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Research Article

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Multiplicity and concentration of semi-classical solutions to nonlinear Dirac-Klein-Gordon systems

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Abstract: In the present article, we study multiplicity of semi-classical solutions of a Yukawa-coupled massive Dirac-Klein-Gordon system with the general nonlinear self-coupling, which is either subcritical or critical growth. The number of solutions obtained is described by the ratio of maximum and behavior at infinity of the potentials. We use the variational method that relies upon a delicate cutting off technique. It allows us to overcome the lack of convexity of the nonlinearities.

Keywords: Dirac-Klein-Gordon system, semi-classical states, multiplicity, subcritical and critical nonlinearities

MSC 2020: 35Q40, 49J35

1 Introduction and main results

In this article, we study the solitary wave solutions of the massive Dirac-Klein-Gordon system involving an external self-coupling:

$$\begin{cases} i\hbar\partial_t\psi + ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - \lambda(x)\phi\beta\psi = g(x, |\psi|)\psi, \\ \frac{\hbar^2}{c^2}\partial_t^2\phi - \hbar^2\Delta\phi + Mc^2\phi = 4\pi\lambda(x)(\beta\psi)\cdot\,\psi \end{cases} \tag{1.1}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, where c is the speed of light, $\hbar$ is Planck's constant, $\lambda(x) > 0$ is the coupling potential, m is the mass of the electron, and M is the mass of the meson (we use the notation $u \cdot v$ to express the inner product of $u, v \in \mathbb{C}^4$). Here α_1, α_2, α_3, and β are 4×4 complex Pauli matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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System (1.1) has been derived as mathematical models of particle physics, especially in nonlinear topics. This system is inspired by approximate descriptions of the external force involving only functions of fields, which describes the Dirac and Klein-Gordon equations coupled through the Yukawa interaction between a Dirac field $\psi \in \mathbb{C}^4$ and a scalar field $\phi \in \mathbb{R}$ (see [5]). The nonlinear self-coupling $g(x, |\psi|)\psi$ describes a self-interaction in Quantum electrodynamics and gives a closer description of many particles found in the real world, which has been widely considered in the literature (see [23,24,26] etc. and references therein).

There has been many works on the existence of solutions of system (1.1). When there is no external self-coupling, i.e., $g \equiv 0$, (1.1) has been studied for a long time and results are available concerning the cauchy problem (see [6,9,10, 28,32]). In [9], Chadam obtained the first result on the global existence and uniqueness of solutions of (1.1) (in one space dimension) under suitable assumptions on the initial data. For later developments, Chadam and Glassey [10] yielded the existence of a global solution in three space dimensions. By using classical Strichartz-type time-space estimates, low regularity solutions of the Dirac-Klein-Gordon system were obtained by Bournaveas in [6]. With respect to the nonautonomous system (1.1) with external self-coupling, Ding and Xu [13] were devoted to the existence and concentration phenomenon of stationary semi-classical solutions for the subcritical nonlinearities. In addition, for the critical nonlinearities, Ding et al. [16] obtained the same results. Here, stationary solution means a solution of the type

$$\begin{cases} \psi(t,x) = \varphi(x)e^{-i\xi t/\hbar},\ \xi \in \mathbb{R},\ \varphi : \mathbb{R}^3 \to \mathbb{C}^4 \\ \phi = \phi(x). \end{cases} \tag{1.2}$$

As far as the multiplicity of solutions of system (1.1) is concerned, there is a pioneering work by Esteban et al. (see [21]) in which a multiplicity result is studied by using the variational arguments. We emphasize that this work was mainly concerned with the autonomous system. Besides, limited work has been done in the semi-classical approximation. For small $\hbar$, the solitary waves are referred to as semi-classical states.

Motivated by the works just described above, we are interested in the multiplicity of stationary semi-classical solutions to system (1.1) with some general subcritical self-coupling nonlinearity $f(x,|\varphi|)\varphi = W(x)|\varphi|^{p-2}\varphi$, $p \in (2,3)$ or critical self-coupling nonlinearity $f(x,|\varphi|) = W_1(x)|\varphi|^{p-2}\varphi + W_2(x)|\varphi|\varphi$. We shall see that one of the difficulties with nonlinear Dirac-Klein-Gordon system (1.1) is caused by the spectra of Dirac operator which are unbounded and consist of essential spectra. A further problem is caused by the Klein-Gordon equations. In [14], Ding and Ruf overcame the first difficulty with nonlinear Maxwell-Dirac system by variational methods for strongly indefinite problems. However, because the Klein-Gordon equations are also influenced by Planck's constant $\hbar$, the limit equation is different between the Dirac-Klein-Gordon systems and the Maxwell-Dirac systems. As a result, the method of nonlinear Maxwell-Dirac systems cannot be directly applied to the nonlinear Dirac-Klein-Gordon systems (1.1). Last but not the least, (1.1) involves several different potentials which bring a competition between the potentials $W(x)$ and $\lambda(x)$ or between the potentials $W_1(x)$, $W_2(x)$ and $\lambda(x)$. This will affect our important result – the number of solutions to Dirac-Klein-Gordon systems.

For notational convenience, denoted by $c = 1$, $\varepsilon = \hbar$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, we are concerned with (substitute (1.1) in (1.2)) the following stationary nonlinear Dirac-Klein-Gordon system:

$$\begin{cases} ic\varepsilon\alpha \cdot \nabla\varphi - a\beta\varphi + \omega\varphi - \lambda(x)\phi\beta\varphi = f(x,|\varphi|)\varphi, \\ -\varepsilon^2\Delta\phi + Mc^2\phi = 4\pi\lambda(x)(\beta\varphi)\cdot\varphi, \end{cases} \tag{1.3}$$

where $a = mc^2 > 0$ and $\omega \in \mathbb{R}$.

Case (A): The subcritical case

First we deal with the subcritical case:

$$f(x,|\varphi|)\varphi = W(x)|\varphi|^{p-2}\varphi, \quad p \in (2,3).$$

Setting

$$\kappa := \max_{x\in\mathbb{R}^3} W(x), \quad \kappa_\infty := \limsup_{|x|\to\infty} W(x), \quad \mathcal{W} := \{x \in \mathbb{R}^3 : W(x) = \kappa\},$$

we assume the nonlinear potential satisfies:

(W) $W \in C^{0,1}(\mathbb{R}^3)$ with $\inf W > 0$ and $\kappa_\infty < \kappa$.

In addition, setting

$$\bar{\lambda} := \max_{x \in \mathbb{R}^3} \lambda(x), \quad \lambda_\infty := \limsup_{|x| \to \infty} \lambda(x), \quad \Gamma := \{x \in \mathbb{R}^3 : \lambda(x) = \bar{\lambda}\},$$

we make the following hypotheses:

(λ) $\lambda(x) \in C^{0,1}(\mathbb{R}^3)$ with $\inf \lambda(x) > 0$ and $\lambda_\infty < \bar{\lambda}$.

Theorem 1.1. *Assume that $\omega \in (-a, a)$, $\mathcal{W} \cap \Gamma \neq \emptyset$, and (W), (λ) are satisfied. Then there is a constant $\lambda_0 > 0$, such that for any $\bar{\lambda} \le \lambda_0$ and $m \in \mathbb{N}$ with*

$$m \le m(\infty) := \min\left\{\frac{\bar{\lambda}}{\lambda_\infty}, \left(\frac{\kappa}{\kappa_\infty}\right)^{\frac{2}{p-2}}\right\},$$

there is $\varepsilon_m > 0$ such that (1.3) *has at least m pairs of semi-classical solutions $(\varphi_\varepsilon, \phi_\varepsilon) \in \bigcap_{q \ge 2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \times C^2(\mathbb{R}^3, \mathbb{R})$ provided $\varepsilon \le \varepsilon_m$.*

Remark 1.1. If additionally $\nabla\lambda(x)$ and $\nabla W\ (x)$ are bounded, then among the solutions, the ground state (the least energy solution) denoted by w_ε satisfies (see [13]):

(i) There is a maximum point x_ε of $|\varphi_\varepsilon|$ with $\lim_{\varepsilon \to 0} \mathrm{dist}(x_\varepsilon, \mathcal{W} \cap \Gamma) = 0$ such that the pair $(u_\varepsilon, V_\varepsilon)$, where $u_\varepsilon(x) := \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$ and $V_\varepsilon := \phi_\varepsilon(\varepsilon x + x_\varepsilon)$, converges in $H^1 \times H^1$ to a ground state solution of (the limit equation)

$$\begin{cases} ic\alpha \cdot \nabla u - a\beta u + \omega u - \bar{\lambda} V \beta u = \kappa |u|^{p-2} u \\ -\Delta V + Mc^2 V = 4\pi\bar{\lambda}(\beta u) \cdot u; \end{cases} \tag{1.4}$$

(ii) There exist C_1, C_2 such that $|\varphi_\varepsilon(x)| \le C_1 \exp\left(-\frac{C_2}{\varepsilon}|x - x_\varepsilon|\right)$.

Moreover, for large c, it is called the nonrelativistic limit problem. With Theorem 1.1 and [18,20], we obtain the following corollary.

Corollary 1.1. *Let $\nu > 0$, $p \in \left[\frac{12}{5}, \frac{8}{3}\right]$. Assume $0 < -\omega < mc^2$, $mc^2 + \omega \to \frac{\nu}{m}$, as $c \to +\infty$, $\omega \to -\infty$. Under the assumption of* Theorem 1.1 *and* Remark 1.1, *the ground solution (φ_c, ϕ_c) where $\varphi_c = (u_c, v_c)$ of equation* (1.4) *have the following properties*:

(1) *u_c and ϕ_c converge in H^1 to 0;*

(2) *v_c converges in H^1 to a solution of a coupled system of nonlinear Schrödinger-type equations*

$$\begin{cases} -\Delta v_1 + 2\nu v_1 = 2m\kappa |v|^{p-2} v_1, \\ -\Delta v_2 + 2\nu v_2 = 2m\kappa |v|^{p-2} v_2, \end{cases}$$

as $c \to +\infty$, $\omega \to -\infty$, where $v = (v_1, v_2) : \mathbb{R}^3 \to \mathbb{C}^2$ is the wave function corresponding to positive energy values, and $\nu > 0$ is a constant.

Remark 1.2. Corollary 1.1 states that the semi-classical limit equation of Dirac-Klein-Gordon systems is the autonomuous Dirac-Klein-Gordon systems and the nonrelativistic limit equation of the autonomuous Dirac-Klein-Gordon systems is a coupled system of nonlinear Schrödinger-type equations. However, we do not know whether the semi-classical limit equation of the nonrelativistic limit equation of Dirac-Klein-Gordon systems is a coupled system of nonlinear Schrödinger-type equations or not.

Case (B): The critical case

Next we deal with the critical case:

$$f(x, |\varphi|)\varphi = W_1(x)|\varphi|^{p-2}\varphi + W_2(x)|\varphi|\varphi, \quad p \in (2, 3).$$

Denoting, for $j = 1, 2$,

$$\kappa_j := \max_{x\in\mathbb{R}^3} W_j(x), \quad \kappa_{j\infty} := \limsup_{|x|\to\infty} W_j(x), \quad \mathcal{W}_j := \{x \in \mathbb{R}^3 : W_j(x) = \kappa_j\},$$

we assume the nonlinear potentials satisfy:

($\overrightarrow{W_1}$) $W_j \in C^{0,1}(\mathbb{R}^3)$ with $\inf W_j > 0$, $\kappa_{j\infty} < \kappa_j$.

Let S be the best Sobolev constant: $S|u|_6^2 \le |\nabla u|_2^2$ for all $u \in H^1(\mathbb{R}^3, \mathbb{R})$ and τ_p be the least energy of the following subcritical equation:

$$i\alpha\cdot\nabla u - a\beta u - \omega u = |u|^{p-2}u$$

(which always exists, see [11]). Assume furthermore that

($\overrightarrow{W_2}$) There holds

$$\kappa_{1\infty}^{-1}\kappa_{2\infty}^{p-2} < \left(\frac{S^{\frac{3}{2}}}{6\tau_p}\right)^{\frac{p-2}{2}}.$$

Setting

$$m(\overrightarrow{\infty}) := \min\left\{\frac{\bar\lambda}{\lambda_\infty}, \left(\frac{\kappa_1}{\kappa_{1\infty}}\right)^{\frac{2}{p-2}}, \left(\frac{\kappa_2}{\kappa_{2\infty}}\right)^2\right\}.$$

Theorem 1.2. *Assume that $\omega \in (-a, a)$, $\mathcal{W}_1 \cap \mathcal{W}_2 \cap \Gamma \ne \emptyset$, and ($\lambda$), ($\overrightarrow{W_1}$), ($\overrightarrow{W_2}$) are satisfied. Then there is a constant $\lambda_0 > 0$, such that for any $\bar\lambda \le \lambda_0$, $m \in \mathbb{N}$ with $m \le m(\overrightarrow{\infty})$ there is $\varepsilon_m > 0$ such that* (1.3) *has at least m pairs of semi-classical solutions $(\varphi_\varepsilon, \phi_\varepsilon) \in \bigcap_{q\ge2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \times C^2(\mathbb{R}^3, \mathbb{R})$ provided $\varepsilon \le \varepsilon_m$.*

Remark 1.3. If additionally $\nabla\lambda$ and $\nabla W_j (j = 1, 2)$ are bounded, then among the solutions, the ground state (the least energy solution) denoted by w_ε satisfies (see [16]):

(i) There is a maximum point x_ε of $|\varphi_\varepsilon|$ with $\lim_{\varepsilon\to0}\mathrm{dist}(x_\varepsilon, \mathcal{W}_1 \cap \mathcal{W}_2 \cap \Gamma) = 0$ such that the pair $(u_\varepsilon, V_\varepsilon)$, where $u_\varepsilon(x) := \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$ and $V_\varepsilon := \phi_\varepsilon(\varepsilon x + x_\varepsilon)$, converges in $H^1 \times H^1$ to a ground state solution of (the limit equation)

$$\begin{cases} i\alpha\cdot\nabla u - a\beta u + \omega u - \bar\lambda V\beta u = \kappa_1|u|^{p-2}u + \kappa_2|u|u \\ -\Delta V + MV = 4\pi\bar\lambda(\beta u)\cdot u; \end{cases} \tag{1.5}$$

(ii) There exist C_1, C_2 such that $|\varphi_\varepsilon(x)| \le C_1 \exp\left(-\frac{C_2}{\varepsilon}|x - x_\varepsilon|\right)$.

