

Research Article

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Existence of normalized solutions for the coupled elliptic system with quadratic nonlinearity

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Abstract: In the present paper, we study the existence of the normalized solutions for the following coupled elliptic system with quadratic nonlinearity

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 |u|u + \beta uv & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2 v = \mu_2 |v|v + \frac{\beta}{2} u^2 & \text{in } \mathbb{R}^N, \end{cases}$$

where u, v satisfying the additional condition

$$\int_{\mathbb{R}^N} u^2 dx = a_1, \quad \int_{\mathbb{R}^N} v^2 dx = a_2.$$

On the one hand, we prove the existence of minimizer for the system with L^2 -subcritical growth ($N \leq 3$). On the other hand, we prove the existence results for different ranges of the coupling parameter $\beta > 0$ with L^2 -supercritical growth ($N = 5$). Our argument is based on the rearrangement techniques and the minimax construction.

Keywords: normalized solutions, quadratic nonlinearity, variational methods

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1 Introduction

In the present paper, we study the existence of solutions $(\lambda_1, \lambda_2, \tilde{u}, \tilde{v})$ to the coupled elliptic equations:

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 |u|u + \beta uv & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2 v = \mu_2 |v|v + \frac{\beta}{2} u^2 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

satisfying the constraint

$$\int_{\mathbb{R}^N} u^2 dx = a_1, \quad \int_{\mathbb{R}^N} v^2 dx = a_2. \quad (1.2)$$

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Here, $N \leq 3$ or $N = 5$, a_1, a_2 are fixed and $\beta > 0$. Prescribing the L^2 masses from the beginning, we call this type of solutions as normalized solutions. The frequencies λ_1, λ_2 appearing as Lagrange multipliers are included in the unknown. The mass $\|\cdot\|_{L^2}$ often has a clear physical meaning; on this account, normalized solutions are particularly interesting from a physical point of view.

The system (1.1) has important physic meaning. That is, we consider the following generalized three-coupled system:

$$\begin{cases} i\partial_t V_1 = -\Delta V_1 - \mu_1 |V_1|^{p-2} V_1 - \beta \overline{V_2} V_3, \\ i\partial_t V_2 = -\Delta V_2 - \mu_2 |V_2|^{p-2} V_2 - \beta \overline{V_1} V_3, \\ i\partial_t V_3 = -\Delta V_3 - \mu_3 |V_3|^{p-2} V_3 - \beta V_1 V_2, \end{cases} \quad (1.3)$$

where V_i ($i = 1, 2, 3$) are complex valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $p > 2$, $N \leq 3$, $\mu_i > 0$ ($i = 1, 2, 3$), and $\beta \in \mathbb{R}$. The system (1.3) is a reduced system studied in [12–14, 33] and related to the Raman amplification in a plasma. It is an instability phenomenon taking place when an incident laser field propagates into a plasma. This kind of model was first introduced by the previous paper [35] to describe the Raman scattering in a plasma. In [9], a modified model was derived to describe the nonlinear interaction between a laser beam and a plasma. From the physical point of view, when an incident laser field enters a plasma, it is back-scattered by a Raman type process. These two waves interact to create an electronic plasma wave. The three waves combine to create a variation of the density of the ions, which have itself an influence on the three proceedings waves. The system describing this phenomenon is composed of three Schrödinger equations coupled to a wave equation and reads in a suitable dimensionless form. For a complete description of this model as well as a precise description of the physical coefficients, we refer to [9–11, 14] and the references therein. Recently, the previous paper [13] studied the case $V_1 = V_2$. That is, the system

$$\begin{cases} i\partial_t V_1 = -\Delta V_1 - \mu_1 |V_1|^{p-2} V_1 - \beta \overline{V_2} V_1, \\ i\partial_t V_2 = -2\Delta V_2 - 2\mu_2 |V_2|^{p-2} V_2 - \beta V_1^2. \end{cases} \quad (1.4)$$

Correspondingly, the stationary system is given by

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{p-2} u_1 + \beta u_1 u_2, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{p-2} u_2 + \beta / 2 u_1^2. \end{cases} \quad (1.5)$$

For the case $p = 3$, $\lambda_1 = \lambda_2$, and $\mu_2 = 1$, [13] proved the existence of multiple synchronous solutions for (1.5). On the other hand, they prove the stability and instability results for the synchronous solution and $(0, e^{i\omega t} \psi_w)$, where ψ_w denotes the unique positive solution $-\Delta u + \omega u = |u|^{p-2} u$, $u \in H^1(\mathbb{R}^N)$. For more general results on this direction, one can refer to the recent papers [38, 40, 41].

In the present paper, we study the existence of $L^2(\mathbb{R}^N)$ -normalized solutions of (1.5) with $p = 3$. On the other hand, the problem under investigation can be seen from the following generalized nonlinear elliptic system:

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + f_1(u_1) + \partial_1 F_1(u_1, u_2), \\ -\Delta u_2 = \lambda_2 u_2 + f_2(u_2) + \partial_2 F_1(u_1, u_2), \\ u_1, u_2 \in H^1(\mathbb{R}^N), \quad N \geq 2, \end{cases} \quad (1.6)$$

under the condition (1.2). This problem is associated with models for binary mixtures of ultracold quantum gases. In the last decades, many authors investigated the case of homogeneous nonlinearities, i.e., $f_i(u_i) = \mu_i |u_i|^{p_i-2} u_i$, $F(u_1, u_2) = \beta |u_1|^{r_1} |u_2|^{r_2}$, with positive constants β, μ_i, p_i, r_i . That is, the existence of L^2 -normalized of the system:

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} |u_2|^{r_2} u_1, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_2|^{r_2-2} |u_1|^{r_1} u_2. \end{cases} \quad (1.7)$$

This kind of the system comes from mean field models for binary mixtures of Bose-Einstein condensates or for binary gases of fermion atoms in degenerate quantum states. For information about the physical background of the problem (1.7), we can refer to [1–3, 7, 8, 15, 17, 24, 26, 27, 29–32, 36, 37, 39, 42, 44, 46] and reference therein. Let $2^* = \frac{2N}{N-2}$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$. For fix $N \geq 2$, $p_1, p_2 \in (2, 2^*)$ and $\beta, \mu_1, \mu_2, r_1, r_2, a_1, a_2 > 0$ with $2 \leq r_1 + r_2 < 2^*$, [3] proved the existence of the normalized solution of (1.7). In [17], the authors proved the existence of multiple normalized solutions. Recently, many papers considered the existence of normalized solutions for the following coupled cubic elliptic system:

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a_1^2 \text{ and } \int_{\mathbb{R}^N} v^2 dx = a_2^2. \end{cases} \quad (1.8)$$

For example, see [4–6, 28] and the references therein. In 2016, Bartsch et al. [4] proved that there exist $\beta_1, \beta_2 > 0$ such that if $\beta \in (0, \beta_1) \cup (\beta_2, \infty)$, then the aforementioned systems have a solution $(\lambda_1, \lambda_2, u, v)$ in $N = 3$. Later, [5] proved the existence of the normalized solutions for the focusing-repulsive case $\mu_i > 0, \beta < 0$. Nearly, [6] proved the existence of normalized solutions using bifurcation methods.

Motivated by previous works [4, 5], in this paper, we consider the existence of normalized solutions for the system (1.1) with quadratic nonlinearity. On the one hand, we prove the existence of minimizer for the system with L^2 -subcritical growth ($N \leq 3$). On the other hand, we prove the existence results for different ranges of the coupling parameter $\beta > 0$ with L^2 -supercritical growth ($N = 5$). Compared to the previous cubic elliptic system, we have some difficulties in studying the existence of the normalized solution of (1.1). That is, since the coupled term $\int_{\mathbb{R}^N} u^2 v dx$ in the energy functional is not symmetric, we need make different estimates to dealing with it. Moreover, this term is not positive defined and leads us to add another constraint to get the positive normalized solution (see our forthcoming paper [43]).

We first consider the L^2 -subcritical case $N \leq 3$. Then, the following results hold.

Theorem 1.1. *For $N \leq 3$, assume that $\beta > 0$, $\mu_1, \mu_2, a_1, a_2 > 0$ are given. Then, there exists a solution $(\lambda_1, \lambda_2, \tilde{u}, \tilde{v})$ to equations (1.1) and (1.2) with $\lambda_1, \lambda_2 < 0$, and $\tilde{u}, \tilde{v} \in C^2(\mathbb{R}^N)$ are positive, radially symmetric, and decrease with $r = |x|$.*

Next we consider the L^2 -supercritical case $N = 5$. Obviously, solutions of (1.1) and (1.2) will be obtained as critical points of the energy functional $J : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ defined by

$$J(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u^2 v dx \quad (1.9)$$

constrained on the L^2 -spheres in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Set

$$S(a_1, a_2) := \left\{ (u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = a_1, \int_{\mathbb{R}^N} v^2 dx = a_2 \right\}. \quad (1.10)$$

To state our results, we shall introduce the minimization problem:

$$m(a_1, a_2) := \inf_{(u, v) \in S(a_1, a_2)} J(u, v). \quad (1.11)$$

It is natural that the minimizers of the problem (1.11) are critical points of $J|_{S(a, b)}$. Then, we have the following results.

Theorem 1.2. *Let $a_1, a_2, \mu_1, \mu_2 > 0$ be fixed, $N = 5$ and let $\beta_1 > 0$ be defined by*

$$\max\left\{\frac{1}{a_1\mu_1^4}, \frac{1}{a_2\mu_2^4}\right\} = \frac{1}{a_1(\mu_1 + \beta_1)^4} + \frac{1}{a_2(\mu_2 + \beta_1)^4}. \quad (1.12)$$

If $0 < \beta < \beta_1$, then (1.1)–(1.2) has a solution $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{u}, \tilde{v})$ such that $\tilde{\lambda}_1, \tilde{\lambda}_2 < 0$, and \tilde{u} and \tilde{v} are both positive and radial.

Finally, we consider the existence normalized solutions for large $\beta > 0$. To state the main results, we introduce a Pohozaev-type constraint as follows:

$$V := \{(u, v) \in H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5) : G(u, v) = 0\}, \quad (1.13)$$

where

$$G(u, v) := \int_{\mathbb{R}^5} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{5}{6} \int_{\mathbb{R}^5} (\mu_1 |u|^3 + \mu_2 |v|^3) dx - \frac{5}{4} \beta \int_{\mathbb{R}^5} u^2 v dx. \quad (1.14)$$

Then, we have the following results.

Theorem 1.3. For $N = 5$, let $a_1, a_2, \mu_1, \mu_2 > 0$ be fixed, and let $\beta_2 > 0$ be defined by

$$\frac{(a_1 + a_2)^5}{12 \left(\mu_1 a_1^{\frac{3}{2}} + \frac{3}{2} \beta_2 a_1 a_2^{\frac{1}{2}} + \mu_2 a_2^{\frac{3}{2}} \right)^4} = \min \left\{ \frac{1}{a_1 \mu_1^4}, \frac{1}{a_2 \mu_2^4} \right\}. \quad (1.15)$$

If $\beta > \beta_2$, then (1.1) and (1.2) has a solution $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{u}, \bar{v})$ such that $\bar{\lambda}_1, \bar{\lambda}_2 < 0$, and \bar{u} and \bar{v} are both positive and radial. Moreover, $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{u}, \bar{v})$ is a ground state solution in the sense that

$$\begin{aligned} J(\bar{u}, \bar{v}) &= \inf\{J(u, v) : (u, v) \in V\} \\ &= \inf\{J(u, v) : (u, v) \text{ is a solution of (1.1)–(1.2) for some } \lambda_1, \lambda_2\} \end{aligned} \quad (1.16)$$

holds.

Remark 1.4.

