DE GRUYTER Advanced Nonlinear Studies 2022; 22: 289–307

Research Article

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Existence of solutions to contact mean-field games of first order

https://doi.org/10.1515/ans-2022-0012
received November 9, 2021; accepted May 13, 2022

Abstract: This paper deals with the existence of solutions of a class of contact mean-field game systems of first order consisting of a contact Hamilton-Jacobi equation and a continuity equation. Evans found a connection between Hamilton-Jacobi equations and continuity equations from the weak KAM point of view, where the coupling term is zero. Inspired by his work, we prove the main existence result by analyzing the properties of the Mather set for contact Hamiltonian systems.

Keywords: mean-field games, weak KAM theory, contact Hamiltonian systems, existence

MSC 2020: 37J51, 35Q89

1 Introduction

The mean-field game system was introduced by Lasry and Lions [21–23] and Caines, Huang, and Malhamé [18,19]. In this paper, we only discuss first-order mean-field game systems. It is a coupled system of partial differential equations, one Hamilton-Jacobi equation and one continuity equation. From the view of control theory, a Hamilton-Jacobi equation with an external mean-field term is standard. The mean-field term involves a probability distribution governed by a continuity equation, which depends on the feedback and the viscosity solution [11] of the Hamilton-Jacobi equation. The idea of equilibrium states in the mean-field games theory, which are distributed along optimal trajectories generated from the feedback strategy, is quite enlightening.

In some situations, the ergodic mean-field game system

$$\begin{cases} K(x, Du) = F(x, m) + c(m) & \text{in} \quad X, \\ \operatorname{div}\left(m\dfrac{\partial K}{\partial p}(x, Du)\right) = 0 & \text{in} \quad X, \\ \displaystyle\int_X m\mathrm{d}x = 1 \end{cases} \tag{1.1}$$

can be described as the limit system of a finite time ($T > 0$) horizon mean-field game system

$$\begin{cases} -\partial_t u^T + K(x, Du^T) = F(x, m^T(t)) & \text{in} \quad (0, T) \times X, \\ \partial_t m^T - \operatorname{div}\left(m^T\dfrac{\partial K}{\partial p}(x, Du^T)\right) = 0 & \text{in} \quad (0, T) \times X, \\ m^T(0) = m_0, \quad u^T(T, x) = u^f(x), & x \in X, \end{cases}$$

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as T goes to infinity. See [4,5,7] for this kind of results, where the state space X is $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$. Obviously, ensuring the existence of solutions to the ergodic mean-field game system is an important issue.

We point out that the works of Evans [12] prior to mean-field games revealed the connection between a Hamilton-Jacobi equation and a continuity equation with $F = 0$. Let us see how to obtain a solution of (1.1) from the weak KAM point of view.

In this paper, we always use M to denote a connected, closed (compact, without boundary), and smooth manifold endowed with a Riemannian metric. We choose, once and for all, a C^∞ Riemannian metric g. Since all the Riemannian metrics are equivalent on the compact manifold M, the superlinearity assumption ((H2) below) imposed on the Hamiltonian does not depend on the choice of the Riemannian metric. A simple example is $M = \mathbb{T}^n$. Denote by $\operatorname{diam}(M)$ the diameter of M. We will denote by (x, v) a point of the tangent bundle TM with $x \in M$ and v a vector tangent at x. The projection $\pi : TM \to M$ is $(x, v) \to x$. The notation (x, p) will designate a point of the cotangent bundle T^*M with $p \in T_x^*M$. Consider a Hamiltonian $K = K(x, p) : T^*M \to \mathbb{R}$, which is C^2, superlinear and strictly convex in p. Such a Hamiltonian is called a Tonelli Hamiltonian. We can associate with K a Lagrangian, as a function on $TM : l(x, v) = \sup_{p \in T_x^*M}\{\langle p, v\rangle_x - K(x, p)\}$, where $\langle \cdot, \cdot \rangle_x$ represents the canonical pairing between the tangent and cotangent space. Sometimes, we use $p \cdot v$ to denote $\langle p, v\rangle_x$ for simplicity.

Let μ be a Mather measure [26] for the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l}{\partial v} = \frac{\partial l}{\partial x}. \tag{1.2}$$

Then μ is a closed measure (see, for instance, [2]), i.e.,

$$\int_{TM} v \cdot D\varphi(x)\mathrm{d}\mu = 0, \quad \forall \varphi \in C^1(M),$$

where $C^k(M)$ $(k \in \mathbb{N})$ stands for the function space of continuously differentiable functions on M. Recall that

$$\operatorname{supp}(\mu) \subset \tilde{\mathcal{A}} = \bigcap_{(u_-, u_+)} \left\{(x, v) \in TM : u_-(x) = u_+(x), Du_-(x) = Du_+(x) = \frac{\partial l}{\partial v}(x, v)\right\},$$

where $\tilde{\mathcal{A}}$ is the Aubry set for Lagrangian system (1.2), and the intersection is taken on the pairs (u_-, u_+) of conjugate functions, i.e., u_- (resp. u_+) is a backward (resp. forward) weak KAM solution of

$$K(x, Du) = c(K) \tag{1.3}$$

and $u_- = u_+$ on the projected Mather set $\mathcal{M}$ of system (1.2). Here, $\mathcal{M} := \pi\tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is the union of supports of Euler-Lagrange flow Φ_t^l-invariant probability measures supported in $\tilde{\mathcal{A}}$, called the Mather set. The symbol $c(K)$ denotes the Mañé critical value of K. See Section 2 for the definition and representation formulas of Mañé's critical value. Let u_- be an arbitrary backward weak KAM solution (or equivalently [13], viscosity solution) of equation (1.3), and let $\sigma := \pi\sharp\mu$, where $\pi\sharp\mu$ denotes the push-forward of μ through π. Then for each $\varphi \in C^1(M)$,

$$0 = \int_{\operatorname{supp}(\mu)} v \cdot D\varphi(x)\mathrm{d}\mu = \int_{\operatorname{supp}(\sigma)} \frac{\partial K}{\partial p}(x, Du_-(x)) \cdot D\varphi(x)\mathrm{d}\sigma = \int_M \frac{\partial K}{\partial p}(x, Du_-(x)) \cdot D\varphi(x)\mathrm{d}\sigma,$$

which means that σ is a solution of the continuity equation

$$\operatorname{div}\left(\sigma\frac{\partial K}{\partial p}(x, Du_-)\right) = 0$$

in the sense of distributions.

In view of the aforementioned arguments, one can deduce that if there is a Borel probability measure m on M such that $l(x, v) + F(x, m)$ admits a Mather measure η_m with

$$m = \pi\sharp\eta_m, \tag{1.4}$$

then for any viscosity solution u of

$$K(x, Du) = F(x, m) + c(m),$$

where $c(m)$ is the Mañé critical value of $K(x, p) - F(x, m)$, the pair (u, m) is a solution of (1.1), i.e., the Hamilton-Jacobi equation is satisfied in viscosity sense and the continuity equation is satisfied in the sense of distributions. So, to find such a solution of (1.1), it suffices to find a probability measure m satisfying (1.4).

In this paper, we aim to prove the existence of solutions of the following contact mean-field game system:

$$\begin{cases} H(x, u, Du) = F(x, m) \quad \text{in} \quad M, & (1.5a) \\ \operatorname{div}\left(m\frac{\partial H}{\partial p}(x, u, Du)\right) = 0 \quad \text{in} \quad M, & (1.5b) \\ \int_M m\mathrm{d}x = 1 & (1.5c) \end{cases}$$

using dynamical approaches. Note that the Hamiltonian $H = H(x, u, p)$ in (1.5) is defined on $T^*M \times \mathbb{R}$, where $(x, p) \in T^*M$ and $u \in \mathbb{R}$. Since the characteristic equations of (1.5a) is a contact Hamiltonian system, we call (1.5) a contact mean-field game system. In the view of the essential differences between weak KAM results for Hamiltonian and contact Hamiltonian systems, we cannot use the aforementioned idea directly to obtain the existence of solutions. A more careful analysis of the structure of Mather sets of contact Hamiltonian systems is needed.

