Research Article

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On the L_q-reflector problem in $\mathbb{R}^n$ with non-Euclidean norm

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Abstract: In this article, we introduce a class of geometric optics measure-the L_q Σ-reflector measure which arises from an L_q extension of the Σ-reflector measure. And we ask a Minkowski-type problem for this class of measure, called the L_q Σ-reflector problem. It is shown that the foundations of such measure have been laid by Caffarelli et al. (*Reflector problem in endowed with non-Euclidean norm*, Arch. Ration. Mech. Anal. **193** (2009), no. 2, 445–473) (for the Σ-reflector measure). Inspired by Alexandrov, we present some variational arguments and existence results of solutions to the L_q Σ-reflector problem.

Keywords: radial function, the Σ-reflector measure, the L_q Σ-reflector measure, variational formulas, Minkowski-type problems

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1 Introduction

The Σ-reflector measure emerged recently [4] as a geometric optics measure which plays a significant role in the reflector problem in Euclidean n-space $\mathbb{R}^n (n \geq 2)$ endowed with non-Euclidean norm. In this article, we will concentrate on the basic structural facts about the L_q Σ-reflector measure with parameter $q \in \mathbb{R}$ and devote to solving the existence part of its associated Minkowski-type problem.

We begin by demonstrating the initial reflector measure proposed by Oliker, together with a related analysis on the reflection phenomenon. Throughout the article, we denote the standard Euclidean norm of x by $|x|$ and the Euclidean unit sphere by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. In $\mathbb{R}^n$, suppose an isotropic point source of light is located at the origin O, and light rays emanate from O with intensity $f(x)$, $x \in S^{n-1}$. Every ray from O incident on reflector R is reflected according to Snell's law. That is to say, the reflected direction

$$m = x - 2\langle x, \nu\rangle \nu, \tag{1.1}$$

where ν is the unit outer normal to R at the point where the light ray hits R. It indicates a set-valued map $\gamma_R : x \to m$, namely the reflector map. For any Borel set $\omega \subset (S^{n-1}, f(x)\mathrm{d}x)$, the function

$$G(R, \omega) = \int_{\gamma_R(\omega)} g(m)\mathrm{d}m, \tag{1.2}$$

where dm denotes the area measure on S^{n-1} and $g(m)$ is a non-negative integrable function on S^{n-1}, is a finite, non-negative and countably additive measure on Borel subsets of S^{n-1} [25]. It is called the reflector measure of R.

The generalized reflector problem is to find necessary and sufficient conditions that for a given non-zero finite Borel measure μ on the standard unit sphere S^{n-1} there exists a reflector R such that

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$$\mu = G(R, \cdot). \tag{1.3}$$

For the special case, in which μ has a density $u(x) \in C(S^{n-1})$, the solvability of this problem amounts to deal with the following Monge–Ampère-type equation:

$$g(p, \nabla p)\frac{\det[\nabla_{ij}p + (p - \eta)e_{ij}]}{\eta^{n-1}\det(e_{ij})} = u(x), \tag{1.4}$$

where $p = \frac{1}{\rho}$ is the unknown function on S^{n-1}, e_{ij} is the matrix of coefficients of the standard metric on S^{n-1}, $\eta = (p^2 + |\nabla p|^2)/2p$, and $\nabla_{ij}p$ denotes the second covariant derivative with respect to the metric e_{ij}. This fully non-linear equation on S^{n-1} was derived by Guan and Wang [11].

When $d\mu = f(x)dx$, problem (1.3) is the Far-Field reflector problem in geometric optics. Dating back to the 1960s, there are various works contributed to specific versions of the Far-Field reflector problem, equipped with the practical applications in engineering, electromagnetics, and optics; see, e.g., [7–10,16, 22–24,26,27–30].

In the case of $n = 3$, this problem was considered by Caffarelli and Oliker [5], in which, the existence of the weak solution was proved in terms of the discrete reflector problem. An alternative approach to treating this problem was independently given in [27], where the uniqueness and regularity were also studied. It is worth mentioning that Caffarelli et al. [3] have done some regularity works in higher dimension. Further results on geometric optics problems for refraction have been shown in [6,12–15,19].

However, as a natural generalization of the geometric optics problems on S^{n-1}, the mathematical literature for these problems in anisotropic medium is lacking. Caffarelli et al. [4] have discussed the existence and uniqueness of the Far-Field reflector problem in a homogeneous non-isotropic medium for which light wavefronts are given by strictly convex norm spheres, through approaches of paraboloid approximation and optimal mass transport. Here part of the theory, in particular the study of their light wavefronts, is best formulated in the more general norm sphere. An incidental benefit of this more complex reflection is that it leads us to the surprising discovery of a finitely σ-additive reflector measure on the strictly convex norm sphere. In a similar manner as in [25], the Σ-reflector measure in this medium is defined by Caffarelli et al. [4].