- (1) In Theorem 1.3, we cannot use the Rayleigh-type quotient to describe the ground state of (1.1) as in [4]. This is because we cannot find the unique maximum of the functional J if the term $\int_{\mathbb{R}^N} u^2 v dx$ is negative.
- (2) It is an interesting issue to prove the existence of normalized solutions of (1.1)–(1.2) when $\beta < 0$.
- (3) In Theorems 1.1–1.3, we can only study the existence of the normalized solutions of (1.1) and (1.2) if $N \leq 3$ or $N = 4$. For the L^2 -critical case $N = 4$, the additional assumptions and different arguments are needed. We give the result in the forthcoming paper [43].

This paper is organized as follows. In Section 2, we establish some basic results. The proof of Theorem 1.1 is given in Section 3. Sections 4 and 5 are devoted to giving the proof of Theorems 1.2 and 1.3.

2 Preliminaries

Throughout this paper, we shall use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx$.
- $H_{\text{rad}}^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ is a radial symmetry function}\}$.
- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm $\|u\|_p^p = \int_{\mathbb{R}^N} |u|^p dx$.

In this section, we recall some basic conclusions. First, we have the following Gagliardo-Nirenberg inequality (see [3]):

$$|u|_p \leq \left(\frac{p}{2|Q|_2^{p-2}} \right)^{\frac{1}{p}} |\nabla u|_2^{N(p-2)/(2p)} |u|_2^{1-N(p-2)/(2p)}, \quad (2.1)$$

where $p \in [2, 2^*)$, and the function Q is the unique ground state solution to

$$-\frac{N(p-2)}{4} \Delta Q + \left(1 + \frac{p-2}{4} (2-N) \right) Q = |Q|^{p-2} Q, \quad x \in \mathbb{R}^N.$$

Next we give some rearrangement inequalities results for system (1.1). For $N \leq 3$, let u be a measurable function on \mathbb{R}^N . Let $\text{mes}(A)$ be the N -dimensional Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}^N$. It is said to vanish at infinity if $\text{mes}(x \in \mathbb{R}^N : |u(x)| > t) < \infty$ for every $t > 0$. In view of two measurable functions u, v , which vanish at infinity with $t > 0$, we define $A^*(u, v, t) := \{x \in \mathbb{R}^N : |x| < r\}$, where $r > 0$ is chosen such that

$$\text{mes}(x \in \mathbb{R}^N : |x| < r) = \text{mes}(x \in \mathbb{R}^N : |u(x)| > t) + \text{mes}(x \in \mathbb{R}^N : |v(x)| > t)$$

and $\{u, v\}^*$ defined by

$$\{u, v\}^*(x) := \int_0^\infty \chi_{A^*(u, v, t)}(x) dt,$$

where χ_A is a characteristic function of the set $A \subset \mathbb{R}^N$. Then, we have the following properties for the aforementioned rearrangement. For the details of the proof, one can refer to [18, Lemma A.1].

Lemma 2.1.

- (i) The function $\{u, v\}^*$ is radially symmetric, nonincreasing, and lower semi-continuous. Moreover, for each $t > 0$, there holds $\{x \in \mathbb{R}^N : \{u, v\}^* > t\} = A^*(u, v, t)$.
- (ii) Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be nondecreasing, lower semi-continuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^* = \Phi(\{u, v\}^*)$.
- (iii) $\|\{u, v\}^*\|_p^p = \|u\|_p^p + \|v\|_p^p$ for $1 \leq p < \infty$.
- (iv) If $u, v \in H^1(\mathbb{R}^N)$, then $\{u, v\}^* \in H^1(\mathbb{R}^N)$ and $|\nabla \{u, v\}^*|_2^2 \leq |\nabla u|_2^2 + |\nabla v|_2^2$. In addition, if $u, v \in (H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)) \setminus \{0\}$ are radially symmetric, positive, and nonincreasing, then

$$\int_{\mathbb{R}^N} |\nabla \{u, v\}^*|^2 dx < \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx.$$

- (v) Let $u_1, u_2, v_1, v_2 \geq 0$ be Borel measurable functions, which vanish at infinity, then

$$\int_{\mathbb{R}^N} (u_1 u_2 + v_1 v_2) dx \leq \int_{\mathbb{R}^N} \{u_1, v_1\}^* \{u_2, v_2\}^* dx.$$

To this end, we introduce some facts concerning the following scalar equation:

$$\begin{cases} -\Delta w + w = |w|w, & w > 0, & \text{in } \mathbb{R}^5, \\ w(0) = \max w & \text{and } w \in H^1(\mathbb{R}^5). \end{cases} \quad (2.2)$$

It is well known that (2.2) has a unique positive solution w_0 . Moreover, w_0 is radial symmetric and decays at infinity. Set

$$C_0 := \int_{\mathbb{R}^5} w_0^2 dx \quad \text{and} \quad C_1 := \int_{\mathbb{R}^5} |w_0|^3 dx. \quad (2.3)$$

For $a, \mu \in \mathbb{R}$ fixed, let us search for $(\lambda, w) \in \mathbb{R} \times H^1(\mathbb{R}^5)$, with $\lambda < 0$ in \mathbb{R} , solving

$$\begin{cases} -\Delta w - \lambda w = \mu |w|w, & \text{in } \mathbb{R}^5, \\ w(0) = \max w \quad \text{and} \quad \int_{\mathbb{R}^5} w^2 = a. \end{cases} \quad (2.4)$$

Solution w of (2.4) can be found as critical points of $I_\mu : H^1(\mathbb{R}^5) \mapsto \mathbb{R}$, defined by

$$I_\mu(w) := \frac{1}{2} \int_{\mathbb{R}^5} |\nabla w|^2 dx - \frac{\mu}{3} \int_{\mathbb{R}^5} |w|^3 dx, \quad (2.5)$$

constrained on the L^2 -sphere $T_a = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = a\}$, and λ appears as Lagrange multipliers. It is easy to see that this solution can be obtained by the scaling of solutions of (2.2). Let

$$\mathcal{P}(a, \mu) := \left\{ w \in T_a : \int_{\mathbb{R}^5} |\nabla w|^2 = \frac{5\mu}{6} \int_{\mathbb{R}^5} w^3 \right\}. \quad (2.6)$$

Then, we have the following results for $\mathcal{P}(a, \mu)$.

Lemma 2.2. *If w is a solution of (2.4), then $w \in \mathcal{P}(a, \mu)$. Furthermore, the positive solution w of (2.4) minimizes I_μ on $\mathcal{P}(a, \mu)$.*

Proof. The proof is similar to [19, Lemmas 2.7, 2.10] (also see [17]). We omit the details here. \square

Proposition 2.3. *Problem (2.4) has a unique positive solution $(\lambda_{a,\mu}, w_{a,\mu})$ defined by*

$$\lambda_{a,\mu}(x) := -\frac{C_0^2}{\mu^4 a^2} \quad \text{and} \quad w_{a,\mu}(x) := \frac{C_0^2}{\mu^5 a^2} w_0\left(\frac{C_0}{\mu^2 a} x\right). \quad (2.7)$$

The function $w_{a,\mu}$ satisfies

$$\int_{\mathbb{R}^5} |\nabla w_{a,\mu}|^2 = \frac{5C_0 C_1}{6\mu^4 a} \quad (2.8)$$

and

$$\int_{\mathbb{R}^5} |w_{a,\mu}|^3 = \frac{C_0 C_1}{\mu^5 a}. \quad (2.9)$$

$$l(a, \mu) := \inf_{u \in \mathcal{P}(a, \mu)} I_\mu(u) = I_\mu(w_{a,\mu}) = \frac{C_0 C_1}{12\mu^4 a}. \quad (2.10)$$

The value $l(a, \mu)$ is called least energy level of problem (2.4).

Proof. Since w_0 is the solution of (2.2), it is straightforward to check that $w_{a,\mu}$ is a solution of (2.4) for $\lambda = \lambda_{a,\mu} < 0$ (here $w_{a,\mu}$ is given in [4] in the same way). By [21], it is the only positive solution. To obtain (2.8) and (2.9), we can use the explicit expression of $w_{a,\mu}$. With a change of variables,

$$\int_{\mathbb{R}^5} |\nabla w_{a,\mu}|^2 dx = \frac{C_0^4}{\mu^{10} a^4} \cdot \frac{C_0^2}{\mu^4 a^2} \int_{\mathbb{R}^5} \left| \nabla w_0\left(\frac{C_0}{\mu^2 a} x\right) \right|^2 dx = \frac{C_0^6}{\mu^{14} a^6} \cdot \left(\frac{C_0}{\mu^2 a}\right)^{-5} \int_{\mathbb{R}^5} |w_0(x)|^3 dx = \frac{5C_0 C_1}{6\mu^4 a}, \quad (2.11)$$

$$\int_{\mathbb{R}^5} |w_{a,\mu}|^3 dx = \frac{C_0^6}{\mu^{15} a^6} \cdot \left(\frac{C_0}{\mu^2 a}\right)^{-5} \int_{\mathbb{R}^5} |w_0(x)|^3 dx = \frac{C_0 C_1}{\mu^5 a}, \quad (2.12)$$

where the last equality follows by Lemma 2.2 with $a = C_0$ and $\mu = 1$. It is not difficult to check (2.9) and the least energy level of I_μ on $\mathcal{P}(a, \mu)$. \square

3 Proof of Theorem 1.1

In this section, we are devoted to giving the proof of Theorem 1.1. To accomplish this, we need the following main results for the least energy level of (1.11).

Lemma 3.1. Assume $N \leq 3$ and $\alpha > 0$. Then, we have the following results.

- (i) For $a, b \geq 0$, one has that $m(a, b)$ is well defined, where $m(a, b)$ is given in (1.11).
- (ii) If $ab > 0$, then $m(a, b) < 0$.
- (iii) If $ab = 0$, then $m(a, b) = 0$.
- (iv) If $(a_n, b_n) \rightarrow (a, b)$ as $n \rightarrow \infty$ with $a, b > 0$, then $m(a_n, b_n) \rightarrow m(a, b)$ as $n \rightarrow \infty$.
- (v) If $0 \leq c \leq a, 0 \leq d \leq b$, then $m(a, b) \leq m(c, d) + m(a - c, b - d)$.

Proof. (i) For any $a, b \geq 0$ and $(u, v) \in S(a, b)$, we infer from Hölder's inequality, Young's inequality, and inequality (2.1) that

$$\begin{aligned}
 J(u, v) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u^2 v dx \\
 &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v|^3 dx - \frac{\beta}{2} \left(\int_{\mathbb{R}^N} |u|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |v|^3 dx \right)^{\frac{1}{3}} \\
 &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v|^3 dx - \frac{\beta}{3} \int_{\mathbb{R}^N} |u|^3 dx - \frac{\beta}{6} \int_{\mathbb{R}^N} |v|^3 dx \quad (3.1) \\
 &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{1}{3} (\mu_1 + \beta) \int_{\mathbb{R}^N} |u|^3 dx - \frac{1}{3} \left(\mu_2 + \frac{\beta}{2} \right) \int_{\mathbb{R}^N} |v|^3 dx \\
 &\geq \frac{1}{2} (|\nabla u|_2^2 + |\nabla v|_2^2) - \frac{(\mu_1 + \beta) a^{\frac{6-N}{4}}}{2|Q|_2} |\nabla u|_2^{\frac{N}{2}} - \frac{(\mu_2 + \frac{\beta}{2}) b^{\frac{6-N}{4}}}{2|Q|_2} |\nabla v|_2^{\frac{N}{2}}.
 \end{aligned}$$

We infer from $N \leq 3$ that J is bounded from below on $S(a, b)$. That is, $m(a, b)$ is well defined.