We now list some basic assumptions on H and F, which will be made in most of the results of this paper. Assume that the contact Hamiltonian H is of class C^3 and satisfies:

(H1) **Positive definiteness**: For every $(x, u, p) \in T^*M \times \mathbb{R}$, the second partial derivative $\partial^2 H / \partial p^2(x, u, p)$ is positive definite as a quadratic form;

(H2) **Superlinearity**: For each $x \in M$, the function $p \mapsto H(x, 0, p)$ is superlinear on the fiber T_x^*M;

(H3) **Strict monotonicity**: There are constants $\delta > 0$ and $\lambda > 0$ such that for every $(x, u, p) \in T^*M \times \mathbb{R}$,

$$\delta < \frac{\partial H}{\partial u}(x, u, p) \le \lambda;$$

(H4) **Reversibility**: $H(x, u, p) = H(x, u, -p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$.

Remark 1. In view of (H2) and (H3), one can deduce that for every $(x, u) \in M \times \mathbb{R}$, $H(x, u, p)$ is superlinear in p. Here, we assume that H is of class C^3, since we will use some dynamical results on the Aubry-Mather theory for contact Hamiltonian systems obtained in [32] under the C^3 assumption.

Example 1. Discounted Hamiltonians

$$H(x, u, p) = \xi u + K(x, p), \quad \xi > 0,$$

where $K(x, p)$ is a smooth reversible Tonelli Hamiltonian and satisfies assumptions (H1)–(H4). Discounted Hamilton-Jacobi equations appear in some optimal control problems (see, e.g., [1]).

We denote by $\mathcal{P}(M)$ the set of Borel probability measures on M, and by $\mathcal{P}(T^*M)$ the set of Borel probability measures on T^*M. Both sets are endowed with the weak-* convergence. A sequence $\{\mu_k\}_{k\in\mathbb{N}} \in \mathcal{P}(X)$ is weakly-* convergent to $\mu \in \mathcal{P}(X)$, denoted by $\mu_k \xrightarrow{w^*} \mu$, if

$$\lim_{k\to\infty} \int_X f(x)\mathrm{d}\mu_k = \int_X f(x)\mathrm{d}\mu, \quad f \in C_b(X),$$

where $C_b(X)$ denotes the function space of bounded uniformly continuous functions on X with $X = M, T^*M$. Let us recall that $\mathcal{P}(M)$ is compact for this topology. We shall work with the Monge-Wasserstein distance defined, for any $m_1, m_2 \in \mathcal{P}(M)$, by

$$d_1(m_1, m_2) = \sup_h \int_M h d(m_1 - m_2),$$

where the supremum is taken over all the maps $h : M \to \mathbb{R}$, which are 1-Lipschitz continuous. $\mathcal{P}_1(T^*M)$ denotes the Wasserstein space of order 1, the space of probability measures with the finite moment of order 1.

Let $F : M \times \mathcal{P}(M) \to \mathbb{R}$ be a function, satisfying the following assumptions:

(F1) For every measure $m \in \mathcal{P}(M)$, the function $x \mapsto F(x, m)$ is of class $C^2(M)$ and

$$F_\infty := \sup_{m \in \mathcal{P}(M)} \sum_{|\alpha| \le 1} \|D^\alpha F(\cdot, m)\|_\infty < +\infty,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $D^\alpha = D^{\alpha_1}_{x_1} \cdots D^{\alpha_n}_{x_n}$, and $\|\cdot\|_\infty$ denotes the supremum norm;

(F2) $F(\cdot,\cdot)$ and $D_x F(\cdot,\cdot)$ are continuous on $M \times \mathcal{P}(M)$;

(F3) For every $x \in M$, the function $m \mapsto F(x, m)$ is Lipschitz continuous and

$$\sup_{\substack{x \in M \\ m_1, m_2 \in \mathcal{P}(M) \\ m_1 \neq m_2}} \frac{|F(x, m_1) - F(x, m_2)|}{d_1(m_1, m_2)} < +\infty.$$

Example 2. Let $F(x, m) = f(x)q(m)$, where $f : M \to \mathbb{R}$ is of class C^2, and $q : \mathcal{P}(M) \to \mathbb{R}$ is Lipschitz. Then

$$\|F(\cdot, m)\|_\infty = \|f(\cdot)q(m)\|_\infty, \quad \|D_x F(\cdot, m)\|_\infty = \|Df(\cdot)q(m)\|_\infty \leqslant F_\infty, \quad \forall m \in \mathcal{P}(M),$$

for some $F_\infty > 0$; $F(\cdot,\cdot) = f(\cdot)q(\cdot)$ and $D_x F(\cdot,\cdot) = Df(\cdot)q(\cdot)$ are continuous on $M \times \mathcal{P}(M)$; for each $x \in M$, each $m_1, m_2 \in \mathcal{P}(M)$,

$$|F(x, m_1) - F(x, m_2)| = |f(x)(q(m_1) - q(m_2))| \leqslant \|f\|_\infty \mathrm{Lip}(q) d_1(m_1, m_2).$$

Definition 1. A solution of the contact mean-field game system (1.5) is a couple $(u, m) \in C(M) \times \mathcal{P}(M)$ such that (1.5b) is satisfied in distribution sense and (1.5a) is satisfied in viscosity sense.

The main result is stated as follows.

Theorem 1. *Assume* (H1)–(H4) *and* (F1)–(F3). *There exists at least one solution* (u, m) *of the contact mean-field game system* (1.5), *which has clear dynamical meaning. More precisely, there is a Mather measure* μ_m *for the contact Hamiltonian system* (1.6) *such that* $m = \pi_x \sharp \mu_m$, *where* $\pi_x : T^*M \times \mathbb{R} \to M$ *denotes the canonical projection.*

Remark 2.

(i) Our methods depend on the analysis of dynamical behavior of the following contact Hamiltonian system:

$$\begin{cases} \dot{x} = \dfrac{\partial H_m}{\partial p}(x, u, p), \\ \dot{p} = -\dfrac{\partial H_m}{\partial x}(x, u, p) - \dfrac{\partial H_m}{\partial u}(x, u, p)p, \\ \dot{u} = \dfrac{\partial H_m}{\partial p}(x, u, p) \cdot p - H_m(x, u, p), \end{cases} \tag{1.6}$$

where $H_m(x, u, p) := H(x, u, p) - F(x, m)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$.

(ii) The notion of Mather measures was introduced by Mather in [26] for convex Hamiltonian systems, while the one for convex contact Hamiltonian systems was introduced in [32], where part of Aubry-Mather and weak KAM theories for Hamiltonian systems was extended to contact Hamiltonian systems under assumptions (H1) and (H2) and strict increasing condition in the argument u. See [25,27] for Aubry-Mather and weak KAM results for discounted Hamiltonian systems.

(iii) u is Lipschitz and thus, differentiable almost everywhere. Furthermore, u is of class $C^{1,1}$ on the Mather set of system (1.6), and thus, $Du(x)$ exists for m-a.e. $x \in M$ [32, Proposition 4.2].

(iv) m satisfies (1.5b) in the sense of distributions, that is,

$$\int_M \left\langle D\varphi(x), \frac{\partial H}{\partial p}(x, u(x), Du(x)) \right\rangle_x \mathrm{d}m(x) = 0, \quad \forall \varphi \in C^1(M).$$

(v) Let $K(x,p)$ be a Tonelli Hamiltonian. Mean-field game systems where the Hamiltonian has the following form:

$$\bar{H}(x,u,p) = u + K(x,p),$$

which appears in certain free-market economy models, see, for instance, [16]. It is clear that $\bar{H}(x,u,p)$ is a specific example of the Hamiltonians satisfying (H1)–(H3). To the best of our knowledge, Theorem 1 is the first step toward understanding contact mean-field game systems from a dynamical point of view.

See, for example, [8–10,17,20] for recent progresses on first-order mean-field games. [14] by Gomes and Saude is a good survey on mean-field games. We also refer the readers to [15] by Gomes et al. for many interesting aspects of mean-field games.

2 Weak KAM results for Hamiltonian and contact Hamiltonian systems

We recall some weak KAM type results for Tonelli Hamiltonian systems and Tonelli contact Hamiltonian systems. Results presented in Section 2.1 come from [13], and the ones in Section 2.2 come from [31,32].