Consider a norm $I(x) = \|x\|$ in $\mathbb{R}^n (n \geq 2)$, and let $\Sigma = \{x \in \mathbb{R}^n : I(x) = 1\}$ be the unit norm sphere such that Σ is C^1 strictly convex and the wavefronts of lights emanated from a point source of light in an anisotropic medium are represented by Σ. In this sense, all lights from O with intensity $f(x)$, $x \in \Sigma$ are reflected by a Σ-reflector (or a reflector for simplicity) R according to Fermat's principle. More precisely, if Y is a point in the reflected direction, then the optical path from O to Y through the reflector R is the path OX_0Y, that is,

$$\|X_0\| + \|Y - X_0\| = \min\{\|X\| + \|Y - X\| : X \in R\}, \tag{1.5}$$

where $X_0 \in R$. From (1.5), it is easily verified that

$$\nabla\|X_0\| - \nabla\|Y - X_0\| = \lambda\nu, \lambda \in \mathbb{R}, \tag{1.6}$$

where ν is the normal field to R. Let $y = \frac{Y - X_0}{\|Y - X_0\|} \in \Sigma$. With the aid of (1.6), it is possible to extract a Σ-reflector map $\boldsymbol{N}_R : x \to y$, which is multivalued. Let us denote by $\mathcal{R}^n$ the class of all non-degenerate Σ-reflectors in Euclidean space $\mathbb{R}^n$. Thus, if we take a Borel set $\omega \subset (\Sigma, f(x)dx)$, the Σ-reflector measure of a reflector $R \in \mathcal{R}^n$ can be expressed as

$$\widetilde{G}(R, \omega) = \int\limits_{\boldsymbol{N}_R(\omega)} g(y)dy, \tag{1.7}$$

where dy denotes the area measure on Σ and $g(y)$ is a non-negative integrable function on Σ.

One of the aims of this work is to study the Σ-reflector problem for measure (1.7). It states: Suppose that the unit norm sphere Σ is C^1 strictly convex. Given a non-zero finite Borel measure μ on Σ, what are the necessary and sufficient conditions on μ to guarantee the existence of a reflector $R \in \mathcal{R}^n$ such that

$$\widetilde{G}(R, \cdot) = \mu? \tag{1.8}$$

Let us give a short description of our main ideas. A main difficulty is to lay down the maps, measures, and reflectors in anisotropic media with regard to setting norms. To overcome this shortcoming, we first formulate those notions and properties in Section 2 by using Fermat's principle and convex analysis. We see that this type of statement implies that the relationship between variables x and m below is in dual norm spheres Σ and Σ^*. For a reflector $R \in \mathcal{R}^n$, we define the radial function of R as $\rho_R(x) = \max\{t \geq 0 : tx \in R\}$ for $x \in \Sigma$. One can write the Legendre transform of $\rho_R(x)$ in the form

$$\rho_{R^*}(m^*) = \inf_{x\in\Sigma} \frac{1}{\rho_R(x)} \frac{1}{1 - m^*x}, \quad m^* \in \Sigma^*. \tag{1.9}$$

Thus, we refer to $R^* = \{\rho_{R^*}(m^*)m^* : m^* \in \Sigma^*\}$ as the dual reflector of R. Based on the variational approach in the classical Minkowski problem [1,2], the next procedure is to construct a new direct variational proof demonstrating the existence of a solution for the Σ-reflector problem. Here we deal with an entropy functional for R^*, that is,

$$\mathcal{E}(R^*) = \int_{\Sigma^*} \log \rho_{R^*}(m^*) g^*(m^*) \mathrm{d}m^*,$$

where dm^* denotes the area measure on Σ^*, $g^*(m^*) = g((\nabla I)^{-1}(m^*))|J((\nabla I)^{-1}(m^*))|$, and $J((\nabla I)^{-1})$ is the Jacobian determinant of the inverse dual map $(\nabla I)^{-1} : \Sigma^* \to \Sigma$ (as will be shown in Section 2). If $\langle R, u, t\rangle$ is a logarithmic family of convex hulls formed by (ρ, u, o), the above preparations will lead to a variational formula (see Lemma 3.1):

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\langle R, u, t\rangle^*)\bigg|_{t=0} = -\int_{\Sigma} u(x) \mathrm{d}\widetilde{G}(R, x). \tag{1.10}$$

As we will see, this entropy functional is essential to obtain the necessary and sufficient conditions of solutions to the Σ-reflector problem (see Theorem 5.1).

A natural, and probably even more interesting, question that arises is to study the L_q Σ-reflector problem with respect to the L_q Σ-reflector measure. Treatments of measure and problem below are mainly inspired by the works of the L_p-surface area measures for convex bodies (see, e.g., [17,18,20,21]).

Before we exploit those skills, we must take a precaution. For each real q, the L_q Σ-reflector combination is defined by

$$\rho_{a\cdot R +_q b\cdot S} = (a\rho_R^q(y) + b\rho_S^q(y))^{\frac{1}{q}},$$

for $a, b \geq 0$ and $R, S \in \mathcal{R}^n$. We also present the L_q Σ-reflector measure $\mathrm{d}\widetilde{G}_q(R, \cdot) = \rho_R^q \mathrm{d}\widetilde{G}(R, \cdot)$ for all $q \in \mathbb{R}$, where $\widetilde{G}_0(R, \cdot) = \widetilde{G}(R, \cdot)$ $(q = 0)$ is the Σ-reflector measure. Those simple techniques allow us to bring in the variational formula. For $t \geq 0$, if R and Q are Σ-reflectors, there is

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(R^* +_q t \cdot Q^*)\bigg|_{t=0} = \frac{1}{q} \int_{\Sigma} \rho_Q(x)^{-q} \mathrm{d}\widetilde{G}_q(R, x), \tag{1.11}$$

where $\widetilde{G}_q(R, \cdot)$ is the L_q Σ-reflector measure.

The L_q Σ-reflector problem we pose asks: Suppose μ is a non-zero finite Borel measure defined on the unit norm sphere Σ and $q \in \mathbb{R}$, what are the necessary and sufficient conditions on μ so that there exists a convex reflector $R \in \mathcal{R}^n$ such that

$$\mu = \widetilde{G}_q(R, \cdot)? \tag{1.12}$$

The focus on this work will be attacking the general problem (1.12). What will be required are the delicate estimates in terms of geometric invariants for treating the associated maximization problems, which will be shown in Section 5.

We are in position to elaborate the main results of this article from the perspective of the value on real q. In the case of $q = 0$, the following theorem gives a complete solution to the existence part of the Σ-reflector problem.