(ii) Given $u, v \in S(a, b)$ with $u > 0, v > 0$, set $u_t(x) = tu\left(t^{\frac{2}{N}}x\right)$, $v_t(x) = tv\left(t^{\frac{2}{N}}x\right)$ for $t > 0$. Then, we have $|u_t|_2^2 = |u|_2^2 = a$ and $|v_t|_2^2 = |v|_2^2 = b$. Moreover, a direct computation shows that

$$\begin{aligned}
 J(u_t, v_t) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_t|^2 + |\nabla v_t|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u_t|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v_t|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u_t^2 v_t dx \\
 &= \frac{t^{\frac{4}{N}}}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{3} t \int_{\mathbb{R}^N} |u|^3 dx - \frac{\mu_2}{3} t \int_{\mathbb{R}^N} |v|^3 dx - \frac{\beta}{2} t \int_{\mathbb{R}^N} u^2 v dx. \quad (3.2)
 \end{aligned}$$

Then, one sees $J(u_t, v_t) < 0$ for $t > 0$ small enough. By the definition of $m(a, b)$, one gets that $m(a, b) < 0$.

(iii) If $a = b = 0$, then the fact $m(a, b) = 0$ is trivial. In the following, we only prove the case $a > 0$ and $b = 0$. (The proof of the case $a = 0, b > 0$ can be obtained in a similar way.) Given $(u, 0) \in S(a, 0)$, set $u_t(x) = tu\left(t^{\frac{2}{N}}x\right)$ for $t > 0$, then equality (3.2) implies that

$$0 \leq m(a, 0) = \frac{t^{\frac{4}{N}}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu_1}{3} t \int_{\mathbb{R}^N} |u|^3 dx \leq 0, \quad (3.3)$$

since t can be arbitrarily small. One has $m(a, 0) = 0$.

(iv) Without the loss of generality, assume that $a_n, b_n > 0$. By definition of $m(a_n, b_n)$, there exists for any $0 < \varepsilon < 1$, $(u_n, v_n) \in S(a_n, b_n)$ such that

$$J(u_n, v_n) \leq m(a_n, b_n) + \varepsilon. \quad (3.4)$$

Noting that $0 < \varepsilon < 1$ and $m(a_n, b_n) < 0$. We infer from (i), the inequalities (3.1) and (3.4) that $J(u_n, v_n)$ is bounded. Then, there exists $C > 0$ such that

$$|\nabla u_n|_2 \leq C, \quad |\nabla v_n|_2 \leq C, \quad |u_n|_3 \leq C, \quad |v_n|_3 \leq C, \quad \left| \int_{\mathbb{R}^N} u_n^2 v_n dx \right| \leq C. \quad (3.5)$$

Set

$$w_n := \frac{u_n}{|u_n|_2} a_n^{\frac{1}{2}} \quad \text{and} \quad z_n := \frac{v_n}{|v_n|_2} b_n^{\frac{1}{2}}.$$

Then, $(w_n, z_n) \in S(a, b)$. We infer from (3.5) that

$$\begin{aligned} m(a, b) &\leq J(w_n, z_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |w_n|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |z_n|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} w_n^2 z_n dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{a}{a_n} |\nabla u_n|^2 + \frac{b}{b_n} |\nabla v_n|^2 \right) dx - \frac{\mu_1}{3} \left(\frac{a}{a_n} \right)^{\frac{3}{2}} \int_{\mathbb{R}^N} |u_n|^3 dx - \frac{\mu_2}{3} \left(\frac{b}{b_n} \right)^{\frac{3}{2}} \int_{\mathbb{R}^N} |v_n|^3 dx \\ &\quad - \frac{\beta}{2} \left(\frac{a\sqrt{bb_n}}{a_n b_n} \right) \int_{\mathbb{R}^N} u_n^2 v_n dx \\ &= J(u_n, v_n) + \frac{1}{2} \left(\frac{a}{a_n} - 1 \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2} \left(\frac{b}{b_n} - 1 \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{\mu_1}{3} \left(\left(\frac{a}{a_n} \right)^{\frac{3}{2}} - 1 \right) \int_{\mathbb{R}^N} |u_n|^3 dx \\ &\quad - \frac{\mu_2}{3} \left(\left(\frac{b}{b_n} \right)^{\frac{3}{2}} - 1 \right) \int_{\mathbb{R}^N} |v_n|^3 dx - \frac{\beta}{2} \left(\frac{a\sqrt{bb_n}}{a_n b_n} - 1 \right) \int_{\mathbb{R}^N} u_n^2 v_n dx \\ &= J(u_n, v_n) + o(1), \end{aligned} \quad (3.6)$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Hence, we deduce from (3.4) and (3.6) that

$$m(a, b) \leq m(a_n, b_n) + \varepsilon + o(1).$$

By using the similar arguments, one obtains

$$m(a_n, b_n) \leq m(a, b) + \varepsilon + o(1).$$

Hence, the desired results follow from the fact that $\varepsilon > 0$ is arbitrary small.

(v) For any $\varepsilon > 0$, there exist $(u_1, v_1) \in S(c, d)$ and $(u_2, v_2) \in S(a - c, b - d)$ such that

$$J(u_1, v_1) \leq m(c, d) + \frac{\varepsilon}{2}, \quad J(u_2, v_2) \leq m(a - c, b - d) + \frac{\varepsilon}{2}. \quad (3.7)$$

Letting $w_1 := \{u_1, u_2\}^*$ and $w_2 := \{v_1, v_2\}^*$. According to Lemma 2.1(iii)–(iv), one obtains

$$|\{u_1, u_2\}^*|_2^2 = |u_1|_2^2 + |u_2|_2^2 = c + (a - c) = a \quad \text{and} \quad |\{v_1, v_2\}^*|_2^2 = |v_1|_2^2 + |v_2|_2^2 = d + (b - d) = b.$$

That is, $(w_1, w_2) \in S(a, b)$. We infer from the definition of $m(a, b)$ and Lemma 2.1(ii), (iv), and (v) that

$$\begin{aligned}
m(a, b) &\leq J(w_1, w_2) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_1|^2 + |\nabla w_2|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |w_1|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |w_2|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} w_1^2 w_2 dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \{u_1, u_2\}^*|^2 + |\nabla \{v_1, v_2\}^*|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |\{u_1, u_2\}^*|^3 dx \\
&\quad - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |\{v_1, v_2\}^*|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} \{u_1^2, u_2^2\}^* \{v_1, v_2\}^* dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u_1|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v_1|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u_1^2 v_1 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_2|^2 + |\nabla v_2|^2) dx - \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u_2|^3 dx - \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v_2|^3 dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u_2^2 v_2 dx \\
&= J(u_1, v_1) + J(u_2, v_2) \\
&\leq m(c, d) + m(a - c, b - d) + \varepsilon.
\end{aligned}$$

In light of the arbitrariness of ε , we deduce that

$$m(a, b) \leq m(c, d) + m(a - c, b - d).$$

This finishes the proof. \square

The next lemma states the Brézis-Lieb type results. For the proof, one can refer to [22, Lemma 3.2].

Lemma 3.2. Assume $(u_n, v_n) \rightharpoonup (u, v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for $N \leq 3$. Then

$$\int_{\mathbb{R}^N} (u_n^2 v_n - (u_n - u)^2 (v_n - v)) dx = \int_{\mathbb{R}^N} u^2 v dx + o(1),$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$.

In the next lemma, we study the compactness of the minimizing sequences of (1.11).

Lemma 3.3. Every minimizing sequence for problem (1.11) is, up to translation, strongly convergent in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ with $N \leq 3$ and $p \in (2, 2^*)$.

Proof. Let $\{(u_n, v_n)\} \in S(a, b)$ be a minimizing sequence for $m(a, b)$. We infer from inequality (3.1) and $m(a, b) < 0$ that $\{(u_n, v_n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Assume that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} (|u_n|^2 + |v_n|^2) dx \rightarrow 0, \quad n \rightarrow \infty.$$

Then, we have $u_n \rightarrow 0$ and $v_n \rightarrow 0$ in $L^3(\mathbb{R}^N)$ (see [25, Lemma I.1]). From Hölder's inequality, one sees that

$$\begin{aligned}
J(u_n, v_n) &+ \frac{\mu_1}{3} \int_{\mathbb{R}^N} |u_n|^3 dx + \frac{\mu_2}{3} \int_{\mathbb{R}^N} |v_n|^3 dx + \frac{\beta}{2} \left(\int_{\mathbb{R}^N} |u_n|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |v_n|^3 dx \right)^{\frac{1}{3}} \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) dx - \frac{\beta}{2} \int_{\mathbb{R}^3} u_n^2 v_n dx + \frac{\beta}{2} \left(\int_{\mathbb{R}^N} |u_n|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |v_n|^3 dx \right)^{\frac{1}{3}} \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \geq 0.
\end{aligned}$$

One has $m(a, b) \geq 0$ as $n \rightarrow \infty$. This contradicts the fact that $m(a, b) < 0$ due to Lemma 3.1(ii). Consequently, for $\alpha > 0$, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n, 1)} (|u_n|^2 + |v_n|^2) dx \geq \alpha > 0.$$

Hence, we have $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $(u, v) \neq (0, 0)$ such that $(u_n(\cdot - y_n), v_n(\cdot - y_n)) \rightharpoonup (u, v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We define

$$\varphi_n(\cdot) := u_n(\cdot) - u(\cdot + y_n) \quad \text{and} \quad \psi_n(\cdot) := v_n(\cdot) - v(\cdot + y_n).$$

Thus, it suffices to show that $\varphi_n(\cdot) \rightarrow 0, \psi_n(\cdot) \rightarrow 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*)$. Suppose by contradiction that there exists $p \in (2, 2^*)$ such that $(\varphi_n, \psi_n) \not\rightarrow (0, 0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. By using [25, Lemma I.1] again, we know that there exists a sequence $\{z_n\} \in \mathbb{R}^N$ such that

$$(\varphi_n(\cdot - z_n), \psi_n(\cdot - z_n)) \rightharpoonup (\varphi, \psi) \neq (0, 0) \quad \text{in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).$$

Now, we infer from the translations invariance, the Brézis-Lieb Lemma (see [45, Lemma 1.32]) and Lemma 3.2 that

$$\begin{aligned} J(u_n, v_n) &= J(u_n(\cdot - y_n), v_n(\cdot - y_n)) \\ &= J(u_n(\cdot - y_n) - u, v_n(\cdot - y_n) - v) + J(u, v) + o(1) \\ &= J(\varphi_n(\cdot - y_n), \psi_n(\cdot - y_n)) + J(u, v) + o(1) \\ &= J(\varphi_n(\cdot - z_n), \psi_n(\cdot - z_n)) + J(u, v) + o(1) \\ &= J(\varphi_n(\cdot - z_n) - \varphi, \psi_n(\cdot - z_n) - \psi) + J(\varphi, \psi) + J(u, v) + o(1) \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} |u_n(\cdot - y_n)|^2 &= |\varphi_n(\cdot - y_n)|^2 + |u|^2 + o(1) \\ &= |\varphi_n(\cdot - z_n)|^2 + |u|^2 + o(1) \\ &= |\varphi_n(\cdot - y_n) - \varphi|^2 + |\varphi|^2 + |u|^2 + o(1) \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} |v_n(\cdot - y_n)|^2 &= |v_n(\cdot - y_n)|^2 + |v|^2 + o(1) \\ &= |\psi_n(\cdot - z_n)|^2 + |v|^2 + o(1) \\ &= |\psi_n(\cdot - y_n) - \psi|^2 + |\psi|^2 + |v|^2 + o(1), \end{aligned} \quad (3.10)$$

where and subsequently $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$|\varphi_n(\cdot - z_n) - \varphi|^2 = |u_n(\cdot - y_n)|^2 - |\varphi|^2 - |u|^2 + o(1) = a - |\varphi|^2 - |u|^2 + o(1) = c + o(1) \quad (3.11)$$

and

$$|\psi_n(\cdot - z_n) - \psi|^2 = |v_n(\cdot - y_n)|^2 - |\psi|^2 - |v|^2 + o(1) = b - |\psi|^2 - |v|^2 + o(1) = d + o(1), \quad (3.12)$$

where $c := a - |\varphi|^2 - |u|^2 \geq 0$, $d := b - |\psi|^2 - |v|^2 \geq 0$. Hence, $(\varphi_n(\cdot - z_n) - \varphi, \psi_n(\cdot - z_n) - \psi) \in S(c, d)$. One infers from (3.9)–(3.12) and Lemma 3.1(iii) that

$$m(c, d) \leq J(\varphi_n(\cdot - z_n) - \varphi, \psi_n(\cdot - z_n) - \psi). \quad (3.13)$$