It is standard that (1.3) is equivalent to, letting $(u(x), V(x)) = (\varphi(\varepsilon x), \phi(\varepsilon x))$,

$$\begin{cases} i\alpha\cdot\nabla u - a\beta u + \omega u - \lambda_\varepsilon(x)V\beta u = f(\varepsilon x, |u|)u, \\ -\Delta V + MV = 4\pi\lambda_\varepsilon(x)(\beta u)\cdot u, \end{cases}$$

where $\lambda_\varepsilon(x) = \lambda(\varepsilon x)$, $W_\varepsilon(x) = W(\varepsilon x)$. We will in the sequel focus on this equivalent problem. Our proofs are variational: the semi-classical solutions that are associated with the equivalent problem (1) are obtained as critical points of an energy functional Φ_ε.

Remark 1.4. Every pair of semi-classical solutions above that has different energy by using variational methods for strongly indefinite problems converge to a ground state solution of the limit equation (1.4) or (1.5), respectively.

There have been many works on multiplicity of semi-classical states of nonlinear Schrödinger-Poisson systems arising in nonrelativistic quantum mechanics, see for example [1–3] and their references. Later, Ding and Ruf [14] studied nonlinear Maxwell-Dirac systems, which obtained the multiplicity of semi-classical states under the assumptions of subcritical or critical nonlinearities, respectively. Recently, Ding et al. yielded the multiplicity of semi-classical states for nonlinear Dirac equations of space-dimension n in [17]. It is worth noting that the limit equation of the nonlinear Maxwell-Dirac systems or Dirac equations is a single equation. The problems in Dirac-Klein-Gordon systems are difficult because the limit equation is a system rather than a single equation, which is going to limit the number of solutions. As far as the authors know, there have been few results on the multiplicity of semiclassical solutions to nonlinear Dirac-Klein-Gordon systems.

An outline of this article is as follows: In Section 2, we treat the linking argument which gives us a min-max scheme. In Section 3, we study the limit equation and introduce the cut-off arguments. Finally, in Section 4, the combination of the results in Sections 2 and 3 proves Theorem 1.1, Corollary 1.1, and Theorem 1.2.

2 The variational framework

In the absence of nonrelativistic limit, the speed of light c is a constant. Without loss of generality, we assume $c = 1$. Our arguments for the problem are variational, so we start from discussing the proper variational setting. Throughout the article we always let the hypotheses of Theorem 1.1 be satisfied. Since $\mathcal{W} \cap \Gamma \neq \emptyset$, without loss of generality, we assume that $0 \in \mathcal{W} \cap \Gamma$ in the subcritical case and $0 \in \mathcal{W}_1 \cap \mathcal{W}_2 \cap \Gamma$ in the critical case.

In the sequel, we denote by $|\cdot|_q$ the usual L^q-norm and by $(\cdot,\cdot)_q$ the usual L^q-inner product. Let $H_\omega = i\alpha\cdot\nabla - a\beta + \omega$ denote the self-adjoint operator on $L^2 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_\omega) = H^1 \equiv H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is well known that $\sigma(H_\omega) = \sigma_c(H_\omega) = \mathbb{R}\backslash(-a+\omega, a+\omega)$, where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and the continuous spectrum. For $\omega \in (-a, a)$, the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^+ \oplus L^-, \quad u = u^+ + u^-, \tag{2.1}$$

so that H_ω is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $E := \mathcal{D}(|H_\omega|^{\frac{1}{2}}) = H^{\frac{1}{2}}$ be equipped with the inner product

$$(u, v) = \Re(|H_\omega|^{\frac{1}{2}}u, |H_\omega|^{\frac{1}{2}}v)_2$$

and the induced norm $\|u\| = (u, v)^{\frac{1}{2}}$, where $|H_\omega|$ and $|H_\omega|^{\frac{1}{2}}$, denote, respectively, the absolute value of H_ω and the square root of $|H_\omega|$. Since $\sigma(H_\omega) = \mathbb{R}\backslash(-a+\omega, a+\omega)$, one has

$$(a - |\omega|)|u|_2^2 \leq \|u\|^2 \quad \text{for all} \quad u \in E. \tag{2.2}$$

Note that this norm is equivalent to the usual $H^{\frac{1}{2}}$-norm, hence E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L^q_{loc} for all $q \in [1, 3)$. It is clear that E possesses the following orthogonal decomposition:

$$E = E^+ \oplus E^- \quad \text{where} \quad E^\pm = E \cap L^\pm, \tag{2.3}$$

with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products. This decomposition induces also a natural decomposition of L^p, hence there is $d_p > 0$ such that

$$d_p|u^{\pm}|_p^p \leq |u|_p^p \quad \text{for all} \quad u \in E. \tag{2.4}$$

Let $H^1(\mathbb{R}^3, \mathbb{R})$ be equipped with the equivalent norm

$$\|v\|_{H^1} = \left(\int_{\mathbb{R}^3} |\nabla v|^2 + Mv^2 \mathrm{d}x \right)^{1/2}, \quad \forall v \in H^1(\mathbb{R}^3, \mathbb{R}).$$

Then (1.5) can be reduced to a single equation with a nonlocal term. Actually, for any $v \in H^1(\mathbb{R}^3, \mathbb{R})$,

$$\left| 4\pi \int_{\mathbb{R}^3} \lambda_\varepsilon(x)(\beta u)u \cdot v \mathrm{d}x \right| \leqslant 4\pi\bar{\lambda} \int_{\mathbb{R}^3} |u|^2|v| \mathrm{d}x \leqslant 4\pi\bar{\lambda}|u|_{12/5}^2|v|_6 \leqslant 4\pi\bar{\lambda}S^{-1/2}|u|_{12/5}^2\|v\|_{H^1}. \tag{2.5}$$

Hence, there exists a unique $V_{\varepsilon,u} \in H^1(\mathbb{R}^3, \mathbb{R})$ such that

$$\int_{\mathbb{R}^3} \nabla V_{\varepsilon,u} \cdot \nabla z + M \cdot V_{\varepsilon,u} z \mathrm{d}x = 4\pi \int_{\mathbb{R}^3} \lambda_\varepsilon(x)(\beta u)u \cdot z \mathrm{d}x$$

for all $z \in H^1(\mathbb{R}^3, \mathbb{R})$. It follows that $V_{\varepsilon,u}$ satisfies the Schrödinger-type equation

$$-\Delta V_{\varepsilon,u} + M \cdot V_{\varepsilon,u} = 4\pi\lambda_\varepsilon(x)(\beta u)u$$

and there holds

$$V_{\varepsilon,u}(x) = \int_{\mathbb{R}^3} \lambda_\varepsilon(y) \frac{[(\beta u)u](y)}{|x-y|} e^{-\sqrt{M}|x-y|} \mathrm{d}y. \tag{2.6}$$

Substituting $V_{\varepsilon,u}$ in (1.5), we are led to the equation

$$H_\omega u - \lambda_\varepsilon(x) V_{\varepsilon,u} \beta u = f(\varepsilon x, u). \tag{2.7}$$

On E we define the functional

$$\Phi_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma_{\lambda_\varepsilon}(u) - \Psi_\varepsilon(u)$$

for $u = u^+ + u^-$, where

$$\Gamma_{\lambda_\varepsilon}(u) = \frac{1}{4} \int_{\mathbb{R}^3} \lambda_\varepsilon(x) V_{\varepsilon,u} \cdot (\beta u) u \mathrm{d}x = \frac{1}{4} \iint \frac{\lambda_\varepsilon(x)[(\beta u)u](x)\lambda_\varepsilon(y)[(\beta u)u](y)}{|x-y|} e^{-\sqrt{M}|x-y|} \mathrm{d}y\mathrm{d}x$$

and

$$\Psi_\varepsilon(u) = \int_{\mathbb{R}^3} F(\varepsilon x, |u|) \mathrm{d}x \quad \text{where} \quad F(x, |u|) = \int_0^{|u|} f(x, s) s \mathrm{d}s.$$

It follows by standard arguments that $\Phi_\varepsilon \in C^2(E, \mathbb{R})$ and any critical point of Φ_ε is a solution of (1).

Before going on we observe the following [13]:

Lemma 2.1. *One has: for any $u, v \in E$*

(1) *For every $\varepsilon > 0$, $\Gamma_{\lambda_\varepsilon}$ is nonnegative and weakly sequentially lower semi-continuous.*

(2)

$$\Gamma'_{\lambda_\varepsilon}(u)v = \frac{1}{2}\Re \iint \lambda_\varepsilon(x)\lambda_\varepsilon(y) \frac{e^{-\sqrt{M}|x-y|}}{|x-y|} ([(\beta u)u](x)[(\beta u)v](y) + [(\beta u)u](y)[(\beta u)v](x)) \mathrm{d}y\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \lambda_\varepsilon(x) V_{\varepsilon,u} \cdot \Re(\beta u)v \mathrm{d}x;$$

(3) *If* $u_n \rightharpoonup u$ *in* E, *then* $\Gamma_{\lambda_\varepsilon}(u_n) - \Gamma_{\lambda_\varepsilon}(u_n - u) \to \Gamma_{\lambda_\varepsilon}(u)$ *and* $\Gamma'_{\lambda_\varepsilon}(u_n) - \Gamma'_{\lambda_\varepsilon}(u_n - u) \to \Gamma'_{\lambda_\varepsilon}(u)$;

(4)

$$|\Gamma_{\lambda_\varepsilon}(u)| \le S^{-1}\bar{\lambda}^2 |u|^2_{\frac{12}{5}} \le C_1\bar{\lambda}^2\|u\|^4;$$

$$|\Gamma'_{\lambda_\varepsilon}(u)v| \le 4\pi S^{-1}\bar{\lambda}^2 |u|_3^2 |u|_2 |v|_2 \le C_2\bar{\lambda}^2\|u\|^3\|v\|;$$

$$|\Gamma''_{\lambda_\varepsilon}(u)[v,v]| \leqslant C_3\bar{\lambda}^2\|u\|^2\|v\|^2,$$

where here (and below) by C_j *we denote a generic positive constant.*

We will study the multiplicity of critical points of Φ_ε via linking arguments. Setting $E_m = E^- \oplus H_m$ for any finite dimensional subspace $H_m \subset E^+$ with $\dim H_m = m$.

Lemma 2.2. *The following conclusions are true:*

1∘ *There are* $r > 0$ *and* $\tau > 0$, *both independent of* ε, *such that* $\Phi_\varepsilon|_{B_r^+} \ge 0$ *and* $\Phi_\varepsilon|_{S_r^+} \ge \tau$ *where* $B_r^+ = B_r \cap E^+ = \{u \in E^+ : \|u\| \le r\}$ *and* $S_r^+ = \partial B_r^+ = \{u \in E^+ : \|u\| = r\}$;

2∘ *For any finite dimensional subspace* $H_m \subset E^+$, *there exist* $C = C_m > 0$ *and* $R = R_m > r$ *both independent of* ε, *such that, for all* $\varepsilon > 0$, $\max \Phi_\varepsilon(E_m) \le C$, *and* $\Phi_\varepsilon(u) < 0$ *for all* $u \in E_m \backslash B_R$.

Proof. The subcritical case can be checked easily (see [13]). We verify the critical case. Recall that $|u|_p^p \le \bar{d}_p\|u\|^p$ for all $u \in E$ by Sobolev embedding theorem. 1° follows easily because, for $u \in E^+$ and $\delta > 0$ small enough

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|u\|^2 - \Gamma_{\lambda_\varepsilon}(u) - \Psi_\varepsilon(u)\\ &\ge \frac{1}{2}\|u\|^2 - c_1\bar{\lambda}^2\|u\|^4 - \frac{|W_1|_\infty}{p}|u|_p^p - \frac{|W_2|_\infty}{3}|u|_3^3\\ &\ge \frac{1}{2}\|u\|^2 - c_1\bar{\lambda}^2\|u\|^4 - \frac{\bar{d}_p|W_1|_\infty}{p}\|u\|^p - \frac{\bar{d}_3|W_2|_\infty}{3}\|u\|^3,\end{aligned}$$

with c_1 independent of u and $p > 2$.

For checking 2°, take a finite dimensional subspace $H_m \subset E^+$. In virtue of (2.4), for $u = u^- + u^+ \in E_m = E^- + H_m$ where $u^+ \in H_m$, $u^- \in E^-$, one obtains

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \Gamma_{\lambda_\varepsilon}(u) - \Psi_\varepsilon(u)\\ &\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{\inf W_2}{3}|u^+|_3^3\\ &\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{\bar{d}_3 \inf W_2}{3}c\|u^+\|^3,\end{aligned}$$

proving the conclusions. □

In particular, for any $e \in E^+\backslash\{0\}$, setting $H_1 = \mathbb{R}e$ and $E_e = E^- \oplus H_1$, the conclusion 2° holds. Based on this lemma, for any $\varepsilon \ge 0$, let c_ε denote the minimax level of Φ_ε deduced by the linking structure [30]:

$$c_\varepsilon := \inf_{e\in E^+\backslash\{0\}} \max_{u\in E_e} \Phi_\varepsilon(u) = \inf_{e\in E^+\backslash\{0\}} \max_{u\in \hat{E}_e} \Phi_\varepsilon(u), \tag{2.8}$$

where $\hat{E}_e = E^- \oplus \mathbb{R}^+ e$.

Recall that a sequence $\{u_n\} \subset E$ is called to be a $(C)_c$-sequence for $\Phi \in C^1(E,\mathbb{R})$ if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. We say that Φ satisfies the $(C)_c$-condition if any $(C)_c$-sequence for Φ has a convergent subsequence. Below we are going to study $(C)_c$-sequences for Φ_ε.

Lemma 2.3. *For every pair of constants* $c_1, c_2 > 0$, *there exists a constant* $\Lambda > 0$, *depending only on* $c_1, c_2, \bar{\lambda}$, *such that for any* $u \in E$ *with*

$$|\Phi_\varepsilon(u)| \le c_1 \quad \text{and} \quad \|u\| \cdot \|\Phi'_\varepsilon(u)\| \le c_2, \tag{2.9}$$

we have

$$\|u\| \le \Lambda.$$

Furthermore, Λ is an increasing function with respect to $\bar{\lambda} > 0$.

Proof. Again we only check the critical case because the subcritical case is easier than the critical case and is similar to [13].