Theorem 1.1. *In the case of $q = 0$, let the unit norm sphere Σ be strictly convex and C^1. There exists $R_0 \in \mathcal{R}^n$ such that $\mu = \widetilde{G}(R_0, \cdot)$ if and only if μ is a non-zero finite Borel measure on Σ and $|\mu| = \int_{\Sigma} g(y)\mathrm{d}y = \int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*$.*

On the other hand, for $q \neq 0$, it is revealed in the following.

Theorem 1.2. *Let $q \neq 0$, and the unit norm sphere Σ is strictly convex and C^1. There exists $R_0 \in \mathcal{R}^n$ such that $\mu = \widetilde{G}_q(R_0, \cdot)$ if and only if μ is a non-zero finite Borel measure on Σ.*

2 Preliminaries

In this section, we collect some relevant definitions and properties about Σ-reflectors. We recommend Caffarelli et al. [4] as a good reference.

2.1 The norms and the convexity

Throughout this article, a convex body $K \subset \mathbb{R}^n (n \geq 2)$ is always assumed to be an origin-symmetric compact convex set enclosing the origin O. The norm $I(\cdot) = \|\cdot\|$ in $\mathbb{R}^n$ is given by

$$\|\cdot\| = \inf\left\{t > 0 : \frac{x}{t} \in K\right\}.$$

For each $x \in \mathbb{R}^n$, the polar body K^* of K is defined by

$$K^* = \{u \in \mathbb{R}^n : u \cdot x \leq 1 \text{ for each } x \in K\}.$$

Let h_K denote the support function of K and h_{K^*} the support function of K^*. For the simplicity of notations, we also denote the boundary of K and K^* by Σ and Σ^*. It is clear that

$$\begin{aligned} h_K(u) &= 1, \quad \text{for each } u \in \Sigma^*, \\ h_{K^*}(x) &= 1, \quad \text{for each } x \in \Sigma. \end{aligned}$$

Then, we have a unit norm sphere $\Sigma = \{x \in \mathbb{R}^n : \|x\| = h_{K^*}(x) = 1\}$ and a unit norm ball $K = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. In a dual way, the dual norm sphere of Σ is defined by $\Sigma^* = \{u \in \mathbb{R}^n : \|u\|^* = h_K(u) = 1\}$. For the set of all continuous functions defined on $\Sigma(\Sigma^*)$, we will write $C(\Sigma)(C(\Sigma^*))$.

The following properties can be found in [4].

Lemma 2.1.
(i) *Σ has a continuous normal field $\nu(x)$ if and only if $h_{K^*} \in C^1(\mathbb{R}^n \backslash \{0\})$. Moreover, $\nabla h_{K^*}(x) = \nu(x)/h_K(\nu(x))$.*
(ii) *If Σ is C^1, then Σ^* is strictly convex. Also, if Σ is strictly convex, then $\Sigma^* \in C^1$.*
(iii) *If Σ is strictly convex and C^1, then $\nabla h_{K^*} : \Sigma \to \Sigma^*$ is a homeomorphism and $\nabla h_{K^*} \circ \nabla h_K = Id$. The symbol "∘" is called the multiplication of composite mapping.*

In what follows, we assume that Σ is strictly convex and C^1.

2.2 The Σ-reflector

Given $m^* \in \Sigma^*$ and a real number b, let $\Pi_{m^*,b}$ be the hyperplane in $\mathbb{R}^n$ with the equation

$$m^*x + b = 0. \tag{2.1}$$

Definition 1. For $m^* \in \Sigma^*$ and $b > 0$, Σ-paraboloid $P(m^*, b)$ with focus O and directrix $\Pi_{m^*,b}$ of equation (2.2) is defined by

$$P(m^*, b) = \Big\{X \in \mathbb{R}^n : \|X\| = \mathrm{dist}\big(X, \Pi_{m^*,b}\big)\Big\}. \tag{2.2}$$

If we write $X = \rho x$ for $x \in \Sigma$ and $\rho = \|X\|$, then $P(m^*, b)$ can be represented by the radial function $\rho(x)$, that is,

$$P(m^*, b) = \left\{\rho(x)x : x \in \Sigma \backslash \nabla h_K(m^*) \quad \text{and} \quad \rho(x) = \frac{b}{1 - m^*x}\right\}, \tag{2.3}$$

where $\nabla h_K(m^*) \in \Sigma$ is called the axial direction of $P(m^*, b)$, and $\rho(\nabla h_K(m^*)) = \infty$.

Definition 2. A closed surface R in $\mathbb{R}^n$ is a Σ-reflector if for any $X_0 \in R$ there exists a Σ-paraboloid $P(m^*, b)$ such that $X_0 \in P(m^*, b)$ and R is contained in the convex body limited by $P(m^*, b)$. $P(m^*, b)$ is called a supporting Σ-paraboloid of R at X_0.

It follows directly from those definitions that any Σ-reflector R is strictly convex. We shall denote by $\mathcal{R}^n$ the family of all convex Σ-reflectors with respect to O in $\mathbb{R}^n$.

As explained before, if $R \in \mathcal{R}^n$, we denote by $\rho_R : \Sigma \to \mathbb{R}$ its radial function, that is,

$$\rho_R(x) = \max\{\lambda : \lambda x \in R\}.$$

Clearly, the radial function of a Σ-reflector is continuous and positive.

For $m^* \in \Sigma^*$, we note that if

$$b_0 = \inf\left\{b > 0 : \rho(x) \le \frac{b}{1 - m^*x} \ \text{for} \ x \in \Sigma\right\}, \tag{2.4}$$

then $P(m^*, b_0)$ is a supporting Σ-paraboloid to Σ-reflector R.