Recalling that $J(u_n, v_n) \rightarrow m(a, b)$, one obtains

$$m(a, b) \geq m(c, d) + J(\varphi, \psi) + J(u, v). \quad (3.14)$$

If $J(\varphi, \psi) > m(|\varphi|_2^2, |\psi|_2^2)$ or $J(u, v) > m(|u|_2^2, |v|_2^2)$, then we infer from the estimate (3.14) and Lemma 3.1(v) that

$$\begin{aligned}
m(a, b) &\geq m(c, d) + J(\varphi, \psi) + J(u, v) \\
&> m(c, d) + m(|\varphi|_2^2, |\psi|_2^2) + m(|u|_2^2, |v|_2^2) \\
&\geq m(a, b),
\end{aligned} \tag{3.15}$$

which is incompatible. Hence, we have $J(\varphi, \psi) = m(|\varphi|_2^2, |\psi|_2^2)$ and $J(u, v) = m(|u|_2^2, |v|_2^2)$. Since $J(|u|, |v|) \leq J(u, v)$, without the loss of generality, we may assume that $u, v, \varphi, \psi \geq 0$. Since $(u, v) \neq (0, 0)$ is a solution to system (1.1) with some (λ_1, λ_2) , by the elliptic regularity theory and the maximum principle, we infer that $u, v \in C^2(\mathbb{R}^N)$ and $v > 0$. Similarly, we have $\varphi, \psi \in C^2(\mathbb{R}^N)$ and $\psi > 0$. We denote by $\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\psi}$ the classical Schwarz symmetric-decreasing rearrangement of u, v, φ, ψ . By the properties of the Schwarz rearrangement, one has

$$\int_{\mathbb{R}^N} \tilde{u}^2 dx = |u|_2^2, \quad \int_{\mathbb{R}^N} \tilde{v}^2 dx = |v|_2^2, \quad \int_{\mathbb{R}^N} \tilde{\varphi}^2 dx = |\varphi|_2^2, \quad \int_{\mathbb{R}^N} \tilde{\psi}^2 dx = |\psi|_2^2.$$

Thus, we know that

$$m(|u|_2^2, |v|_2^2) \leq J(\tilde{u}, \tilde{v}) \leq J(u, v), \quad m(|\varphi|_2^2, |\psi|_2^2) \leq J(\tilde{\varphi}, \tilde{\psi}) \leq J(\varphi, \psi).$$

This implies that

$$J(\tilde{u}, \tilde{v}) = m(|u|_2^2, |v|_2^2), \quad J(\tilde{\varphi}, \tilde{\psi}) = m(|\varphi|_2^2, |\psi|_2^2).$$

In view of Lemma 2.1(ii), (iv), (v) and [23, Lemma 3.4] (also see [20]), we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla \tilde{u}, \tilde{\varphi}^*|^2) dx &< \int_{\mathbb{R}^N} (|\nabla \tilde{u}|^2 + |\nabla \tilde{\varphi}|^2) dx \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla \varphi|^2) dx, \\
\int_{\mathbb{R}^N} (|\nabla \tilde{v}, \tilde{\psi}^*|^2) dx &< \int_{\mathbb{R}^N} (|\nabla \tilde{v}|^2 + |\nabla \tilde{\psi}|^2) dx \leq \int_{\mathbb{R}^N} (|\nabla v|^2 + |\nabla \psi|^2) dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} (\{\tilde{u}, \tilde{\varphi}^*\}^2 \{\tilde{v}, \tilde{\psi}^*\}^2) dx &= \int_{\mathbb{R}^N} \{\tilde{u}^2, \tilde{\varphi}^2\}^* \{\tilde{v}, \tilde{\psi}^*\}^2 dx \\
&\geq \int_{\mathbb{R}^N} (\tilde{u}^2 \tilde{v} + \tilde{\varphi}^2 \tilde{\psi}) dx \\
&= \int_{\mathbb{R}^N} (\tilde{u}^2 \tilde{v} + \tilde{\varphi}^2 \tilde{\psi}) dx \\
&\geq \int_{\mathbb{R}^N} (u^2 v + \varphi^2 \psi) dx.
\end{aligned}$$

Thus,

$$J(u, v) + J(\varphi, \psi) > J(\{\tilde{u}, \tilde{\varphi}^*\}^2, \{\tilde{v}, \tilde{\psi}^*\}^2). \tag{3.16}$$

Moreover, we infer from Lemma 2.1(iii) that

$$\begin{aligned}
\int_{\mathbb{R}^N} (\{\tilde{u}, \tilde{\varphi}^*\}^2) dx &= \int_{\mathbb{R}^N} (\tilde{u}^2, \tilde{\varphi}^2) dx = \int_{\mathbb{R}^N} (u^2 + \varphi^2) dx, \\
\int_{\mathbb{R}^N} (\{\tilde{v}, \tilde{\psi}^*\}^2) dx &= \int_{\mathbb{R}^N} (\tilde{v}^2, \tilde{\psi}^2) dx = \int_{\mathbb{R}^N} (v^2 + \psi^2) dx.
\end{aligned} \tag{3.17}$$

From (3.15)–(3.17) and Lemma 3.1(v), one deduces that

$$\begin{aligned}
m(a, b) &\geq m(c, d) + J(\varphi, \psi) + J(u, v) \\
&> m(c, d) + J(\{\tilde{u}, \tilde{\varphi}\}^*, \{\tilde{v}, \tilde{\psi}\}^*) \\
&\geq m(c, d) + m(a - c, b - d) \geq m(a, b).
\end{aligned} \tag{3.18}$$

This is a contradiction. \square

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\{(u_n, v_n)\}$ be a minimizing sequence for the functional J on $S(a_1, a_2)$. In consideration of inequality (3.1) and Lemma 3.3, we may assume that there exist $\{y_n\} \in \mathbb{R}^N$ and $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}
(u_n(\cdot - y_n), v_n(\cdot - y_n)) &\rightharpoonup (u, v) \quad \text{in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \\
(u_n(\cdot - y_n), v_n(\cdot - y_n)) &\rightarrow (u, v) \quad \text{in } L^3(\mathbb{R}^N) \times L^3(\mathbb{R}^N)
\end{aligned} \tag{3.19}$$

for $n \rightarrow \infty$. For simplify the notation, we denote $\alpha_n = u_n(\cdot - y_n)$ and $\beta_n = v_n(\cdot - y_n)$. Then,

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} (\alpha_n^2 \beta_n - u^2 v) dx \right| &= \left| \int_{\mathbb{R}^N} (\alpha_n^2 \beta_n - \alpha_n^2 v + \alpha_n^2 v - u^2 v) dx \right| \\
&\leq \int_{\mathbb{R}^N} (|\alpha_n^2 \beta_n - \alpha_n^2 v| + |\alpha_n^2 v - u^2 v|) dx \\
&= \int_{\mathbb{R}^N} \alpha_n^2 |\beta_n - v| dx + \int_{\mathbb{R}^N} |(\alpha_n + u)(\alpha_n - u)v| dx \\
&\leq \left(\int_{\mathbb{R}^N} |\alpha_n|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |\beta_n - v|^3 dx \right)^{\frac{1}{3}} + \left(\int_{\mathbb{R}^N} |(\alpha_n + u)(\alpha_n - u)|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |v|^3 dx \right)^{\frac{1}{3}} \\
&\leq |\alpha_n|_3^2 |\beta_n - v|_3 + |\alpha_n + u|_3 |\alpha_n - u|_3 |v|_3 \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned} \tag{3.20}$$

Due to the weak lower semi-continuity of the norm, (3.20) induces that

$$J(u, v) \leq \liminf_{n \rightarrow \infty} J(u_n, v_n) = m(a_1, a_2) < 0. \tag{3.21}$$

Thus, recalling (3.1), we conclude that $u \neq 0$, $v \neq 0$. Furthermore,

$$\begin{aligned}
|u|_2^2 &\leq \liminf_{n \rightarrow \infty} |u_n|_2^2 = \lim_{n \rightarrow \infty} |u_n|_2^2 = a_1, \\
|v|_2^2 &\leq \liminf_{n \rightarrow \infty} |v_n|_2^2 = \lim_{n \rightarrow \infty} |v_n|_2^2 = a_2.
\end{aligned}$$

Now we focus on showing $|u|_2^2 = a_1$ and $|v|_2^2 = a_2$. Assume by contradiction that $|u|_2^2 := c < a_1$ or $|v|_2^2 := d < a_2$. By the definition of $m(c, d)$ and inequality (3.21), it follows $m(c, d) \leq J(u, v) \leq m(a_1, a_2)$. From Lemma 3.1(ii) and (v), we deduce

$$J(u, v) \leq m(c, d) \leq m(a_1, a_2) + m(c - a_1, d - a_2) < m(a_1, a_2) \leq J(u, v).$$

This is contradiction. This leads to the fact that $(u, v) \in S(a_1, a_2)$.

Let \tilde{u} and \tilde{v} denote the Schwarz spherical rearrangement of $|u|$ and $|v|$. Note that $\tilde{u}, \tilde{v} \in H^1(\mathbb{R}^N)$, $|\tilde{u}|_2^2 = a_1$, $|\tilde{v}|_2^2 = a_2$, and $J(\tilde{u}, \tilde{v}) \leq J(u, v)$. Thus, (\tilde{u}, \tilde{v}) is a weak solution of equations (1.1) and (1.2), where the parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ are determined by the Lagrange's multiplier rule. As a consequence of the elliptic regularity theory and the maximum principle in [22], we know that $\tilde{u}, \tilde{v} \in C^2(\mathbb{R}^N)$ and $\tilde{u}, \tilde{v} > 0$. According to Lemma 2.3 in [22], $\lambda_1 < 0$, $\lambda_2 < 0$. The proof is completed. \square

4 Proof of Theorem 1.2

In this section, we follow the idea of [4] to give the proof of Theorem 1.2. To avoid compactness issue, we work in the standard radial space $H_{\text{rad}}^1(\mathbb{R}^5) \times H_{\text{rad}}^1(\mathbb{R}^5)$. That is, we look for the critical points of J constrained on $H_{\text{rad}}^1(\mathbb{R}^5) \times H_{\text{rad}}^1(\mathbb{R}^5)$. For any $a > 0$, the set S_a is defined by

$$S_a := \left\{ w \in H_{\text{rad}}^1(\mathbb{R}^5) : \int_{\mathbb{R}^5} w^2 dx = a \right\}. \quad (4.1)$$

For $s \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^5)$, we define the dilation

$$(s \star w)(x) := e^{\frac{5s}{2}} w(e^s x). \quad (4.2)$$

It is easy to check that $s \star w \in S_a$ for every $s \in \mathbb{R}$ and $w \in S_a$. Moreover, we also have the following conclusions for the functional I_μ (I_μ denotes the functional for the scalar equation (2.5)).