Take $u \in E$ such that (2.9) is satisfied. Without loss of generality we may assume that $\|u\| \ge 1$. The form of Φ_ε and Lemma 2.1 imply that

$$\begin{aligned} c_1 + c_2 &\ge \Phi_\varepsilon(u) - \frac{1}{2}\Phi'_\varepsilon(u)u \\ &= \Gamma_{\lambda_\varepsilon}(u) - \left(\frac{1}{p} - \frac{1}{2}\right)\int_{\mathbb{R}^3} W_1(\varepsilon x)|u|^p \mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^3} W_2(\varepsilon x)|u|^3 \mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\inf W_1 |u|_p^p + \frac{\inf W_2}{6}|u|_3^3, \end{aligned}$$

that is,

$$|u|_3 \le C_1 \quad \text{and} \quad |u|_p \le C_2. \tag{2.10}$$

By [13, Lemma 2.4], we have $\left|\frac{V_u}{\bar{\lambda}\,\|u\|}\right|_6 \le |u|_p$, $p \in (2,3)$, which, together with the Hölder inequality, implies that

$$\begin{aligned} \left|\Re\int \lambda_\varepsilon(x)V_{\varepsilon,u}(\beta u)\cdot(u^+ - u^-)\right| &\le \bar{\lambda}\|u\|\left|\Re\int \frac{V_{\varepsilon,u}}{\|u\|}(\beta u)\cdot(u^+ - u^-)\right| \\ &\le \bar{\lambda}^2\|u\|\left|\frac{V_{\varepsilon,u}}{\bar{\lambda}\|u\|}\right|_6 |u|_p\cdot|u^+ - u^-|_q \\ &\le \bar{\lambda}^2 C_3\|u\|\cdot|u|_q. \end{aligned} \tag{2.11}$$

Let $q = \frac{6p}{5p-6}$. Then $2 < q < 3$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{6} = 1$. Set

$$\xi = \begin{cases} 0, & \text{if } q = p, \\ \dfrac{2(p-q)}{q(p-2)} & \text{if } q < p, \\ \dfrac{3(q-p)}{q(3-p)} & \text{if } q > p, \end{cases}$$

we deduce that $\xi < 1$ and

$$|u|_q \le \begin{cases} |u|_2^\xi \cdot |u|_p^{1-\xi}, & \text{if } 2 < q \le p, \\ |u|_3^\xi \cdot |u|_p^{1-\xi} & \text{if } p < q < 3. \end{cases}$$

This, together with Lemma 2.1 and (2.11), implies that

$$\Gamma'_{\lambda_\varepsilon}(u)(u^+ - u^-)| \le \bar{\lambda}^2 C_4\|u\|^{1+\xi}. \tag{2.12}$$

Then (2.10) and (2.12) imply that

$$\begin{aligned} c_2 &\geq \Phi'_\varepsilon(u)(u^+ - u^-) \\ &= \|u\|^2 - \Gamma'_{\lambda_\varepsilon}(u)(u^+ - u^-) - \Re \int_{\mathbb{R}^3} W_1(\varepsilon x)|u|^{p-2}u(u^+ - u^-)\mathrm{d}x - \Re \int_{\mathbb{R}^3} W_2(\varepsilon x)|u|u(u^+ - u^-)\mathrm{d}x \\ &\geq \|u\|^2 - \bar{\lambda}^2 C_4\|u\|^{1+\xi} - \kappa_1 C_2^p - \kappa_2 C_1^3. \end{aligned}$$

That is,

$$\|u\|^2 \leq \bar{\lambda}^2 C_4\|u\|^{1+\xi} + C_5. \tag{2.13}$$

Therefore, there is $\Lambda = \Lambda(c_1, c_2, \lambda)$ such that

$$\|u\| \leq \Lambda.$$

Moreover, (2.13) implies Λ is increasing in $\bar{\lambda}$. □

Lemma 2.3 has an immediate consequence, which implies the boundedness of a $(C)_c$-sequence:

Corollary 2.1. *Consider $\varepsilon \in (0, 1]$, and $\{u_n^\varepsilon\}$ is the corresponding $(C)_c$-sequence for Φ_ε. If there exists $C > 0$ such that $|c_\varepsilon| \leqslant C$ for all ε, then we have (up to a subsequence if necessary)*

$$\|u_n^\varepsilon\| \leq \Lambda,$$

where Λ found in Lemma 2.3 *depends on λ and the pair $c_1 = C$ and $c_2 = 1$.*

Additionally, for later aims we define the operator $\mathcal{V} : E \to H^1(\mathbb{R}^3, \mathbb{R})$ by $\mathcal{V}(u) = V_u$. According to [13], we have the following lemma.

Lemma 2.4.
(1) *$\mathcal{V}$ maps bounded sets into bounded sets.*
(2) *$\mathcal{V}$ is continuous.*

In order to establish our multiplicity results, we recall an abstract critical point theorem, see [7,12]. Let X, Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a countable dense subset. Let $\mathcal{P}$ be the family of semi-norms on E consisting of all semi-norms

$$p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x + y) = |s(x)| + \|y\|, \ s \in \mathcal{S}.$$

Denote by $\mathcal{T}_\mathcal{P}$ the topology on E induced by $\mathcal{P}$. Let $\mathcal{T}_{w^*}$ be the weak*-topology of E^*.

For a functional $\Phi : E \to \mathbb{R}$ and numbers $a, b \in \mathbb{R}$, we write $\Phi^a := \{u \in E : \Phi(u) \leq a\}$, $\Phi_a := \{u \in E : \Phi(u) \geq a\}$, and $\Phi_a^b := \Phi_a \cap \Phi^b$. Assume

(Φ_1) $\Phi \in C^1(E, \mathbb{R})$; $\Phi : (E, \mathcal{T}_\mathcal{P}) \to \mathbb{R}$ is upper semi-continuous, and $\Phi' : (\Phi_a, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$;

(Φ_2) There exists $r > 0$ with $\rho := \inf\Phi(S_rY) > \Phi(0) = 0$, where $S_rY := \{y \in Y : \|y\| = r\}$;

(Φ_3) There exist a finite-dimensional subspace $Y_0 \subset Y$ and $R > r$ such that we have for $E_0 := X \times Y_0$ and $B_0 := \{u \in E_0 : \|u\| \leq R\}$, $b := \sup\Phi(E_0) < \infty$ and $\sup\Phi(E_0 \backslash B_0) < \inf\Phi(B_rY)$.

We consider the set $\mathcal{M}(\Phi^c)$ of maps $g : \Phi^c \to E$ with the properties
(i) g is $\mathcal{P}$-continuous and odd;
(ii) $g(\Phi^a) \subset \Phi^a$ for all $a \in [\rho, b]$;
(iii) Each $u \in \Phi^c$ has a $\mathcal{P}$-open neighborhood $O \subset E$ such that the set $(id - g)(O \cap \Phi^c)$ is contained in a finite-dimensional linear subspace.

The pseudo-index of Φ^c is defined by

$$\psi(c) := \min\{\mathrm{gen}(g(\Phi^c) \cap S_rY) : g \in \mathcal{M}(\Phi^c)\} \in \mathbb{N}_0 \cup \{\infty\},$$

where gen$(\cdot)$ denotes the usual symmetric index. Additionally, set for $d > 0$ fixed.

$$\mathcal{M}_0(\Phi^d) := \{g \in \mathcal{M}(\Phi^d) : g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d)\}.$$

Then we define for $c \in [0, d]$

$$\psi_d(c) := \min\{\mathrm{gen}(g(\Phi^c) \cap S_r Y) : g \in \mathcal{M}_0(\Phi^d)\}.$$

Note that, by definition, $\psi(c) \le \psi_d(c)$ for all $c \in [0, d]$.

Theorem 2.1. [7,12] *Let* (Φ_1)–(Φ_3) *be satisfied and assume that* Φ *is even and satisfies the* $(C)_c$*-condition for* $c \in [\rho, b]$. *Then* Φ *has at least* $n := \dim Y_0$ *pairs of critical points with critical values given by*

$$c_i := \inf\{c \ge 0 : \psi(c) \ge i\} \in [\rho, b], \quad i = 1, \ldots, n.$$

If Φ *has only finitely many critical points in* Φ_ρ^b, *then* $\rho < c_1 < c_2 < \cdots < c_n \le b$.

Remark 2.1. Setting $X = E^-$ and $Y = E^+$, it follows from the definition and Lemma 2.2 that the functional $\Phi = \Phi_\varepsilon$ is even and satisfies (Φ_1) and (Φ_2). In the following, we are focusing on checking the assumption (Φ_3) and the $(C)_c$-condition.

3 Cut-off arguments

We will describe further the minimax value c_ε by using a Mountain-Pass reduction technique, which depends on the convexity of the nonlinearities for verifying particularly that the second-order derivative of the functional is negative definite. However, the nonlocal term $\Gamma_{\lambda_\varepsilon}$ destroys such a convexity, because its second-order derivative may take positive values (possibly very large) as well as negative values (possibly very large). In fact, the nonlocal power is 4 exceeding the growth of the nonlinear frequency term and dominates the behavior of the functional when $\|u\|$ is very large. Therefore, we will adopt the cut-off argument of [13].

Denote $T := (\Lambda + 1)^2$ and let $\eta : [0, \infty) \to [0, 1]$ be a smooth function with $\eta(t) = 1$ if $0 \le t \le T$, $\eta(t) = 0$ if $t \ge T + 1$, $\max|\eta'(t)| \le C_1$ and $\max|\eta''(t)| \le C_2$. Without loss of generality, we can assume that $\eta(t)$ is nonincreasing. Define

$$\widetilde{\Phi}_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \eta(\|u\|^2)\Gamma_{\lambda_\varepsilon}(u) - \Psi_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_{\lambda_\varepsilon}(u) - \Psi_\varepsilon(u). \tag{3.1}$$

By definition, $\Phi_\varepsilon|_{B_T} = \widetilde{\Phi}_\varepsilon|_{B_T}$ where $B_T := \{u \in E, \|u\| \le T\}$. Clearly,

$$0 \le \mathcal{F}_{\lambda_\varepsilon}(u) \le \Gamma_{\lambda_\varepsilon}(u) \tag{3.2}$$

and

$$\mathcal{F}'_{\lambda_\varepsilon}(u)v = 2\eta'(\|u\|^2)\Gamma_{\lambda_\varepsilon}(u)\langle u, v\rangle + \eta(\|u\|^2)\Gamma'_{\lambda_\varepsilon}(u)v \quad \text{for } u, v \in E.$$

3.1 The limit equation: subcritical case

For any $0 < \mu \le \kappa$, $0 < \lambda \le \bar{\lambda}$, we consider the following constant coefficient system:

$$\begin{cases} i\alpha \cdot \nabla u - a\beta u + \omega u - \lambda V\beta u = \mu|u|^{p-2}u, \\ -\Delta V + M \cdot V = 4\pi\lambda(\beta u)u. \end{cases} \tag{3.3}$$

As before, we consider the modified functional

$$\begin{aligned}\phi_{\lambda,\mu}(u) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \eta(\|u\|^2)\Gamma_\lambda(u) - \Psi_\mu(u)\\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_\lambda(u) - \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x,\end{aligned} \tag{3.4}$$

defined for $u = u^+ + u^- \in E$. Obviously, $\phi_{\lambda,\mu}$ possesses the linking structure. By $\gamma_{\lambda,\mu}$ we denote the linking level of $\phi_{\lambda,\mu}$. Define $\ell_{\lambda,\mu} : E^+ \to E^-$ by, for $u \in E^+$,

$$\phi_{\lambda,\mu}(u + \ell_{\lambda,\mu}(u)) = \max_{v\in E^-}\phi_{\lambda,\mu}(u + v)$$

and $I_{\lambda,\mu}(u) = \phi_{\lambda,\mu}(u + \ell_{\lambda,\mu}(u))$. Then $I_{\lambda,\mu} \in C^2(E^+,\mathbb{R})$ and $u \in E^+$ is a critical point of $I_{\lambda,\mu}$ if and only if $u + \ell_{\lambda,\mu}(u)$ is a critical point of $\phi_{\lambda,\mu}$. Set $\mathcal{M}_{\lambda,\mu} = \{u \in E^+ : I'_{\lambda,\mu}(u)u = 0\}$. We will call $(\ell_{\lambda,\mu}(\cdot), I_{\lambda,\mu}(\cdot), \mathcal{M}_{\lambda,\mu})$ the Mountain-Pass reduction of (3.1). It is known that

(i) $\mathcal{L}_{\lambda,\mu} := \{u \in E : \phi'_{\lambda,\mu}(u) = 0\} \neq \varnothing$ and $\mathcal{L}_{\lambda,\mu} \subset \bigcap_{q\ge2}W^{1,q}$;

(ii) $\gamma_{\lambda,\mu} := \inf\{\phi_{\lambda,\mu}(u) : u \in \mathcal{L}_{\lambda,\mu}\backslash\{0\}\} = \inf_{u\in\mathcal{M}_{\lambda,\mu}} I_{\lambda,\mu}(u) > 0$ and is attained;

(iii) $\mathcal{R}_{\lambda,\mu} := \{u \in \mathcal{L}_{\lambda,\mu} : \phi_{\lambda,\mu}(u) = \gamma_{\lambda,\mu}, |u(0)| = |u|_\infty\}$ is compact in H^1, and there exist $C, c > 0$ such that $|u(x)| \le C\exp(-c|x|)$ for all $x \in \mathbb{R}^3$ and $u \in \mathcal{R}_{\lambda,\mu}$.

