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Research Article

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Multiple solutions to multi-critical Schrödinger equations

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Abstract: In this article, we investigate the existence of multiple positive solutions to the following multi-critical Schrödinger equation:

$$\begin{cases} -\Delta u + \lambda V(x)u = \mu|u|^{p-2}u + \sum_{i=1}^{k}\left(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\right)|u|^{2_i^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{0.1}$$

where $\lambda, \mu \in \mathbb{R}^+, N \geq 4$, and $2_i^* = \frac{N+\alpha_i}{N-2}$ with $N-4 < \alpha_i < N$, $i = 1, 2, \ldots, k$ are critical exponents and $2 < p < 2_{\min}^* = \min\{2_i^* : i = 1, 2, \ldots, k\}$. Suppose that $\Omega = \operatorname{int} V^{-1}(0) \subset \mathbb{R}^N$ is a bounded domain, we show that for λ large, problem (0.1) possesses at least $\operatorname{cat}_\Omega(\Omega)$ positive solutions.

Keywords: multi-critical Schrödinger equation, multiple solutions, Lusternik-Schnirelman theory

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1 Introduction

In this article, we investigate the existence of multiple positive solutions to the following multi-critical Schrödinger equation:

$$\begin{cases} -\Delta u + \lambda V(x)u = \mu|u|^{p-2}u + \sum_{i=1}^{k}\left(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\right)|u|^{2_i^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\lambda, \mu \in \mathbb{R}^+, N \geq 4$, and $2_i^* = \frac{N+\alpha_i}{N-2}$ with $N-4 < \alpha_i < N$, $i = 1, 2, \ldots, k$ are critical Hardy-Littlewood-Sobolev exponents and $2 < p < 2_{\min}^* = \min\{2_i^* : i = 1, 2, \ldots, k\}$.

In the case $k = 1$, problem (1.1) is the so-called Choquard equation. It has been extensively studied for subcritical and critical cases in [11,13,14,18,19,20] and references therein. Particularly, multiple solutions for the nonlinear Choquard problem

$$\begin{cases} -\Delta u + \lambda u = (|x|^{-(N-\alpha)} * |u|^q)|u|^{q-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{1.2}$$

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in the bounded Ω are obtained in [13] by the Lusternik-Schnirelman theory. It was proved that if q is closed to the critical exponent $2_\alpha^* = \frac{N+\alpha}{N-2}$, problem (1.2) possesses at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions; similar results were obtained in [14] for the critical case $q = 2_\alpha^*$. This sort of result extends early works for elliptic problem with local nonlinearities by [1,3,5,6,10,15] etc.

A counterpart for the Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{1.3}$$

was considered in [7] and [8]. A graph of the potential function $V(x)$ affects the existence of number of solutions. Precisely, assume, among other things, that $\Omega := V^{-1}(0) = \{x \in \mathbb{R}^N : V(x) = 0\}$ is bounded, it is proved in [8] for the subcritical case and in [7] for the critical case that problem (1.3) possesses at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions if $\varepsilon > 0$ is small.

Analogous problem for the Schrödinger equation

$$-\Delta u + (\lambda V(x) + 1)u = u^{p-1} \quad u \in H^1(\mathbb{R}^N), \tag{1.4}$$

was investigated in [4] and [9] for subcritical and critical cases, respectively. Under certain conditions on $V(x)$, it is obtained in [4] and [9] $\mathrm{cat}_\Omega(\Omega)$ positive solutions for problem (1.4) if $\lambda > 0$ large. Various extensions of the results in [4] and [9] can be found in [1,2,18].

In this article, we focus on problem (1.1) in the case $k > 2$, multi-critical terms then involved in. In the bounded domain Ω, the existence of positive solution of the multi-critical Sobolev-Hardy problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{|x|^s}u^{p-1} + \displaystyle\sum_{i=1}^{l}\frac{\lambda_i}{|x|^{s_i}}u^{2^*(s_i)-1} + u^{2^*-1} & \text{in}\,\Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.5}$$

was studied in [12], while the number of positive solutions of the Choquard-type problem

$$\begin{cases} -\Delta u = \mu|u|^{p-2}u + \displaystyle\sum_{i=1}^{k}\left(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\right)|u|^{2_i^*-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{1.6}$$

is described in [17] by the Lusternik-Schnirelman category $\mathrm{cat}_\Omega(\Omega)$ of the domain Ω. Such a result seems to be difficult to establish for problem (1.5).

Inspired by [4] and [9], we will show that problem (1.1) has at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions based on the result in [17].

Our hypotheses are as follows.

$(V1)$ $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \geqslant 0$, and $\Omega :=$ int $V^{-1}(0)$ is a nonempty bounded set with smooth boundary, and $\overline{\Omega} = V^{-1}(0)$.

$(V2)$ There exists $M_0 > 0$ such that

$$\mathrm{mes}\{x \in \mathbb{R}^N : V(x) \leqslant M_0\} < \infty.$$

Let μ_1 be the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet condition. We present our first result as follows.

Theorem 1.1. *Suppose* $(V1)$ *and* $(V2)$ *hold, then for every* $0 < \mu < \mu_1$, *there exists* $\lambda(\mu) > 0$ *such that problem* (1.1) *has at least one ground state positive solution for each* $\lambda \geqslant \lambda(\mu)$.

Positive solutions of problem (1.1) are found as critical points of the associated functional

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N}(|\nabla u|^2 + \lambda V(x)u^2)\mathrm{d}x - \frac{\mu}{p}\int\limits_{\mathbb{R}^N}(u^+)^p\mathrm{d}x - \sum_{i=1}^{k}\frac{1}{22_i^*}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y, \tag{1.7}$$

where $u^+ := \max\{u, 0\}$, in the Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < \infty \right\} \tag{1.8}$$

endowed with the norm

$$\|u\|^2 = \|u\|_{H^1}^2 + \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x. \tag{1.9}$$

It is standard to show that a nontrivial critical point of $I_{\lambda,\mu}$ is nonnegative the maximum principle then implies that it is positive.

Theorem 1.1 is proved by the mountain pass theorem. A crucial ingredient is to verify the $(\mathrm{PS})_c$ condition for the functional $I_{\lambda,\mu}(u)$. By a $(\mathrm{PS})_c$ condition we mean a sequence $\{u_n\}$ in E satisfying $I_{\lambda,\mu}(u_n) \to c$, $I'_{\lambda,\mu}(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence.