Lemma 4.1. *For any $w \in H^1(\mathbb{R}^5)$ and for every $\mu > 0$, there holds,*

$$\begin{aligned} I_\mu(s \star w) &= \frac{e^{2s}}{2} \int_{\mathbb{R}^5} |\nabla w|^2 dx - \frac{e^{\frac{5s}{2}}}{3} \mu \int_{\mathbb{R}^5} |w|^3 dx, \\ \frac{\partial}{\partial s} I_\mu(s \star w) &= e^{2s} \int_{\mathbb{R}^5} |\nabla w|^2 dx - \frac{5e^{\frac{5s}{2}}}{6} \mu \int_{\mathbb{R}^5} |w|^3 dx. \end{aligned}$$

In particular, if $w = w_{a,\mu}$, then

$$\frac{\partial}{\partial s} I_\mu(s \star w_{a,\mu}) \text{ is } \begin{cases} > 0 & \text{if } s < 0, \\ = 0 & \text{if } s = 0, \\ < 0 & \text{if } s > 0. \end{cases}$$

Since the proof of Lemma 4.1 is similar to [4, Lemma 3.1], we omit the details here. For $a_1, a_2, \mu_1, \mu_2 > 0$, let $\beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2) > 0$ be defined by (1.12). We have the following conclusions for the functionals J and I .

Lemma 4.2. *For $0 < \beta < \beta_1$, there holds*

$$\inf\{J(u, v) : (u, v) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)\} > \max\{l(a_1, \mu_1), l(a_2, \mu_2)\},$$

where $l(a_i, \mu_i)$ is defined by (2.10).

Proof. From the definition of I_μ in (2.6) and the virtue of Young's inequality, we infer that for $(u, v) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$,

$$\begin{aligned} J(u, v) &= I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{2} \int_{\mathbb{R}^5} u^2 v dx \\ &\geq I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{3} \int_{\mathbb{R}^5} |u|^3 dx - \frac{\beta}{6} \int_{\mathbb{R}^5} |v|^3 dx \\ &\geq I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{3} \int_{\mathbb{R}^5} |u|^3 dx - \frac{\beta}{3} \int_{\mathbb{R}^5} |v|^3 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^5} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1 + \beta}{3} \int_{\mathbb{R}^5} |u|^3 dx - \frac{\mu_2 + \beta}{3} \int_{\mathbb{R}^5} |v|^3 dx \\ &= I_{\mu_1 + \beta}(u) + I_{\mu_2 + \beta}(v) \\ &\geq \inf_{u \in \mathcal{P}(a_1, \mu_1 + \beta)} I_{\mu_1 + \beta}(u) + \inf_{v \in \mathcal{P}(a_2, \mu_2 + \beta)} I_{\mu_2 + \beta}(v) \\ &= l(a_1, \mu_1 + \beta) + l(a_2, \mu_2 + \beta). \end{aligned}$$

The conclusion holds if and only if

$$\max\{l(a_1, \mu_1), l(a_2, \mu_2)\} < l(a_1, \mu_1 + \beta) + l(a_2, \mu_2 + \beta).$$

By the definition of $l(a_1, \mu_1)$, $l(a_2, \mu_2)$, $l(a_1, \mu_1 + \beta)$, $l(a_2, \mu_2 + \beta)$ in Proposition 2.3, one obtains

$$\frac{C_0 C_1}{12a_1(\mu_1 + \beta)^4} + \frac{C_0 C_1}{12a_2(\mu_2 + \beta)^4} > \max\left\{\frac{C_0 C_1}{12a_1\mu_1^4}, \frac{C_0 C_1}{12a_2\mu_2^4}\right\}. \quad (4.3)$$

This ends the proof. \square

Now we fix $0 < \beta < \beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2)$ and choose $\varepsilon > 0$ such that

$$\inf\{J(u, v) : (u, v) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)\} > \max\{l(a_1, \mu_1), l(a_2, \mu_2)\} + \varepsilon. \quad (4.4)$$

We define the following

$$w_1 := w_{a_1, \mu_1 + \beta} \quad \text{and} \quad w_2 := w_{a_2, \mu_2 + \beta}, \quad (4.5)$$

and for $i = 1, 2$,

$$\varphi_i(s) := I_{\mu_i}(s \star w_i) \quad \text{and} \quad \psi_i(s) := \frac{\partial}{\partial s} I_{\mu_i + \beta}(s \star w_i). \quad (4.6)$$

The next lemma is a major building block for the proof of the linking structure of J .

Lemma 4.3. *For $i = 1, 2$, there exists $\rho_i < 0$ and $R_i > 0$, depending on ε and on β , such that*

- (i) $0 < \varphi_i(\rho_i) < \varepsilon$ and $\varphi_i(R_i) \leq 0$;
- (ii) $\psi_i(s) > 0$ for any $s < 0$ and $\psi_i(s) < 0$ for every $s > 0$. In particular, $\psi_i(\rho_i) > 0$ and $\psi_i(R_i) < 0$.

Proof. By using the change of variables as in Lemma 4.1, we deduce that for $w_{a, \mu} \in \mathcal{P}(a, \mu)$

$$\varphi_i = I_{\mu_i}(s \star w_{a_i, \mu_i + \beta}) = \frac{e^{2s}}{2} \int_{\mathbb{R}^5} |\nabla w_{a_i, \mu_i + \beta}|^2 dx - \frac{e^{\frac{5}{2}s}}{3} \mu_i \int_{\mathbb{R}^5} |w_{a_i, \mu_i + \beta}|^3 dx.$$

We deduce that $\varphi_i(s) \rightarrow 0^+$ as $s \rightarrow -\infty$, and $\varphi_i(s) \rightarrow -\infty$ as $s \rightarrow +\infty$. Thus, there exist ρ_i and R_i satisfying (i). Conclusion (ii) follows directly from Lemma 4.1. \square

Set $Q := [\rho_1, R_1] \times [\rho_2, R_2]$. We define

$$\gamma_0(t_1, t_2) := (t_1 \star w_1, t_2 \star w_2) \in S_{a_1} \times S_{a_2}, \quad \forall (t_1, t_2) \in \bar{Q}$$

and

$$\Gamma := \{\gamma \in C(\bar{Q}, S_{a_1} \times S_{a_2}) : \gamma = \gamma_0 \quad \text{on} \quad \partial Q\}.$$

Then, the minimax structure of the functional J is based on (4.4) and the following two lemmas.

Lemma 4.4. *There holds*

$$\sup_{\partial Q} J(\gamma_0) \leq \max\{l(a_1, \mu_1), l(a_2, \mu_2)\} + \varepsilon.$$

Proof. We infer from $u, v \geq 0$ and $\beta > 0$ that

$$J(u, v) = I_{\mu_1}(u) + I_{\mu_2}(v) - \frac{\beta}{2} \int_{\mathbb{R}^5} u^2 v dx \leq I_{\mu_1}(u) + I_{\mu_2}(v)$$

for every $(u, v) \in S_{a_1} \times S_{a_2}$. This and Lemma 4.3 imply that

$$\begin{aligned}
J(t_1 * w_1, \rho_2 * w_2) &\leq I_{\mu_1}(t_1 * w_1) + I_{\mu_2}(\rho_2 * w_2) \\
&= I_{\mu_1}(t_1 * w_1) + \varphi_2(\rho_2) \\
&\leq I_{\mu_1}(t_1 * w_1) + \varepsilon \\
&\leq \sup_{s \in \mathbb{R}} I_{\mu_1}(s * w_1) + \varepsilon \\
&= \sup_{s \in \mathbb{R}} I_{\mu_1}(s * w_{a_1 \mu_1}) + \varepsilon.
\end{aligned}$$

To get the estimates of the last term, we infer from Proposition 2.3 that

$$w_{a_i, \mu_i} = \bar{s}_i * w_i \quad \text{for } e^{\frac{s_i}{2}} := \frac{6 \int_{\mathbb{R}^5} |\nabla w_i|^2 dx}{5 \int_{\mathbb{R}^5} \mu_i w_i^3 dx} = \frac{\mu_i + \beta}{\mu_i}.$$

By the definition of $s * w$, it is easy to check that $s_1 * (s_2 * w) = (s_1 + s_2) * w$ for every $s_1, s_2 \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^5)$. Thus, we have

$$\sup_{s \in \mathbb{R}} I_{\mu_1}(s * w_1) = \sup_{s \in \mathbb{R}} I_{\mu_1}(s * w_{a_1, \mu_1}).$$

By using Lemma 4.1, the supremum on the right hand side is achieved for $s = 0$, and hence,

$$J(t_1 * w_1, \rho_2 * w_2) \leq l(a_1, \mu_1) + \varepsilon \quad \forall t_1 \in [\rho_1, R_1]. \quad (4.7)$$

Similarly, we have

$$J(\rho_1 * w_1, t_2 * w_2) \leq l(a_2, \mu_2) + \varepsilon \quad \forall t_2 \in [\rho_2, R_2]. \quad (4.8)$$

The value of $J(\gamma_0)$ on the remaining sides of ∂Q is smaller. Indeed, by Lemma 4.3 and similar calculations to the derivation mentioned earlier, one can obtain

$$J(t_1 * w_1, R_2 * w_2) \leq I_{\mu_1}(t_1 * w_1) + I_{\mu_2}(R_2 * w_2) \leq \sup_{s \in \mathbb{R}} I_{\mu_1}(s * w_1) = l(a_1, \mu_1) \quad (4.9)$$

for every $t_1 \in [\rho_1, R_1]$. Notice that, in a similar way,

$$J(R_1 * w_1, t_2 * w_2) \leq l(a_2, \mu_2), \quad \forall t_2 \in [\rho_2, R_2]. \quad (4.10)$$

This finishes the proof. \square

In the following, we show that the class Γ links with $\mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$.

Lemma 4.5. *For every $\gamma \in \Gamma$, there exists $(t_{1,\gamma}, t_{2,\gamma}) \in Q$ such that $\gamma(t_{1,\gamma}, t_{2,\gamma}) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$.*

Proof. For $\gamma \in \Gamma$, we use the notation $\gamma(t_1, t_2) = (\gamma_1(t_1, t_2), \gamma_2(t_1, t_2)) \in S_{a_1} \times S_{a_2}$. Let us consider the map $F_\gamma : Q \rightarrow \mathbb{R}^2$ defined by

$$F_\gamma(t_1, t_2) := \left(\frac{\partial}{\partial s} I_{\mu_1 + \beta}(s * \gamma_1(t_1, t_2)) \Big|_{s=0}, \frac{\partial}{\partial s} I_{\mu_2 + \beta}(s * \gamma_2(t_1, t_2)) \Big|_{s=0} \right),$$

where

$$\begin{aligned}
\frac{\partial}{\partial s} I_{\mu_i + \beta}(s * \gamma_i(t_1, t_2)) \Big|_{s=0} &= \frac{\partial}{\partial s} \left(\frac{e^{2s}}{2} \int_{\mathbb{R}^5} |\nabla \gamma_i(t_1, t_2)|^2 dx - \frac{e^{\frac{5}{2}s}}{3} (\mu_i + \beta) \int_{\mathbb{R}^5} \gamma_i^3(t_1, t_2) dx \right) \Big|_{s=0} \\
&= \int_{\mathbb{R}^5} |\nabla \gamma_i(t_1, t_2)|^2 dx - \frac{5}{6} (\mu_i + \beta) \int_{\mathbb{R}^5} \gamma_i^3(t_1, t_2) dx.
\end{aligned}$$

We deduce that

$$F_\gamma(t_1, t_2) = (0, 0) \quad \text{if and only if } \gamma(t_1, t_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta).$$

Now we focus on the solution of $F_\gamma(t_1, t_2) = (0, 0)$ in Q for every $\gamma \in \Gamma$. The results can be obtained by the standard degree theory (see [4]). We shall show that the oriented path $F_\gamma(\partial^+ Q)$ has winding number equal to 1 with respect to the origin of \mathbb{R}^2 . In this way, we observe that $F_\gamma(\partial^+ Q) = F_{\gamma_0}(\partial^+ Q)$ depends only on the choice of γ_0 and not on γ . Recalling the definition of ψ_i given in (4.6), it is natural to show that

$$F_{\gamma_0}(t_1, t_2) = \left(e^{2t_1} \left(\int_{\mathbb{R}^5} |\nabla w_1|^2 dx - \frac{5e^{\frac{5}{2}t_1}}{6} (\mu_1 + \beta) \int_{\mathbb{R}^5} |w_1|^3 dx \right), \right. \\ \left. e^{2t_2} \left(\int_{\mathbb{R}^5} |\nabla w_2|^2 dx - \frac{5e^{\frac{5}{2}t_2}}{6} (\mu_2 + \beta) \int_{\mathbb{R}^5} |w_2|^3 dx \right) \right) = (\psi_1(t_1), \psi_2(t_2)).$$

Hence, Lemma 4.3(ii) completely describes the restriction of F_{γ_0} on ∂Q . Denote $\iota(\rho, P)$ as the winding number of the curve ρ with respect to the point P . Particularly, by virtue of the topological degree,

$$\deg(F_{\gamma_0}, Q, (0, 0)) = \iota(F_{\gamma_0}(\partial^+ Q), (0, 0)) = 1.$$

Thus, there exists $\gamma(t_1, \gamma, t_2, \gamma) \in P$ such that $F_\gamma(t_1, \gamma, t_2, \gamma) = (0, 0)$. Thus, the desired result is obtained. \square

From Lemmas 4.4 and 4.5, we can apply the minimax principle [16, Theorem 3.2] to J on Γ . Then, we obtain the Palais-Smale sequence for the constrained functional J on $S_{a_1} \times S_{a_2}$. However, we do not know the boundedness of a Palais-Smale sequence. To accomplish this, we borrow the idea of [19].