Lemma 3.1. *Let γ_∞ denote the least energy of* (3.4) *with $\lambda = \lambda_\infty$, $\mu = \kappa_\infty$. For any $\lambda_\infty < \lambda \le \bar\lambda$, $\kappa_\infty < \mu \le \kappa$, there holds $m(\lambda,\mu)\gamma_{\lambda,\mu} < \gamma_\infty$, where $m(\lambda,\mu) := \min\left\{\left(\frac{\lambda}{\lambda_\infty}\right)^2, \left(\frac{\mu}{\kappa_\infty}\right)^{\frac{2}{p-2}}\right\}$.*

Proof. Let u be a least energy critical point of

$$\begin{aligned}\phi_{\lambda_\infty,\mu_\infty}(u) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \eta(\|u\|^2)\Gamma_{\lambda_\infty}(u) - \Psi_{\kappa_\infty}(u)\\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_{\lambda_\infty}(u) - \frac{\mu_\infty}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x,\end{aligned}$$

with the energy γ_∞. For any $\lambda_\infty < \lambda \le \bar\lambda$, $\kappa_\infty < \mu \le \kappa$, set

$$v(x) = bu(x) \quad \text{with } 1 > b \ge \max\left\{\frac{\lambda_\infty}{\lambda}, \left(\frac{\kappa_\infty}{\mu}\right)^{\frac{1}{p-2}}\right\}.$$

Then

$$\begin{aligned}\gamma_\infty &= \frac{1}{2b^2}(\|v^+\|^2 - \|v^-\|^2) - \frac{\eta\left(\frac{1}{b^2}\|v\|^2\right)}{b^4}\Gamma_{\lambda_\infty}(v) - \frac{1}{b^p}\Psi_{\kappa_\infty}(v)\\ &> \frac{1}{2b^2}(\|v^+\|^2 - \|v^-\|^2) - \frac{\eta(\|v\|^2)}{b^4}\Gamma_{\lambda_\infty}(v) - \frac{1}{b^p}\Psi_{\kappa_\infty}(v)\\ &\ge \frac{1}{b^2}\left[\frac{1}{2}(\|v^+\|^2 - \|v^-\|^2) - \eta(\|v\|^2)\Gamma_\lambda(v) - \Psi_\kappa(v)\right]\\ &\ge \frac{1}{b^2}\gamma_{\lambda,\mu}.\end{aligned}$$

Here we use the fact that $\frac{\lambda_\infty}{b\lambda} \le 1$ and $\frac{\kappa_\infty}{b^{p-2}\mu} \le 1$. Thus, $m(\lambda,\mu)\gamma_{\lambda,\mu} < \gamma_\infty$. □

3.2 The limit equation: critical case

For any $0 < \lambda \le \bar\lambda$, $0 < \mu_j \le \kappa_j$, $j = 1, 2$, we consider the following system:

$$\begin{cases} i\alpha\cdot\nabla u - a\beta u + \omega u - \lambda V\beta u = \mu_1|u|^{p-2}u + \mu_2|u|u, \\ -\Delta V + M\cdot V = 4\pi\lambda(\beta u)u. \end{cases} \tag{3.5}$$

As before, we consider the modified functional

$$\begin{aligned} \phi_{\lambda,\vec{\mu}}(u) &:= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \eta(\|u\|^2)\Gamma_\lambda(u) - \Psi_\mu(u) \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_\lambda(u) - \frac{\mu_1}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x - \frac{\mu_2}{3}\int_{\mathbb{R}^3}|u|^3\mathrm{d}x, \end{aligned} \tag{3.6}$$

where $\vec{\mu} := (\mu_1, \mu_2)$. As above, let $\gamma_{\lambda,\vec{\mu}}$ denote the linking level of $\phi_{\lambda,\vec{\mu}}$. Define $\ell_{\lambda,\vec{\mu}} : E^+ \to E^-$ by, for $u \in E^+$,

$$\phi_{\lambda,\vec{\mu}}(u + \ell_{\lambda,\vec{\mu}}(u)) = \max_{v\in E^-}\phi_{\lambda,\vec{\mu}}(u + v)$$

and $I_{\lambda,\vec{\mu}}(u) = \phi_{\lambda,\vec{\mu}}(u + \ell_{\lambda,\vec{\mu}}(u))$. Then $I_{\lambda,\vec{\mu}} \in C^2(E^+, \mathbb{R})$ and $u \in E^+$ is a critical point of $I_{\lambda,\vec{\mu}}$ if and only if $u + \ell_{\lambda,\vec{\mu}}(u)$ is a critical point of $\phi_{\lambda,\vec{\mu}}$. Set $\mathcal{M}_{\lambda,\vec{\mu}} = \{u \in E^+ : I'_{\lambda,\vec{\mu}}(u)u = 0\}$. Write as before $\mathcal{L}_{\lambda,\vec{\mu}}$ and $\mathcal{R}_{\lambda,\vec{\mu}}$. One has $\gamma_{\lambda,\vec{\mu}} := \inf\{\phi_{\lambda,\vec{\mu}}(u) : u \in \mathcal{L}_{\lambda,\vec{\mu}}\backslash\{0\}\} = \inf_{u\in\mathcal{M}_{\lambda,\vec{\mu}}} I_{\lambda,\vec{\mu}}(u) > 0$ and $\mathcal{L}_{\lambda,\vec{\mu}} \subset \bigcap_{q\geq 2}W^{1,q}$.

Lemma 3.2. *$\gamma_{\lambda,\vec{\mu}}$ is attained provided $\gamma_{\lambda,\vec{\mu}} < \frac{S^{\frac{3}{2}}}{6\mu_2^2}$.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence with $c = \gamma_{\lambda,\vec{\mu}}$. By the statements in Corollary 2.1, $\{u_n\}$ is bounded in E. By Lion's concentration principle [27], $\{u_n\}$ is either vanishing or nonvanishing.

Assume that $\{u_n\}$ is vanishing. Then $|u_n|_s \to 0$ for $s \in (2, 3)$, together with Lemma 2.1 implies that $\mathcal{F}_\lambda(u_n) \leq S^{-1}\bar{\lambda}^2|u_n|^4_{\frac{12}{5}} \to 0$. Thus, one obtains

$$\begin{aligned} \gamma_{\lambda,\vec{\mu}} + o(1) &= \phi_{\lambda,\vec{\mu}}(u_n) - \frac{1}{2}\phi'_{\lambda,\vec{\mu}}(u_n)u_n \\ &= \mathcal{F}_\lambda(u_n) + \int_{\mathbb{R}^3}\left(\frac{1}{2} - \frac{1}{6}\right)\mu_1|u_n|^p\mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{1}{2} - \frac{1}{3}\right)\mu_2|u_n|^3\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\frac{1}{6}\mu_2|u_n|^3\mathrm{d}x + o(1), \end{aligned}$$

that is, $\int_{\mathbb{R}^3}|u_n|^3\mathrm{d}x = \frac{6\gamma_{\lambda,\vec{\mu}}}{\mu_2} + o(1)$. Similarly, we also have

$$\begin{aligned} o(1) &= \phi'_{\lambda,\vec{\mu}}(u_n)(u_n^+ - u_n^-) \\ &= \|u_n\|^2 - \mathcal{F}'_\lambda(u_n)(u_n^+ - u_n^-) - \Re\int_{\mathbb{R}^3}\mu_1|u_n|^{p-2}u_n(u_n^+ - u_n^-)\mathrm{d}x - \Re\int_{\mathbb{R}^3}\mu_2|u_n|u_n(u_n^+ - u_n^-)\mathrm{d}x \\ &= \|u_n\|^2 - \Re\int_{\mathbb{R}^3}\mu_2|u_n|u_n(u_n^+ - u_n^-)\mathrm{d}x + o(1). \end{aligned}$$

Thus, jointly with the fact $S^{\frac{1}{2}}|u|_3^2 \leq \|u\|^2$ [8], we have

$$\|u_n\|^2 \leq \mu_2|u_n|_3|u_n|_3|u_n^+ - u_n^-|_3 + o(1) \leq \mu_2 S^{-\frac{1}{2}}\|u_n\|\left(\frac{6\gamma_{\lambda,\vec{\mu}}}{\mu_2}\right)^{\frac{1}{3}}\|u_n^+ - u_n^-\| + o(1),$$

which implies $\gamma_{\lambda,\vec{\mu}} \geq \frac{S^{\frac{3}{2}}}{6\mu_2^2}$, a contradiction.

Therefore, $\{u_n\}$ is nonvanishing, that is, there exist $r, \delta > 0$ and $x_n \in \mathbb{R}^3$ such that, setting $v_n(x) = u_n(x + x_n)$, along a subsequence,

$$\int_{B_r(0)}|v_n|^2\mathrm{d}x \geq \delta.$$

Without loss of generality, we assume $v_n \to v$. Then $v \neq 0$ and is a solution of (3.5). And so $\gamma_{\lambda,\vec{\mu}}$ is attained. □

Lemma 3.3. *$\gamma_{\lambda,\vec{\mu}}$ is attained, provided $\mu_1^{-1}\mu_2^{p-2} < \left(\frac{S^{\frac{3}{2}}}{6\tau_p}\right)^{\frac{p-2}{2}}$.*

Proof. By the reduction process and the minmax scheme, we deduce

$$\gamma_{\lambda,\vec{\mu}} \le \gamma_{\mu_1} = \mu_1^{\frac{-2}{p-2}}\tau_p.$$

If $\mu_1^{-1}\mu_2^{p-2} < \left(\frac{S^{\frac{3}{2}}}{6\tau_p}\right)^{\frac{p-2}{2}}$, then $\gamma_{\lambda,\vec{\mu}} \le \mu_1^{\frac{-2}{p-2}}\tau_p \le \frac{S^{\frac{3}{2}}}{6\mu_2^2}$. So $\gamma_{\lambda,\vec{\mu}}$ is attained by Lemma 3.2. □

In the sequel, by $\vec{\mu_1} < \vec{\mu_2}$ we mean that $\min\{\mu_1^2 - \mu_1^1, \mu_2^2 - \mu_2^1\} > 0$ for any vectors $\vec{\mu_j} = (\mu_1^j, \mu_2^j)$. The following lemma will be useful to study our problem.

Lemma 3.4.
(1) *Let $u \in \mathcal{M}_{\lambda,\vec{\mu}}$ be such that $I_{\lambda,\vec{\mu}}(u) = \gamma_{\lambda,\vec{\mu}}$ and set $E_u = E^- \oplus \mathbb{R}^+u$. Then*

$$\max_{w\in E_u} \phi_{\lambda,\vec{\mu}}(w) = I_{\lambda,\vec{\mu}}(u).$$

(2) *If $\lambda_2 < \lambda_1$, $\vec{\mu_1} - \vec{\mu_2} > 0$, then $\gamma_{\lambda_1,\vec{\mu_1}} < \gamma_{\lambda_2,\vec{\mu_2}}$.*

Proof. To prove (1), we note that $u + \ell_{\lambda,\vec{\mu}}(u) \in E_u$ and

$$I_{\lambda,\vec{\mu}}(u) = \phi_{\lambda,\vec{\mu}}(u + \ell_{\lambda,\vec{\mu}}(u)) \le \max_{w\in E_u} \phi_{\lambda,\vec{\mu}}(w).$$

Moreover, since $u \in \mathcal{M}_{\lambda,\vec{\mu}}$,

$$\max_{w\in E_u} \phi_{\lambda,\vec{\mu}}(w) \le \max_{s\ge 0} \phi_{\lambda,\vec{\mu}}(su + \ell_{\lambda,\vec{\mu}}(su)) \le \max_{s\ge 0} I_{\lambda,\vec{\mu}}(su) = I_{\lambda,\vec{\mu}}(u).$$

Therefore, $\max_{w\in E_u}\phi_{\lambda,\vec{\mu}}(w) = I_{\lambda,\vec{\mu}}(u)$.

To obtain (2), let u_2 be a least energy solution of $\phi_{\lambda_2,\vec{\mu_2}}$ and set $e = u_2^+$. Then

$$\gamma_{\lambda_2,\vec{\mu_2}} = \phi_{\lambda_2,\vec{\mu_2}}(u_2) = \max_{w\in E_e} \phi_{\lambda_2,\vec{\mu_2}}(w).$$

Suppose $u_1 \in E_e$ be such that $\phi_{\lambda_1,\vec{\mu_1}}(u_1) = \max_{w\in E_e}\phi_{\lambda_1,\vec{\mu_1}}(w)$. We deduce that

$$\gamma_{\lambda_2,\vec{\mu_2}} = \phi_{\lambda_2,\vec{\mu_2}}(u_2) \ge \phi_{\lambda_2,\vec{\mu_2}}(u_1) > \phi_{\lambda_1,\vec{\mu_1}}(u_1).$$

This ends the proof. □

Lemma 3.5. *Let $\gamma_{\vec{\infty}}$ denote the least energy of* (3.5) *with $\lambda = \lambda_\infty$, $\vec{\mu} = \kappa_{\vec{\infty}}$ with $\kappa_{\vec{\infty}} := (\kappa_{1\infty}, \kappa_{1\infty})$. For any $\lambda_\infty < \lambda \le \bar{\lambda}$, $\kappa_{j\infty} < \mu_j \le \kappa_j$, there holds*

$$m(\lambda, \vec{\mu})\gamma_{\lambda,\vec{\mu}} < \gamma_{\vec{\infty}}, \quad \textit{where } m(\lambda, \vec{\mu}) := \min\left\{\left(\frac{\lambda}{\lambda_\infty}\right)^2, \left(\frac{\mu_1}{\kappa_{1\infty}}\right)^{\frac{2}{p-2}}, \left(\frac{\mu_2}{\kappa_{2\infty}}\right)^2\right\}.$$

Proof. Let u be a least energy critical point of

$$\begin{aligned}\phi_{\lambda_\infty,\kappa_{\vec{\infty}}}(u) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \eta(\|u\|^2)\Gamma_{\lambda_\infty}(u) - \Psi_{\kappa_{\vec{\infty}}}(u)\\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_{\lambda_\infty}(u) - \frac{\mu_{1\infty}}{p}\int_{\mathbb{R}^3}|u|^p dx - \frac{\mu_{2\infty}}{3}\int_{\mathbb{R}^3}|u|^3 dx,\end{aligned}$$

with the energy $\gamma_{\vec{\infty}}$. For any $\lambda_\infty < \lambda \le \bar{\lambda}$, $\kappa_\infty < \mu \le \kappa$, set

$$v(x) = bu(x) \quad \text{with } 1 > b \ge \max\left\{\frac{\lambda}{\lambda_\infty}, \left(\frac{\mu_1}{\kappa_{1\infty}}\right)^{\frac{1}{p-2}}, \frac{\mu_2}{\kappa_{2\infty}}\right\}.$$

Then

$$\begin{aligned}
\gamma_{\vec{\infty}} &= \frac{1}{2b^2}(\|v^+\|^2 - \|v^-\|^2) - \frac{\eta\left(\frac{1}{b^2}\|v\|^2\right)}{b^4}\Gamma_{\lambda_\infty}(v) - \frac{1}{b^p}\Psi_{\kappa_{\vec{\infty}}}(v) \\
&> \frac{1}{2b^2}(\|v^+\|^2 - \|v^-\|^2) - \frac{\eta(\|v\|^2)}{b^4}\Gamma_{\lambda_\infty}(v) - \frac{1}{b^p}\frac{\kappa_{1\infty}}{p}\int_{\mathbb{R}^3}|v|^p\mathrm{d}x - \frac{1}{b^3}\frac{\kappa_{2\infty}}{3}\int_{\mathbb{R}^3}|v|^3\mathrm{d}x \\
&\geq \frac{1}{b^2}\left[\frac{1}{2}(\|v^+\|^2 - \|v^-\|^2) - \eta(\|v\|^2)\Gamma_\lambda(v) - \Psi_{\vec{\mu}}(v)\right] \\
&\geq \frac{1}{b^2}\gamma_{\lambda,\vec{\mu}}.
\end{aligned}$$