2.1 Weak KAM results for Hamiltonian systems

2.1.1 Mañé's critical value

Let K denote a Tonelli Hamiltonian on T^*M and let l denote the associated Tonelli Lagrangian on TM as given in Section 1. If $[a,b]$ is a finite interval and $\gamma : [a,b] \to M$ is an absolutely continuous curve, we define its l action as follows:

$$A_l(\gamma) = \int_a^b l(\gamma(s), \dot{\gamma}(s))\mathrm{d}s.$$

The following estimate for action $A_l(\cdot)$ will be used later.

Proposition 1. ([13, Proposition 4.4.4]) *For every given $t > 0$, there exists a constant $C_t < +\infty$, such that, for each $x, y \in M$, there is a C^∞ curve $\gamma : [0,t] \to M$ with $\gamma(0) = x$, $\gamma(t) = y$, and $A_l(\gamma) \le C_t$.*

The Mañé critical value of the Lagrangian l, which was introduced by Mañé in [24], is defined by $c(l) \coloneqq \sup\{k \in \mathbb{R} : A_{l+k}(\gamma) < 0$ for some absolutely continuous closed curve $\gamma\}$. The Mañé's critical value has several other representation formulas:

$$c(l) = \inf_{u \in C^1(M)} \max_{x \in M} K(x, Du(x)) = -\inf_{\mu} \int_{TM} l(x, v) \mathrm{d}\mu,$$

where the second infimum is taken with respect to all Borel probability measures on TM invariant by the Euler-Lagrange flow Φ_t^l. Furthermore, $c(l)$ is the unique value of e for which $K(x, Du) = e$ admits a viscosity solution. In the following, we also call $c(l)$ the Mañé critical value of the Hamiltonian K, denoted by $c(K)$.

2.1.2 Weak KAM solutions

A backward weak KAM solution of equation (1.3) is a function $u : M \to \mathbb{R}$ such that

(1) $u(x) - u(y) \le \inf_{s>0}\{\inf_\alpha A_l(\alpha) + c(K)s\}$, $\forall x, y \in M$, where the second infimum is taken over all the absolutely continuous curves $\alpha : [0, s] \to M$ with $\alpha(0) = y$ and $\alpha(s) = x$;

(2) for every $x \in M$, there exists a curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ such that

$$u(x) - u(\gamma(t)) = \int_t^0 l(\gamma_x(s), \dot{\gamma}_x(s)) \mathrm{d}s - c(K)t, \quad \forall t \in (-\infty, 0].$$

Fathi introduced this notion and showed that for Tonelli Hamiltonians, backward weak KAM solutions and viscosity solutions of equation (1.3) are the same [13, Theorem 7.6.2]. Similarly, one can define forward weak KAM solutions of equation (1.3).

Proposition 2. ([13], Proposition 4.2.1) *The family of viscosity solutions of equation* (1.3) *is equi-Lipschitz with the Lipschitz constant* $\mathrm{Lip}(u) \le B + c(K)$, *where*

$$B = \sup\{l(x, v) : (x, v) \in TM, \|v\|_x = 1\},$$

where $\|\cdot\|_x$ *denotes the norm on* T_xM *induced by a Riemannian metric.*

2.2 Weak KAM results for contact Hamiltonian systems

2.2.1 Admissibility

Definition 2. (Admissibility). We call a contact Hamiltonian $H(x, u, p)$ is admissible, if there exists $a \in \mathbb{R}$ such that $c(H(x, a, p)) = 0$, where $c(H(x, a, p))$ denotes the Mañé critical value of the classical Hamiltonian $H(x, a, p)$.

Under assumptions (H1), (H2), and $0 \le \frac{\partial H}{\partial u} \le \lambda$, it was proved in [32, Appendix B] that $H(x, u, p)$ is admissible if and only if the equation

$$H(x, u, Du) = 0 \tag{2.1}$$

admits at least a viscosity solution. It was also proved in [32, Remark 1.2] that a contact Hamiltonian $H(x, u, p)$ satisfying (H1), (H2), and (H3) is admissible, and thus, equation (2.1) has at least a viscosity solution. Furthermore, under (H1), (H2), and (H3), equation (2.1) has a unique viscosity solution, see, for instance, [32, Proposition A.1] for a proof. Denote by u_- the unique viscosity solution. Moreover, u_- is Lipschitz on M.

From now on till the end of Section 2, we always assume (H1)–(H3).

2.3 Backward weak KAM solutions and calibrated curves

Let Φ_t^H denote the local flow of

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,u,p), \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,u,p) - \dfrac{\partial H}{\partial u}(x,u,p)p, \\ \dot{u} = \dfrac{\partial H}{\partial p}(x,u,p)\cdot p - H(x,u,p). \end{cases} \tag{2.2}$$

The Legendre transform $\mathcal{L} : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ defined by

$$\mathcal{L} : (x,u,p) \mapsto \left(x, u, \frac{\partial H}{\partial p}(x,u,p)\right)$$

is a diffeomorphism. By using $\mathcal{L}$, we can define the contact Lagrangian $L(x,u,v)$ associated with $H(x,u,p)$ as follows:

$$L(x,u,v) := \sup_{p\in T_x^*M} \{\langle v,p\rangle_x - H(x,u,p)\}.$$

Then $L(x,u,v)$ and $H(x,u,p)$ are Legendre transforms of each other, depending on conjugate variables v and p respectively. Let $\Phi_t^L = \mathcal{L} \circ \Phi_t^H \circ \mathcal{L}^{-1}$. We call Φ_t^L the Euler-Lagrange flow.

Following Fathi, one can define backward weak KAM solutions of equation (2.1) as follows. A function $u \in C(M)$ is called a backward weak KAM solution if: (i) for each continuous piecewise C^1 curve $\gamma : [t_1, t_2] \to M$,

$$u(\gamma(t_2)) - u(\gamma(t_1)) \le \int_{t_1}^{t_2} L(\gamma(s), u(\gamma(s)), \dot{\gamma}(s))ds;$$

(ii) for each $x \in M$, there exists a C^1 curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ such that

$$u(x) - u(\gamma_x(t)) = \int_t^0 L(\gamma_x(s), u(\gamma_x(s)), \dot{\gamma}_x(s))ds, \quad \forall t < 0.$$

Backward weak KAM solutions and viscosity solutions are still the same [32, Proposition 2.7]. Thus, equation (2.1) has a unique backward weak KAM solution u_-. The curves in (ii) are called $(u, L, 0)$-calibrated curves. We can also define forward weak KAM solutions of equation (2.1). Note that backward and forward weak KAM solutions of equation (1.3) always exist in pairs [13, Theorem 5.1.2]. But, this is not the case for equation (2.1), see, for instance, [32, Example 1.1].

Proposition 3. ([32], Proposition 4.1) *Given $x \in M$, if $\gamma : (-\infty, 0] \to M$ is a $(u_-, L, 0)$-calibrated curve with $\gamma(0) = x$, then $(\gamma(t), u_-(\gamma(t)), p(t))$ satisfies equations* (2.2) *on* $(-\infty, 0)$, *where* $p(t) = \frac{\partial L}{\partial v}(\gamma(t), u_-(\gamma(t)), \dot{\gamma}(t))$.

Let us recall two semigroups of operators introduced in [31]. Define a family of nonlinear operators $\{T_t^-\}_{t\ge 0}$ from $C(M)$ to itself as follows. For each $\varphi \in C(M)$, denote by $(x,t) \mapsto T_t^-\varphi(x)$ the unique continuous function on $(x,t) \in M \times [0, +\infty)$ such that

$$T_t^-\varphi(x) = \inf_\gamma \left\{\varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau^-\varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau\right\},$$

where the infimum is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$. $\{T_t^-\}_{t\ge 0}$ is called the backward solution semigroup. The infimum can be achieved. Similarly, one can define another semigroup of operators $\{T_t^+\}_{t\ge 0}$, called the forward solution semigroup, by

$$T_t^+\varphi(x) = \sup_\gamma \left\{ \varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), T_{t-\tau}^+\varphi(\gamma(\tau)), \dot\gamma(\tau))d\tau \right\},$$

where the supremum is taken among absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(0) = x$. These two semigroups can be regarded as the contact counterparts of the Lax-Oleinik semigroups for classical Lagrangians defined on TM.