Lemma 4.6. *There exists a Palais-Smale sequence (u_n, v_n) for J on $S_{a_1} \times S_{a_2}$ at the level*

$$c := \inf_{\gamma \in \Gamma(t_1, t_2) \in Q} \max J(\gamma(t_1, t_2)) > \max\{l(a_1, \mu_1), l(a_2, \mu_2)\} \quad (4.11)$$

satisfying the additional condition

$$\int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx - \frac{5}{6} \int_{\mathbb{R}^5} \left(\mu_1 |u_n|^3 + \mu_2 |v_n|^3 + \frac{3}{2} \beta u_n^2 v_n \right) dx = o(1), \quad (4.12)$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $u_n^-, v_n^- \rightarrow 0$ a.e. in \mathbb{R}^5 as $n \rightarrow \infty$.

Proof. We first define the augmented functional $\tilde{J} : \mathbb{R} \times S_{a_1} \times S_{a_2} \rightarrow \mathbb{R}$ by

$$\tilde{J}(s, u, v) := J(s * u, s * v).$$

Set

$$\tilde{\gamma}_0(t_1, t_2) := (0, \gamma_0(t_1, t_2)) = (0, t_1 * w_1, t_2 * w_2)$$

and

$$\tilde{\Gamma} := \{\tilde{\gamma} \in C(Q, \mathbb{R} \times S_{a_1} \times S_{a_2}) : \tilde{\gamma} = \tilde{\gamma}_0 \text{ on } \partial Q\}$$

by using the minimax principle in [16, Theorem 3.2] to the functional \tilde{J} with the minimax class $\tilde{\Gamma}$, we find a Palais-Smale sequence for \tilde{J} at level

$$\tilde{c} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}(t_1, t_2) \in Q} \tilde{J}(\tilde{\gamma}(t_1, t_2)).$$

Since $\tilde{J}(\tilde{\gamma}_0) = J(\gamma_0)$ on ∂Q , it follows from Lemmas 4.4 and 4.5 that assumptions of the minimax principle will be satisfied if we prove that $\tilde{c} = c$. Next, we use the notation

$$\tilde{\gamma}(t_1, t_2) = (s(t_1, t_2), \gamma_1(t_1, t_2), \gamma_2(t_1, t_2))$$

for any $\tilde{\gamma} \in \tilde{\Gamma}$ and $(t_1, t_2) \in Q$. It follows that

$$\tilde{J}(\tilde{y}(t_1, t_2)) = J(s(t_1, t_2) * \gamma_1(t_1, t_2), s(t_1, t_2) * \gamma_2(t_1, t_2))$$

and $(s(\cdot) * \gamma_1(\cdot), s(\cdot) * \gamma_2(\cdot)) \in \Gamma$. Thus, $\tilde{c} = c$, and the minimax principle is applicable. It is not difficult to directly check that $\tilde{J}(s, u, v) = \tilde{J}(s, |u|, |v|)$ in view of the definition of \tilde{J} . Moreover,

$$\tilde{J}(\tilde{y}(t_1, t_2)) = J(s(t_1, t_2) * \gamma_1(t_1, t_2), s(t_1, t_2) * \gamma_2(t_1, t_2)) = \tilde{J}(0, s(t_1, t_2) * \gamma_1(t_1, t_2), s(t_1, t_2) * \gamma_2(t_1, t_2)).$$

In this way, using the notation of Theorem 3.2 in [16], we can choose the minimizing sequence $\gamma_n = (s_n, \gamma_{1,n}, \gamma_{2,n})$ for \tilde{c} satisfying the additional conditions $s_n \equiv 0$, $\gamma_{1,n}(t_1, t_2) \geq 0$ a.e. in \mathbb{R}^5 for every $(t_1, t_2) \in Q$, $\gamma_{2,n}(t_1, t_2) \geq 0$ a.e. in \mathbb{R}^5 for every $(t_1, t_2) \in Q$. Thus, we follow [16, Theorem 3.2] to deduce that there exists a Palais-Smale sequence $(\tilde{s}_n, \tilde{u}_n, \tilde{v}_n)$ for \tilde{J} on $\mathbb{R} \times S_{a_1} \times S_{a_2}$ at level c , and such that

$$\lim_{n \rightarrow \infty} |\tilde{s}_n| + \text{dist}_{H^1}((\tilde{u}_n, \tilde{v}_n), \tilde{\gamma}_n(Q)) = 0. \quad (4.13)$$

To obtain a Palais-Smale sequence for J at level c satisfying (4.12), we follow closely the approach of [19, Lemma 2.4] with minor changes. Finally, we infer from (4.13) that $u_n^-, v_n^- \rightarrow 0$ a.e. in \mathbb{R}^5 as $n \rightarrow \infty$. Moreover, the lower estimate for c follows from Lemma 4.4. \square

To complete the proof of Theorem 1.2, we want to show that (u_n, v_n) is strongly convergent in $H^1(\mathbb{R}^5, \mathbb{R}^2)$ to a limit (u, v) . This can be accomplished by proving the following:

$$dJ|_{S_{a_1} \times S_{a_2}}(u_n, v_n) \rightarrow 0 \quad \text{and} \quad (u_n, v_n) \in S_{a_1} \times S_{a_2}$$

holds for all n . Now we focus on the following statement to achieve desired results.

Lemma 4.7. *The sequence $\{(u_n, v_n)\}$ is bounded in $H^1(\mathbb{R}^5, \mathbb{R}^2)$. Furthermore, there exists $\tilde{C} > 0$ such that*

$$\int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \geq \tilde{C} \quad \text{for all } n.$$

Proof. We deduce from (4.12) that

$$J(u_n, v_n) = \frac{1}{10} \left(\int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \right) + o(1),$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the desired result is a consequence of $J(u_n, v_n) \rightarrow c > 0$. \square

By the previous results, we know that there exists a subsequence such that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in $H^1(\mathbb{R}^5)$, strongly in $L^3(\mathbb{R}^5)$, and a.e. in \mathbb{R}^5 . Moreover, we have $\tilde{u}, \tilde{v} \geq 0$ in \mathbb{R}^5 . Since the compact embedding $H^1(\mathbb{R}^5) \hookrightarrow L^p(\mathbb{R}^5)$ ($\forall p \in (2, 10/3)$), we cannot conclude that $(\tilde{u}, \tilde{v}) \in S_{a_1} \times S_{a_2}$ directly. Observe that for every $(\varphi, \psi) \in H^1(\mathbb{R}^5, \mathbb{R}^2)$, $dJ|_{S_{a_1} \times S_{a_2}}(u_n, v_n) \rightarrow 0$ implies that there exists two sequences of real numbers $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ such that

$$\begin{aligned} & \int_{\mathbb{R}^5} \left(\nabla u_n \cdot \nabla \varphi + \nabla v_n \cdot \nabla \psi - \mu_1 u_n^2 \varphi - \mu_2 v_n^2 \psi - \beta u_n \left(\frac{u_n}{2} \psi + v_n \varphi \right) \right) dx - \int_{\mathbb{R}^5} (\lambda_{1,n} u_n \varphi + \lambda_{2,n} v_n \psi) dx \\ & = o(1) \|(\varphi, \psi)\|_{H^1}, \end{aligned} \quad (4.14)$$

with $o(1) \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 4.8. *Both $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded sequences, and at least one of them is converging, up to a subsequence, to a strictly negative value.*

Proof. The value of the $\{\lambda_{i,n}\}$ can be found using $(u_n, 0)$ and $(0, v_n)$ as test functions in (4.14):

$$\begin{aligned}\lambda_{1,n}a_1^2 &= \int_{\mathbb{R}^5} (|\nabla u_n|^2 - \mu_1|u_n|^3 - \beta u_n^2 v_n) dx - o(1), \\ \lambda_{2,n}a_2^2 &= \int_{\mathbb{R}^5} (|\nabla v_n|^2 - \mu_2|v_n|^3 - \frac{\beta}{2}u_n^2 v_n) dx - o(1),\end{aligned}$$

with $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Then, in view of the boundedness of $\{(u_n, v_n)\}$ in H^1 , the boundedness of $\{\lambda_{i,n}\}$ is obtained. Furthermore, equality (4.12) and Lemma 4.7 imply that

$$\begin{aligned}\lambda_{1,n}a_1^2 + \lambda_{2,n}a_2^2 &= \int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2 - \mu_1|u_n|^3 - \mu_2|v_n|^3 - \frac{3}{2}\beta u_n^2 v_n) dx - o(1) \\ &= -\frac{1}{5} \int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx + o(1) \leq -\frac{\tilde{C}}{10}\end{aligned}$$

for n sufficiently large. It follows that at least one sequence of $\{\lambda_{i,n}\}$ is negative and bounded away from 0. \square

Next we assume that $\lambda_{1,n} \rightarrow \lambda_1 \in \mathbb{R}$ and $\lambda_{2,n} \rightarrow \lambda_2 \in \mathbb{R}$. The sign of the limit values plays a crucial role in our argument, as the next statement will illustrate.

Lemma 4.9. *If $\lambda_1 < 0$ (resp. $\lambda_2 < 0$), then $u_n \rightarrow \tilde{u}$ (resp. $v_n \rightarrow \tilde{v}$) strongly in $H^1(\mathbb{R}^5)$.*

Proof. Let us suppose that $\lambda_1 < 0$. By weak convergence in $H^1(\mathbb{R}^5)$, strong convergence in $L^3(\mathbb{R}^5)$, and using (4.14), we have that

$$\begin{aligned}o(1) &= (dJ(u_n, v_n) - dJ(\tilde{u}, \tilde{v}))[(u_n - \tilde{u}, 0)] - \lambda_1 \int_{\mathbb{R}^5} (u_n - \tilde{u})^2 dx \\ &= dJ(u_n, v_n)[(u_n - \tilde{u}, 0)] - dJ(\tilde{u}, \tilde{v})[(u_n - \tilde{u}, 0)] - \lambda_1 \int_{\mathbb{R}^5} (u_n - \tilde{u})^2 dx \\ &= \int_{\mathbb{R}^5} \nabla u_n \cdot \nabla (u_n - \tilde{u}) dx - \int_{\mathbb{R}^5} \nabla \tilde{u} \cdot \nabla (u_n - \tilde{u}) dx \\ &\quad - \int_{\mathbb{R}^5} (|u_n|^3 - |\tilde{u}|^3)(u_n - \tilde{u}) dx - \int_{\mathbb{R}^5} (\beta u_n v_n (u_n - \tilde{u}) - \lambda_1 (u_n - \tilde{u})^2) dx \\ &= \int_{\mathbb{R}^5} (|\nabla (u_n - \tilde{u})|^2 - \lambda_1 (u_n - \tilde{u})^2) dx + o(1),\end{aligned}$$

with $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Since $\lambda_1 < 0$, it follows that this is equivalent to the strong convergence in H^1 . One can use the similar arguments to prove the case $\lambda_2 < 0$. \square

Remark 4.10. Note that Lemmas 4.7–4.9 do not depend on the value of β . Hence, we can use them in the proof of Theorem 1.3.