Here we use the fact that $\frac{\lambda_\infty}{b\lambda} \leq 1$, $\frac{\kappa_{1\infty}}{b^{p-2}\mu_1} \leq 1$, and $\frac{\kappa_{2\infty}}{b\mu_2} \leq 1$. Thus, $m(\lambda, \mu)\gamma_{\lambda,\mu} < \gamma_{\vec{\infty}}$. □

3.3 The cut-off functionals

Let $\hat{\lambda} = \inf\lambda(x)$, $b = \inf W(x)$, and $\vec{b} = (b_1, b_2)$ with $b_j = \inf W_j(x)$. Take

$$e_0 \in \begin{cases} \mathcal{R}_{\hat{\lambda},b}, & \text{for the subcritical case}, \\ \mathcal{R}_{\hat{\lambda},\vec{b}}, & \text{for the critical case}, \end{cases}$$

and set

$$c_{\hat{\lambda},\vec{b}} = \begin{cases} \gamma_{\hat{\lambda},b}, & \text{for the subcritical case}, \\ \gamma_{\hat{\lambda},\vec{b}}, & \text{for the critical case}, \end{cases}$$

$$c_\infty = \begin{cases} \gamma_\infty, & \text{for the subcritical case}, \\ \gamma_{\vec{\infty}}, & \text{for the critical case}. \end{cases}$$

Clearly, $c_\varepsilon \leq c_{\hat{\lambda},\vec{b}}$ for all $\varepsilon > 0$ and $c_\infty \leq c_{\hat{\lambda},\vec{b}}$. In addition, one has

Lemma 3.6. *For all $\varepsilon > 0$, $\max_{w\in E_{e_0}}\tilde{\Phi}_\varepsilon(w) \leq c_{\hat{\lambda},\vec{b}}$.*

Proof. It is clear that $\tilde{\Phi}_\varepsilon(u) \leq \phi_{\hat{\lambda},\vec{b}}$ for all $u \in E$, hence, by Lemma 3.4(1)

$$\max_{w\in E_{e_0}}\tilde{\Phi}_\varepsilon(w) \leq \max_{w\in E_{e_0}}\phi_{\hat{\lambda},b}(w) = I_{\hat{\lambda},\vec{b}}(e_0) = c_{\hat{\lambda},\vec{b}}$$

as claimed. □

Lemma 3.7. *There exists $\varepsilon_1 > 0$ and $\lambda_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1)$ and $\bar{\lambda} \in (0, \lambda_0)$, if $\{u_n^\varepsilon\}$ is a $(C)_c$ sequence of $\widetilde{\Phi}_\varepsilon$, then $\|u_n^\varepsilon\| \leq \Lambda + \frac{1}{2}$, and consequently $\tilde{\Phi}_\varepsilon(u_n^\varepsilon) = \Phi_\varepsilon(u_n^\varepsilon)$.*

In particular, replace $\tilde{\Phi}_\varepsilon$ with $\tilde{\phi}_{\lambda,\vec{\mu}}$, we obtain that $\tilde{\phi}_{\lambda,\vec{\mu}}$ shares the same ground state solution with $\phi_{\lambda,\vec{\mu}}$.

Proof. We repeat the arguments of Lemma 2.3. If $\|u\|^2 \geq T + 1$, then $\mathcal{F}_{\lambda_\varepsilon}(u) = 0$. Thus, as proved in Lemma 2.3, one obtains $\|u\|^2 \leq \Lambda$, a contradiction. We assume that $\|u\|^2 \leq T + 1$. Then, using Lemma 2.1, $|\eta'(\|u\|^2)\|u\|^2\Gamma_{\lambda_\varepsilon}(u)| \leq \bar{\lambda}^2 d_\lambda^{(1)}$ (here and in the following, by $d_\lambda^{(j)}$ we denote positive constants depending only on λ and $d_\lambda^{(j)}$ is increasing with respect to λ). Similar to (2.10),

$$\begin{aligned} c_1 + 1 &\geq \tilde{\Phi}_\varepsilon(u) - \frac{1}{2}\tilde{\Phi}_\varepsilon'(u)u \\ &= \eta(\|u\|^2) + 2\eta'(\|u\|^2)\|u\|^2\Gamma_{\lambda_\varepsilon}(u) - \left(\frac{1}{p} - \frac{1}{2}\right)\int_{\mathbb{R}^3} W_1(\varepsilon x)|u|^p \mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^3} W_2(\varepsilon x)|u|^3 \mathrm{d}x \\ &\geq -\bar{\lambda}^2 d_\lambda^{(1)} + \left(\frac{1}{2} - \frac{1}{p}\right)\inf W_1 |u|_p^p + \frac{\inf W_2}{6}|u|_3^3, \end{aligned}$$

that is,

$$|u|_3 \leq d_\lambda^{(2)} \quad \text{and} \quad |u|_p \leq d_\lambda^{(3)}.$$

Similarly, we obtain that

$$\begin{aligned} c_2 &\geq \tilde{\Phi}_\varepsilon'(u)(u^+ - u^-) \\ &\geq \|u\|^2 - \bar{\lambda}^2 d_\lambda^{(1)} - \bar{\lambda}^2 C_4\|u\|^{1+\xi} - \kappa_1|u|_p^p - \kappa_2|u|_3^3 \\ &\geq \|u\|^2 - \bar{\lambda}^2 d_\lambda^{(1)} - \bar{\lambda}^2 C_4\|u\|^{1+\xi} - \kappa_1(d_\lambda^{(2)})^p - \kappa_2(d_\lambda^{(3)})^3. \end{aligned}$$

That is,

$$\|u\|^2 \leq \bar{\lambda}^2 C_4\|u\|^{1+\xi} + \bar{\lambda}^2 d_\lambda^{(1)} + d_\lambda^{(4)}.$$

By monotonicity of $d_\lambda^{(j)}$, we see that, for $\lambda_0 > 0$ being suitably chosen, let $\bar{\lambda} \in (0, \lambda_0]$ then $\|u\| \leq \Lambda + \frac{1}{2}$. The proof is complete. □

Based on this lemma, to prove our main results, it suffices to study $\tilde{\Phi}_\varepsilon$ and obtain its critical points with critical values in $[0, \tilde{c}_{\hat{\lambda}, \vec{b}}]$. This will be done via a series of arguments. The first is to introduce the minmax values of $\tilde{\Phi}_\varepsilon$. It is easy to verify the following lemma by a similar argument of Lemma 2.2.

Lemma 3.8. *$\widetilde{\Phi}_\varepsilon$ possesses a linking structure and the constants in* Lemma 2.2 *are true for $\widetilde{\Phi}_\varepsilon$. In addition,* $\max_{v\in E_{e_0}}\widetilde{\Phi}_\varepsilon(v) \leq \tilde{c}_{\hat{\lambda}, \vec{b}}$.

Let $\tilde{c}_\varepsilon$ denote the minimax level of $\widetilde{\Phi}_\varepsilon$ induced by the linking structure defined by (2.8) with Φ_ε replaced by $\widetilde{\Phi}_\varepsilon$. By (3.6) and the forms of the functionals, one sees $c_\varepsilon \leq \tilde{c}_\varepsilon$. As before, define $h_\varepsilon : E^+ \to E^-$ by

$$\widetilde{\Phi}_\varepsilon(u + h_\varepsilon(u)) = \max_{v\in E^-}\widetilde{\Phi}_\varepsilon(u + v),$$

and $h_{\lambda,\vec{\mu}} : E^+ \to E^-$ by

$$\widetilde{\phi}_{\lambda,\vec{\mu}}(u + h_{\lambda,\vec{\mu}}(u)) = \max_{v\in E^-}\widetilde{\phi}_{\lambda,\vec{\mu}}(u + v).$$

The following is known (see [4])

(1) $h_\varepsilon, h_{\lambda,\vec{\mu}} \in C^1(E^+, E^-)$, $h_\varepsilon(0) = 0$, $h_{\lambda,\vec{\mu}}(0) = 0$;
(2) $h_\varepsilon, h_{\lambda,\vec{\mu}}$ are bounded maps;
(3) If $u_n \rightharpoonup u$ in E^+, then

$$h_\varepsilon(u_n) - h_\varepsilon(u_n - u) \to h_\varepsilon(u) \quad \text{and} \quad h_\varepsilon(u_n) \rightharpoonup h_\varepsilon(u)$$

$$h_{\lambda,\vec{\mu}}(u_n) - h_{\lambda,\vec{\mu}}(u_n - u) \to h_{\lambda,\vec{\mu}}(u) \quad \text{and} \quad h_{\lambda,\vec{\mu}}(u_n) \rightharpoonup h_{\lambda,\vec{\mu}}(u).$$

We now define $I_\varepsilon : E^+ \to \mathbb{R}$ by $I_\varepsilon(u) = \widetilde{\Phi}_\varepsilon(u + h_\varepsilon(u))$, $I_{\lambda,\vec{\mu}} : E^+ \to \mathbb{R}$ by $I_{\lambda,\vec{\mu}}(u) = \widetilde{\Phi}_{\lambda,\vec{\mu}}(u + h_{\lambda,\vec{\mu}}(u))$, and set

$$\mathcal{N}_\varepsilon := \{u \in E^+\backslash\{0\} : I_\varepsilon'(u)u = 0\}, \quad \mathcal{N}_{\lambda,\vec{\mu}} := \{u \in E^+\backslash\{0\} : I_{\lambda,\vec{\mu}}'(u)u = 0\}. \tag{3.7}$$

Clearly,

$$\tilde{c}_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) \le \tilde{c}_{\hat{\lambda},b}. \tag{3.8}$$

This, jointly with (2.8), implies that there is a sequence $\{e_n\} \subset E^+\backslash\{0\}$ such that, denoting $u_n = e_n + h_\varepsilon(e_n)$, $\widetilde{\Phi}_\varepsilon(u_n) \to \tilde{c}_\varepsilon$ and $\widetilde{\Phi}'_\varepsilon(u_n) \to 0$ as $n \to \infty$. Consequently, by Lemma 3.7 one has

$$\tilde{c}_\varepsilon = c_\varepsilon \text{ for all } \varepsilon > 0 \text{ small}.$$

From now on we will only use c_ε.

Recall that $\lambda(\varepsilon x) \to \bar{\lambda}$, $W(\varepsilon x) \to \kappa$, and $(W_1(\varepsilon x), W_2(\varepsilon x)) \to (\kappa_1, \kappa_2) = \overrightarrow{\kappa}$ as $\varepsilon \to 0$ uniformly in bounded sets of x.

By a similar argument to [4,13,15], one has

Lemma 3.9. *For any $u \in E^+\backslash\{0\}$, there is a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon$. Moreover, $\{t_\varepsilon(u)\}_{\varepsilon\le 1}$ is bounded, and if along a subsequence $t_\varepsilon(u) \to t_0(u)$, then $\|h_\varepsilon(t_\varepsilon u) - h_{\bar{\lambda},\overrightarrow{\kappa}}(t_0 u)\| \to 0$. If $u \in \mathcal{N}_{\bar{\lambda},\overrightarrow{\kappa}}$, then $t_0 = 1$.*

In addition, we have

Lemma 3.10. $\lim_{\varepsilon\to 0}\tilde{c}_\varepsilon = \tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}}$.

Proof. First, we show that $\liminf c_\varepsilon \ge \tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}}$. Arguing indirectly, assume that $\liminf c_\varepsilon < \tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}}$. By the definition of c_ε and (3.8), we can choose an $e_j \in \mathcal{N}_\varepsilon$ and $\delta > 0$ such that $\max_{u\in E_{e_j}}\tilde{\Phi}_{\varepsilon_j}(u) \le \tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}} - \delta$, as $\varepsilon_j \to 0$. Clearly, $\tilde{\Phi}_{\varepsilon_j}(u) \ge \tilde{\phi}_{\bar{\lambda},\overrightarrow{\kappa}}(u)$ for all $u \in E$ and ε small. Note also that $\tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}} \le I_{\bar{\lambda},\overrightarrow{\kappa}}(e_j) \le \max_{u\in E_{e_j}}\tilde{\phi}_{\bar{\lambda},\overrightarrow{\kappa}}(u)$. Therefore, we obtain, for all ε_j small,

$$\tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}} - \delta \ge \max_{u\in E_{e_j}}\tilde{\Phi}_{\varepsilon_j}(u) \ge \max_{u\in E_{e_j}}\tilde{\phi}_{\bar{\lambda},\overrightarrow{\kappa}}(u) \ge \tilde{\gamma}_{\bar{\lambda},\overrightarrow{\kappa}},$$

a contradiction. □

Next take $\lambda_\infty < \lambda < \bar{\lambda}$, $\kappa_\infty < \mu < \kappa$, $\kappa_{1\infty} < \mu_1 < \kappa_1$ and define

$$\lambda^\lambda(x) = \min\{\lambda, \lambda(x)\}, \quad W^\mu(x) = \min\{\mu, W(x)\}, \quad W_1^{\mu_1}(x) = \min\{\mu_1, W_1(x)\}.$$

We consider the truncated energy functional

$$\widetilde{\Phi}_\varepsilon^{\lambda,\mu} = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_\varepsilon^\lambda(u) - \Psi_\varepsilon^\mu(u),$$

where

$$\Psi_\varepsilon^\mu = \begin{cases} \int\limits_{\mathbb{R}^3} \frac{1}{p} W_\varepsilon^\mu(x)|u|^p \mathrm{d}x & \text{in the subcritical case} \\ \int\limits_{\mathbb{R}^3} \frac{1}{p} W_{1\varepsilon}^{\mu_1}(x)|u|^p \mathrm{d}x + \int\limits_{\mathbb{R}^3} \frac{1}{3} W_2(\varepsilon x)|u|^3 \mathrm{d}x & \text{in the critical case} \end{cases},$$

and $\mathcal{F}_\varepsilon^\lambda = \frac{1}{4}\eta(\|u\|^2)\Gamma_{\lambda_\varepsilon^\lambda}$, $\lambda_\varepsilon^\lambda(x) = \lambda^\lambda(\varepsilon x)$, $W_\varepsilon^\mu(x) = W^\mu(\varepsilon x)$, $W_{1\varepsilon}^{1\mu}(x) = W_1^\mu(\varepsilon x)$. As before define correspondingly $\tilde{h}_\varepsilon^{\lambda,\mu} : E^+ \to E^-$, $\tilde{I}_\varepsilon^{\lambda,\mu} : E^+ \to \mathbb{R}$, $\tilde{\mathcal{N}}_\varepsilon^{\lambda,\mu}$, $\tilde{c}_\varepsilon^{\lambda,\mu}$, and so on.