Proposition 4. ([32], Propositions 2.9, 2.10) *For each $\varphi \in C(M)$, the uniform limit $\lim_{t\to+\infty} T_t^-\varphi$ exists and $\lim_{t\to+\infty} T_t^-\varphi = u_-$. The uniform limit $\lim_{t\to+\infty} T_t^+ u_-$ exists. Let $u_+ = \lim_{t\to+\infty} T_t^+ u_-$. Then u_+ is a forward weak KAM solution of equation* (2.1).

See [13] for the convergence result for the Lax-Oleinik semigroups associated with autonomous Lagrangian systems $l(x, v)$, and [28–30] for the convergence and rate of convergence results for a kind of modified Lax-Oleinik semigroups associated with time-periodic Lagrangian systems $z(t, x, v)$.

Proposition 5. ([32], Lemma 4.7) *Let u_+ be as in Proposition 4. For any given $x \in M$ with $u_-(x) = u_+(x)$, there exists a curve $\gamma : (-\infty, +\infty) \to M$ with $\gamma(0) = x$ such that $u_-(\gamma(t)) = u_+(\gamma(t))$ for each $t \in \mathbb{R}$, and*

$$u_\pm(\gamma(t')) - u_\pm(\gamma(t)) = \int_t^{t'} L(\gamma(s), u_\pm(\gamma(s)), \dot\gamma(s))ds, \quad \forall t \le t' \in \mathbb{R}.$$

Moreover, $u_\pm$ are differentiable at x with the same derivative $Du_\pm(x) = \frac{\partial L}{\partial v}(x, u_\pm(x), \dot\gamma(0))$.

2.3.1 Aubry and Mather sets

We recall the definitions of Mather sets and Aubry sets for (2.2) now. We define a subset of $T^*M \times \mathbb{R}$ associated with u_- by $G_{u_-} := \mathrm{cl}(\{(x, u, p) : x \text{ is a point of differentiability of } u_-, u = u_-(x), p = Du_-(x)\})$, where $\mathrm{cl}(S)$ denotes the closure of $S \subseteq T^*M \times \mathbb{R}$.

Define the Aubry set for (2.2) by

$$\tilde{\mathcal{A}}_H := \bigcap_{t\ge 0} \Phi_{-t}^H(G_{u_-}).$$

$\tilde{\mathcal{A}}_H$ is nonempty, compact, and Φ_t^H-invariant [32]. Then there exist Borel Φ_t^H -invariant probability measures supported in $\tilde{\mathcal{A}}_H$. We call these measures Mather measures and denote by $\mathfrak{M}$ the set of Mather measures. The Mather set is defined by

$$\tilde{\mathcal{M}}_H = \mathrm{cl}\left(\bigcup_{\mu\in\mathfrak{M}} \mathrm{supp}(\mu)\right).$$

We call $\mathcal{A}_H := \pi_x \tilde{\mathcal{A}}_H$ and $\mathcal{M}_H := \pi_x \tilde{\mathcal{M}}_H$, the projected Aubry set and the projected Mather set, respectively. The projection $\pi_x : T^*M \times \mathbb{R} \to M$ induces a bi-Lipschitz homeomorphism from $\tilde{\mathcal{A}}_H$ to $\mathcal{A}_H$ [32, Theorem 1.3]. We also have that [32, Theorem 1.3, Formula (1.15)]

$$\tilde{\mathcal{A}}_H = G_{u_-} \cap G_{u_+} \subset \{(x, u, p) \in T^*M \times \mathbb{R} : H(x, u, p) = 0, u = u_-(x)\}, \tag{2.3}$$

where u_+ is as in Proposition 4, and $G_{u_+} := \mathrm{cl}(\{(x, u, p) : x$ is a point of differentiability of $u_+, u = u_+(x), p = Du_+(x)\})$.

We will also use the following notations:

$$\tilde{\mathcal{M}}_L := \mathcal{L}(\tilde{\mathcal{M}}_H) \subset TM \times \mathbb{R}, \quad \tilde{\mathcal{A}}_L := \mathcal{L}(\tilde{\mathcal{A}}_H) \subset TM \times \mathbb{R}.$$

Proposition 6. ([32], Theorem 1.5) *Given $x \in M$, let $\gamma : (-\infty, 0] \to M$ be a $(u_-, L, 0)$-calibrated curve with $\gamma(0) = x$. Let $u := u_-(x)$, $p := \frac{\partial L}{\partial \dot{x}}(x, u, \dot{\gamma}(0)_-)$, where $\dot{\gamma}(0)_-$ denotes the left derivative of $\gamma(t)$ at $t = 0$. Let $\alpha(x, u, p)$ be the α-limit set of (x, u, p). Then*

$$\alpha(x, u, p) \subseteq \tilde{\mathcal{A}}_H,$$

where $\tilde{\mathcal{A}}_H$ denotes the Aubry set for system (2.2).

3 Existence of solutions of mean-field game system

3.1 Mather sets of reversible contact Hamiltonian systems

Under the assumptions (H1)–(H4), we can take a closer look at the Mather set of (2.2).

Proposition 7. *Let*

$$\mathcal{K}_H := \{(x, u_-(x), 0) : H(x, u_-(x), 0) = 0\}.$$

Then $\mathcal{K}_H$ is a nonempty compact subset of the Mather set $\tilde{\mathcal{M}}_H$.

Proof. Since H is strictly convex and reversible in p, then by the Legendre transform, we have

$$L(x, u_-(x), 0) = \sup_{p \in T_x^*M} -H(x, u_-(x), p) = -\inf_{p \in T_x^*M} H(x, u_-(x), p) = -H(x, u_-(x), 0), \quad \forall x \in M.$$

For any $(x, u_-(x), 0) \in \mathcal{K}_H$, let $\gamma^*(t) \equiv x$ for $t \leq 0$. Then $u_-(\gamma^*(t)) \equiv u_-(x)$ and $\dot{\gamma}^*(t) \equiv 0$. Note that

$$\int_t^0 L(\gamma^*(s), u_-(\gamma^*(s)), \dot{\gamma}^*(s))\mathrm{d}s = \int_t^0 L(x, u_-(x), 0)\mathrm{d}s = 0, \quad \forall t \leq 0.$$

Thus, we obtain that

$$u_-(x) - u_-(\gamma^*(t)) = \int_t^0 L(\gamma^*(s), u_-(\gamma^*(s)), \dot{\gamma}^*(s))\mathrm{d}s, \quad \forall t \leq 0,$$

implying that γ^* is a $(u_-, L, 0)$-calibrated curve with $\gamma^*(0) = x$. Thus, by Proposition 3, we obtain that

$$\left(\gamma^*(t), u_-(\gamma^*(t)), \frac{\partial L}{\partial v}(\gamma^*(t), u_-(\gamma^*(t)), \dot{\gamma}^*(t))\right) = (x, u_-(x), 0), \quad \forall t \leq 0,$$

which satisfies equation (2.2). It means that $(x, u_-(x), 0)$ is a fixed point of the flow Φ_t^H. Thus, the α-limit set of $(x, u_-(x), 0)$ is a singleton $\{(x, u_-(x), 0)\}$. From Proposition 6, the α-limit set of $(x, u_-(x), 0)$ is contained in the Aubry set $\tilde{\mathcal{A}}_H$, and thus, we deduce that $(x, u_-(x), 0) \in \tilde{\mathcal{A}}_H$. Since each point in $\mathcal{K}_H$ is a fixed point of the flow Φ_t^H, then $\mathcal{K}_H \subset \tilde{\mathcal{M}}_H$.

Next we show that $\mathcal{K}_H$ is nonempty. Assume by contradiction that $\mathcal{K}_H = \varnothing$. There would be two possibilities: (Case I) $H(x, u_-(x), 0) > 0$, $\forall x \in M$; (Case II) $H(x, u_-(x), 0) < 0$, $\forall x \in M$.

