1, \\ b_2(x,t,\mathscr{U},\nabla[\mathscr{U}]_2) = (\nabla\kappa_2 + u_1\nabla\gamma_{21} + 2u_2\nabla\gamma_{22})\cdot\nabla u_2 + u_2\nabla\gamma_{21}\cdot\nabla u_1 + e_2\nabla\varphi\cdot\nabla u_2 \\ \qquad + u_2\Delta\kappa_2 + u_1u_2\Delta\gamma_{21} + u_2^2\Delta\gamma_{22} + u_2 \operatorname{div}(e_2\nabla\varphi) \\ \qquad + [d_{21}(x,t) + d_{22}(x,t)u_1 + d_{23}(x,t)u_2]u_2. \end{cases}$$

Corollary 5.1. *Assume that functions κ_i, γ_{2i}, and γ_{11} $(i = 1, 2)$ belong to $C^{1+\alpha}(D_T)$ for some $\alpha \in (0,1)$, and that functions $\kappa_{ix_kx_k}$, $\gamma_{2ix_kx_k}$, $\gamma_{11x_kx_k}$, $\operatorname{div}(e_i\nabla\varphi)$, and d_{ij} $(i = 1,2;\ j = 1,2,3;\ k = 1,\dots,n)$ belong to $C^1(D_T)$. Let $\mathscr{U} = (u_1,u_2)$ be a bounded nonnegative weak solution of* (1.0). *Then for any $\Omega' \subset\subset \Omega$ and $t' \in (0,T)$, there exists $\alpha' \in (0,1)$, such that u_1, u_2 belong to $C^{2+\alpha',1+\alpha'/2}(\bar{\Omega}' \times [t',T])$.*

Proof. Since $\mathscr{U}$ is a bounded nonnegative weak solution of (1.0), then $\mathcal{S} = [\operatorname{ess\,inf}_{D_T} u_1, \operatorname{ess\,sup}_{D_T} u_1] \times [\operatorname{ess\,inf}_{D_T} u_2, \operatorname{ess\,sup}_{D_T} u_2] \subseteq [0, \operatorname{ess\,sup}_{D_T} u_1] \times [0, \operatorname{ess\,sup}_{D_T} u_2]$. By a direct computation, we find that hypotheses (H_1) and (H_2) all hold. Then employing Theorem 2.2, we obtain this corollary. □

5.2 Regularity of nonnegative weak solutions for a predator–prey model

Consider a predator–prey model with two-species and with cross-diffusion. Let u_1 and u_2 be the population densities of predator and prey, respectively. Assume that the prey exhibits group defense and the cross-diffusion pressure of the predator is zero. If the mankind's influence is taken into account, function $\mathscr{U} = (u_1,u_2)$ satisfies triangular parabolic system

$$\begin{cases} u_{1t} = \Delta[\kappa_1(x,t)u_1] + \operatorname{div}[e_1(x,t)u_1\nabla\varphi] + \dfrac{K_1u_1u_2}{\varrho_1 + u_2^2} - d_{11}(x,t)u_1 + \dfrac{d_{12}(x,t)u_1^{1+\underline{m}_n}}{\varrho_2 + u_1} \quad ((x,t)\in D_T), \\ u_{2t} = \Delta[\kappa_2(x,t)u_2 + \dfrac{\gamma_{21}u_1u_2}{\varrho_0 + u_2}] + \operatorname{div}[e_2(x,t)u_2\nabla\varphi] + d_{21}(x,t)u_2 - \dfrac{K_2u_1u_2}{\varrho_1 + u_2^2} \\ \qquad + \dfrac{d_{22}(x,t)u_2^{1+\underline{m}_n}}{\varrho_2 + u_2} \quad ((x,t)\in D_T), \end{cases} \tag{5.1}$$

where $\kappa_1, \kappa_2 > 0$ are the diffusion rates, and φ is a known outer potential, $d_{11}(x,t) > 0$ is the death rate of the predator, $d_{21}(x,t) > 0$ is the growth rate of the prey, and γ_{21}, K_1, K_2, ϱ_0, ϱ_1, ϱ_2 are positive constants, and where $\gamma_{21}u_1u_2/(\varrho_0 + u_2)$ represents the cross-diffusion pressures of the prey, $u_2/(\varrho_1 + u_2^2)$ represents predator–prey interaction when the prey exhibits group defense (see [7]), and $d_{12}(x,t)u_1^{1+\underline{m}_n}/(\varrho_2 + u_1)$ and $d_{22}(x,t)u_2^{1+\underline{m}_n}/(\varrho_2 + u_2)$ represent the mankind's influence.