Lemma 3.11. *$\tilde{c}_\varepsilon$ is attained for small $\varepsilon > 0$.*

Proof. Given $\varepsilon > 0$, let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a minimization sequence: $I_\varepsilon(u_n) \to c_\varepsilon$. By the Ekeland variational principle, we can assume that $\{u_n\}$ is in fact a $(PS)_{c_\varepsilon}$-sequence for I_ε on E^+. Then $w_n = u_n + h_\varepsilon(u_n)$ is a $(PS)_{c_\varepsilon}$-sequence for $\tilde{\Phi}_\varepsilon$ on E. It is clear that $\{w_n\}$ is bounded, hence is a $(C)_{c_\varepsilon}$-sequence. We can assume without loss of generality that $w_n \rightharpoonup w_\varepsilon$ in E. If $w_\varepsilon \neq 0$, it is easy to check that $\tilde{\Phi}_\varepsilon(w_\varepsilon) = c_\varepsilon$. So we are going to show that $w_\varepsilon \neq 0$ for all small $\varepsilon > 0$. Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$, then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L^q_{loc} for $q \in [1, 3)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$. Let $t_n > 0$ be such that $t_n u_n \in \tilde{\mathcal{N}}^{\lambda,\mu}_\varepsilon$. Since $u_n \in \mathcal{N}_\varepsilon$, it is not difficult to see $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. Remark that $\tilde{h}^{\lambda,\mu}_\varepsilon(t_n u_n) \rightharpoonup 0$ in E and $\tilde{h}^{\lambda,\mu}_\varepsilon(t_n u_n) \rightharpoonup 0$ in L^q_{loc} for $q \in [1, 3)$. $\tilde{w}_n := t_n u_n + \tilde{h}^{\lambda,\mu}_\varepsilon(t_n u_n) \to 0$ in L^q_{loc} for $q \in [1, 3)$. Set $A_\varepsilon := \{x \in \mathbb{R}^3 : \lambda(\varepsilon x) > \lambda\}$ is bounded. Thus,

$$
\begin{aligned}
|\mathcal{F}^\lambda_{\varepsilon_j} - \mathcal{F}_{\varepsilon_j}| &\le C \left| \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \left(\lambda^\lambda_{\varepsilon_j}(x)\lambda^\lambda_{\varepsilon_j}(y) - \lambda_{\varepsilon_j}(x)\lambda_{\varepsilon_j}(y)\right) \frac{[(\beta\tilde{w}_n)\tilde{w}_n](x)[(\beta\tilde{w}_n)\tilde{w}_n](y)}{|x-y|} e^{-\sqrt{M}|x-y|}\mathrm{d}y\mathrm{d}x \right| \\
&= \left| \int\limits_{A_\varepsilon}\int\limits_{A^c_\varepsilon} \left(\lambda - \lambda_{\varepsilon_j}(x)\right)\lambda \frac{[(\beta\tilde{w}_n)\tilde{w}_n](x)[(\beta\tilde{w}_n)\tilde{w}_n](y)}{|x-y|} e^{-\sqrt{M}|x-y|}\mathrm{d}y\mathrm{d}x \right| \\
&\quad + \left| \int\limits_{A^c_\varepsilon}\int\limits_{A_\varepsilon} \left(\lambda - \lambda_{\varepsilon_j}(y)\right)\lambda \frac{[(\beta\tilde{w}_n)\tilde{w}_n](x)[(\beta\tilde{w}_n)\tilde{w}_n](y)}{|x-y|} e^{-\sqrt{M}|x-y|}\mathrm{d}y\mathrm{d}x \right| \\
&\quad + \left| \int\limits_{A_\varepsilon}\int\limits_{A_\varepsilon} \left(\lambda\lambda - \lambda_{\varepsilon_j}(x)\lambda_{\varepsilon_j}(y)\right) \frac{[(\beta\tilde{w}_n)\tilde{w}_n](x)[(\beta\tilde{w}_n)\tilde{w}_n](y)}{|x-y|} e^{-\sqrt{M}|x-y|}\mathrm{d}y\mathrm{d}x \right| \\
&\le C \left(\int\limits_{A_\varepsilon} |\tilde{w}_n|^{\frac{12}{5}} \right)^{\frac{5}{6}} \left(\int\limits_{\mathbb{R}^3} |V_{\tilde{w}_n}|^6 \right)^{\frac{1}{6}} + \left(\int\limits_{A_\varepsilon} |\tilde{w}_n|^{\frac{12}{5}} \right)^{\frac{5}{6}} \\
&\le C \left(\int\limits_{A_\varepsilon} |\tilde{w}_n|^{\frac{12}{5}} \right)^{\frac{5}{6}} \left(\int\limits_{\mathbb{R}^3} |\tilde{w}_n|^{\frac{12}{5}} \right)^{\frac{5}{6}} + C\left(\int\limits_{A_\varepsilon} |\tilde{w}_n|^{\frac{12}{5}} \right)^{\frac{5}{6}} \to 0,
\end{aligned}
$$

where $V_{\tilde{w}_n}(x) = \int_{\mathbb{R}^3} \frac{[(\beta\tilde{w}_n)\tilde{w}_n](y)}{|x-y|} e^{-\sqrt{M}|x-y|}\mathrm{d}y$.

Similarly, since set $\{x \in \mathbb{R}^3 : W_1(\varepsilon x) > \mu\}$ is bounded, we have

$$
\int\limits_{\mathbb{R}^3} \frac{1}{p} W^{\mu_1}_{1\varepsilon}(x)|\tilde{w}_n|^p \mathrm{d}x - \int\limits_{\mathbb{R}^3} \frac{1}{p} W_{1\varepsilon}(x)|\tilde{w}_n|^p \mathrm{d}x = o(1).
$$

Therefore, we obtain

$$
\begin{aligned}
\tilde{c}^{\lambda,\mu}_{\varepsilon_j} &\le \tilde{I}^{\lambda,\mu}_{\varepsilon_j}(t_n u_n) = \tilde{\Phi}^{\lambda,\mu}_{\varepsilon_j}(\tilde{w}_n) \\
&= \tilde{\Phi}_{\varepsilon_j}(\tilde{w}_n) - \mathcal{F}^\lambda_{\varepsilon_j} + \mathcal{F}_{\varepsilon_j} - \int\limits_{\mathbb{R}^3} \frac{1}{p} W^{\mu_1}_{1\varepsilon}(x)|\tilde{w}_n|^p \mathrm{d}x + \int\limits_{\mathbb{R}^3} \frac{1}{p} W_{1\varepsilon}(x)|\tilde{w}_n|^p \mathrm{d}x \\
&\le \tilde{I}_{\varepsilon_j}(u_n) + o(1) = \tilde{c}_{\varepsilon_j} + o(1).
\end{aligned}
$$

For the subcritical case, as done in the proof of Lemma 3.10, $\lim_{\varepsilon\to 0}\tilde{c}^{\lambda,\mu}_{\varepsilon_j} = \tilde{\gamma}_{\lambda,\mu}$. Thus, $\tilde{\gamma}_{\lambda,\mu} \le \tilde{\gamma}_{\bar{\lambda},\kappa}$, which contradicts with $\tilde{\gamma}_{\lambda,\mu} > \tilde{\gamma}_{\bar{\lambda},\kappa}$.

For the critical case, note that $\tilde{\gamma}_{\lambda,\vec{\mu}} \le \tilde{c}^{\lambda,\mu}_{\varepsilon_j} \le \tilde{c}_{\varepsilon_j}$, where $\vec{\mu} := (\mu_1, \kappa_2)$. Thus, by Lemma 3.10, we have $\tilde{\gamma}_{\lambda,\vec{\mu}} \le \tilde{\gamma}_{\bar{\lambda},\vec{\kappa}}$, which contradicts with $\tilde{\gamma}_{\lambda,\vec{\mu}} > \tilde{\gamma}_{\bar{\lambda},\vec{\kappa}}$. □

We now turn to prove the desired conclusion.

Lemma 3.12. *Let $u_n = u_n^+ + u_n^-$ be a $(C)_c$-sequence for $\widetilde{\Phi}_\varepsilon$ and set $v_n = u_n^+ + h_\varepsilon(u_n^+)z_n = u_n^- - h_\varepsilon(u_n^+)$. Then $\|z_n\| \to 0$ and $\{v_n\}$ is also a $(C)_c$-sequence for $\widetilde{\Phi}_\varepsilon$, that is, $\{u_n^+\}$ is a $(C)_c$-sequence for I_ε. Consequently, either $c = 0$ or $c \geq c_\varepsilon$.*

Proof. It suffices to show that $\|z_n\| \to 0$. Note that, by Lemma 3.7, one has $\|u_n\| \leq \Lambda + \frac{1}{2}$, hence $\{u_n\}$ is in fact a bounded $(C)_c$ sequence for $\Phi_\varepsilon : \Phi_\varepsilon(u_n) \to c$ and $\Phi'_\varepsilon(u_n) \to 0$. Observe that

$$0 = \Phi'_\varepsilon(v_n)z_n = -(h_\varepsilon(u_n^+), z_n) - \Gamma'_{\lambda_\varepsilon}(v_n)z_n - \Psi'_\varepsilon(v_n)z_n$$

and

$$o(1) = \Phi'_\varepsilon(u_n)z_n = -(u_n^-, z_n) - \Gamma'_{\lambda_\varepsilon}(u_n)z_n - \Psi'_\varepsilon(u_n)z_n.$$

Following [13, Remark 3.10], one has

$$|\Gamma''_{\lambda_\varepsilon}(v_n + tz_n)[z_n, z_n]| \leq \frac{1}{2}\|z_n\|^2.$$

Thus,

$$o(1) = \|z_n\|^2 + (\Gamma'_{\lambda_\varepsilon}(v_n + z_n) - \Gamma'_{\lambda_\varepsilon}(v_n))z_n + (\Psi'_\varepsilon(v_n + z_n) - \Psi'_\varepsilon(v_n))z_n \geq \|z_n\|^2,$$

which shows $\|z_n\| \to 0$. Finally, it follows from (2.8) that if $c \neq 0$ then $c \geq c_\varepsilon$. □

We turn to study the $(C)_c$-condition of $\widetilde{\Phi}$.

Lemma 3.13. *For all $\varepsilon > 0$ small, $\widetilde{\Phi}_\varepsilon$ satisfies the $(C)_c$ condition for all $c < c_\infty$.*

Proof. For later aims we set

$$\lambda^\infty(x) = \min\{\lambda_\infty, \lambda(x)\}, \quad W^\infty(x) = \min\{\kappa_\infty, W(x)\}, \quad W_j^\infty(x) = \min\{\kappa_{j\infty}, W_j(x)\}, \quad j = 1, 2.$$

Define

$$\Psi_\varepsilon^\infty = \begin{cases} \int_{\mathbb{R}^3} \frac{1}{p} W_\varepsilon^\infty(x)|u|^p \mathrm{d}x, & \text{in the subcritical case}, \\ \int_{\mathbb{R}^3} \frac{1}{p} W_{1\varepsilon}^\infty(x)|u|^p \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{3} W_{2\varepsilon}^\infty(x)|u|^3 \mathrm{d}x, & \text{in the critical case}, \end{cases}$$

$$\mathcal{F}_\varepsilon^\infty = \frac{1}{4}\eta(\|u\|^2)\Gamma_{\lambda_\varepsilon^\infty},$$

and

$$\widetilde{\Phi}_\varepsilon^\infty = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{F}_\varepsilon^\infty(u) - \Psi_\varepsilon^\infty(u), \tag{3.9}$$

where $\lambda_\varepsilon^\infty(x) = \lambda^\infty(\varepsilon x)$, $W_\varepsilon^\infty(x) = W^\infty(\varepsilon x)$, $W_{j\varepsilon}^\infty(x) = W_j^\infty(\varepsilon x)$. It is not difficult to verify that $\widetilde{\Phi}_\varepsilon^\infty$ has the same properties possessed by $\widetilde{\Phi}_\varepsilon$ shown above. In particular, letting c_ε^∞ be the linking level of $\widetilde{\Phi}_\varepsilon^\infty$, we have $c_\varepsilon^\infty \to c_\infty$ as $\varepsilon \to 0$.

Let $\{u_n\}$ be a $(C)_c$-sequence for $\widetilde{\Phi}_\varepsilon$ with $c < c_\infty$. By virtue of Lemma 3.2, $\{u_n\}$ is bounded and $\widetilde{\Phi}_\varepsilon(u_n) = \Phi_\varepsilon(u_n)$, hence it is a $(C)_c$-sequence for Φ_ε. We can assume that $u_n \rightharpoonup u$. Clearly, $\Phi'_\varepsilon(u) = 0$. Set $z_n = u_n - u$. Note that $z_n \rightharpoonup 0$ in E, $z_n \to 0$ in $L^q_{\mathrm{loc}} \in [1, 3)$, and $z_n(x) \to 0$ a.e. in x. Using the Brezis-Lieb lemma [33]:

$$\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x - \int_{\mathbb{R}^3} |z_n|^q \mathrm{d}x \to \int_{\mathbb{R}^3} |u|^q \mathrm{d}x,$$

it is easy to check that $\Phi_\varepsilon^\infty(z_n) \to c - \Phi_\varepsilon(u)$ and $(\Phi_\varepsilon^\infty)'(z_n) \to 0$. If $c = \Phi_\varepsilon(u)$, then $z_n \to 0$ and we are done. If $c - \Phi_\varepsilon(u) \ge c_\varepsilon^\infty$, then $c \ge c_\varepsilon + c_\varepsilon^\infty$, a contradiction. □

Let $\mathcal{K}_\varepsilon := \{u \in E : \widetilde{\Phi}_\varepsilon'(u) = 0\}$ be the critical set of $\widetilde{\Phi}_\varepsilon$. By using the same iterative argument of [22] one obtains easily the following (see [13,16]).