(**Case I**): For any $(x, p) \in T^*M$, since $H(x, u_-(x), p) \geq H(x, u_-(x), 0) > 0$, then the set

$$\{(x, u_-(x), p) : H(x, u_-(x), p) = 0\} = \varnothing.$$

Recall (2.3), i.e., $\tilde{\mathcal{A}}_H \subset \{(x, u_-(x), p) : H(x, u_-(x), p) = 0\}$. Since $\tilde{\mathcal{A}}_H$ is nonempty, then $\{(x, u_-(x), p) : H(x, u_-(x), p) = 0\}$ is nonempty, a contradiction.

(**Case II**): Since

$$L(x, u_-(x), 0) = -H(x, u_-(x), 0) > 0, \quad \forall x \in M,$$

then

$$L(x, u_-(x), v) \geq L(x, u_-(x), 0) > 0, \quad \forall (x, v) \in TM. \tag{3.1}$$

Recall that the Mather set $\tilde{\mathcal{M}}_L \subset \tilde{\mathcal{A}}_L$ is nonempty. Taking an arbitrary $(x_0, u_0, v_0) \in \tilde{\mathcal{M}}_L$, let $(x(t), u(t), \dot{x}(t)) = \Phi_t^L(x_0, u_0, v_0)$ for $t \in \mathbb{R}$. Then $x(t)$ is a $(u_-, L, 0)$-calibrated curve implying that

$$u_-(x(t)) - u_-(x_0) = \int_0^t L(x(s), u_-(x(s)), \dot{x}(s))\mathrm{d}s, \quad \forall t > 0. \tag{3.2}$$

In fact, since $(x_0, u_0, v_0) \in \tilde{\mathcal{M}}_L$, then by the definition of the Aubry set and (2.3), one can deduce that $u_0 = u_-(x_0)$ and $v_0 = \frac{\partial H}{\partial p}(x_0, u_-(x_0), Du_-(x_0))$. In view of Proposition 5, there is a $(u_-, L, 0)$-calibrated curve $\gamma : (-\infty, +\infty) \to M$ with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = \frac{\partial H}{\partial p}(x_0, u_-(x_0), Du_-(x_0))$. Then by Proposition 3,

$$(\gamma(t), u_-(\gamma(t)), \frac{\partial H}{\partial p}(\gamma(t), u_-(\gamma(t)), Du_-(\gamma(t))) = \Phi_t^L(x_0, u_0, v_0)$$

for all $t \in \mathbb{R}$. Thus, $x(t) = \gamma(t)$ for all $t \in \mathbb{R}$.

Since $(x_0, u_0, v_0) \in \tilde{\mathcal{M}}_L$, then (x_0, u_0, v_0) belongs to the support of some Φ_t^L-invariant probability measure, and thus by Poincaré's recurrence theorem, we obtain $(x_0, u_0, v_0) \in \omega(x_0, u_0, v_0)$, where $\omega(x_0, u_0, v_0)$ denotes the ω-limit set of the orbit $(x(t), u(t), \dot{x}(t))$. Thus, there exists $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to +\infty$, such that $|x(t_n) - x_0| \leq \frac{1}{n}$. By (3.2), we deduce that

$$\int_0^{t_n} L(x(s), u_-(x(s)), \dot{x}(s))\mathrm{d}s = |u_-(x(t_n)) - u_-(x_0)| \leq K_{u_-}|x(t_n) - x_0| \leq \frac{K_{u_-}}{n}, \tag{3.3}$$

where $K_{u_-} > 0$ is the Lipschitz constant of u_-. For n large enough, combining (3.1) and (3.3) leads to a contradiction.

Hence, $\mathcal{K}_H$ is nonempty. In view of the compactness of M, it is clear that $\mathcal{K}_H$ is also compact. □

Proposition 8. $\mathcal{K}_H = \tilde{\mathcal{M}}_H$.

Proof. Since $H(x, u, p)$ is strictly convex in p and $H(x, u, p) = H(x, u, -p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$, then it is direct to see that $\frac{\partial H}{\partial p}(x, u, p)\cdot p \geq 0$, where equality holds if and only if $p = 0$.

Let $(x(t), u(t), p(t))$ be an arbitrary trajectory in the Mather set $\tilde{\mathcal{M}}_H$. Then in view of $\tilde{\mathcal{M}}_H \subset \tilde{\mathcal{A}}_H \subset G_{u_-}$ and the differentiability of u_- on $\mathcal{A}_H$, we deduce that

$$(x(t), u(t), p(t)) = (x(t), u_-(x(t)), Du_-(x(t))).$$

Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}u_-(x(t)) &= \frac{\partial H}{\partial p}(x(t), u_-(x(t)), Du_-(x(t)))\cdot Du_-(x(t)) - H(x(t), u_-(x(t)), Du_-(x(t))) \\ &= \frac{\partial H}{\partial p}(x(t), u_-(x(t)), Du_-(x(t)))\cdot Du_-(x(t)). \end{aligned}$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}u_-(x(t)) \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}u_-(x(t)) = 0$ if and only if $Du_-(x(t)) = 0$. If there is $t_0 \in \mathbb{R}$ such that $Du_-(x(t_0)) = 0$, then in view of the proof of the aforementioned proposition, $(x(t_0), u_-(x(t_0)), Du_-(x(t_0)))$ is a fixed point of Φ_t^H. If $Du_-(x(t)) \neq 0$ for all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t}u_-(x(t)) > 0$ for all $t \in \mathbb{R}$, which contradicts the recurrence property of points in the Mather set $\tilde{\mathcal{M}}_H$. Hence, one can deduce that $\tilde{\mathcal{M}}_H$ consists of fixed points, which have the form $(x, u_-(x), Du_-(x))$ with $H(x, u_-(x), Du_-(x)) = 0$. So far, we have proved that $\tilde{\mathcal{M}}_H \subset \mathcal{K}_H$, which together with Proposition 7 finishes the proof. □

3.2 Proof of Theorem 1

For each $m \in \mathcal{P}(M)$, $H_m(x,u,p) := H(x,u,p) - F(x,m)$ is a Hamiltonian on $T^*M \times \mathbb{R}$. When H satisfies (H1)–(H4), so does H_m. Thus, H_m is admissible for all $m \in \mathcal{P}(M)$. Let $a_m \in \mathbb{R}$ be such that the Mañé critical value of $H(x, a_m, p) - F(x,m)$ is 0, that is,

$$0 = \inf_{x\in M}(L(x, a_m, 0) + F(x, m)) = -\sup_{x\in M}(H(x, a_m, 0) - F(x, m)).$$

Lemma 1. *There is a constant $D_1 > 0$ such that $|a_m| \le D_1$ for all $m \in \mathcal{P}(M)$.*

Proof. For any $x \in M$, any $a \in \mathbb{R}$,

$$H(x, a, 0) = H(x, 0, 0) + \frac{\partial H}{\partial u}(x, \theta, 0)a,$$

for some $\theta \in \mathbb{R}$ depending on x and a. Recall that $0 < \delta \le \frac{\partial H}{\partial u}(x,u,p)$ for all $(x,u,p) \in T^*M \times \mathbb{R}$.

If $a > 0$, then $H(x, a, 0) \ge H(x, 0, 0) + \delta a$. So, we deduce that $\lim_{a\to+\infty} H(x, a, 0) = +\infty$ uniformly in $x \in M$. If $a < 0$, then $H(x, a, 0) \le H(x, 0, 0) + \delta a$. So, we deduce that $\lim_{a\to-\infty} H(x, a, 0) = -\infty$ uniformly in $x \in M$.