2.3 The Σ-reflector measure

For a Σ-reflector R in $\mathbb{R}^n$ and $m^* \in \Sigma^*$, the supporting Σ-paraboloid is given by

$$P_R(m^*, b) = \{\rho(x)x \in \mathbb{R}^n : x \in \Sigma \text{ and } \rho(x)(1 - m^*x) = b\}.$$

Let $\sigma \subset \Sigma$ be a Borel set. We denote by $\boldsymbol{F}_R(\sigma)$ the Σ-front image of R at point $\rho(x)x$, i.e.,

$$\boldsymbol{F}_R(\sigma) = \{m^* \in \Sigma^* : \rho(x)x \in P_R(m^*, b) \quad \text{for some } x \in \sigma\} \subset \Sigma^*.$$

Let $\eta \subset \Sigma^*$ be a Borel set. The reverse Σ-front image, $\boldsymbol{F}_R^{-1}(\eta)$, of η is defined by

$$\boldsymbol{F}_R^{-1}(\eta) = \{x \in \Sigma : \rho(x)x \in P_R(m^*, b) \text{ for some } m^* \in \eta\} \subset \Sigma.$$

We usually denote the set $\boldsymbol{F}_R(\{x\})$ simply by $\boldsymbol{F}_R(x)$. Let $\sigma_R \subset \Sigma$ such that $\boldsymbol{F}_R(x)$ contains at least one element. Combining with the fact that $\mathcal{H}^{n-1}(\sigma_R) = 0$, the ∂K-front map of R, denoted by $F_R(x)$, is the map

$$F_R : \Sigma \backslash \sigma_R \to \Sigma^*,$$

for each $x \in \Sigma \backslash \sigma_R$. Let $\eta_R \subset \Sigma^*$ be a Borel set consisting of all $m^* \in \Sigma^*$, for which the set $\boldsymbol{F}_R^{-1}(m^*)$ contains more than a single element. In view of the fact that $\mathcal{H}^{n-1}$-measure is 0, the function

$$F_R^{-1} : \Sigma^* \backslash \eta_R \to \Sigma$$

is well defined. For each $m^* \in \Sigma^* \backslash \eta_R$, let $F_R^{-1}(m^*)$ be the unique element in $\boldsymbol{F}_R^{-1}(m^*)$, which is called the reverse Σ-front map. The functions F_R and F_R^{-1} are continuous.

For a strictly convex body K, we define the dual map

$$\nabla h_K : \Sigma^* \to \Sigma \text{ by } \nabla h_K(m^*) = \frac{\nu(m^*)}{h_K(\nu(m^*))} \in \Sigma$$

for $m^* \in \Sigma^*$. Note that h_K and h_{K^*} are $C^2(\mathbb{R}^n \backslash \{0\})$, equivalently ∇h_K is a C^1 diffeomorphism from Σ^* to Σ and ∇h_{K^*} is a C^1 diffeomorphism from Σ to Σ^*.

Let $\omega \subset \Sigma$ be a Borel set. The Σ-reflector image of ω is defined by

$$\boldsymbol{N}_R(\omega) = \nabla h_K(\boldsymbol{F}_R(\omega)) \subset \Sigma.$$

Thus, for $x \subset \Sigma$, one has

$$\boldsymbol{N}_R(x) = \Big\{ y \in \Sigma : \rho(x)x \in P_R\big(\nabla h_{K^*}(y), b\big)\Big\}.$$

Define the Σ-reflector map of the Σ-reflector R by

$$N_R : \Sigma \backslash \omega_R \to \Sigma \text{ by } N_R = \nabla h_K \circ F_R,$$

where $\omega_R = \nabla h_K(\sigma_R)$. Since ∇h_K is a C^1 diffeomorphism between the spaces Σ^* and Σ, it follows that ω_R has Lebesgue measure 0. If $x \in \Sigma \backslash \omega_R$, then $\boldsymbol{N}_R(\{x\})$ contains only the element $N_R(x)$. Since F_R and ∇h_K are continuous, N_R is continuous.

For $\eta \subset \Sigma$, define the reverse Σ-reflector image of η by

$$\boldsymbol{N}_R^{-1}(\eta) = \boldsymbol{F}_R^{-1}\big(\nabla h_{K^*}(\eta)\big) \subset \Sigma.$$

Thus, for $\eta \subset \Sigma$, there is

$$\boldsymbol{N}_R^{-1}(y) = \Big\{ x \in \Sigma : \rho(x)x \in P_R\big(\nabla h_{K^*}(y), b\big)\Big\}.$$

In light of the fact that F_R^{-1} and ∇h_{K^*} are continuous, N_R^{-1} is continuous. Define the reverse Σ-reflector map of the Σ-reflector R by

$$N_R^{-1} : \Sigma \backslash \eta_R \to \Sigma \text{ by } N_R^{-1} = F_R^{-1}\big(\nabla h_{K^*}\big).$$

If $\eta_R \subset \Sigma$ is a Borel set, then $\boldsymbol{N}_R^{-1}(\eta) = \boldsymbol{F}_R^{-1}(\nabla h_{K^*}(\eta)) \subset \Sigma$ is Σ-reflector Lebesgue measurable. It is easily shown that if $y \notin \eta_R$ and $\omega \subset \Sigma$, then

$$y \in \boldsymbol{N}_R(\omega) \text{ if and only if } N_R^{-1}(y) \in \omega, \tag{2.5}$$

for almost all $y \in \Sigma$, with respect to Lebesgue measure.