We will show that $I_{\lambda,\mu}(u)$ satisfies $(\mathrm{PS})_c$ condition if $c < m(\mathbb{R}^N)$ with

$$m(\mathbb{R}^N) := \inf\left\{ J(u) : u \in \mathcal{N}_{\mathbb{R}^N} \right\}, \tag{1.10}$$

where

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \sum_{i=1}^{k} \frac{1}{22_i^*} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y \tag{1.11}$$

and

$$\mathcal{N}_{\mathbb{R}^N} := \{u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\} : \langle J'(u), u\rangle = 0\}. \tag{1.12}$$

It is shown in [17] that $m(\mathbb{R}^N)$ is uniquely achieved up to translations and dilations by the function

$$U(x) = C\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}},$$

which is a solution for the limit problem

$$-\Delta u = \sum_{i=1}^{k} \left(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\right)|u|^{2_i^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.13}$$

Next, we consider the limit behavior of solutions (1.1). Let $\lambda_n \to \infty$ and $\{u_n\}$ be the corresponding solutions of (1.1). Then, we will show that $\{u_n\}$ concentrates at a solution of the limit problem

$$\begin{cases} -\Delta u = \mu|u|^{p-2}u + \sum_{i=1}^{k} \left(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\right)|u|^{2_i^*-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.14}$$

Theorem 1.2. *Suppose* (*V*1) *and* (*V*2) *hold. Let* $\{u_n\}$ *be a sequence of solutions of* (1.1) *such that* $0 < \mu < \mu_1$, $\lambda_n \to \infty$, *and* $I_{\lambda_n,\mu}(u_n) \to c < m(\mathbb{R}^N)$ *as* $n \to \infty$. *Then,* $\{u_n\}$ *concentrates at a solution* u *of* (1.14), *that is,* $u_n \to u$ *in* E as $n \to \infty$.

Finally, we derive from the Lusternik-Schnirelman theory a multiplicity result for problem (1.1).

Theorem 1.3. *Suppose* (*V*1) *and* (*V*2) *hold and* $N \geqslant 4$. *Then there exists* $0 < \mu^* < \mu_1$, *for each* $0 < \mu \leqslant \mu^*$, *there is* $\Lambda(\mu) > 0$ *such that problem* (1.1) *has at least* $\mathrm{cat}_\Omega(\Omega)$ *positive solutions whenever* $\lambda \geqslant \Lambda(\mu)$.

The article is organized as follows. In Section 2, we present preliminary results. Then we prove Theorems 1.1 and 1.2 in Section 3. The multiple result in Theorem 1.3 is established in Section 4.

2 Preliminaries

In the sequel, we denote by $|u|_q$ the norm of u in $L^q(\mathbb{R}^N)$ and

$$\|u\|_\lambda^2 = \|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x$$

for $\lambda > 0$, which is equivalent to that defined in (1.9).

Now, by the Hardy-Littlewood-Sobolev inequality in [16], we see that the functionals $I_{\lambda,\mu}(u)$ and $J(u)$ are well-defined and differentiable in E and $D^{1,2}(\mathbb{R}^N)$, respectively.

Lemma 2.1. *Let $\lambda_n \geqslant 1$ and $u_n \in E$ be such that $\|u_n\|_{\lambda_n}^2 \leq K$ for $\lambda_n \to \infty$. Then there is a $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Moreover, if $u = 0$, we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 \leq p < \frac{2N}{N-2}$ as $n \to \infty$.*

Proof. Since for $\lambda_n \geqslant 1$, $\|u_n\|^2 \leqslant \|u_n\|_{\lambda_n}^2 \leq K$, we may assume $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Set $C_m := \left\{x : V(x) \geqslant \frac{1}{m}\right\}$ for $m \in \mathbb{N}$. Then

$$\mathbb{R}^N \backslash \overline{\Omega} = \bigcup_{m=1}^{+\infty} C_m,$$

and for all $m \in \mathbb{N}$,

$$\int_{C_m} |u_n|^2 \mathrm{d}x \leqslant m \int_{C_m} V|u_n|^2 \mathrm{d}x = \frac{m}{\lambda_n} \int_{C_m} \lambda_n V |u_n|^2 \mathrm{d}x \leqslant \frac{m}{\lambda_n} \|u_n\|_{\lambda_n}^2 \leq \frac{m}{\lambda_n} K.$$

Hence, $u_n \to 0$ in $L^2(C_m)$ as $n \to \infty$. By a diagonal process, we may assume that $u_n \to 0$ in $L^2(\mathbb{R}^N \backslash \overline{\Omega})$ as $n \to \infty$.

In order to show that $u_n \to u$ in $L^2(\mathbb{R}^N)$, we set for $R > 0$ that

$$A(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) \geqslant M_0\}$$

and

$$B(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) < M_0\},$$

where M_0 is given in $(V2)$. Since $A(R) \subset C_m$ if m large, we have

$$\int_{A(R)} |u_n|^2 \mathrm{d}x \leqslant \int_{C_m} |u_n|^2 \mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{2.1}$$

By the Hölder inequality, for $1 < p < \frac{N}{N-2}$, $\frac{1}{p} + \frac{1}{p'} = 1$ there holds

$$\int_{B(R)} |u_n - u|^2 \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N} |u_n - u|^{2p} \mathrm{d}x \right)^{\frac{1}{p}} |B(R)|^{\frac{1}{p'}} \leqslant C \|u_n - u\|_{\lambda_n}^2 |B(R)|^{\frac{1}{p'}} = o_R(1),$$

where $o_R(1) \to 0$ as $R \to \infty$. Then, for $\varepsilon > 0$, there exists $N > 0$ such that for $n > N$,

$$\int_{\mathbb{R}^N} |u_n - u|^2 \mathrm{d}x \leqslant \int_{B_R} |u_n - u|^2 \mathrm{d}x + \int_{A(R)} |u_n|^2 \mathrm{d}x + \int_{A(R)} |u|^2 \mathrm{d}x < \varepsilon.$$

It yields that $u_n \to u$ in $L^2(\mathbb{R}^N)$.

Now, we turn to the last assertion. If $u_n \rightharpoonup 0$, for $2 < p < \frac{2N}{N-2}$ and any subset $\tilde{\Omega} \subset \mathbb{R}^N$, the interpolation inequality yields

$$\int_{\tilde{\Omega}} |u_n|^p \,dx \leqslant C\left(\int_{\tilde{\Omega}} |\nabla u_n|^2 \,dx\right)^{\frac{\theta p}{2}} \left(\int_{\tilde{\Omega}} |u_n|^2 \,dx\right)^{\frac{(1-\theta)p}{2}} \leqslant C\|u_n\|_{\lambda_n}^{\theta p}\left(\int_{A(R)} u_n^2 dx + \int_{B(R)} u_n^2 dx\right)^{\frac{(1-\theta)p}{2}} < \varepsilon, \tag{2.2}$$

for n large, where $\theta = \frac{N(p-2)}{2p}$. Choosing $\tilde{\Omega}$ to be $A(R)$ and $B(R)$, respectively, we obtain $u_n \to 0$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. □

Let $L_\lambda := -\Delta + \lambda V$ be the self-adjoint operator acting on $L^2(\mathbb{R})$ with the domain E for all $\lambda \geqslant 0$. We denote by $(\cdot,\cdot)$ the L^2-inner product and write

$$(L_\lambda u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x) uv) dx$$

for $u, v \in E$. It implies that

$$(L_\lambda u, u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x) u^2) dx \leqslant \|u\|_\lambda^2. \tag{2.3}$$

Denote by $a_\lambda := \inf \sigma(L_\lambda)$ the infimum of the spectrum of L_λ. It follows that

$$0 \leqslant a_\lambda = \inf\{(L_\lambda u, u) : u \in E, |u|_2 = 1\},$$

which is nondecreasing about the parameter λ. It is proved in [9] the following result.