Corollary 5.2. *Let n be in $\{1,2,3\}$, and let $\underline{m}_1 = 2$, $\underline{m}_2 = 9/5$, and $\underline{m}_3 = 6/5$. Assume that functions κ_1 and κ_2 belong to $C^{1+\alpha}(D_T)$ for some $\alpha \in (0,1)$, and that functions $\kappa_{ix_kx_k}$, $\operatorname{div}(e_i\nabla\varphi)$ and d_{ij} $(i,j = 1,2;\ k = 1,\dots,n)$ belong to $C^1(D_T)$. Let $\mathscr{U} = (u_1,u_2)$ be a nonnegative weak solution of* (5.1). *Then for any $\Omega' \subset\subset \Omega$ and $t' \in (0,T)$, there exists $\alpha' \in (0,1)$, such that u_1, u_2 belong to $C^{2+\alpha',1+\alpha'/2}(\bar{\Omega}' \times [t',T])$.*

Proof. System (5.1) is a special case of (1.1) with

$$a_{11}(x,t,[\mathscr{U}]_1) = \kappa_1(x,t), \quad a_{21}(x,t,[\mathscr{U}]_2) = \frac{\gamma_{21}u_2}{\varrho_0 + u_2}, \quad a_{22}(x,t,[\mathscr{U}]_2) = \kappa_2(x,t) + \frac{\gamma_{21}\varrho_0 u_1}{(\varrho_0 + u_2)^2}$$

and

$$\begin{cases} b_1(x,t,\mathscr{U},\nabla u_1) = \nabla\kappa_1\cdot\nabla u_1 + e_1\nabla\varphi\cdot\nabla u_1 + u_1\Delta\kappa_1 + u_1\ \mathrm{div}(e_1\nabla\varphi) + \dfrac{K_1u_1u_2}{\varrho_1 + u_2^2} - d_{11}(x,t)u_1 + \dfrac{d_{12}(x,t)u_1^{1+\underline{m}_n}}{\varrho_2 + u_1}, \\ b_2(x,t,\mathscr{U},\nabla\mathscr{U}) = \nabla\kappa_2\cdot\nabla u_2 + e_2\nabla\varphi\cdot\nabla u_2 + u_2\Delta\kappa_2 + u_2\ \mathrm{div}(e_2\nabla\varphi) + d_{21}(x,t)u_2 - \dfrac{K_2u_1u_2}{\varrho_1 + u_2^2} + \dfrac{d_{22}(x,t)u_2^{1+\underline{m}_n}}{\varrho_2 + u_2}. \end{cases}$$

Since $\mathscr{U}$ is a nonnegative weak solution of (5.1), then $\mathcal{S} = [\mathrm{ess\,inf}_{D_T}u_1, +\infty) \times [\mathrm{ess\,inf}_{D_T}u_2, +\infty) \subseteq [0,+\infty)\times[0,+\infty)$. We can take

$$\begin{cases} q_0 = 6, & q_1 = 2, & \chi_1 = 1/4, & \text{for } n = 1, \\ q_0 = 4, & q_1 = 20/9, & \chi_1 = 1/10, & \text{for } n = 2, \\ q_0 = 10/3, & q_1 = 25/9, & \chi_1 = 1/10, & \text{for } n = 3. \end{cases}$$

Then hypotheses (H_1) and (H_2) are satisfied with

$$\delta_l = 0, \quad \sigma_{ll} = \underline{m}_n, \quad \sigma_{12} = 0, \quad \tilde{\sigma}_{1r} = 0, \quad \phi_l = \tilde{\phi}_{lr}(x,t) = 0, \quad r = 1,\dots,4;\ l = 1,2.$$

By Theorem 2.2, we obtain the result of the corollary. □

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References

[1] H. Amann, *Dynamic theory of quasilinear parabolic equations -I. Abstract evolution equations*, Nonlinear Anal. **12** (1988), no. 9, 895–919.
[2] H. Amann, *Dynamic theory of quasilinear parabolic systems-III. Global existence*, Math. Z. **202** (1989), no. 2, 219–250.
[3] H. Amann, *Dynamic theory of quasilinear parabolic equations-II. Reaction-diffusion systems*, Differ. Integral Equ. **25** (1990), no. 1, 13–75.
[4] H. Amann, *Abstract methods in differential equations*, Rev. R. Acad. Cienc. Ser. A Mat. **97** (2003), no. 1, 89–105.
[5] M. Bendahmane and M. Langlais, *A reaction-diffusion system with cross-diffusion modeling the spread of an epidemic disease*, J. Evol. Equ. **10** (2010), no. 4, 883–904.
[6] L. Chen, and A. Jüngel, *Analysis of a multi-dimensional parabolic population model with strong crossdiffusion*, SIAM J. Math. Anal. **36** (2004), no. 1, 301–322.
[7] X. Chen, Y. Qi, and M. Wang, *A strongly coupled predator–prey system with non-monotonic functional response*, Nonlinear Anal. **67** (2007), no. 6, 1966–1979.
[8] Y. S. Choi, Z. Huan, and R. Lui, *Global existence of solutions of a stronglycoupled quasilinear parabolic system with applications to electrochemistry*, J. Differ. Equ. **194** (2003), no. 2, 406–432.
[9] Y. S. Choi, R. Lui, and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross diffusion*, Discrete Contin. Dyn. Syst. **10** (2004), no. 3, 719–730.