Definition 3. Let $R \in \mathcal{R}^n$. Suppose that $g(y)$ is a non-negative integrable function on Σ. The Σ-reflector measure, $\widetilde{G}(R, \cdot)$ of R is a Borel measure on Σ defined by

$$\widetilde{G}(R, \omega) = \int_{\boldsymbol{N}_R(\omega)} g(y)\mathrm{d}y, \tag{2.6}$$

where $\mathrm{d}y$ is the area measure on Σ.

Since for $\lambda > 0$ obviously $\boldsymbol{N}_{\lambda R} = \boldsymbol{N}_R$, it follows from (2.6) that

$$\widetilde{G}(\lambda R, \cdot) = \widetilde{G}(R, \cdot). \tag{2.7}$$

Definition 4. The Legendre transform of $\rho(x)$, with respect to the function $\frac{1}{1-m^*x}$, is a function ρ^* defined on Σ^*, given by

$$\rho^*(m^*) = \inf_{x\in\Sigma} \frac{1}{\rho(x)}\frac{1}{1-m^*x}. \tag{2.8}$$

Let $E^* = F_R(E)$. For any $m_0^* \in E^*$, we choose $x_0 \in E$ so that $F_R(x_0) = m_0^*$. Let $\rho^*(m_0^*) = \frac{b_0^*}{1-m_0^*x_0}$ be a paraboloid $P(x_0, b_0^*)$ with axis $\nabla h_{K^*}(x_0)$ so that

$$\rho^*(m_0^*) = \frac{b_0^*}{1-m_0^*x_0} = \frac{1}{\rho(x_0)}\frac{1}{1-x_0F_R(x_0)},$$

namely $b_0^* = \frac{1}{\rho(x_0)}$. Hence,

$$\rho^*(m^*) \le \frac{1}{\rho(x_0)}\frac{1}{1-m^*x_0}, \quad \text{for } m^* \in \Sigma^*, \tag{2.9}$$

$$\rho^*(m_0^*) \le \frac{1}{\rho(x)}\frac{1}{1-m_0^*x}, \quad \text{for } x \in \Sigma. \tag{2.10}$$

From (2.8) and (2.9) we obtain that $\rho^*(m^*) = \frac{b^*}{1-m^*x_0}\left(b^* = \frac{1}{\rho(x_0)}\right)$ is a supporting paraboloid of ρ^* at m_0^*, where $\rho^*(m^*) = \frac{b^*}{1-m^*x_0}\left(b^* = \frac{1}{\rho(x_0)}\right)$. By combining (2.8) and (2.10) one has that

$$\rho(x) = \frac{b}{1-m_0^*x}\left(b = \frac{1}{\rho^*(m_0^*)}\right)$$

is a supporting paraboloid of ρ at x_0.

Note also that $m_0^* \in F_R(x_0)$ if and only if $x_0 \in F_{R^*}(m_0^*)$. We shall denote the inverse of F_R by F_{R^*}. Therefore, F_{R^*} is also a diffeomorphism from E^* to E. Similar to (2.8), we can define

$$\rho^{**}(x) = \inf_{m^*\in E^*} \frac{1}{\rho^*(m^*)}\frac{1}{1-xm^*}, \quad x \in E.$$

For any $x_0 \in E$, the infimum is attained at the point m_0^* so that $F_{R^*}(m_0^*) = x_0$. In particular,

$$\rho^{**}(x_0) = \frac{1}{\rho^*(m_0^*)}\frac{1}{1-x_0m_0^*} = \rho(x_0).$$

Then we may regard $R^* = \{\rho^*(m^*)m^* : m^* \in \Sigma^*\}$ as the dual of R. That is to say the following lemma.

Lemma 2.2. *If R is a Σ-reflector and R^{**} denotes the Legendre transform of R^*, then $R^{**} = R$.*

Some weak compactness results to the Σ-reflector map under uniform convergence of Σ-reflectors shall be given as the following.

Lemma 2.3. *Let $R_j = \{\rho_j(x)x : x \in \Sigma\}$, $j \ge 1$, be Σ-reflectors. Suppose that $0 < a_1 \le \rho_j \le a_2$ and $\rho_j \longrightarrow \rho$ uniformly on Σ. Then $R = \{\rho(x)x : x \in \Sigma\}$ is a Σ-reflector. Furthermore,*
- *(i) For any compact set $F \subset \Sigma$, $\limsup_{j\to\infty} \boldsymbol{N}_{R_j}(F) \subset \boldsymbol{N}_R(F)$.*
- *(ii) For any open set $J \subset \Sigma$, $\boldsymbol{N}_R(J) \subset \liminf_{j\to\infty} \boldsymbol{N}_{R_j}(J) \cup S$, where $S = \{y \in \Sigma : R$ has no tangent plane at point $\rho(x)x\}$ is the singular set of R with the area measure 0.*

With the aid of Lemma 2.3, we can show that $\widetilde{G}(R_j, \omega) \rightharpoonup \widetilde{G}(R, \omega)$ weakly. For the proofs of Lemmas 2.3 and 2.4, please refer to [4].