By the definition of a_m, we obtain that

$$0 = \sup_{x\in M}(H(x, a_m, 0) - F(x, m)). \tag{3.4}$$

Then one can deduce that the set $\{a_m\}_{m\in\mathcal{P}(M)}$ is bounded. In fact, if it is unbounded from above, then there would be a sequence $a_{m_i} \to +\infty$ as $i \to +\infty$. By the aforementioned arguments we obtain that $\lim_{i\to+\infty} H(x, a_{m_i}, 0) = +\infty$ uniformly in $x \in M$, which together with the boundedness of F implies for i large enough (3.4) cannot hold true, a contradiction. If the set $\{a_m\}_{m\in\mathcal{P}(M)}$ is unbounded from below, we can obtain a contradiction in a similar manner. □

Remark 3. Lemma 1 still holds ture when H satisfies (H1)–(H3). In fact, since a_m satisfies

$$0 = \inf_{u\in C^1(M)} \max_{x\in M}(H(x, a_m, Du(x)) - F(x, m)),$$

then

$$0 \le \max_{x\in M}(H(x, a_m, 0) - F(x, m)). \tag{3.5}$$

Conversely, for any $u \in C^1(M)$, there must be a point $x_u \in M$ such that $Du(x_u) = 0$ since M is compact and closed. Thus, we have that

$$\max_{x\in M}(H(x, a_m, Du(x)) - F(x, m)) \ge H(x_u, a_m, 0) - F(x_u, m),$$

implying

$$0 \ge \min_{x\in M}(H(x, a_m, 0) - F(x, m)). \tag{3.6}$$

By similar arguments used in the proof of Lemma 1, we can deduce from (3.5), (3.6), and (H3) that $\{a_m\}_{m\in\mathcal{P}(M)}$ is bounded.

The following result is a direct consequence of Proposition 2 and Lemma 1.

Lemma 2. *For each $m \in \mathcal{P}(M)$, let w_m denote an arbitrary viscosity solution of*

$$H(x, a_m, Dw) - F(x, m) = 0. \tag{3.7}$$

Then $\{w_m\}_{m\in\mathcal{P}(M)}$ is equi-Lipschitz with a Lipschitz constant $D_2 > 0$ given by

$$D_2 := \sup\{L(x,u,v) + F_\infty : (x,u,v) \in TM \times \mathbb{R}, \quad |u| \le D_1, \|v\|_x = 1\}.$$

Define

$$h_t^m(x,y) := \inf_\gamma \int_0^t (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))ds, \quad \forall x, y \in M,$$

where the infimum is taken among the absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$. By definition and Lemma 2, for any $x, y \in M$ and any $t > 0$, we deduce that

$$h_t^m(x,y) \ge w_m(y) - w_m(x) \ge -D_2 \mathrm{diam}(M), \quad \forall m \in \mathcal{P}(M), \tag{3.8}$$

which means that $h_t^m(x,y)$ is bounded from below.

The proof of the following lemma is quite similar to the one of Proposition 1; thus, we omit it here.

Lemma 3. *For each given $t > 0$, there is a constant $E_t \in \mathbb{R}$ such that for any $x, y \in M$, there is a C^∞ curve $\gamma : [0,t] \to M$ with $\gamma(0) = x$, $\gamma(t) = y$ and*

$$\int_0^t (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))ds \le E_t, \quad \forall m \in \mathcal{P}(M),$$

where E_t is given by

$$E_t := t\tilde{E}_t, \quad \tilde{E}_t := \sup\left\{L(x,u,v) + F_\infty : (x,u,v) \in TM \times \mathbb{R}, |u| \le D_1, \|v\|_x \le \frac{\mathrm{diam}(M)}{t}\right\}.$$

Let $w_m'(x) := w_m(x) - w_m(0)$, where w_m is as in Lemma 2. Then w_m' is still a viscosity solution of (3.7) and $w_m'(0) = 0$. From Lemma 2, Lemma 3, and [13, Lemma 5.3.2 (4)], one can deduce that for any given $t_0 > 0$, if $t \ge t_0$, then

$$h_t^m(x,y) \le E_{t_0} + 2\|w_m'\|_\infty \le E_{t_0} + 2D_2 \mathrm{diam}(M), \quad \forall x, \quad y \in M, \quad \forall m \in \mathcal{P}(M). \tag{3.9}$$

We provide a proof of (3.9) in the Appendix.

Based on (3.8) and (3.9), we can obtain the following result.

Proposition 9. *Given any $t_0 > 0$, for any $\phi \in C(M)$, there is a constant $D_{t_0,\phi} > 0$ such that*

$$|T_t^m \phi(x)| \le D_{t_0,\phi}, \quad \forall (x,t) \in M \times [t_0, +\infty), \quad \forall m \in \mathcal{P}(M),$$

where $\{T_t^m\}_{t\ge0}$ denotes the backward solution semigroup associated with $L(x,u,v) + F(x,m)$.

Proof. Let $T_t^{a_m}$ denote the backward Lax-Oleinik operator associated with $L(x, a_m, v) + F(x,m)$, i.e., for each $\varphi \in C(M)$ and each $t \ge 0$,

$$T_t^{a_m}\varphi(x) := \inf_\gamma \left\{\varphi(\gamma(0)) + \int_0^t (L(\gamma(\tau), a_m, \dot\gamma(\tau)) + F(\gamma(\tau), m))d\tau\right\},$$

where the infimum is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$. The infimum can be achieved.

Boundedness from above: for $(x,t) \in M \times [t_0, +\infty)$ with $T_t^m\phi(x) > a_m$, let $\gamma : [0,t] \to M$ be a minimizer of $T_t^{a_m}\phi(x)$. Consider the function $s \mapsto T_s^m\phi(\gamma(s))$ for $s \in (0,t]$. Since $T_0^m\phi(\gamma(0)) = \phi(\gamma(0))$ and $T_t^m\phi(x) > a_m$, then there exists $s_0 \in [0,t)$ such that $T_{s_0}^m\phi(\gamma(s_0)) \le \max\{\phi(\gamma(0)), a_m\}$ and $T_s^m\phi(\gamma(s)) > a_m$ for $s \in (s_0, t]$. Hence, by (H3), (3.8), and (3.9), we have that

$$\begin{aligned} T_t^m\phi(x) &\le T_{s_0}^m\phi(\gamma(s_0)) + \int_{s_0}^{t} (L(\gamma(s), T_s^m\phi(\gamma(s)), \dot\gamma(s)) + F(\gamma(s), m))ds \\ &\le \max\{\phi(\gamma(0)), a_m\} + \int_{s_0}^{t} (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))ds \\ &\le \|\phi\|_\infty + D_1 + h_{t-s_0}^m(\gamma(s_0), x) \\ &\le \|\phi\|_\infty + D_1 + E_{t_0} + 3D_2\mathrm{diam}(M). \end{aligned}$$

We have proved that $T_t^m\phi(x)$ is bounded from above by $\|\phi\|_\infty + 2D_1 + E_{t_0} + 3D_2\mathrm{diam}(M)$ on $M \times [t_0, +\infty)$.

Boundedness from below: For $(x, t) \in M \times [t_0, +\infty)$ with $T_t^m\phi(x) < a_m$, let $\alpha : [0, t] \to M$ be a minimizer of $T_t^m\phi(x)$. Consider the function $s \mapsto T_s^m\phi(\alpha(s))$ for $s \in (0, t]$. Since $T_0^m\phi(\alpha(0)) = \phi(\alpha(0))$ and $T_t^m\phi(x) < a_m$, then there exists $s_0 \in [0, t)$ such that $T_{s_0}^m\phi(\alpha(s_0)) \ge \min\{\phi(\alpha(s_0)), a_m\}$ and $T_s^m\phi(\alpha(s)) < a_m$ for $s \in (s_0, t]$. Hence, by (H3) and (3.8), we have that

$$\begin{aligned} T_t^m\phi(x) &= T_{s_0}^m\phi(\alpha(s_0)) + \int_{s_0}^{t} (L(\alpha(s), T_s^m\phi(\alpha(s)), \dot\alpha(s)) + F(\alpha(s), m))ds \\ &\ge \min\{\phi(\alpha(s_0)), a_m\} + \int_{s_0}^{t} (L(\alpha(s), a_m, \dot\alpha(s)) + F(\alpha(s), m))ds \\ &\ge -\|\phi\|_\infty - D_1 + h_{t-s_0}^m(\alpha(s_0), x) \\ &\ge -\|\phi\|_\infty - D_1 - D_2\mathrm{diam}(M), \end{aligned}$$

which shows that $T_t^m\phi(x)$ is bounded from below by $-\|\phi\|_\infty - 2D_1 - D_2\mathrm{diam}(M)$ on $M \times [t_0, +\infty)$. □