Lemma 2.2. *For every $0 < \mu < \mu_1$, there exists $\lambda(\mu) > 0$ such that*

$$a_\lambda \geqslant \frac{\mu + \mu_1}{2}$$

if $\lambda \geqslant \lambda(\mu)$. Moreover, for all $u \in E$ and $\lambda \geqslant \lambda(\mu)$ there exists a constant $C > 0$ such that

$$C\|u\|_\lambda^2 \leqslant (L_\lambda u, u),$$

where $C \le (\mu + \mu_1)/(2 + \mu + \mu_1)$.

3 Positive solutions and concentration

In this section, we will prove the existence of positive solution for problem (1.1) and investigate the limit behavior of the solution as the parameter $\lambda \to \infty$.

Lemma 3.1. *If $\mu \in (0, \mu_1)$ and $\lambda \geqslant \lambda(\mu)$, then every $(PS)_c$ sequence $\{u_n\} \subset E$ for $I_{\lambda,\mu}$ is bounded in E.*

Proof. Since $2 < p < 2^*_{\min}$, by Lemma 2.2 we have

$$\begin{aligned} c + 1 + o(1)\|u_n\|_\lambda &\geqslant I_{\lambda,\mu}(u_n) - \frac{1}{p}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)(L_\lambda u_n, u_n) + \sum_{i=1}^{k}\left(\frac{1}{p} - \frac{1}{2 \cdot 2^*_i}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^+(x)^{2^*_i} u_n^+(y)^{2^*_i}}{|x-y|^{N-\alpha_i}} dx dy \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right) C\|u_n\|_\lambda^2. \end{aligned}$$

This implies that $\{u_n\}$ is bounded in E. □

Now we verify that $I_{\lambda,\mu}(u)$ has the mountain pass geometry.

Lemma 3.2. *For all $\mu \in (0, \mu_1)$ and $\lambda \geqslant \lambda(\mu)$, the functional $I_{\lambda,\mu}$ satisfies that*
(*i*) *$I_{\lambda,\mu}(0) = 0$ and there exist $\rho, \delta > 0$, such that $\inf_{\|u\|_\lambda=\rho} I_{\lambda,\mu}(u) \geq \delta > 0$;*
(*ii*) *There exists $e \in E$ such that $\|e\|_\lambda > \rho$ and $I_{\lambda,\mu}(e) \leq 0$.*

Proof. By the Hardy-Littlewood-Sobolev inequality, we have

$$I_{\lambda,\mu}(u) \geqslant C\left(\|u\|_\lambda^2 - \|u\|_\lambda^p - \sum_{i=1}^{k} \|u\|_\lambda^{2\cdot 2_i^*}\right).$$

Then, (*i*) follows by choosing $\|u\|_\lambda = \rho$ sufficiently small. (*ii*) is valid since for any fixed positive $u \in E\backslash\{0\}$, $I_{\lambda,\mu}(tu) \to -\infty$ as $t \to +\infty$. □

By Lemma 3.2 and the mountain pass theorem, we know that there exists a $(\mathrm{PS})_c$ sequence of the functional $I_{\lambda,\mu}(u)$ at the mountain pass level

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

where

$$\Gamma_\lambda = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, I_{\lambda,\mu}(\gamma(1)) \leqslant 0, \gamma(1) \neq 0\}.$$

Define

$$m_{\lambda,\mu} := \inf_{u\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u)$$

and

$$c_{\lambda,\mu}^s = \inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}} \sup_{t>0} I_{\lambda,\mu}(tu).$$

We may show as [21] that

$$m_{\lambda,\mu} = c_{\lambda,\mu} = c_{\lambda,\mu}^s. \tag{3.1}$$

In the same spirit, we may define for the functional

$$I_{\mu,\Omega}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p}\int_\Omega \mu(u^+)^p \mathrm{d}x - \sum_{i=1}^{k} \frac{1}{22_i^*}\int_\Omega\int_\Omega \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y \tag{3.2}$$

associated with problem (1.14) the mountain pass level

$$c_{\mu,\Omega} = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_{\mu,\Omega}(\gamma(t)),$$

where

$$\Gamma_0 = \{\gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0, I_{\mu,\Omega}(\gamma(1)) \leqslant 0, \gamma(1) \neq 0\},$$

and

$$c_{\mu,\Omega}^s = \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \sup_{t>0} I_{\mu,\Omega}(tu)$$

as well as

$$m_{\mu,\Omega} := \inf\{I_{\mu,\Omega}(u) : u \in \mathcal{N}_{\mu,\Omega}\}, \tag{3.3}$$

where

$$\mathcal{N}_{\mu,\Omega} := \{u \in H_0^1(\Omega)\backslash\{0\} : \langle I'_{\mu,\Omega}(u), u\rangle = 0\} \tag{3.4}$$

is the corresponding Nehari manifold. Similarly,

$$m_{\mu,\Omega} = c_{\mu,\Omega} = c^s_{\mu,\Omega}.$$

Lemma 3.3. *There exist $\tau = \tau(\mu) > 0$ and a constant $\sigma(\mu) > 0$ such that the mountain pass level $c_{\lambda,\mu}$ of $I_{\lambda,\mu}$ satisfies that*

$$\sigma(\mu) \le c_{\lambda,\mu} < m(\mathbb{R}^N) - \tau$$

for all $\lambda > 0$.

Proof. It is proved in [17] that $c_{\mu,\Omega} < m(\mathbb{R}^N)$ and that there exists a critical point $u \in H_0^1(\Omega)$ of $I_{\mu,\Omega}$ such that $I_{\mu,\Omega}(u) = c_{\mu,\Omega}$. Extend the function u to $\mathbb{R}^N$ such that $u = 0$ outside Ω, then $u \in \mathcal{N}_{\lambda,\mu}$, which implies

$$c_{\lambda,\mu} \leqslant I_{\lambda,\mu}(u) = I_{\mu,\Omega}(u) = c_{\mu,\Omega}.$$

So there exists $\tau = \tau(\mu) > 0$ such that

$$c_{\lambda,\mu} \le c_{\mu,\Omega} < m(\mathbb{R}^N) - \tau$$

for all $\lambda > 0$.