Lemma 3.14. *If $u \in \mathcal{K}_\varepsilon$ with $|\widetilde{\Phi}_\varepsilon(u)| \le C_1$, then, for any $q \in [2, +\infty)$, $\|u\|_{W^{1,q}} \le \Lambda_q$ where Λ_q depends only on C_1.*

4 Proof of main results

4.1 Proof of main results: the subcritical case

Without loss of generality, we may assume that $0 \in \mathcal{W} \cap \Gamma$, hence, $\bar{\lambda} = \lambda(0)$, $\kappa = W(0)$. Solutions of (2.7) are critical points of the functional $\Phi_\varepsilon(u) = \Phi_\varepsilon^{\bar{\lambda},\kappa}(u)$. For notational convenience we denote $\Phi_0(u) = \phi_{\bar{\lambda},\kappa}(u)$. We will utilize Theorem 2.1. Obviously, Φ_ε is even, and in virtue of Remark 2.1 the conditions (Φ_1) and (Φ_2) are satisfied. It remains to verify (Φ_3).

Let $u \in \mathcal{R}_{\bar{\lambda},\kappa}$ and let $\chi_r \in C_0^\infty(\mathbb{R}^+)$ be such that $\chi_r(s) = 1$ if $s \le r$, $\chi_r(s) = 0$ if $s \ge r+1$, and $\chi_r'(s) \le 0$. Set $u_r(x) = \chi_r(|x|)u(x)$. Recall that $|u(x)| \le Ce^{-c|x|}$ for some $C, c > 0$ and all $x \in \mathbb{R}^3$, hence $\|u_r - u\| \to 0$ as $r \to \infty$. Then $\|u_r^\pm - u^\pm\| \le \|u_r - u\| \to 0$, $\Phi_0(u_r) \to \gamma_{\bar{\lambda},\kappa}$ and $\Phi_0'(u_r) \to 0$, as $r \to \infty$. Let $h_0 : E^+ \to E^-$ be defined so that $\Phi_0(u + h_0(u)) = \max_{v\in E^-}\Phi_0(u+v)$. Clearly, $\|u_r^- - h_0(u_r^+)\| \to 0$, and $\|u_r - \hat{u}_r\| \to 0$, where $\hat{u}_r = u_r^+ + h_0(u_r^+)$ (see Lemma 3.12). Therefore,

$$\max_{v\in E^-}\Phi_0(u_r^+ + v) = \Phi_0(\hat{u}_r) = \Phi_0(u_r) + o(1) = \gamma_{\bar{\lambda},\kappa} + o(1), \tag{4.1}$$

as $\varepsilon \to 0$. Observe that since, as $\varepsilon \to 0$, $\lambda_\varepsilon \to \bar{\lambda}$ and $W(\varepsilon x) \to \kappa$ uniformly in $|x| \le r+1$ we have that, for any $\delta > 0$, there are $r_\delta > 0$ and $\varepsilon_\delta > 0$ such that

$$\max_{w\in E^-\oplus\mathbb{R}u_r}\Phi_\varepsilon(w) < \gamma_{\bar{\lambda},\kappa} + \delta, \tag{4.2}$$

for all $r \ge r_\delta$ and $\varepsilon \le \varepsilon_\delta$.

Let $y^j = (2j(r+1), 0, 0)$, define $u_j(x) = u(x - y^j) = u(x_1 - 2j(r+1), x_2, x_3)$, $u_{rj}(x) = u_r(x - y^j)$ for $j = 0, 1, \dots, m-1$. Setting $r_m = (2m-1)(r+1)$, it is clear that supp $u_{rj} \subset B_{r_m}(0)$. Obviously $\{u_{rj}^+\}_{j=0}^{m-1}$ are linearly independent. Indeed, if $w^+ = \sum_{j=0}^{m-1}c_ju_{rj}^+ = 0$, denoting $w = \sum_{j=0}^{m-1}c_ju_{rj}$, one has $w = w^- + w^+$ and

$$-\|w^-\|^2 = a(w) = \sum_j c_j^2 a(u_{rj}^+) = a(u_r)\sum_j c_j^2,$$

which implies $c_j = 0$, $j = 0, 1, \dots, m-1$. Now set

$$E_m = E^- \oplus \operatorname{span}\{u_{rj} : j = 0, \dots, m-1\} = E^- \oplus \operatorname{span}\{u_{rj}^+ : j = 0, \dots, m-1\}.$$

By virtue of Lemma 3.9, let $t_{\varepsilon rj} > 0$ be such that $t_{\varepsilon rj}u_{rj}^+ \in \mathcal{N}_\varepsilon$. Observe that

$$\lim_{\varepsilon\to0}\lim_{r\to\infty} t_{\varepsilon rj} = \lim_{\varepsilon\to0} t_{\varepsilon j} = 1; \tag{4.3}$$

$$\lim_{\varepsilon\to0}\lim_{r\to\infty} h_\varepsilon(t_{\varepsilon rj}u_{rj}^+) = \lim_{\varepsilon\to0} h_\varepsilon(t_{\varepsilon j}u_{rj}^+) = h_0(u^+) = u^-; \tag{4.4}$$

$$\lim_{\varepsilon\to0}\lim_{r\to\infty}\|h_\varepsilon(t_{\varepsilon rj}u_{rj}^+) - t_{\varepsilon rj}u_{rj}^-\| = \lim_{\varepsilon\to0}\|h_\varepsilon(t_{\varepsilon j}u^+) - t_{\varepsilon j}u^-\| = 0. \tag{4.5}$$

It is not difficult to check the following:

$$\begin{aligned}\max_{w\in E_m}\Phi_\varepsilon(w) &= \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)\right) = \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-\right) + o(1_r)\\ &= \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}\right) + o(1_r) = \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}) + o(1_r)\\ &= \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-) + o(1_r) = \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)) + o(1_r)\\ &= \sum_{j=0}^{m-1}\Phi_0(t_{0j}u_{rj}^+ + h_0(t_{0j}u_{rj}^+)) + o(1_{r\varepsilon}) = \sum_{j=0}^{m-1}\Phi_0(u) + o(1_{r\varepsilon})\\ &= m\gamma_{\bar\lambda,\kappa} + o(1_{r\varepsilon}),\end{aligned}$$

where $o(1_r)$ means arbitrary small as $r\to\infty$, and $o(1_{r\varepsilon})$ means arbitrary small as r is sufficiently large and ε is sufficiently small. Now, by assumptions and Lemma 3.1, for any $0<\delta<\gamma_\infty - m\gamma_{\bar\lambda,\kappa}$, one may choose $r>0$ large and then $\varepsilon_m>0$ small such that, for all $\varepsilon\le\varepsilon_m$, $\max_{w\in E_m}\Phi_\varepsilon(w)\le\gamma_\infty-\delta$. Now by Theorem 2.1 one obtains the multiplicity conclusion.

Let $\mathcal{L}_\varepsilon$ denote the set of all least energy solutions of $\tilde\Phi_\varepsilon$. Let $\varepsilon_j\to 0$, $u_j\in\mathcal{L}_j$, where $\mathcal{L}_j=\mathcal{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument (see [27]) shows that there exist a sequence $\{x_j\}\subset\mathbb{R}^3$ and constants $R>0$, $\delta>0$ such that

$$\liminf_{j\to\infty}\int_{B(x_j,R)}|u_j|^2\mathrm{d}x\ge\delta.$$

Set

$$v_j = u_j(x+x_j)$$

and denoted by $\hat\lambda_j(x)=\lambda(\varepsilon_j(x+x_j))$, $\widehat{W}_j(x)=W(\varepsilon_j(x+x_j))$, one easily checks that v_j solves

$$H_\omega v_j - \hat\lambda_j(x)V_{\varepsilon_j,v_j}\beta v_j = \widehat{W}_j(x)\cdot|v_j|^{p-2}v_j,$$

with energy

$$\begin{aligned}S(v_j) &:= \frac12(\|v_j^+\|^2-\|v_j^-\|^2) - \Gamma_{\hat\lambda_j(x)}(v_j) - \frac1p\int_{\mathbb{R}^3}\widehat{W}_j(x)|v_j|^p\mathrm{d}x\\ &= \widetilde{\Phi}_j(v_j) = \Phi_j(v_j) = \Gamma_{\hat\lambda_j(x)}(v_j) + \left(\frac12-\frac1p\right)\int_{\mathbb{R}^3}\widehat{W}_j(x)|v_j|^p\mathrm{d}x = c_{\varepsilon_j}.\end{aligned}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j\to v$ in L^q_{loc} for $q\in[1,3)$. We now turn to prove that $\{\varepsilon_j x_j\}$ is bounded. Arguing indirectly we assume $\varepsilon_j|x_j|\to\infty$ and obtain a contradiction. Without loss of generality assume $\lambda(\varepsilon_j x_j)\to\lambda_\infty$, $W(\varepsilon_j x_j)\to W_\infty$. By the boundedness of $\nabla\lambda$ and ∇W, one sees that $\hat\lambda_j(x)\to\lambda_\infty$, $\widehat{W}_j(x)\to W_\infty$ uniformly on bounded sets of x. Since for any $\psi\in C_c^\infty$,

$$\begin{aligned}0 &= \lim_{j\to\infty}\int_{\mathbb{R}^3}(H_\omega v_j - \hat\lambda_j(x)V_{\varepsilon_j,v_j}\beta v_j - \widehat{W}_j\,|v_j|^{p-2}v_j)\overline{\psi}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(H_\omega v - \lambda_\infty V_v\beta v - W_\infty|v|^{p-2}v)\overline{\psi}\,\mathrm{d}x,\end{aligned}$$

hence v solves

$$i\alpha\cdot\nabla v - a\beta v + \omega v - \lambda_\infty V_v\beta v = W_\infty|v|^{p-2}v.$$

Therefore,

$$S_\infty(v) := \frac{1}{2}(\|v^+\|^2 - \|v^-\|^2) - \Gamma_{\lambda_\infty}(v) - W_\infty \int_{\mathbb{R}^3} |v|^p \mathrm{d}x \geq \tilde{\gamma}_\infty.$$

It follows from $\bar{\lambda} > \lambda_\infty$, $\kappa > W_\infty$, one has $\tilde{\gamma}_{\bar{\lambda},\kappa} < \tilde{\gamma}_\infty$. Moreover, by Fatou's lemma, we have

$$\tilde{\gamma}_{\bar{\lambda},\kappa} < \tilde{\gamma}_\infty \leqslant S_\infty(v) \leqslant \lim_{j\to\infty} c_{\varepsilon_j} = \tilde{\gamma}_{\bar{\lambda},\kappa},$$

a contradiction. Thus, $\{\varepsilon_j x_j\}$ is bounded. And hence, we can assume $y_j = \varepsilon_j x_j \to y_0$. Then repeating the proof of [13] gives the concentration and the exponential decay.

Finally, by Lemma 3.14 we see that the solutions are in $\bigcap_{q\geq 2} W^{1,q}$.

4.2 Proof of Corollary 1.1

By Theorem 1.1 and Remark 1.1, we obtain the ground solution of Dirac-Klein-Gordon systems which converges to a ground state solution of the autonomuous Dirac-Klein-Gordon systems equation (1.4) for every $c > 0$. Let $\{c_n\}$, $\{\omega_n\}$ be two real sequences such that

$$\begin{aligned} &0 < c_n, -\omega_n \to +\infty,\\ &0 < -\omega_n < mc_n^2,\\ &mc_n^2 + \omega_n \to \frac{\nu}{m}, \end{aligned}$$

as $n \to \infty$. Let $\{(\psi_n, \phi_n)\}$ (where $\psi_n := (u_n, v_n) \in \mathbb{C}^4$) denote a sequence of solutions for system equation (1.4) with frequency ω_n at speed of light c_n, then according to [20], there exists a mass m_0, λ_0, such that for $m \leq m_0$, $\lambda \leq \lambda_0$, up to a subsequence,

$$u_n \to 0, \quad v_n \to v, \quad \text{in} \quad H^1(\mathbb{R}^3, \mathbb{C}^2),$$
$$\phi_n \to 0, \quad \text{in } H^1(\mathbb{R}^3, \mathbb{R}),$$

as $n \to \infty$, where $v : \mathbb{R}^3 \to \mathbb{C}^2$ is a solution for the following coupled system of nonlinear Schrödinger-type equations

$$\begin{cases} -\Delta v_1 + 2\nu v_1 = 2m\kappa |v|^{p-2} v_1,\\ -\Delta v_2 + 2\nu v_2 = 2m\kappa |v|^{p-2} v_2. \end{cases}$$

4.3 Proof of main results: the critical case

Part 1. Multiplicity

Without loss of generality, we may assume that $0 \in \mathcal{W}_1 \cap \mathcal{W}_2 \cap \Gamma$, hence, $\bar{\lambda} = \lambda(0)$, $\kappa_1 = W_1(0)$, $\kappa_2 = W_2(0)$. Solutions of (2.7) are critical points of the functional $\Phi_\varepsilon(u) = \Phi_\varepsilon^{\bar{\lambda},\vec{\kappa}}(u)$. We will utilize Theorem 2.1. Obviously, Φ_ε is even, and in virtue of Remark 2.1 the conditions (Φ_1) and (Φ_2) are satisfied. It remains to verify (Φ_3).

Let u be a critical point $\Phi_0 := \phi_{\bar{\lambda},\vec{\kappa}}$, with $\Phi_0(u) = \gamma_{\bar{\lambda},\vec{\kappa}}$. Define u_r, u_{rj}, $\quad j = 0, \ldots, m-1$, and set E_m as before.

By virtue of Lemma 3.9, let $t_{\varepsilon rj} > 0$ be such that $t_{\varepsilon rj} u_{rj}^+ \in \mathcal{N}_\varepsilon$. Observe that (4.3), (4.4), and (4.5) keep true. One then checks easily the following

$$\begin{aligned}\max_{w\in E_m}\Phi_\varepsilon(w) &= \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)\right) = \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-\right) + o(1_r)\\ &= \Phi_\varepsilon\left(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}\right) + o(1_r) = \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}) + o(1_r)\\ &= \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-) + o(1_r) = \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)) + o(1_r)\\ &= \sum_{j=0}^{m-1}\Phi_0(t_{0j}u_{rj}^+ + h_0(t_{0j}u_{rj}^+)) + o(1_{r\varepsilon}) = \sum_{j=0}^{m-1}\Phi_0(u) + o(1_{r\varepsilon})\\ &= m\gamma_{\bar\lambda,\overrightarrow{\kappa}} + o(1_{r\varepsilon}),\end{aligned}$$

where $o(1_r)$ means arbitrary small as $r \to \infty$, and $o(1_{r\varepsilon})$ means arbitrary small as r is sufficiently large and ε is sufficiently small. Now, by assumptions and Lemma 3.1, for any $0 < \delta < \gamma_{\overrightarrow{\infty}} - m\gamma_{\bar\lambda,\overrightarrow{\kappa}}$, one may choose $r > 0$ large and then $\varepsilon_m > 0$ small such that, for all $\varepsilon \le \varepsilon_m$, $\max_{w\in E_m}\Phi_\varepsilon(w) \le \gamma_{\overrightarrow{\infty}} - \delta$. Now by Theorem 2.1, one obtains the multiplicity conclusion.