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{(u_n, v_n)\} \in S_{a_1} \times S_{a_2}$ be given by Lemma 4.7. By using (4.14), the weak convergence $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$, and the convergence of $\lambda_{1,n} \rightarrow \lambda_1 \in \mathbb{R}$ and $\lambda_{2,n} \rightarrow \lambda_2 \in \mathbb{R}$, we deduce that (\tilde{u}, \tilde{v}) is a solution of (1.1). It remains to prove that $(\tilde{u}, \tilde{v}) \in S_{a_1} \times S_{a_2}$. Without the loss of generality, we may suppose that $\lambda_1 < 0$ by Lemma 4.8. Moreover, Lemmas 4.6 and 4.9 imply that $u_n \rightarrow \tilde{u}$ strongly in $H^1(\mathbb{R}^5)$ and $\tilde{u} \in S_{a_1}$. If $\lambda_2 < 0$, we can deduce in the same way that $v_n \rightarrow \tilde{v}$ strongly in $H^1(\mathbb{R}^5)$ and $\tilde{v} \in S_{a_2}$. Then, the proof is completed. We use the contradiction arguments. Suppose that $\lambda_2 \geq 0$ and $v_n \rightharpoonup \tilde{v}$ strongly in $H^1(\mathbb{R}^5)$.

Note that any weak solution of (1.1) is smooth by regularity. Moreover, we know that $\tilde{v} > 0$ by using maximum principle. Hence, we deduce that

$$-\Delta \tilde{v} = \lambda_2 \tilde{v} + \mu_2 \tilde{v}^2 + \frac{\beta}{2} \tilde{u}^2 \geq c \tilde{v}^{\frac{3}{2}} \quad \text{in } \mathbb{R}^5, \quad (4.15)$$

where $c > 0$ is a constant. Then, it follows from [34, Theorem 8.4] that $\tilde{v} \equiv 0$. Particularly, this implies that \tilde{u} solves

$$\begin{cases} -\Delta \tilde{u} - \lambda_1 \tilde{u} = \mu_1 |\tilde{u}| \tilde{u}, & \text{in } \mathbb{R}^5, \\ \tilde{u} > 0, & \text{in } \mathbb{R}^5, \int_{\mathbb{R}^5} \tilde{u}^2 = a_1. \end{cases}$$

So that $\tilde{u} \in \mathcal{P}(a_1, \mu_1)$ and $I_{\mu_1}(\tilde{u}) = l(a_1, \mu_1)$. On the other hand, we infer from (4.12) that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} J(u_n, v_n) = \lim_{n \rightarrow \infty} \frac{1}{12} \int_{\mathbb{R}^5} \left(\mu_1 |u_n|^3 + \frac{3}{2} \beta u_n^2 v_n + \mu_2 |v_n|^3 \right) dx \\ &= \frac{\mu_1}{12} \int_{\mathbb{R}^5} |\tilde{u}|^3 dx = I_{\mu_1}(\tilde{u}) = l(a_1, \mu_1). \end{aligned} \quad (4.16)$$

This contradicts Lemma 4.6. □

5 Proof of Theorem 1.3

In this section, we shall follow the idea of [4] to give the proof of Theorem 1.3. To accomplish this, we first show the existence of a positive solution (\bar{u}, \bar{v}) . Then, we characterize it as a ground state in the second one, in the sense that

$$\begin{aligned} J(\bar{u}, \bar{v}) &= \inf \{ J(u, v) : (u, v) \in V \} \\ &= \inf \{ J(u, v) : (u, v) \text{ is a solution of (1.1)–(1.2) for some } \lambda_1, \lambda_2 \}. \end{aligned}$$

To give the proof of Theorem 1.3, we shall use a mountain pass lemma arguments, which has been developed in [4]. For $(u, v) \in S_{a_1} \times S_{a_2}$, we define the functional

$$J(s \star (u, v)) = \frac{e^{2s}}{2} \int_{\mathbb{R}^5} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{e^{\frac{5}{2}s}}{3} \int_{\mathbb{R}^5} \left(\mu_1 |u|^3 + \frac{3}{2} \beta u^2 v + \mu_2 |v|^3 \right) dx,$$

where $s \star u$ is defined in (4.2). We use the notation $s \star (u, v) = (s \star u, s \star v)$ for simplicity. Then, it is easy to check the following conclusions hold.

Lemma 5.1.

$$\lim_{s \rightarrow -\infty} \int_{\mathbb{R}^5} |\nabla(s \star u)|^2 + |\nabla(s \star v)|^2 = 0, \quad \lim_{s \rightarrow +\infty} \int_{\mathbb{R}^5} |\nabla(s \star u)|^2 + |\nabla(s \star v)|^2 = +\infty$$

and

$$\lim_{s \rightarrow -\infty} J(s \star (u, v)) = 0^+, \quad \lim_{s \rightarrow +\infty} J(s \star (u, v)) = -\infty.$$

The next lemma states a mountain pass structure of the problem.

Lemma 5.2. *There exists $K > 0$ sufficiently small such that for the sets*

$$A := \left\{ (u, v) \in S_{a_1} \times S_{a_2} : \int_{\mathbb{R}^5} |\nabla u|^2 + |\nabla v|^2 \leq K \right\}$$

and

$$B := \left\{ (u, v) \in S_{a_1} \times S_{a_2} : \int_{\mathbb{R}^5} |\nabla u|^2 + |\nabla v|^2 = 2K \right\},$$

and there holds

$$J(u, v) > 0 \text{ on } A \text{ and } \sup_A J < \inf_B J.$$

Proof. By using the Gagliardo-Nirenberg inequality (2.1), we have

$$\int_{\mathbb{R}^5} \left(\mu_1 |u|^3 + \frac{3}{2} \beta u^2 v + \mu_2 |v|^3 \right) dx \leq C \int_{\mathbb{R}^5} (|u|^3 + |v|^3) dx \leq C \left(\int_{\mathbb{R}^5} |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{5}{4}} \quad (5.1)$$

for every $(u, v) \in S_{a_1} \times S_{a_2}$, provided $C > 0$ depends on $\mu_1, \mu_2, \beta, a_1, a_2 > 0$ but not on the particular choice of (u, v) . Now taking K to be determined, if $(u_1, v_1) \in B$ and $(u_2, v_2) \in A$, we have

$$\begin{aligned} J(u_1, v_1) - J(u_2, v_2) &\geq \frac{1}{2} \left(\int_{\mathbb{R}^5} |\nabla u_1|^2 + |\nabla v_1|^2 - \int_{\mathbb{R}^5} |\nabla u_2|^2 + |\nabla v_2|^2 \right) - \frac{1}{3} \int_{\mathbb{R}^5} \left(\mu_1 |u_1|^3 + \frac{3}{2} \beta u_1^2 v_1 + \mu_2 |v_1|^3 \right) \\ &\geq \frac{K}{2} - \frac{C}{3} (2K)^{\frac{5}{4}} \geq \frac{K}{4}, \end{aligned}$$

where $K > 0$ is sufficiently small. Moreover, we take K smaller such that

$$J(u_2, v_2) \geq \frac{1}{2} \left(\int_{\mathbb{R}^5} |\nabla u_2|^2 + |\nabla v_2|^2 \right) dx - \frac{C}{3} \left(\int_{\mathbb{R}^5} |\nabla u_2|^2 + |\nabla v_2|^2 \right)^{\frac{5}{4}} dx > 0$$

for every $(u_2, v_2) \in A$. □

Next our aim is to introduce a suitable minimax class. Recalling the definition of $w_{a,\mu}$ in Proposition 2.3. Then, we define

$$C := \left\{ (u, v) \in S_{a_1} \times S_{a_2} : \int_{\mathbb{R}^5} |\nabla u|^2 + |\nabla v|^2 \geq 3K \text{ and } J(u, v) \leq 0 \right\}. \quad (5.2)$$

It follows from Lemma 5.1 that there exist $s_1 < 0$ and $s_2 > 0$, such that

$$s_1^* \left(w_{a_1, \left(\frac{c_0}{a_1}\right)^{\frac{1}{2}}}, w_{a_2, \left(\frac{c_0}{a_2}\right)^{\frac{1}{2}}} \right) = (\bar{u}_1, \bar{v}_1) \in A \quad \text{and} \quad s_2^* \left(w_{a_1, \left(\frac{c_0}{a_1}\right)^{\frac{1}{2}}}, w_{a_2, \left(\frac{c_0}{a_2}\right)^{\frac{1}{2}}} \right) = (\bar{u}_2, \bar{v}_2) \in C.$$

We define

$$\Gamma := \{ \gamma \in C([0, 1], S_{a_1} \times S_{a_2}) : \gamma(0) = (\bar{u}_1, \bar{v}_1) \text{ and } \gamma(1) = (\bar{u}_2, \bar{v}_2) \}. \quad (5.3)$$

The mountain pass lemma is applicable for J on the minimax class Γ , which comes from Lemma 5.2 and the continuity of the L^2 -norm of the gradient in the topology of H^1 . With similar calculations as in Lemma 4.6, we deduce the following conclusions.

Lemma 5.3. *There exists a Palais-Smale sequence (u_n, v_n) for J on $S_{a_1} \times S_{a_2}$ at the level*

$$d := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

satisfying the additional condition (4.12):

$$\int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx - \frac{5}{6} \left(\int_{\mathbb{R}^5} \mu_1 |u_n|^3 + \frac{3}{2} \beta u_n^2 v_n + \mu_2 |v_n|^3 \right) dx = o(1),$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $u_n^-, v_n^- \rightarrow 0$ a.e. in \mathbb{R}^5 as $n \rightarrow \infty$.

Similar to the previous section, we want to show that $(u_n, v_n) \rightarrow (\bar{u}, \bar{v})$ in $H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$. As in the Remark 4.10 and $\beta > 0$ small, we know that (\bar{u}, \bar{v}) is a solution of (1.1) and (1.2). By Lemma 4.6, we know that up to a subsequence, $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v})$ weakly in $H^1(\mathbb{R}^5, \mathbb{R}^2)$, strongly in $L^3(\mathbb{R}^5, \mathbb{R}^2)$, a.e. in \mathbb{R}^5 . In addition, we can also suppose that one of them, say λ_1 is strictly negative. Therefore, as Lemma 4.9, we know that $u_n \rightarrow \bar{u}$ strongly in $H^1(\mathbb{R}^5)$. If by contradiction $v_n \not\rightarrow \bar{v}$ strongly in $H^1(\mathbb{R}^5)$, then $\lambda_2 \geq 0$. By using the similar argument as in the proof of Theorem 1.2, we infer that $\bar{v} \equiv 0$. Thus, as (4.16), we know that $d = l(a_1, \mu_1)$ (see Proposition 2.3). In the following, we need find the contradiction. To accomplish this, we recall the definition of $\beta_2 = \beta_2(a_1, a_2, \mu_1, \mu_2) > 0$, see (1.15).