Lemma 2.4. *Let ω be a Borel set on Σ. The set function $\widetilde{G}(R, \omega)$ is a non-negative and σ-additive measure on Borel sets of Σ. Furthermore, if R_j, $j = 1, 2, \dots$, is a sequence of Σ-reflectors in $\mathcal{R}^n$ converging to a closed convex hypersurface R in the Hausdorff metric, then the hypersurface R is also a Σ-reflector with the source O and the measures $\widetilde{G}(R_j, \omega)$ converge weakly to $\widetilde{G}(R, \omega)$.*

Proof. Obviously, $\widetilde{G}(R,\omega)$ is non-negative. Let

$$\mathcal{B}=\{E\subset\Sigma:\boldsymbol{N}_R(E)\ \text{is Lebesgue measurable}\}.$$

Note that $\mathcal{B}$ is a σ-algebra containing all Borel sets in Σ. Let $\{\omega_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{B}$. Let $H_1=\boldsymbol{N}_R(\omega_1)$, and $H_k=\boldsymbol{N}_R(\omega_k)\setminus\cup_{i=1}^{k-1}\boldsymbol{N}_R(\omega_i)$, for $k\geq 2$. Since $H_i\cap H_j=\emptyset$ for $i\neq j$ and $\cup_{k=1}^{\infty}H_k=\cup_{k=1}^{\infty}\boldsymbol{N}_R(\omega_k)$, it is easy to obtain

$$\widetilde{G}(R,\cup_{k=1}^{\infty}\omega_k)=\int\limits_{\cup_{k=1}^{\infty}H_k}g\mathrm{d}y=\sum\nolimits_{k=1}^{\infty}\int\limits_{H_k}g\mathrm{d}y.$$

Observe that $\nabla h_{K^*}[\boldsymbol{N}_R(\omega_k)-H_k]=\nabla h_{K^*}[\boldsymbol{N}_R(\omega_k)\cap(\cup_{i=1}^{k-1}\boldsymbol{N}_R(\omega_i))]$ has area measure 0 for $k\geq 2$. We obtain $\int_{H_k}g\mathrm{d}y=\widetilde{G}(R,\omega_k)$ and the σ-additivity of $\widetilde{G}(R,\omega)$ is proved.

We recall (see Lemma 2.3) that

$$\overline{\lim_{j\to\infty}}\,\widetilde{G}(R_j,F)\geq\int\limits_{\lim_{j\to\infty}\boldsymbol{N}_{R_j}(F)}g\mathrm{d}y\leq\int\limits_{\boldsymbol{N}_R(F)}g\mathrm{d}y=\widetilde{G}(R,F),$$

for any compact set $F\subset\Sigma$. On the other hand, for any open set $J\subset\Sigma$,

$$\lim_{j\to\infty}\widetilde{G}(R_j,J)\geq\widetilde{G}(R,J).$$

In summary, $\widetilde{G}(R_j,\omega)\rightharpoonup\widetilde{G}(R,\omega)$ weakly. □

Let ω be a Borel set on Σ such that $\mathbb{1}_\omega$ is the indicator function of ω. An interesting consequence is that

$$\int\limits_{\Sigma}\mathbb{1}_\omega(x)\mathrm{d}\widetilde{G}(R,x)=\int\limits_{\Sigma}\mathbb{1}_{\boldsymbol{N}_R(\omega)}(y)g(y)\mathrm{d}y=\int\limits_{\Sigma}\mathbb{1}_\omega(N_R^{-1}(y))g(y)\mathrm{d}y.\tag{2.11}$$

The last identity comes from the fact that $y\in\boldsymbol{N}_R(\omega)$, if and only if $N_R^{-1}(y)\in\omega$, for almost $y\in\Sigma$ with respect to Lebesgue measure.

For each $u\in C(\Sigma)$, by (2.11), it is easy to show that

$$\int\limits_{\Sigma}u(x)\mathrm{d}\widetilde{G}(R,x)=\int\limits_{\Sigma}u(N_R^{-1}(y))g(y)\mathrm{d}y.\tag{2.12}$$

Equivalently, if $R\in\mathcal{R}^n$, then for every Borel set $\omega\subset\Sigma$, we have

$$\int\limits_{\Sigma}u(x)\mathrm{d}\widetilde{G}_q(R,x)=\int\limits_{\Sigma}u(N_R^{-1}(y))\rho_R^q(N_R^{-1}(y))g(y)\mathrm{d}y,\tag{2.13}$$

where $\widetilde{G}_q(R,\cdot)$ is the L_q Σ-reflector measure, for each $u\in C(\Sigma)$ and $q\in\mathbb{R}$.

Finally, together with (2.12) and (2.13), we obtain

$$\mathrm{d}\widetilde{G}_q(R,\cdot)=\rho_R^q\mathrm{d}\widetilde{G}(R,\cdot).\tag{2.14}$$

Combining (2.7) and (2.14) we see that

$$\widetilde{G}_q(\lambda R,\cdot)=\lambda^q\widetilde{G}_q(R,\cdot).\tag{2.15}$$

3 Variational formulas to L_q Σ-reflector problem

3.1 The Σ-reflector shapes and Σ-convex hulls

Let $\Omega^*\subset\Sigma^*$. Assume $v^*:\Omega^*\to\mathbb{R}$ is an arbitrary continuous function and δ is a sufficiently small positive number. For $t\in(-\delta,\delta)$, $b(m^*,t)$ is a positive continuous function on Σ^* defined by

$$\log b(m^*, t) = \log b(m^*) + tv^*(m^*) + o(t, m^*),$$

where $o(t, \cdot) : \Sigma^* \to \mathbb{R}$ is continuous and $\lim_{t\to 0}\frac{o(t,\cdot)}{t} = 0$ uniformly on Ω^*. The Σ-reflector shape $[b_t]$ associated with $b(m^*, t)$ is determined by

$$[b_t] = \bigcap_{m^*\in\Omega^*} \{X \in \mathbb{R}^n \| \|X\| - m^*X \le b(m^*, t) \text{ for all } m^* \in \Omega^*\}.$$

We shall denote $[b, v^*, t]$ by $[b_t]$. Let b be the focal function of a Σ-reflector R as $[b, v^*, t]$. We say that $[R, v^*, t]$ is a logarithmic family of Σ-reflector shapes generated by (R, v^*, o).