Part 2. Concentration

Let $\mathcal{L}_\varepsilon$ denote the set of all least energy solutions of $\tilde\Phi_\varepsilon$. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{L}_j$, where $\mathcal{L}_j = \mathcal{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument (see [27]) shows that there exist a sequence $\{x_j\} \subset \mathbb{R}^3$ and constant $R > 0$, $\delta > 0$ such that

$$\liminf_{j\to\infty}\int_{B(x_j,R)}|u_j|^2\mathrm{d}x \ge \delta.$$

Set

$$v_j = u_j(x + x_j)$$

and denoted by $\hat\lambda_j(x) = \lambda(\varepsilon_j(x + x_j))$, $\widehat{W}_j^1(x) = W_1(\varepsilon_j(x + x_j))$, $\widehat{W}_j^2(x) = W_2(\varepsilon_j(x + x_j))$ one easily checks that v_j solves

$$H_\omega v_j - \hat\lambda_j(x)V_{\varepsilon_j,v_j}\beta v_j = \widehat{W}_j^1(x)|v_j|^{p-2}v_j + \widehat{W}_j^2(x)|v_j|v_j, \tag{4.6}$$

with energy

$$\begin{aligned}S(v_j) &:= \frac{1}{2}(\|v_j^+\|^2 - \|v_j^-\|^2) - \Gamma_{\hat\lambda_j(x)}(v_j) - \int_{\mathbb{R}^3}\widehat{W}_j(x)|v_j|^p\mathrm{d}x + \int_{\mathbb{R}^3}\widehat{W}_j^2(x)|v_j|^3\mathrm{d}x\\ &= \widetilde{\Phi}_j(v_j) = \Phi_j(v_j)\\ &= c_{\varepsilon_j}.\end{aligned}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1,3)$. We now turn to prove that $\{\varepsilon_j x_j\}$ is bounded. Arguing indirectly we assume $\varepsilon_j|x_j| \to \infty$ and obtain a contradiction. Without loss of generality assume $\lambda(\varepsilon_j x_j) \to \lambda_\infty$, $W_1(\varepsilon_j x_j) \to W_{1\infty}$, $\widehat{W}_j^2(\varepsilon_j x_j) \to W_{2\infty}$. By the boundedness of $\nabla\lambda$ and ∇W, one sees that $\hat\lambda_j(x) \to \lambda_\infty$, $\widehat{W}_j^1(x) \to W_{1\infty}$, $\widehat{W}_j^2(x) \to W_{2\infty}$ uniformly on bounded sets of x. Since for any $\psi \in C_c^\infty$,

$$\begin{aligned}0 &= \lim_{j\to\infty}\int_{\mathbb{R}^3}(H_\omega v_j - \hat\lambda_j(x)V_{\varepsilon_j,v_j}\beta v_j - \widehat{W}_j^1(x)|v_j|^{p-2}v_j + \widehat{W}_j^2(x)|v_j|v_j)\overline{\psi}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(H_\omega v - \lambda_\infty V_v\beta v - W_{1\infty}|v|^{p-2}v - W_{2\infty}|v|v)\overline{\psi}\,\mathrm{d}x,\end{aligned}$$

hence v solves

$$i\alpha\cdot\nabla v - a\beta v + \omega v - \lambda_\infty V_v\beta v = W_{1\infty}|v|^{p-2}v + W_{2\infty}|v|v.$$

Therefore,

$$S_\infty(v) := \frac{1}{2}(\|v^+\|^2 - \|v^-\|^2) - \Gamma_{\lambda_\infty}(v) - W_{1\infty}\int_{\mathbb{R}^3}|v|^p \mathrm{d}x - W_{2\infty}\int_{\mathbb{R}^3}|v|^3 \mathrm{d}x \geq \tilde{\gamma}_\infty.$$

It follows from $\bar{\lambda} > \lambda_\infty$, $\kappa > W_\infty$, one has $\tilde{\gamma}_{\bar{\lambda},\kappa} < \tilde{\gamma}_\infty$. Moreover, by Fatou's lemma, we have

$$\tilde{\gamma}_{\bar{\lambda},\kappa} < \tilde{\gamma}_\infty \leq S_\infty(v) \leq \lim_{j\to\infty} c_{\varepsilon_j} = \tilde{\gamma}_{\bar{\lambda},\kappa},$$

a contradiction. Thus, $\{\varepsilon_j x_j\}$ is bounded. And hence, we can assume $y_j = \varepsilon_j x_j \to y_0$. Then v solves

$$i\alpha\cdot\nabla v - a\beta v + \omega v - \lambda_\infty V_v \beta v = W_1(y_0)|v|^{p-2}v + W_2(y_0)|v|v. \tag{4.7}$$

By Lemma 3.10, it is easy to check that $\lim_{\varepsilon\to 0}\mathrm{dist}(\varepsilon y_\varepsilon, \mathcal{W}_1 \cap \mathcal{W}_2 \cap \Gamma) = 0$.

In order to prove $v_j \to v$ in E, recall that as the argument shows

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\widehat{W}_j^2(x)|v_j|^3\mathrm{d}x = \int_{\mathbb{R}^3} W(y_0)|v|^3\mathrm{d}x.$$

By the decay of v, using the Brezis-Lieb lemma, one obtains $|v_j - v|_3 \to 0$. Hence, using the interpolation inequality and the boundedness of v_j in E yields $v_j \to v$ in $L^t(\mathbb{R}^3, \mathbb{C}^4)$ for $t \in (2, 3]$. Denote $z_j = v_j - v$. The scalar product of (4.6) with z_j^+ yields

$$(v_j^+, z_j^+) = o(1).$$

Similarly, using the decay of v together with the fact that $z_j^\pm \to 0$ in L^q_{loc} for $q \in [1, 3)$, it follows from (4.7) that

$$(v^+, z_j^+) = o(1).$$

Thus,

$$\|z_j^+\| = o(1),$$

and the same arguments show

$$\|z_j^-\| = o(1),$$

we then obtain $v_j \to v$ in E, and the arguments in [13] show that $v_j \to v$ in H^1.

Part 3. Exponential decay

For the later use denote $D = i\alpha\cdot\nabla$ and for $u \in \mathcal{L}_\varepsilon$ rewrite (2.7) as

$$Du = a\beta u - \omega u + \lambda_\varepsilon(x)V_{\varepsilon,u}\beta u + W_{1\varepsilon}(x)|u|^{p-2}u + W_{2\varepsilon}(x)|u|u.$$

Acting the operator D on the two sides and noting that $D^2 = -\Delta$, we obtain

$$\Delta u = (a + \lambda_\varepsilon(x)V_{\varepsilon,u})^2 u - (\omega - W_{1\varepsilon}|u|^{p-2} - W_{2\varepsilon}|u|)^2 u - D(\lambda_\varepsilon(x)V_{\varepsilon,u})\beta u - D(W_{1\varepsilon}|u|^{p-2} + W_{2\varepsilon}|u|)u. \tag{4.8}$$

Now define

$$\mathrm{sgn}\, u = \begin{cases} \dfrac{u}{|u|} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

By Kato's inequality [19] there holds

$$\Delta|u| \geqslant \Re[\Delta u(\mathrm{sgn}\, u)].$$

Note that

$$\Re[D(W_{1\varepsilon}|u|^{p-2} + W_{2\varepsilon}|u|)u(\mathrm{sgn}\, u)] = 0.$$

Then, we obtain

$$\Delta|u| \geqslant (a + \lambda_\varepsilon(x)V_{\varepsilon,u})^2|u| - (\omega - W_{1\varepsilon}|u|^{p-2} - W_{2\varepsilon}|u|)^2|u| - |D(\lambda_\varepsilon(x)V_{\varepsilon,u})|\cdot|u|. \tag{4.9}$$

It follows from Hölder inequality and $u \in L^\infty(\mathbb{R}^3, \mathbb{C}^4)$ that

$$|D(\lambda_\varepsilon(x)V_{\varepsilon,u})| \leq C.$$

So, due to (4.9), there exists a constant $M > 0$ such that

$$\Delta|u| \geq -M|u|.$$

It then follows from the sub-solution estimate [25,31] that

$$|u(x)| \leq C_0 \int_{B_1(x)} |u(y)|\mathrm{d}y,$$

where C_0 is independent of x and ε.

To obtain the uniformly decay estimate for the semi-classical states, we first need the following result:

Lemma 4.1. *Let v_ε and V_{v_ε} be given in the proof of Part 2. Then $|v_\varepsilon(x)|$ and $|V_{v_\varepsilon}(x)|$ vanish at infinity uniformly in $\varepsilon > 0$ small.*

Proof. Similar to [13, 16], we can easily prove this lemma. We omit it here. □

And at this point, applying the maximum principle (see [29]), we easily have

Lemma 4.2. *Let $v_\varepsilon \in E$ be given in the proof of Part 2, then v_ε exponentially decays at infinity uniformly in $\varepsilon > 0$ small. More specifically, there exist $C, c > 0$ independent of ε such that*

$$|v_\varepsilon(x)| \leqslant Ce^{-c|x|}.$$

Consequently, we infer that

$$|u_\varepsilon(x)| \leqslant Ce^{-c|x-x_\varepsilon|}.$$

With the above arguments, we can easily prove Theorem 1.2.

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References

[1] A. Ambrosetti, *On Schrödinger-Poisson systems*, Milan J. Math. **76** (2008), 257–274.
[2] A. Ambrosetti and D. Ruiz, *Multiple bound states for the Schrödinger-Poisson equation*, Commun. Contemp. Math. **10** (2008), 1–14.
[3] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J.Math. Anal. Appl. **345** (2008), 90–108.

[4] N. Ackermann, *A nonlinear superposition principle and multibump solutions of periodic Schrödinger equations*, J. Funct. Anal. **234** (2006), 423–443.
[5] J. Bjorken and S. Drell, *Relativistic Quantum Fields*, McGraw-Hill Book Co., New York, 1965.
[6] N. Bournaveas, *Low regularity solutions of the Dirac Klein-Gordon equations in two space dimensions*, Comm. Partial Differ. Equ. **26** (2001), no. 7–8, 1345–1366.
[7] T. Bartsch and Y. Ding, *Deformation theorems on non-metrizable vector spaces and applications to critical point theory*, Math. Nachr. **279** (2006), 1267–1288.
[8] T. Bartsch and Y. Ding, *Solutions of nonlinear Dirac equations*, J. Differ. Equ. **226** (2006), 210–249.
[9] J. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac system in one space dimension*, J. Funct. Anal. **13** (1973), 173–184.
[10] J. Chadam and R. Glassey, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions*, Arch. Ration. Mech. Anal. **54** (1974), 223–237.
[11] Y. Ding, *Semi-classical ground states concentrating on the nonlinear potential for a Dirac equation*, J. Differ. Equ. **249** (2010), 1015–1034.
[12] Y. Ding, *Variational methods for strongly indefinite problems*, Interdisciplinary Mathematical Sciences, vol. 7, World Scientific Publications, Singapore, 2007.
[13] Y. Ding and T. Xu, *On the concentration of semi-classical states for a nonlinear Dirac-Klein-Gordon system*, J. Differ. Equ. **256** (2014), 1264–1294.
[14] Y. Ding and B. Ruf, *On multiplicity of semi-classical solutions to a nonlinear Maxwell-Dirac system*, J. Differ. Equ. **260** (2016), 5565–5588.
[15] Y. Ding and J. Wei, *Stationary states of nonlinear Dirac equations with general potentials*, Rev. Math. Phys. **20** (2008), 1007–1032.
[16] Y. Ding, Q. Guo, and Y. Yu, *Semiclassical states for Dirac-Klein-Gordon system with critical growth*, J. Math. Anal. Appl. **488** (2020), 124092.
[17] Y. Ding, X. Dong, and Q. Guo, *On multiplicity of semi-classical solutions to nonlinear Dirac equations of space-dimension n*, Discrete Contin. Dyn. Syst. **41** (2021), no. 9, 4105–4123.
[18] Y. Ding, X. Dong, and Q. Guo, *Nonrelativistic limit and some properties of solutions for nonlinear Dirac equations*, Calc. Var. Partial Differ. Equ. **60** (2021), no. 4, 23.
[19] R. Dautray and J. Lions, *Mathematical analysis and numerical methods for science and technology*, vol. 3, Spectral Theory and Applications, Springer, Berlin, Heidelberg, 2000.
[20] X. Dong and Z. Tang, *Nonrelativistic limit of ground state solutions for nonlinear Dirac-Klein-Gordon systems*, Minimax Theory Appl. **7** (2022), no. 2, 253–276.
[21] M. Esteban, V. Georgiev and E. Séré, *Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 3, 265–281.
[22] M. Esteban and E. Séré, *Stationary states of the nonlinear Dirac equation: a variational approach*, Comm. Math. Phys. **171** (1995), 323–350.
[23] R. Finkelstein, R. LeLevier, and M. Ruderman, *Nonlinear spinor fields*, Phys. Rev. **83** (1951), no. 2, 326–332.
[24] R. Finkelstein, C. Fronsdal, and P. Kaus, *Nonlinear spinor field*, Phys. Rev. **103** (1956), no. 5, 1571–1579.
[25] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.
[26] D. Ivanenko, *Notes to the theory of interaction via particles*, Zh.É ksp. Teor. Fiz. **8** (1938), 260–266.
[27] P. Lions, *The concentration-compactness principle in the calculus of variations: the locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. NonLinéaire. **1** (1984), 223–283.
[28] S. Machihara and T. Omoso, *The explicit solutions to the nonlinear Dirac equation and Dirac-Klein-Gordon equation*, Ric. Mat. **56** (2007), no. 1, 19–30.
[29] P. Rabier and C. Stuart, *Exponential decay of the solutions of quasilinear second-order equations and Pohozaev identities*, J. Differ. Equ. **165** (2000), no. 1, 199–234.
[30] A. Szulkin and T. Weth, *Groundstate solutions for some indefinite variational problems*, J. Funct. Anal. **257** (2009), 3802–3822.
[31] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 447–526
[32] S. Selberg and A. Tesfahun, *Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension*, Commun. Contemp. Math. **10** (2008), no. 2, 181–194.
[33] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.