Lemma 5.4. *If $\beta > \beta_2$, then*

$$\sup_{s \in \mathbb{R}} J \left(s \star w_{a_1, \left(\frac{C_0}{a_1}\right)^{\frac{1}{2}}}, w_{a_2, \left(\frac{C_0}{a_2}\right)^{\frac{1}{2}}} \right) < \min\{l(a_1, \mu_1), l(a_2, \mu_2)\}.$$

Proof. From Proposition 2.3 and (2.3), a direct computation shows that

$$\begin{aligned} \int_{\mathbb{R}^5} \left(s \star w_{a_1, \left(\frac{C_0}{a_1}\right)^{\frac{1}{2}}} \right)^2 \left(s \star w_{a_2, \left(\frac{C_0}{a_2}\right)^{\frac{1}{2}}} \right) dx &= \int_{\mathbb{R}^5} e^{\frac{15}{2}s} \left(w_{a_1, \left(\frac{C_0}{a_1}\right)^{\frac{1}{2}}} (e^s x) \right)^2 \cdot \left(w_{a_2, \left(\frac{C_0}{a_2}\right)^{\frac{1}{2}}} (e^s x) \right) dx \\ &= e^{\frac{5}{2}s} \int_{\mathbb{R}^5} \left(\frac{C_0^2}{\left(\frac{C_0}{a_1}\right)^{\frac{5}{2}} a_1^2} \right) \cdot \left(\frac{C_0^2}{\left(\frac{C_0}{a_2}\right)^{\frac{5}{2}} a_2^2} \right) w_0^3 dx \\ &= e^{\frac{5}{2}s} \frac{a_1 a_2^{\frac{1}{2}}}{C_0^{\frac{3}{2}}} \int_{\mathbb{R}^5} w_0^3 dx = e^{\frac{5}{2}s} \frac{C_1 a_1 a_2^{\frac{1}{2}}}{C_0^{\frac{3}{2}}}. \end{aligned} \quad (5.4)$$

By using Proposition 2.3 again, we can explicitly compute the maximum in s of the function

$$J(s \star (w_{a_1, C_0/a_1}, w_{a_2, C_0/a_2})) = \frac{5e^{2s}}{12} \left(\frac{C_1 a_1}{C_0} + \frac{C_1 a_2}{C_0} \right) - \frac{e^{\frac{5}{2}s}}{3} \left(\frac{\mu_1 C_1 a_1^{\frac{3}{2}}}{C_0^{\frac{3}{2}}} + \frac{3\beta C_1 a_1 a_2^{\frac{1}{2}}}{2C_0^{\frac{3}{2}}} + \frac{\mu_2 C_1 a_2^{\frac{3}{2}}}{C_0^{\frac{3}{2}}} \right).$$

That is, the maximum is given by

$$\max_{s \in \mathbb{R}} J(s \star (w_{a_1, C_0/a_1}, w_{a_2, C_0/a_2})) = \frac{C_1 C_0 (a_1 + a_2)^5}{12 \left(\mu_1 a_1^{\frac{3}{2}} + \frac{3}{2} \beta a_1 a_2^{\frac{1}{2}} + \mu_2 a_2^{\frac{3}{2}} \right)^4}.$$

Recall the definitions of β_2 , $l(a_1, \mu_1)$ and $l(a_2, \mu_2)$, then the conclusion holds if $\beta > \beta_2$. \square

Next we prove the existence of a positive solution at level d .

Existence of a positive solution at level d . Now we argue by contradiction. Suppose that $v_n \rightharpoonup \bar{v}$ strongly in $H^1(\mathbb{R}^5)$. Then, we have $\bar{v} \equiv 0$ and $d = l(a_1, \mu_1)$. We define the path

$$\gamma(t) := (((1-t)s_1 + ts_2) \star (w_{a_1, \mu_1}, w_{a_2, \mu_2})).$$

It is easy to show that $\gamma \in \Gamma$, and as a result of Lemma 5.4,

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \leq \sup_{t \in [0,1]} J(\gamma(t)) \leq \sup_{s \in \mathbb{R}} J(s \star (w_{a_1, \mu_1}, w_{a_2, \mu_2})) < l(a_1, \mu_1).$$

This is a contradiction.

Next we introduce the variational characterization for (\bar{u}, \bar{v}) . That is, we shall prove the following conclusion.

$$\begin{aligned} J(\bar{u}, \bar{v}) &= \inf \{ J(u, v) : (u, v) \in V \} \\ &= \inf \{ J(u, v) : (u, v) \text{ is a solution of (1.1)–(1.2) for some } \lambda_1, \lambda_2 \}, \end{aligned}$$

where V is given in (1.13). We also recall the definitions of A and C (see Lemma 5.2 and (5.2)). Set

$$A^+ := \{(u, v) \in A : u, v \geq 0 \text{ a.e. in } \mathbb{R}^5\}$$

and

$$C^+ := \{(u, v) \in C : u, v \geq 0 \text{ a.e. in } \mathbb{R}^5\}.$$

For any $(u_1, v_1) \in A^+$ and $(u_2, v_2) \in C^+$, let

$$\Gamma(u_1, v_1, u_2, v_2) := \{\gamma \in C([0, 1], S_{a_1} \times S_{a_2}) : \gamma(0) = (u_1, v_1) \text{ and } \gamma(1) = (u_2, v_2)\}.$$

The next lemma we give another characterization of d . This can be done by using similar arguments as in [4, Lemma 4.5]. Here, we omit the details.

Lemma 5.5. *The sets A^+ and C^+ are connected by arcs, so that*

$$d := \inf_{\gamma \in \Gamma(u_1, v_1, u_2, v_2)} \max_{t \in [0,1]} J(\gamma(t)) \quad (5.5)$$

for every $(u_1, v_1) \in A^+$ and $(u_2, v_2) \in C^+$.

Let us recall the set

$$V := \{(u, v) \in T_{a_1} \times T_{a_2} : G(u, v) = 0\}.$$

Its radial subset is given by

$$V_{\text{rad}} := \{(u, v) \in S_{a_1} \times S_{a_2} : G(u, v) = 0\}, \quad (5.6)$$

where

$$G(u, v) = \int_{\mathbb{R}^5} (|\nabla u_n|^2 + |\nabla v_n|^2) dx - \frac{5}{6} \left(\int_{\mathbb{R}^5} \mu_1 |u_n|^3 + \mu_2 |v_n|^3 + \frac{3}{2} \beta u_n^2 v_n \right) dx.$$

The next lemma proves some properties for the set V .

Lemma 5.6. *If (u, v) is a solution of (1.1) and (1.2) for some $\lambda_1, \lambda_2 \in \mathbb{R}$, then $(u, v) \in V$.*

Proof. We use (u, v) as test function in (1.1). Then, we obtain

$$\begin{aligned} \int_{\mathbb{R}^5} |\nabla u|^2 dx - \lambda_1 \int_{\mathbb{R}^5} u^2 dx &= \int_{\mathbb{R}^5} (\mu_1 |u|^3 + \beta u^2 v) dx, \\ \int_{\mathbb{R}^5} |\nabla v|^2 dx - \lambda_2 \int_{\mathbb{R}^5} v^2 dx &= \int_{\mathbb{R}^5} \left(\frac{\beta}{2} u^2 v + \mu_2 |v|^3 \right) dx. \end{aligned} \quad (5.7)$$

Hence, we infer that

$$\lambda_1 \int_{\mathbb{R}^5} u^2 dx + \lambda_2 \int_{\mathbb{R}^5} v^2 dx = \int_{\mathbb{R}^5} |\nabla u|^2 dx + \int_{\mathbb{R}^5} |\nabla v|^2 dx - \int_{\mathbb{R}^5} \left(\mu_1 |u|^3 + \frac{3}{2} \beta u^2 v + \mu_2 |v|^3 \right) dx. \quad (5.8)$$

Moreover, it is easy to deduce that the Pohozaev identity for system (1.1) is given by

$$3 \int_{\mathbb{R}^5} (|\nabla u|^2 + |\nabla v|^2) dx = 5 \int_{\mathbb{R}^5} (\lambda_1 u^2 + \lambda_2 v^2) + \frac{10}{3} \left(\mu_1 |u|^3 + \frac{3}{2} \beta u^2 v + \mu_2 |v|^3 \right) dx. \quad (5.9)$$

From (5.8) and (5.9), we obtain the desired results. \square

Now we define

$$\Psi_{(u,v)}(s) := J(s \star (u, v)),$$

where $(u, v) \in T_{a_1} \times T_{a_2}$. Then, it is easy to check the following results.

Lemma 5.7. *For every $(u, v) \in T_{a_1} \times T_{a_2}$ and $v \geq 0$, then there exists a unique $s_{(u,v)} \in \mathbb{R}$ such that $s_{(u,v)} \star (u, v) \in V$. Moreover, $s_{(u,v)}$ is the unique critical point of $\Psi(u, v)$, which is a strict maximum.*

Lemma 5.8. *There holds $\inf_V J = \inf_{V_{\text{rad}}} J$.*

Proof. To prove this lemma by contradiction, we suppose there exists $(u, v) \in V$ such that

$$0 < J(u, v) < \inf_{V_{\text{rad}}} J. \quad (5.10)$$

For $(u, v) \in H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$, let $\{u, v\}^*$ denotes its Schwarz spherical rearrangement as before. The properties of Schwarz symmetrization leads to $J(\{u, v\}^*) \leq J(u, v)$ and $G(\{u, v\}^*) \leq G(u, v) = 0$. Thus, there exists $s_0 \leq 0$ such that $G(s_0 \star (\{u, v\}^*)) = 0$. One implies that

$$J(s_0 \star (\{u, v\}^*)) \leq e^{2s_0} J(\{u, v\}^*).$$

But in fact, by using that $G(s_0 \star (\{u, v\}^*)) = G(u, v) = 0$, we have

$$J(s_0 \star (\{u, v\}^*)) = \frac{e^{2s_0}}{10} \int_{\mathbb{R}^5} (|\nabla u^*|^2 + |\nabla v^*|^2) dx \leq \frac{e^{2s_0}}{10} \int_{\mathbb{R}^5} (|\nabla u|^2 + |\nabla v|^2) dx = e^{2s_0} J(u, v). \quad (5.11)$$

Thus, with the assumption of (5.10) and applying (5.11),

$$0 < J(u, v) < \inf_{V_{\text{rad}}} J \leq J(s_0 \star (\{u, v\}^*)) \leq e^{2s_0} J(u, v),$$

which contradicts $s_0 \leq 0$. \square

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Recalling that any solution of (1.1) and (1.2) stays in V . If we show

$$J(\bar{u}, \bar{v}) = d \leq \inf \{J(u, v) : (u, v) \in V_{\text{rad}}\}, \quad (5.12)$$

then $J(\bar{u}, \bar{v}) = \inf_{V} J$ by using Lemma 5.8. To prove (5.12), we choose an arbitrary $(u, v) \in V_{\text{rad}}$ and show that $J(u, v) \geq d$. First, we choose an arbitrary $(u, v) \in V_{\text{rad}}$, since $(|u|, |v|) \in V_{\text{rad}}$ and $J(u, v) = J(|u|, |v|)$, it is possible to suppose that $u, v \geq 0$ a.e. in \mathbb{R}^5 . Now let us consider the function $\Psi(u, v)$. By Lemma 5.1, there exists $s_0 \gg 1$ such that $(-s_0) \star (u, v) \in A^+$ and $s_0 \star (u, v) \in C^+$. Thus, the continuous path

$$\gamma(t) := ((2t - 1)s_0) \star (u, v) \quad t \in [0, 1]$$

connects A^+ with C^+ , and by Lemmas 5.5 and 5.7, we infer that

$$d \leq \max_{t \in [0, 1]} J(\gamma(t)) = J(u, v).$$

Consequently, this holds for all the elements in V_{rad} . Thus, inequality (5.12) holds. \square

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