Next, for each $u \in \mathcal{N}_{\lambda,\mu}$,

$$\begin{aligned} 0 = \langle I'_{\lambda,\mu}(u), u\rangle &= (L_\lambda u, u) - \mu|u^+|_p^p - \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \\ &\geqslant \|u\|_\lambda^2 - C\|u\|_\lambda^p - C\sum_{i=1}^k \|u\|_\lambda^{22_i^*}, \end{aligned}$$

where C depending only on μ, yields that there exists $\sigma > 0$ independent of λ such that $\|u\|_\lambda \geqslant \sigma$, and then

$$C\sigma \leqslant C\|u\|_\lambda^2 \leqslant (L_\lambda u, u) = \mu|u^+|_p^p + \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y.$$

Hence,

$$\begin{aligned} c_{\lambda,\mu} = m_{\lambda,\mu} &= \inf_{u\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u) \\ &\geqslant \inf_{u\in\mathcal{N}_{\lambda,\mu}} \left(\frac{1}{2} - \frac{1}{p}\right)\left[\mu|u^+|_p^p + \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\right] \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)C\sigma, \end{aligned}$$

the conclusion follows. □

Now we show that $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition for certain c.

Proposition 3.4. *There exist $\mu \in (0, \mu_1)$ and $\lambda(\mu) > 0$ such that $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition for*

$$c < m(\mathbb{R}^N) - \tau,$$

whenever $\lambda \geqslant \lambda(\mu)$.

Proof. It is known from Lemma 3.1 that the $(PS)_c$ sequence $\{u_n\}$ of $I_{\lambda,\mu}$ is bounded in E. We may assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } E, \\ u_n &\to u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \\ u_n(x) &\to u_0(x) \quad \text{a.e. on } \mathbb{R}^N \end{aligned}$$

as $n \to \infty$. By the Hardy-Littlewood-Sobolev inequality and Hölder inequality,

$$\left(|x|^{-(N-\alpha_i)} * |u_n|^{2_i^*}\right)|u_n|^{2_i^*-1} \rightharpoonup \left(|x|^{-(N-\alpha_i)} * |u_0|^{2_i^*}\right)|u_0|^{2_i^*-1} \quad \text{in } L^{\frac{2N}{N+\alpha_i}}(\mathbb{R}^N), \tag{3.5}$$

for $i = 1, 2, \dots, k$. Thus, for every $\varphi \in H^1(\mathbb{R}^N)$,

$$0 = \lim_{n\to\infty} \langle I'_{\lambda,\mu}(u_n), \varphi\rangle = \langle I'_{\lambda,\mu}(u_0), \varphi\rangle,$$

that is, u_0 is a weak solution of problem (1.1).

Let $w_n = u_n - u_0$. The Brézis-Lieb-type lemma in [11]

$$\lim_{n\to\infty}\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*}w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\right] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_0^+(x)^{2_i^*}u_0^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \tag{3.6}$$

enables us to deduce

$$o(1) = \langle I'_{\lambda,\mu}(u_n), u_n\rangle = \langle I'_{\lambda,\mu}(u_0), u_0\rangle + \langle I'_{\lambda,\mu}(w_n), w_n\rangle + o(1),$$

namely,

$$\langle I'_{\lambda,\mu}(w_n), w_n\rangle = o(1). \tag{3.7}$$

Similarly,

$$c + o(1) = I_{\lambda,\mu}(u_n) = I_{\lambda,\mu}(u_0) + I_{\lambda,\mu}(w_n) \geqslant I_{\lambda,\mu}(w_n), \tag{3.8}$$

since $I_{\lambda,\mu}(u_0) \geqslant 0$.

In order to prove that u_n converges strongly to u_0 in E, we claim that for each nontrivial nonnegative $u \in E$ the function $f_u(t) = I_{\lambda,\mu}(tu)$ has a unique critical point t_u such that $f_u(t_u) = \max_{t\geq 0} f(t)$ and $t_u u \in \mathcal{N}_{\lambda,\mu}$. Indeed,

$$f'_u(t) = t(L_\lambda u, u) - t^{p-1}\mu|u^+|_p^p - \sum_{i=1}^k t^{22_i^*-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y = t[(L_\lambda u, u) - m_u(t)],$$

where the function

$$m_u(t) := t^{p-2}\mu|u^+|_p^p + \sum_{i=1}^k t^{22_i^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)^{2_i^*}u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y > 0$$

satisfies $\lim_{t\to+\infty} m_u(t) = +\infty$ and $m_u(0) = 0$. The claim follows readily.

Now we show u_n converge strongly to u_0 in E. Suppose, by contradiction, that u_n does not converge strongly to u_0, then $w_n \nrightarrow 0$ in E as $n \to \infty$. So there exists a unique $t_n > 0$ such that $t_n w_n \in \mathcal{N}_{\lambda,\mu}$ and $f_{w_n}(t)$ achieves the maximum at $t = t_n$. By (3.7) and

$$\int_{\mathbb{R}^N} (|\nabla w_n|^2 + \lambda V(x)w_n^2)\mathrm{d}x = t_n^{p-2}\mu|w_n^+|_p^p + \sum_{i=1}^k t_n^{22_i^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*}w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y, \tag{3.9}$$

we find

$$(t_n^{p-2} - 1)\mu|w_n^+|_p^p + \sum_{i=1}^k \left(t_n^{22_i^*-2} - 1\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*}w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y = o(1),$$

which implies $t_n \to 1$ since $w_n \not\to 0$ in E. Therefore,

$$I_{\lambda,\mu}(w_n) = I_{\lambda,\mu}(t_n w_n) + o(1). \tag{3.10}$$

For any $\tilde{\Omega} \subset \mathbb{R}^N$, we define

$$J_{\tilde{\Omega}}(u) = \frac{1}{2}\int_{\tilde{\Omega}} |\nabla u|^2 \,dx - \sum_{i=1}^{k} \frac{1}{22_i^*} \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dx dy \tag{3.11}$$

and denote $J(u) = J_{\mathbb{R}^N}(u)$. Set $v_n = t_n w_n$. We may show similarly that the function $g_u(s) = J(su)$ for each $u \in E\backslash\{0\}$ has unique critical point $s_u > 0$ which is the maximum point of $g_u(s)$. Particularly, there exists a unique $s_n \in \mathbb{R}^+$ such that $g'_{v_n}(s_n) = 0$ and $s_n v_n \in \mathcal{N}_{\mathbb{R}^N}$. Since $v_n \in \mathcal{N}_{\lambda,\mu}$, we obtain

$$I_{\lambda,\mu}(v_n) = \sup_{s \geqslant 0} I_{\lambda,\mu}(s v_n).$$

By Lemma 2.1, (3.8), and (3.10),

$$\begin{aligned} m(\mathbb{R}^N) - \tau \geqslant I_{\lambda,\mu}(u_n) \geqslant I_{\lambda,\mu}(w_n) = I_{\lambda,\mu}(t_n w_n) + o(1) \geqslant I_{\lambda,\mu}(s_n v_n) + o_n(1) \\ \geqslant J(s_n v_n) + o_n(1) + o_\lambda(1) \geqslant m(\mathbb{R}^N) + o_n(1) + o_\lambda(1). \end{aligned}$$

This yields a contradiction by choosing λ large enough. Hence, $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_n\}$ be a minimizing sequence of $I_{\lambda,\mu}$ restricting on $\mathcal{N}_{\lambda,\mu}$. By the Ekeland variational principle, we may assume that $\{u_n\}$ is a $(PS)_{c_{\lambda,\mu}}$ sequence, that is,

$$I_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} \quad \text{and} \quad I'_{\lambda,\mu}(u_n) \to 0.$$

It follows from Proposition 3.4 and Lemma 3.3 that $\{u_n\}$ has a subsequence converging to a least energy solution u_λ of (1.1).