Fixed $\Omega \subset \Sigma$, and consider an arbitrary continuous function $u : \Omega \to \mathbb{R}$. For each $t \in (-\delta, \delta)$, suppose $\rho(x, t) : \Omega \to \mathbb{R}$ is a continuous function defined by

$$\log\rho(x, t) = \log\rho(x) + tu(x) + o(t, x),$$

where $o(t, \cdot) : \Sigma \to \mathbb{R}$ is continuous and satisfies $\lim_{t\to 0}\frac{o(t,\cdot)}{t} = 0$ uniformly on Ω. The Σ-convex hull $\langle\rho_t\rangle$ is denoted by

$$\langle\rho_t\rangle = \mathrm{conv}\{\rho(x, t)x : x \in \Omega\}.$$

Let ρ be the radial function of a Σ-reflector R as $\langle R, u, t\rangle$. We call $\langle R, u, t\rangle$ a logarithmic family of Σ-convex hulls generated by (R, u, o). We also write $\langle\rho, u, t\rangle$ as $\langle\rho_t\rangle$ and say that $\langle\rho_t\rangle$ is a logarithmic family of Σ-convex hulls generated by (ρ, u, o).

We describe the L_q Σ-reflector combination, using a similar method of the L_p-combination in the L_p-Brunn-Minkowski theory.

Fix $q \in \mathbb{R}$. For $a, b \ge 0$, define the L_q Σ-reflector combination of two sets $R, S \in \mathcal{R}^n$ by

$$a \cdot R +_q b \cdot S = \{X \in \mathbb{R}^n \| \|X\| \le [a\rho_R^q(x) + b\rho_S^q(x)]^{1/q} \text{ for all } x \in \Omega\},$$

where $q \ne 0$. With this we can say that either a or b may be negative, as long as the function $a\rho_R^q + b\rho_S^q$ is strictly positive on Σ. Besides, when $q = 0$, $a \cdot R +_0 b \cdot S = \langle\rho_R^a\rho_S^b\rangle$.

The L_q Σ-reflector harmonic combination $a \cdot R \hat{+}_q b \cdot S$ is defined by

$$a \cdot R \hat{+}_q b \cdot S = (a \cdot R^* +_q b \cdot S^*)^*.$$

3.2 Variational formula for entropy of Σ-reflectors and Σ*-reflectors

Let Σ-reflector $R \in \mathcal{R}^n$, $x \in \Sigma$. The entropy of the dual Σ^*-reflector R^* is defined by

$$\mathcal{E}(R^*) = \int_{\Sigma^*} \log\rho^*(m^*)g^*(m^*)\mathrm{d}m^*,$$

where dm^* denotes the area measure on Σ^*, $g^*(m^*) = g(\nabla h_K(m^*))|J(\nabla h_K(m^*))|$ and $J(\nabla h_K)$ is the Jacobian determinant of the inverse dual map ∇h_K.

Now we are going to calculate some variational formulas.

Lemma 3.1. *Let $R \in \mathcal{R}^n$. Suppose that $u : \Sigma \to \mathbb{R}$ and $v^* : \Sigma^* \to \mathbb{R}$ are arbitrary continuous functions. When $\langle R, u, t\rangle$ happens to be a logarithmic family of Σ-convex hulls generated by (R, u, o),*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\langle R, u, t\rangle^*)\bigg|_{t=0} = -\int_{\Sigma} u(x)\mathrm{d}\widetilde{G}(R, x), \tag{3.1}$$

where $\widetilde{G}(R, x)$ is the Σ-reflector measure of R. When $\langle R^*, v^*, t\rangle$ happens to be a logarithmic family of Σ^*-convex hulls formed by $\langle R^*, v^*, o\rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\langle R^*, v^*, t\rangle^*)\Big|_{t=0} = -\int_{\Sigma^*} v^*(m^*)\mathrm{d}\widetilde{G}(R^*, m^*), \tag{3.2}$$

where $\widetilde{G}(R^*, m^*)$ *is the* Σ^**-reflector measure of* R^*.

Proof. It is enough to consider only (3.1). By virtue of the assumption that $\langle R, u, t\rangle$ is the Σ-convex hulls associated with the radial function $\rho(x, t)$, there holds $\langle R, u, t\rangle = R_t$.

Suppose $u \in C(\Sigma)$ is fixed. Define

$$\log\rho(x, t) = \log\rho(x) + tu(x).$$

As in (2.8), we can write

$$\log\rho^*(m^*, t) \le -\log\rho_{R_t}(x) - \log(1 - m^*x) \le -\log\rho(x, t) - \log(1 - m^*x),\ x \in \Sigma.$$

Also, if it happens that

$$\log\rho(x_t, t) = -\log\rho^*(m^*, t) - \log(1 - m^*x_t),$$

for $m^* \in \Sigma^*$, then simple calculations show that

$$\begin{aligned}\liminf_{t\to 0}\frac{\log\rho^*(m^*, t) - \log\rho^*(m^*)}{t} &= \liminf_{t\to 0}\frac{-\log\rho(x_t, t) - \log(1 - m^*x_t) - \log\rho^*(m^*)}{t}\\ &\ge \liminf_{t\to 0}\frac{-\log\rho(x_t, t) + \log\rho(x_t)}{t}\\ &= \liminf_{t\to 0} -u(x_t).\end{aligned} \tag{3.3}$$

We have $x_t \to x$ as $t \to 0$, hence (3.3) gives

$$\liminf_{t\to 0} -u(x_t) = -u(x). \tag{3.4}$$

In addition, we recall the fact that if $x \in \Sigma$, then for every $m^* \in \Sigma^*$, $\log\rho(x) + \log\rho^*(m^*) = -\log(1 - m^*x)$. It follows that

$$\begin{aligned}\limsup_{t\to 0}\frac{\log\rho^*(m^*, t) - \log\rho^*(m^*)}{t} &= \limsup_{t\to 0}\frac{\log\rho^*(m^*, t) + \log(1 - m^*x) + \log\rho(x)}{t}\\ &\le \limsup_{t\to 0}\frac{-\log\rho(x, t) + \log\rho(x)}{t}\\ &= -u(x).\end{aligned} \tag{3.5}$$