[10] G. Galiano, *On a cross-diffusion population model deduced from mutation and splitting of a single species*, Comput. Math. Appl. **64** (2012), no. 6, 1927–1936.

[11] G. Galiano, A. Jüngel, and J. Velasco, *A parabolic cross-diffusion system for granular materials*, SIAM J. Math. Anal. **35** (2003), no. 3, 561–578.

[12] O. A. Ladyzenskaya, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, USA, 1968.

[13] O. A. Ladyzenskaya and N. N. Ural'ceva, *Linear and Quasi-linear Elliptic Equations*, Academic Press, New York, 1968.

[14] D. Le, *Hölder regularity for certain strongly coupled parabolic systems*, J. Differ. Equ. **151** (1999), no. 2, 313–344.

[15] D. Le, *Global existence for a class of strongly coupled parabolic systems*, Ann. Mat. Pura Appl. **185** (2006), no. 1, 133–154.

[16] D. Le, *Regularity of BMO weak solutions to nonlinear parabolic systems via homotopy*, Trans. Am. Math. Soc. **365** (2013), no. 5, 2723–2753.

[17] D. Le, *Partial regularity of solutions to a class of strongly coupled degenerate parabolic systems*, Discrete Contin. Dyn. Syst. **2005** (2005) (suppl.), 576–586.

[18] Y. Lou and W. M. Ni, *Diffusion, vs. cross-diffusion: an elliptic approach*, J. Differ. Equ. **154** (1999), no. 1, 157–190.

[19] Y. Lou, W. M. Ni, and Y. P. Wu, *On the global existence of a cross-diffusion system*, Discrete Contin. Dyn. Syst. **4** (1998), no. 2, 193–203.

[20] W. M. Ni, Y. Wu, and Q. Xu, *The existence and stability of nontrivial steady states for S-K-T competition model with cross diffusion*, Discrete Contin. Dyn. Syst. Ser. A **34** (2014), no. 12, 5271–5298.

[21] N. Shigesada, K. Kawasaki, and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biol. **79** (1979), no. 1, 83–99.

[22] S. A. Shim, *Uniform boundedness and convergence of solutions to cross-diffusion systems*, J. Differ. Equ. **185** (2002), no. 1, 281–305.

[23] S. A. Shim, *Uniform boundedness and convergence of solutions to the systems with a single nonzero cross-diffusion*, J. Math. Anal. Appl. **279** (2003), no. 1, 1–21.

[24] J. Stara, O. John, and J. Maly, *Counterexample to the regularity of weak solution of the quasilinear parabolic system*, Comment. Math. Univ. Carolina **27** (1986), no. 1, 123–136.

[25] Q. J. Tan, *An elliptic cross-diffusion system describing two-species models on a bounded domain with different natural conditions*, J. Math. Anal. Appl. **437** (2016), no. 1, 853–869.

[26] Q. J. Tan, *A free boundary problem describing S-K-T competition ecological model with cross-diffusion*, Nonlinear Anal. Real World Appl. **45** (2019), 53–82.

[27] Q. J. Tan, *Global existence of classical solutions for a class of diffusive ecological models with two free boundaries and cross-diffusion*, Nonlinear Anal. Real World Appl. **60** (2021), no. 6, 103302.

[28] P. V. Tuoc, *On global existence of solutions to a cross-diffusion system*, J. Math. Anal. Appl. **343** (2008), no. 1, 826–834.

[29] F. J. Vermolen and C. Vuik, *Solution of vector Stefan problems with cross-diffusion*, J. Comput. Appl. Math. **176** (2005), no. 1, 179–201.

[30] M. Wang, *Stationary patterns caused by cross-diffusion for a three-species prey-predator model*, Comput. Appl. Math. **52** (2006), no. 5, 707–720.