By Theorem 1.1, for all $\lambda_n \geqslant \lambda(\mu)$, the problem

$$-\Delta u + \lambda_n V(x)u = \mu|u|^{p-2}u + \sum_{i=1}^{k}\Big(|x|^{-(N-\alpha_i)} * |u|^{2_i^*}\Big)|u|^{2_i^*-2}u \quad \text{in } \mathbb{R}^N \tag{3.12}$$

has at least one positive solution u_{λ_n}. □

Proof of Theorem 1.2. Let u_{λ_n} be the positive solution to (3.12) for every $\lambda_n \geqslant \lambda(\mu)$. Then we have $I_{\lambda_n,\mu}(u_{\lambda_n}) = c_{\lambda_n,\mu}$ and $I'_{\lambda_n,\mu}(u_{\lambda_n}) = 0$. By Lemma 3.3, $c_{\lambda_n,\mu}$ is uniformly bounded. We may show as Lemma 3.1 that there is $C > 0$ such that $\|u_{\lambda_n}\|_{\lambda_n}^2 \le C$. By Lemma 2.1, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_{\lambda_n} \rightharpoonup u$ weakly in E, $u_{\lambda_n} \to u$ in $L^2(\mathbb{R}^N)$ and $u_{\lambda_n} \to u$ in $L^p(\mathbb{R}^N)$.

For each $\varphi \in H_0^1(\Omega)$, we extend φ to $\mathbb{R}^N$ by setting $\varphi = 0$ outside Ω. Then we have

$$0 = \lim_{n\to\infty} \langle I'_{\lambda_n,\mu}\big(u_{\lambda_n}\big), \varphi\rangle = \langle I'_{\mu,\Omega}(u), \varphi\rangle. \tag{3.13}$$

This means that u is a weak solution of problem (1.14).

Now we show u_{λ_n} converges to u in E. Suppose on the contrary that $v_n := u_{\lambda_n} - u \nrightarrow 0$ in E as $n \to \infty$. By (3.6) and (3.13),

$$\lim_{n\to\infty} \langle I'_{\lambda_n,\mu}(v_n), v_n\rangle = \lim_{n\to\infty} \langle I'_{\lambda_n,\mu}\big(u_{\lambda_n}\big), u_{\lambda_n}\rangle - \langle I'_{\mu,\Omega}(u), u\rangle = 0.$$

Arguing as the proof of Proposition 3.4, there is unique $t_n > 0$ such that $t_n v_n \in \mathcal{N}_{\lambda_n,\mu}$ and $t_n \to 1$ as $n \to \infty$. Hence,

$$I_{\lambda_n,\mu}(v_n) + o(1) = I_{\lambda_n,\mu}(t_n v_n) = \sup_{t\geqslant 0} I_{\lambda_n,\mu}(t v_n).$$

On the other hand, there exists $s_n > 0$ such that $s_n t_n v_n \in \mathcal{N}_{\mathbb{R}^N}$. We derive from Proposition 3.4 that

$$\begin{aligned} m(\mathbb{R}^N) - \tau > c_{\lambda_n,\mu} + o(1) = I_{\lambda_n,\mu}\big(u_{\lambda_n}\big) \geqslant I_{\lambda_n,\mu}(v_n) = I_{\lambda_n,\mu}(t_n v_n) + o(1) \\ \geqslant I_{\lambda_n,\mu}(s_n t_n v_n) + o(1) \geq J(s_n t_n v_n) + o(1) \geq m(\mathbb{R}^N) + o(1), \end{aligned}$$

a contradiction. The assertion follows. □

Corollary 3.5. *If* $\mu \in (0, \mu_1)$, *then* $\lim_{\lambda_n\to\infty} c_{\lambda_n,\mu} = c_{\mu,\Omega}$.

Proof. Obviously, $c_{\lambda_n,\mu} \leqslant c_{\mu,\Omega}$. On the other hand, by Theorems 1.1 and 1.2, there exists $\{u_n\}$ such that $c_{\lambda_n,\mu} = I_{\lambda_n,\mu}(u_n)$ and $\lim_{\lambda_n\to\infty} I_{\lambda_n,\mu}(u_n) = I_{\mu,\Omega}(u) \geqslant m_{\mu,\Omega} = c_{\mu,\Omega}$. □

4 Multiple solutions

In this section, we show that problem (1.1) has multiple solutions in connection with the domain $V^{-1}(0) = \Omega$ by the Lusternik-Schnirelman theory. Such a problem is actually related to the limit problem (1.14). We know from [17] that (1.14) possesses at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions, and eventually so does problem (1.1) as we will show.

Let us fix $r > 0$ small enough so that

$$\Omega_{2r}^+ := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < 2r\}$$

and

$$\Omega_r^- := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}$$

are homotopically equivalent to Ω. Denote $B_r(0) := \{x \in \mathbb{R}^N : |x| < r\} \subset \Omega$ and define $c_{\mu,r} := c_{\mu,B_r(0)}$. Therefore, we may verify as Lemma 3.3 that for $\mu \in (0, \mu_1)$,

$$c_{\mu,\Omega} < c_{\mu,r} < m(\mathbb{R}^N) - \tau.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta(x) = x$ for all $x \in \Omega$. We introduce the barycenter of a function $0 \neq u \in H^1(\mathbb{R}^N)$ as

$$\beta(u) := \frac{\int_{\mathbb{R}^N} \eta(x)|\nabla u|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x}.$$

Denote $I_{\mu,\Omega}^c = \{u \in \mathcal{N}_{\mu,\Omega} : I_{\mu,\Omega} \leqslant c\}$ the level set. It is proved in [17] the following result.

Lemma 4.1. *There exists* $\mu^* \in (0, \mu_1)$ *such that if* $0 < \mu < \mu^*$, *then*

$$\beta(u) \in \Omega_r^+ \quad \text{for every } u \in \cap I_{\mu,\Omega}^{c_{\mu,r}}.$$

For any domain $\tilde{\Omega} \subset \mathbb{R}^N$, we define

$$m(\tilde{\Omega}) := \inf_{u\in\mathcal{N}_{\tilde{\Omega}}} J_{\tilde{\Omega}}(u),$$

where

$$\mathcal{N}_{\tilde{\Omega}} := \{u \in D_0^{1,2}(\tilde{\Omega})\backslash\{0\} : \langle J_{\tilde{\Omega}}'(u), u\rangle = 0\}.$$

Lemma 4.2. *There holds*

$$m(\tilde{\Omega}) = m(\mathbb{R}^N),$$

and $m(\tilde{\Omega})$ *is never achieved unless* $\tilde{\Omega} = \mathbb{R}^N$.