By Proposition 4, for each $m \in \mathcal{P}(M)$, the uniform limit of $T_t^m\phi$ as $t \to \infty$ exists and the limit function is a viscosity solution of

$$H(x, u, Du) - F(x, m) = 0.$$

Note that for any fixed m the aforementioned equation has a unique viscosity solution under (H1)–(H3) and (F1)–(F3) (see, for instance, [32, Proposition A.1] for a proof). Denote by

$$u_m(x) := \lim_{t\to+\infty} T_t^m\phi(x)$$

the unique viscosity solution. Therefore, by Proposition 9, there is a constant $D_3 > 0$ such that

$$\|u_m\|_\infty \le D_3, \quad \forall m \in \mathcal{P}(M). \tag{3.10}$$

Note that F is bounded, then by the aforementioned estimate and [32, Lemma 4.1], we deduce that $\{u_m\}_{m\in\mathcal{P}(M)}$ is equi-Lipschitz with a Lipschitz constant

$$\sup\{L(x, u, v) + F_\infty : (x, u, v) \in TM \times \mathbb{R}, \quad |u| \le D_3, \quad \|v\|_x = 1\}.$$

See Lemma 4 in the Appendix for the proof of the equi-Lipschitz property of $\{u_m\}_{m\in\mathcal{P}(M)}$.

Remark 4. Let us point out that the aforementioned Lemmas 1–3 and Propositions 9 and 10 still hold true under assumptions (H1)–(H3).

Proposition 10. *For any $m \in \mathcal{P}(M)$, let u_m denote the unique viscosity solution of*

$$H(x, u, Du) = F(x, m), \quad x \in M.$$

Let $m_j, m_0 \in \mathcal{P}(M)$, $j \in \mathbb{N}$. If $m_j \xrightarrow{w^*} m_0$, as $j \to \infty$, then u_{m_j} converges uniformly to u_{m_0} on M, as $j \to \infty$.

Proof. Let $H_m(x,u,p) := H(x,u,p) - F(x,m)$. Then by (F3) and $m_j \xrightarrow{w^*} m_0$ as $j \to \infty$, H_{m_j} converges uniformly to H_{m_0} on compact subsets of $T^*M \times \mathbb{R}$, as $j \to \infty$. Since $\{u_m\}_{m \in \mathcal{P}(M)}$ is uniformly bounded and equi-Lipschitz, then by the stability of viscosity solutions and the uniqueness of viscosity solutions of

$$H(x,u,Du) = F(x,m_0), \quad x \in M,$$

we conclude that u_{m_j} converges uniformly to u_{m_0} on M, as $j \to \infty$. □

Now we prove Theorem 1.

Proof. (Proof of Theorem 1) For any $m \in \mathcal{P}(M)$, in view of Proposition 8, we know that

$$\tilde{\mathcal{M}}_{H_m} = \mathcal{K}_{H_m} = \{(x, u_m(x), 0) : H_m(x, u_m(x), 0) = 0\}$$

and that each point in $\mathcal{K}_{H_m}$ is a fixed point of $\Phi_t^{H_m}$. So, any convex combination of atomic measures supported in $\mathcal{K}_{H_m}$ is a Mather measure for H_m. We use $\mathfrak{M}_m$ to denote the set of all convex combinations of atomic measures supported in $\mathcal{K}_{H_m}$.

Define the set-valued map

$$\Psi : \mathcal{P}(M) \rightrightarrows \mathcal{P}(M), \quad m \mapsto \Psi(m),$$

where

$$\Psi(m) := \{\pi_x \sharp \eta_m : \eta_m \in \mathfrak{M}_m\}.$$

In view of the arguments in Section 1, it is important to show that there exists a fixed point $\bar{m}$ of Ψ.

Note that the metric space $(\mathcal{P}(M), d_1)$ is convex and compact due to Prokhorov's theorem (see, for instance, [3]). Since Ψ has nonempty convex values, the only hypothesis of Kakutani's theorem we need to check is that Ψ has a closed graph: for any pair of sequences $\{m_j\}_{j\in\mathbb{N}} \subset \mathcal{P}(M)$, $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathcal{P}(M)$ such that

$$m_j \xrightarrow{w^*} m, \quad \mu_j \xrightarrow{w^*} \mu, \quad \text{as} \quad j \to +\infty \quad \text{and} \quad \mu_j \in \Psi(m_j) \quad \text{for all } j \in \mathbb{N},$$

we aim to prove that $\mu \in \Psi(m)$.

Since $\mu_j \in \Psi(m_j)$, there are measures $\eta_{m_j} \in \mathfrak{M}_{m_j}$ such that $\mu_j = \pi \sharp \eta_{m_j}$. From (3.10), we have that

$$\|u_{m_j}\|_\infty, \quad \|u_m\|_\infty \le D_3, \quad \forall j \in \mathbb{N}.$$

By Proposition 7, we obtain that

$$\operatorname{supp}\big(\eta_{m_j}\big) \subset M \times [-D_3, D_3] \times \{0\} =: K_0, \quad \forall j \in \mathbb{N}.$$

Thus, the sequence $\{\eta_{m_j}\}_{j\in\mathbb{N}}$ is tight. By Prokhorov's theorem again, passing to a subsequence if necessary, we may suppose that

$$\eta_{m_j} \xrightarrow{w^*} \eta, \quad \text{as} \quad j \to +\infty \quad \text{and} \quad \mu = \pi \sharp \eta,$$

where $\eta \in \mathcal{P}_1(T^*M)$. It suffices to show that $\eta \in \mathfrak{M}_m$. We first show that η is a $\Phi_t^{H_m}$-invariant measure. Since η_{m_j} are $\Phi_t^{H_{m_j}}$-invariant measures, then we deduce that, for any given $t \in \mathbb{R}$,

$$\int_{K_0} f\Big(\Phi_t^{H_{m_j}}(x,u,p)\Big) d\eta_{m_j} = \int_{K_0} f(x,u,p) d\eta_{m_j}, \quad \forall f \in C(K_0), \quad \forall j \in \mathbb{N}. \tag{3.11}$$

Note that H is of class C^3, F satisfies (F1), (F2), and (F3). Since K_0 is compact, then by the continuous dependence of the solutions on the initial condition and a parameter, we obtain that

$$\lim_{j\to\infty} f\Big(\Phi_t^{H_{m_j}}(x,u,p)\Big) = f\Big(\Phi_t^{H_m}(x,u,p)\Big)$$

uniformly on K_0. Thus, by (3.11), we deduce that

$$\int_{K_0} f\Big(\Phi_t^{H_m}(x,u,p)\Big)\mathrm{d}\eta = \int_{K_0} f(x,u,p)\mathrm{d}\eta, \quad \forall f \in C(K_0),$$

which shows that η is $\Phi_t^{H_m}$-invariant. Next, we show that $\mathrm{supp}(\eta) \subset \mathcal{K}_{H_m} = \{(x, u_m(x), 0) : H_m(x, u_m(x), 0) = 0\}$. Since $\eta_{m_j} \xrightarrow{w^*} \eta$ as $j \to \infty$, for any $(x_0, u_0, p_0) \in \mathrm{supp}(\eta)$, there is a sequence of points $(x_j, u_j, p_j) \in \mathrm{supp}(\eta_{m_j})$ with $(x_j, u_j, p_j) \to (x_0, u_0, p_0)$ as $j \to \infty$. By Proposition 7 and $\eta_{m_j} \in \mathfrak{M}_{m_j}$, we deduce that $u_j = u_{m_j}(x_j)$, $p_j = 0$, and that

$$H\Big(x_j, u_{m_j}(x_j), 0\Big) - F(x_j, m_j) = 0$$

for all $j \in \mathbb{N}$. By Proposition 10, the equi-Lipschitz property of $\{u_{m_j}\}$ and (F3), we obtain that

$$H(x_0, u_m(x_0), 0) - F(x_0, m) = 0,$$

which shows that $\mathrm{supp}(\eta) \subset \mathcal{K}_{H_m}$. Thus, $\eta \in \mathfrak{M}_m$. So far, we have proved that Ψ has a closed graph. By Kakutani's theorem, there exists $\bar{m} \in \mathcal{P}(M)$ such that $\bar{m} \in \Psi(\bar{m})$, i.e., there is $\eta_{\bar{m}} \in \mathfrak{M}_{\bar{m}}$ such that $\bar{m} = \pi_x \sharp \eta_{\bar{m}}$.