Proof. Obviously, $m(\tilde{\Omega}) \geq m(\mathbb{R}^N)$. To prove the inverse inequality, let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a minimizing sequence of $m(\mathbb{R}^N)$. Choosing $y_n \in \mathbb{R}^N$ and $\lambda_n > 0$ so that $v_n(x) = \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n) \in C_0^\infty(\tilde{\Omega})$, hence we obtain $m(\tilde{\Omega}) \leq m(\mathbb{R}^N)$. □

Lemma 4.3. *For* $\mu \in (0, \mu_1)$, *we have* $\lim_{\mu\to 0} c_{\mu,r} = \lim_{\mu\to 0} c_{\mu,\Omega}$.

Proof. For any $\mu_n \in (0, \mu_1)$ and $\mu_n \to 0$ as $n \to \infty$, by [17] there exists a solution u_n of (1.14) such that $I_{\mu_n,\Omega}(u_n) = c_{\mu_n,\Omega}$. We may show as Lemma 3.1 that $\{u_n\}$ is uniformly bounded in $H_0^1(\Omega)$. Moreover, there is a unique $t_n \in \mathbb{R}^+$ such that $t_n u_n \in \mathcal{N}_\Omega$, that is,

$$\int_\Omega |\nabla u_n|^2 \mathrm{d}x = \sum_{i=1}^k t_n^{22_i^*-2} \int_\Omega \int_\Omega \frac{u_n^+(x)^{2_i^*} u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y.$$

Because u_n is bounded, so does t_n. Since $u_n \in \mathcal{N}_{\mu_n,\Omega}$, we deduce

$$\begin{aligned}\sum_{i=1}^k t_n^{22_i^*-2} \int_\Omega \int_\Omega \frac{u_n^+(x)^{2_i^*} u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y &= \mu_n \int_\Omega (u_n^+)^p \mathrm{d}x + \sum_{i=1}^k \int_\Omega \int_\Omega \frac{u_n^+(x)^{2_i^*} u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y \\ &\geq \sum_{i=1}^k \int_\Omega \int_\Omega \frac{u_n^+(x)^{2_i^*} u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y\end{aligned}$$

implying $t_n \geq 1$.

Note that for $q > 2$, the function $h(t) = \frac{1}{2}t^2 - \frac{1}{q}t^q$ is decreasing if $t \geq 1$ and increasing if $t \leq 1$. In particular, $h(t)$ is decreasing for both $q = p$ and $q = 22_i^*$, $i = 1, 2, \ldots k$ whenever $t \geqslant 1$. Therefore,

$$\begin{aligned}I_{\mu_n,\Omega}(u_n) - I_{\mu_n,\Omega}(t_n u_n) = &\left[\left(\frac{1}{2} - \frac{1}{p}\right) - \left(\frac{t_n^2}{2} - \frac{t_n^p}{p}\right)\right] \mu_n \int_\Omega (u_n^+)^p \mathrm{d}x \\ &+ \sum_{i=1}^k \left[\left(\frac{1}{2} - \frac{1}{22_i^*}\right) - \left(\frac{t_n^2}{2} - \frac{t_n^{22_i^*}}{22_i^*}\right)\right] \int_\Omega \int_\Omega \frac{u_n^+(x)^{2_i^*} u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y\end{aligned}$$

yields

$$m(\Omega) \leq I_{\mu_n,\Omega}(t_n u_n) \leqslant I_{\mu_n,\Omega}(u_n) \leqslant c_{\mu_n,\Omega} + \frac{\mu_n}{p} \int_\Omega (t_n u_n^+)^p \mathrm{d}x,$$

and then

$$m(\Omega) \leqslant \liminf_{n\to\infty} c_{\mu_n,\Omega}.$$

On the other hand, for each $u \in \mathcal{N}_\Omega$, there exists $s_n > 0$ such that $s_n u \in \mathcal{N}_{\mu_n,\Omega}$, alternatively,

$$\begin{aligned}\sum_{i=1}^k \int_\Omega \int_\Omega \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y &= \mu_n s_n^{p-2} \int_\Omega (u^+)^p \mathrm{d}x + \sum_{i=1}^k s_n^{22_i^*-2} \int_\Omega \int_\Omega \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y \\ &\geq \sum_{i=1}^k s_n^{22_i^*-2} \int_\Omega \int_\Omega \frac{u^+(x)^{2_i^*} u^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y\end{aligned}$$

yielding $s_n \le 1$. As a result,

$$c_{\mu_n,\Omega} = m_{\mu_n,\Omega} \le I_{\mu_n,\Omega}(s_n u) \le I_{\mu_n,\Omega}(u) \le J(u),$$

which yields

$$\limsup_{n\to\infty} c_{\mu_n,\Omega} \le m(\Omega).$$

Consequently, $m(\Omega) = \lim_{n\to\infty} c_{\mu_n,\Omega}$. Similarly, $m(B_r) = \lim_{n\to\infty} c_{\mu_n,B_r}$. Lemma 4.2 then implies

$$\lim_{\mu\to 0} c_{\mu,r} = \lim_{\mu\to 0} c_{\mu,\Omega}.$$ □

Similarly, denote $I^c_{\lambda,\mu} = \{u \in \mathcal{N}_{\lambda,\mu} : I_{\lambda,\mu} \leqslant c\}$ the level set.

Proposition 4.4. *There exists $\mu^* \in (0, \mu_1)$ such that for each $0 < \mu < \mu^*$ there is $\Lambda(\mu) > 0$ so that if $u \in I^{c_{\mu,r}}_{\lambda,\mu}$ there holds $\beta(u) \in \Omega^+_{2r}$, provided that $\lambda \geqslant \Lambda(\mu)$.*

Proof. We argue indirectly. Suppose on the contrary that for μ arbitrarily small, there is a sequence $\{u_n\}$ such that $u_n \in \mathcal{N}_{\lambda_n,\mu}$, $\lambda_n \to \infty$, $I_{\lambda_n,\mu}(u_n) \to c \leqslant c_{\mu,r}$ as $n \to \infty$ and $\beta(u_n) \notin \Omega^+_{2r}$. We may show as Lemma 3.1 that $\{u_n\}$ is uniformly bounded in E. By Lemma 2.1, there is $u_\mu \in H^1_0(\Omega)$ such that $u_n \rightharpoonup u_\mu$ weakly in E and $u_n \to u_\mu$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. The interpolation inequality yields $u_n \to u_\mu$ in $L^p(\mathbb{R}^N)$. Now, we distinguish two cases to discuss.