Denote by $\bar{u}$ the unique viscosity solution of $u + H_{\bar{m}}(x, Du) = 0$. From the arguments in Section 2, $\bar{u}$ is differentiable $\bar{m}$ -a.e since $\bar{m}$ is supported on a subset of the projected Mather set $\mathcal{M}_{H_{\bar{m}}}$.

For any $x \in \mathrm{supp}(\bar{m})$, let $\gamma_t(x) = \pi_x \circ \Phi_t^{H_{\bar{m}}}(x, \bar{u}(x), D\bar{u}(x))$. Then, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(x) = \frac{\partial H_{\bar{m}}}{\partial p}(\gamma_t(x), \bar{u}(\gamma_t(x)), D\bar{u}(\gamma_t(x))).$$

Since the map $\pi_x : \mathrm{supp}(\eta_{\bar{m}}) \to \mathrm{supp}(\bar{m})$ is one to one and its inverse is given by $x \mapsto (x, \bar{u}(x), D\bar{u}(x))$ on $\mathrm{supp}(\bar{m})$, then $\gamma_t : \mathrm{supp}(\bar{m}) \to \mathrm{supp}(\bar{m})$ is a bijection for each $t \in \mathbb{R}$. Note that, for each $t \in \mathbb{R}$ and any function $f \in C^1(M)$, we obtain that

$$\begin{aligned}
\int_{\mathrm{supp}(\bar{m})} f(\gamma_t(x))\mathrm{d}\bar{m} &= \int_{\mathrm{supp}(\bar{m})} f \circ \gamma_t(x)\mathrm{d}\pi_x\sharp\eta_{\bar{m}} \\
&= \int_{\mathrm{supp}(\eta_{\bar{m}})} f \circ \gamma_t(\pi_x(x,u,p))\mathrm{d}\eta_{\bar{m}} \\
&= \int_{\mathrm{supp}(\eta_{\bar{m}})} f\Big(\pi_x \circ \Phi_t^{H_{\bar{m}}}(x,u,p)\Big)\mathrm{d}\eta_{\bar{m}} \\
&= \int_{\mathrm{supp}(\eta_{\bar{m}})} f(\pi_x(x,u,p))\mathrm{d}\eta_{\bar{m}} \\
&= \int_{\mathrm{supp}(\bar{m})} f(x)\mathrm{d}\bar{m}.
\end{aligned}$$

Here, the first equality holds since $\bar{m}$ is a fixed point of Ψ, the second one holds by the property of the push-forward, the third holds since γ_t is a bijection, the fourth one comes from the $\Phi_t^{H_{\bar{m}}}$-invariance property of $\eta_{\bar{m}}$, and the last one is again due to the property of the push-forward. So, for any function $f \in C^1(M)$ and any $t \in \mathbb{R}$, one can deduce that

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M f(\gamma_t(x))\mathrm{d}\bar{m}(x) \\
&= \int_M \left\langle Df(\gamma_t(x)), \frac{\partial H_{\bar{m}}}{\partial p}(\gamma_t(x), \bar{u}(\gamma_t(x)), D\bar{u}(\gamma_t(x)))\right\rangle_x \mathrm{d}\bar{m}(x) \\
&= \int_M \left\langle Df(x), \frac{\partial H_{\bar{m}}}{\partial p}(x, \bar{u}(x), D\bar{u}(x))\right\rangle_x \mathrm{d}\bar{m}(x).
\end{aligned}$$

Hence, $\bar{m}$ satisfies the continuity equation, which completes the proof. □

Acknowledgments: The authors sincerely thank the referees for their careful reading of the manuscript and invaluable comments, which were very useful in improving this paper. Kaizhi Wang is supported by NSFC Grant Nos. 12171315 and 11931016 and Innovation Program of Shanghai Municipal Education Commission No. 2021-01-07-00-02-E00087.

Conflict of interest: The authors state no conflict of interest.

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Appendix

Proof. Proof of (3.9) Recall that

$$h_t^m(x,y) := \inf_\gamma \int_0^t (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))\mathrm{d}s, \quad \forall x, y \in M,$$

where the infimum is taken among the absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$.

Lemma 3 guarantees that for each given $t_0 > 0$, there is a constant $E_{t_0} \in \mathbb{R}$ such that for any $x, y \in M$, there is a C^∞ curve $\gamma_{x,y} : [0, t_0] \to M$ with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(t_0) = y$ and

$$\int_0^{t_0} (L(\gamma_{x,y}(s), a_m, \dot\gamma_{x,y}(s)) + F(\gamma_{x,y}(s), m))\mathrm{d}s \le E_{t_0}, \quad \forall m \in \mathcal{P}(M),$$

where E_{t_0} is given by

$$E_{t_0} := t_0\tilde E_{t_0}, \quad \tilde E_{t_0} := \sup\left\{L(x,u,v) + F_\infty : (x,u,v) \in TM \times \mathbb{R}, \quad |u| \le D_1, \quad \|v\|_x \le \frac{\mathrm{diam}(M)}{t_0}\right\}.$$

Recall that $w_m'(x) := w_m(x) - w_m(0)$, where w_m is as in Lemma 2. Since w_m' is a viscosity solution (or equivalently, a backward weak KAM solution) of (3.7), then there is a curve $\gamma^y : (-\infty, 0] \to M$ with $\gamma^y(0) = y$, and

$$w_m'(y) - w_m'(\gamma^y(-s)) = \int_{-s}^0 (L(\gamma^y(\tau), a_m, \dot\gamma^y(\tau)) + F(\gamma^y(\tau), m))\mathrm{d}\tau, \quad \forall s \geqslant 0.$$

For any $t \geqslant t_0$, define a curve $\gamma : [0,t] \to M$ by $\gamma(s) = \gamma_{x,\gamma^y(t_0-t)}(s)$, for $s \in [0, t_0]$, and $\gamma(s) = \gamma^y(s-t)$, for $s \in [t_0, t]$. This curve connects x and y. Hence, we have

$$\begin{aligned}
&\int_0^t (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))\mathrm{d}s \\
&= \int_0^{t_0} (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))\mathrm{d}s + \int_{t_0}^t (L(\gamma(s), a_m, \dot\gamma(s)) + F(\gamma(s), m))\mathrm{d}s \\
&\leqslant E_{t_0} + w_m'(y) - w_m'(\gamma^y(t_0 - t)) \\
&\leqslant E_{t_0} + 2\|w_m'\|_\infty \\
&\leqslant E_{t_0} + 2\|w_m(x) - w_m(0)\|_\infty \\
&\leqslant E_{t_0} + 2D_2\mathrm{Diam}(M),
\end{aligned}$$

where the last inequality comes from Lemma 2. □

Lemma 4. *u_m is equi-Lipschitz continuous on M.*

Proof. For each x, $y \in M$, let $\gamma : [0, d(x,y)] \to M$ be a geodesic of length $d(x,y)$, parameterized by arclength and connecting x to y, where $d(\cdot,\cdot)$ denotes the distance function defined by the Riemannian metric g on M. Let

$$A_1 := \sup\{L(x,u,v) + F_\infty : x \in M, \quad |u| \leqslant D_3, \|v\|_x = 1\},$$

where D_3 is as in (3.10). Since $\|\dot\gamma(s)\|_{\gamma(s)} = 1$ for each $s \in [0, d(x,y)]$, we have $L(\gamma(s), u_m(\gamma(s)), \dot\gamma(s)) + F(\gamma(s), m) \leqslant A_1$. Since u_m is a backward weak KAM solution, then

$$u_m(\gamma(d(x,y))) - u_m(\gamma(0)) \leq \int_0^{d(x,y)} (L(\gamma(s), u_m(\gamma(s)), \dot{\gamma}(s)) + F(\gamma(s), m))\mathrm{d}s \leq \int_0^{d(x,y)} A_1 \mathrm{d}s = A_1 d(x,y).$$

We finish the proof by exchanging the roles of x and y. □