Case 1: $\sum_{i=1}^k \int_\Omega \int_\Omega \frac{u_\mu^+(x)^{2_i^*} u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy \leqslant \int_\Omega |\nabla u_\mu|^2 dx - \mu \int_\Omega (u_\mu^+)^p dx.$

Let $w_n = u_n - u_\mu$. First, we show $w_n \to 0$ in E as $n \to \infty$. Indeed, were it not the case, we would have $w_n \nrightarrow 0$ in E as $n \to \infty$. By the Brézis-Lieb-type lemma,

$$c + o(1) = I_{\lambda_n,\mu}(u_n) = I_{\lambda_n,\mu}(w_n) + I_{\lambda_n,\mu}(u_\mu) + o(1) \geqslant I_{\lambda_n,\mu}(w_n) + o(1) \tag{4.1}$$

and by the assumption,

$$\begin{aligned} o(1) &= \langle I'_{\lambda_n,\mu}(u_n), u_n\rangle \\ &= \langle I'_{\lambda_n,\mu}(w_n), w_n\rangle + \int_\Omega |\nabla u_\mu|^2 dx - \mu \int_\Omega (u_\mu^+)^p dx - \sum_{i=1}^k \int_\Omega\int_\Omega \frac{u_\mu^+(x)^{2_i^*} u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy + o(1) \\ &\ge \langle I'_{\lambda_n,\mu}(w_n), w_n\rangle + o(1). \end{aligned} \tag{4.2}$$

That is,

$$\int_{\mathbb{R}^N} (|\nabla w_n|^2 + \lambda_n V(x) w_n^2) dx \leqslant \mu \int_{\mathbb{R}^N} (w_n^+)^p dx + \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*} w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy + o(1). \tag{4.3}$$

As before, we can find a unique constant $t_n > 0$ such that $t_n w_n \in \mathcal{N}_{\lambda_n,\mu}$, which and (4.3) yield

$$t_n^{p-2}\mu|w_n^+|_p^p + \sum_{i=1}^k t_n^{22_i^*-2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*} w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy \leqslant \mu|w_n^+|_p^p + \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*} w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy + o(1), \tag{4.4}$$

then $t_n \leqslant 1$. By (4.1) and (4.4),

$$\begin{aligned} c + o(1) &= I_{\lambda_n,\mu}(u_n) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\mu \int_{\mathbb{R}^N} (w_n^+)^p dx + \sum_{i=1}^k \left(\frac{1}{2} - \frac{1}{22_i^*}\right) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*} w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy + o(1) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\mu \int_{\mathbb{R}^N} (t_n w_n^+)^p dx + \sum_{i=1}^k \left(\frac{1}{2} - \frac{1}{22_i^*}\right) t_n^{22_i^*} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w_n^+(x)^{2_i^*} w_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}} dxdy + o(1) \\ &= I_{\lambda_n,\mu}(t_n w_n) + o(1). \end{aligned}$$

Since $v_n := t_n w_n \in \mathcal{N}_{\lambda_n,\mu}$, we have

$$I_{\lambda_n,\mu}(v_n) = \sup_{s\geq 0} I_{\lambda_n,\mu}(sv_n). \tag{4.5}$$

Similarly, there exists a unique $s_n \in \mathbb{R}^+$ such that $s_n v_n \in \mathcal{N}_{\mathbb{R}^N}$. It follows that

$$\begin{aligned}\sum_{i=1}^{k} s_n^{22_i^*-2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_n^+(x)^{2_i^*} v_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \\ = \int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \lambda_n V v_n^2)\mathrm{d}x \\ = \mu \int_{\mathbb{R}^N} (v_n^+)^p \mathrm{d}x + \sum_{i=1}^{k} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_n^+(x)^{2_i^*} v_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{4.6}$$

which implies $\{s_n\}$ is bounded. Therefore, by (4.5) and the fact that $u_n \to u_\mu$ in $L^p(\mathbb{R}^N)$ as $n\to\infty$ we deduce

$$m(\mathbb{R}^N) - \tau > I_{\lambda_n,\mu}(v_n) + o_n(1) \geqslant I_{\lambda_n,\mu}(s_n v_n) + o_n(1) \geqslant J(s_n v_n) + o_n(1) \geqslant m(\mathbb{R}^N) + o_n(1),$$

a contradiction. Hence, $u_n \to u_\mu$ in E as $n\to\infty$. As a result, $\beta(u_n)\to\beta(u_\mu)$ as $n\to\infty$. Moreover, by Lemma 2.1,

$$I_{\mu,\Omega}(u_\mu) \leqslant \lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) \leqslant c_{\mu,r}.$$

Whence by Lemma 4.1, $\beta(u_\mu)\in\Omega_r^+$, which is a contradiction to $\beta(u_n)\notin\Omega_{2r}^+$.

Case 2: $\sum_{i=1}^k \int_\Omega\int_\Omega \frac{u_\mu^+(x)^{2_i^*}u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y > \int_\Omega |\nabla u_\mu|^2\,\mathrm{d}x - \mu\int_\Omega (u_\mu^+)^p\mathrm{d}x$.

It is known that there exists $t_\mu>0$ such that $t_\mu u_\mu \in \mathcal{N}_{\mu,\Omega}$. By the assumption,

$$\begin{aligned} t_\mu^{p-2}\mu\int_\Omega (u_\mu^+)^p\mathrm{d}x + \sum_{i=1}^k t_\mu^{22_i^*-2}\int_\Omega\int_\Omega \frac{u_\mu^+(x)^{2_i^*}u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \\ = \int_\Omega |\nabla u_\mu|^2\,\mathrm{d}x < \mu\int_\Omega (u_\mu^+)^p\mathrm{d}x + \sum_{i=1}^k \int_\Omega\int_\Omega \frac{u_\mu^+(x)^{2_i^*}u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

which implies $t_\mu\in(0,1)$. Since $p<2^*_{\min}$, we have

$$\begin{aligned} c_{\mu,\Omega} &\leq I_{\mu,\Omega}(t_\mu u_\mu) \\ &\leqslant t_\mu^p\left[\left(\frac{1}{2}-\frac{1}{p}\right)\mu\int_\Omega (u_\mu^+)^p\mathrm{d}x + \sum_{i=1}^k\left(\frac{1}{2}-\frac{1}{22_i^*}\right)\sum_{i=1}^k\int_\Omega\int_\Omega \frac{u_\mu^+(x)^{2_i^*}u_\mu^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\right] \\ &\leqslant t_\mu^p \lim_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\mu\int_{\mathbb{R}^N} (u_n^+)^p\mathrm{d}x + \sum_{i=1}^k\left(\frac{1}{2}-\frac{1}{22_i^*}\right)\sum_{i=1}^k\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\right] \\ &\leq \lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) \leqslant c_{\mu,r}.\end{aligned}$$

Thus, by Lemma 4.3, for n large we obtain

$$|I_{\lambda_n,\mu}(u_n) - I_{\mu,\Omega}(t_\mu u_\mu)| \leqslant c_{\mu,r} - c_{\mu,\Omega} \to 0 \tag{4.7}$$

as $\mu\to 0$. Therefore,

$$\begin{aligned} I_{\lambda_n,\mu}(u_n) - I_{\mu,\Omega}(t_\mu u_\mu) = \left(\frac{1}{2}-\frac{1}{p}\right)\mu\left(\int_{\mathbb{R}^N}(u_n^+)^p\mathrm{d}x - t_\mu^p\int_\Omega (u_\mu^+)^p\mathrm{d}x\right) \\ + \sum_{i=1}^k\left(\frac{1}{2}-\frac{1}{22_i^*}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y - t_\mu^{22_i^*}\int_\Omega\int_\Omega \frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\right)\end{aligned}$$

and (4.7) yield that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y - t_\mu^{22_i^*}\int_\Omega\int_\Omega\frac{u_n^+(x)^{2_i^*}u_n^+(y)^{2_i^*}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \to 0 \tag{4.8}$$

for n large enough and as $\mu \to 0$, $i = 1, 2, \dots, k$.

Extending u_μ to $\mathbb{R}^N$ by setting $u_\mu = 0$ outside Ω, we deduce from $\langle I'_{\lambda_n,\mu}(u_n), u_n\rangle - \langle I'_{\mu,\Omega}(t_\mu u_\mu), t_\mu u_\mu\rangle = 0$ and (4.8) that

$$\int_{\mathbb{R}^N}(|\nabla u_n|^2 - |t_\mu \nabla u_\mu|^2)\mathrm{d}x = o_\mu(1).$$

Hence,

$$\int_{\mathbb{R}^N}|\nabla u_\mu|^2\,\mathrm{d}x \le \lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x = \int_{\mathbb{R}^N}|t_\mu\nabla u_\mu|^2\,\mathrm{d}x + o_\mu(1).$$

That is, $t_\mu \to 1_-$ as $\mu \to 0$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x = \int_{\mathbb{R}^N}|\nabla u_\mu|^2\,\mathrm{d}x + o_\mu(1).$$

Since $u_n \rightharpoonup u_\mu$ in E as $n \to \infty$, we have

$$\int_{\mathbb{R}^N}|\nabla(u_n - u_\mu)|^2\mathrm{d}x = \int_{\mathbb{R}^N}(|\nabla u_n|^2 - 2\nabla u_n\nabla u_\mu + |\nabla u_\mu|^2)\mathrm{d}x = o_n(1) + o_\mu(1).$$

This readily yields for $\mu > 0$ small enough and n large that

$$|\beta(u_n) - \beta(t_\mu u_\mu)| < r.$$

It is known from Lemma 4.1 that $\beta(t_\mu u_\mu) \in \Omega_r^+$, whereas $\beta(u_n) \notin \Omega_{2r}^+$. The contradiction completes the proof. □

We know from [17] and [19] that there exists a radially symmetric minimize $v > 0$ of I_{μ,B_r} on $\mathcal{N}_{\mu,B_r}$ for $\mu \in (0, \mu_1)$. It allows us to estimate the category of the level set of $I_{\lambda,\mu}$.

Lemma 4.5. *If $N \geqslant 4$ and $\mu \in (0, \mu^*)$, for $\lambda \geqslant \Lambda(\mu)$, then*

$$\operatorname{cat}_{I_{\lambda,\mu}^{c_{\mu,r}}}\big(I_{\lambda,\mu}^{c_{\mu,r}}\big) \geqslant \operatorname{cat}_\Omega(\Omega).$$

Proof. Define $\gamma : \Omega_r^- \to I_{\lambda,\mu}^{c_{\mu,r}}$ by

$$\gamma(y)(x) = \begin{cases} v(x-y), & x \in B_r(y),\\ 0, & x \notin B_r(y).\end{cases}$$

We may verify that $\gamma(y)(x) \in \mathcal{N}_{\lambda,\mu}$, $I_{\lambda,\mu}(\gamma(y)(x)) \leqslant c_{\mu,r}$, and $\beta \circ \gamma = id : \Omega_r^- \to \Omega_r^-$.

Assume that $\operatorname{cat}_{I_{\lambda,\mu}^{c_{\mu,r}}}(I_{\lambda,\mu}^{c_{\mu,r}}) = n$, and

$$I_{\lambda,\mu}^{c_{\mu,r}} = \bigcup_{j=1}^n A_j,$$

where A_j, $j = 1, 2, \dots, n$, is closed and contractible in $I_{\lambda,\mu}^{c_{\mu,r}}$, i.e., there exists $h_j \in C([0,1] \times A_j, I_{\lambda,\mu}^{c_{\mu,r}})$ such that, for every $u, v \in A_j$,

$$h_j(0,u) = u, \quad h_j(1,u) = h_j(1,v).$$

Let $B_j := \gamma^{-1}(A_j)$, $j = 1, 2, \dots, n$. For each $x \in \Omega_r^-$,

$$\gamma(x) \in I_{\lambda,\mu}^{c_{\mu,r}} \subset \bigcup_{j=1}^{n} A_j.$$

So there exists j_0 such that $\gamma(x) \in A_{j_0}$, that is, $x \in \gamma^{-1}(A_{j_0}) = B_{j_0}$. Therefore,

$$\Omega_r^- \subset \bigcup_{j=1}^{n} B_j.$$

For $x, y \in B_j$, $\gamma(x), \gamma(y) \in A_j$, the deformation

$$g_j(t, x) = \beta_0(h_j(t, \gamma(x))), \quad j = 1, 2, \ldots, n,$$

fulfills

$$g_j(0, x) = \beta_0(h_j(0, \gamma(x))) = \beta_0(\gamma(x)) = x$$

and

$$g_j(1, x) = \beta_0(h_j(1, \gamma(x))) = \beta_0(h_j(1, \gamma(y))) = g_j(1, y).$$

Hence, B_j is contractible in Ω_{2r}^+. It follows that

$$\mathrm{cat}_{\Omega}(\Omega) = \mathrm{cat}_{\Omega_{2r}^+}(\Omega_r^-) \leqslant \sum_{k=1}^{n} \mathrm{cat}_{\Omega_{2r}^+}(B_k) = n. \qquad \square$$

Lemma 4.6. *If $I_{\lambda,\mu}$ constraint to $\mathcal{N}_{\lambda,\mu}$ denoted by $I_{\lambda,\mu}|_{\mathcal{N}_{\lambda,\mu}}$ is bounded from below and satisfies the* $(\mathrm{PS})_c$ *condition for any $c \in [c_{\lambda,\mu}, c_{\mu,r}]$, then $I_{\lambda,\mu}|_{\mathcal{N}_{\lambda,\mu}}$ has a minimum and $I_{\lambda,\mu}^{c_{\mu,r}}$ contains at least $\mathrm{cat}_{I_{\lambda,\mu}^{c_{\mu,r}}}(I_{\lambda,\mu}^{c_{\mu,r}})$ critical points of $I_{\lambda,\mu}|_{\mathcal{N}_{\lambda,\mu}}$.*

Proof of Theorem 1.3. For $0 < \mu \leqslant \mu^*$ and $\lambda \geqslant \Lambda(\mu)$, we defined two maps

$$\Omega_r^- \xrightarrow{\gamma} I_{\lambda,\mu}^{c_{\mu,r}} \xrightarrow{\beta} \Omega_{2r}^+.$$

The conclusion follows from Proposition 3.4, Proposition 4.4, Lemma 4.5, and Lemma 4.6. $\square$

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