This inequality, combining (3.4), yields

$$\lim_{t\to 0}\frac{\log\rho^*(m^*, t) - \log\rho^*(m^*)}{t} = -u(x) = -u(F_R^{-1}(m^*)) = -u(N_R^{-1}(y)), \tag{3.6}$$

provided that

$$\left|\frac{\log\rho^*(m^*, t) - \log\rho^*(m^*)}{t}\right| \le M_0,$$

where $M_0 = \max_{x\in\Sigma} u(x)$.

At the end, by using control convergence theorem and definition (2.6), we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\langle R, u, t\rangle^*)\Big|_{t=0} &= \lim_{t\to 0}\frac{\int_{\Sigma^*}\log\rho^*(m^*, t)g^*(m^*)\mathrm{d}m^* - \int_{\Sigma^*}\log\rho^*(m^*)g^*(m^*)\mathrm{d}m^*}{t}\\ &= -\int_{\Sigma^*} u(F_R^{-1}(m^*))g^*(m^*)\mathrm{d}m^*\\ &= -\int_{\Sigma} u(N_R^{-1}(y))g(y)\mathrm{d}y\\ &= -\int_{\Sigma} u(x)\mathrm{d}\widetilde{G}(R, x).\end{aligned}$$

In the same way, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\langle R^*, v^*, t\rangle^*)\bigg|_{t=0} = -\int_{\Sigma^*} v^*(m^*)\mathrm{d}\widetilde{G}(R^*, m^*) \tag{3.7}$$

□

Lemma 3.2. *Let $R, Q \in \mathcal{R}^n$. Then, for every $q \neq 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R \hat{+}_q t \cdot Q)\bigg|_{t=0} = \frac{1}{q}\int_{\Sigma^*} \rho_{Q^*}(m^*)^{-q}\mathrm{d}\widetilde{G}_q(R^*, m^*), \tag{3.8}$$

while for $q = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R \hat{+}_0 t \cdot Q)\bigg|_{t=0} = -\int_{\Sigma^*} \log\rho_{Q^*}(m^*)\mathrm{d}\widetilde{G}(R^*, m^*). \tag{3.9}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R^* +_q t \cdot Q^*)\bigg|_{t=0} = \frac{1}{q}\int_{\Sigma} \rho_Q(x)^{-q}\mathrm{d}\widetilde{G}_q(R, x), \tag{3.10}$$

while for $q = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R^* +_0 t \cdot Q^*)\bigg|_{t=0} = -\int_{\Sigma} \log\rho_Q(x)\mathrm{d}\widetilde{G}(R, x). \tag{3.11}$$

Proof. In order to show (3.8) and (3.9) we divide the proof into two cases: $q = 0$ and $q \neq 0$.

In the first case of $q = 0$, letting

$$\rho_t^* = \rho_{R^*}\rho_{Q^*}^t,$$

we obtain

$$\log\rho_t^* = \log\rho_{R^*} + t\log\rho_{Q^*}.$$

Choose $v^* = \log\rho_{Q^*}$, thus $R^* +_0 t \cdot Q^* = \langle R^*, v^*, t\rangle$. The formula (3.2) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R \hat{+}_0 t \cdot Q)\bigg|_{t=0} = -\int_{\Sigma^*} \log\rho_{Q^*}(m^*)\mathrm{d}\widetilde{G}(R^*, m^*).$$

In the second case of $q \neq 0$, letting

$$\rho_t^* = \left(\rho_{R^*}^{-q} + t\rho_{Q^*}^{-q}\right)^{-1/q},$$

then

$$\log\rho_t^* = \log\rho_{R^*} - \frac{1}{q}\left(\frac{\rho_{Q^*}}{\rho_{R^*}}\right)^{-q} t + o_q(t, \cdot),$$

where $o_q(t, \cdot) : \Sigma^* \to \mathbb{R}$ is continuous and $\lim_{t\to 0}\frac{o(t,\cdot)}{t} = 0$, uniformly on Σ^*. Then we write $R^* +_q t \cdot Q^* = \langle\rho_t^*\rangle = \langle R^*, v^*, t\rangle$ where

$$v^* = -\frac{1}{q}\left(\frac{\rho_{Q^*}}{\rho_{R^*}}\right)^{-q}.$$

Using the method in the proof of Lemma 3.1 and (2.14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R \hat{+}_q t \cdot Q)\bigg|_{t=0} = \frac{1}{q}\int_{\Sigma^*} \rho_{Q^*}^{-q}(m^*)\mathrm{d}\widetilde{G}_q(R^*, m^*).$$

In (3.8) and (3.9) replacing R^*, Q^*, by R, Q we finally obtain (3.10) and (3.11). □

4 Maximizing the entropy of Σ-reflectors

In this section, we provide a proof of existence result on a maximization problem which is associated with the L_q Σ-reflector problem.

Let μ be a non-zero finite Borel measure on Σ. For every function $f \in C^+(\Sigma)$, we fix $q \in \mathbb{R}$ and set

$$\|f : \mu\|_q = \left(\frac{1}{|\mu|} \int_{\Sigma} f^q \mathrm{d}\mu \right)^{1/q}, \quad q \neq 0,$$

$$\|f : \mu\|_0 = \|f\|_0 = \exp\left(\frac{1}{|\mu|} \int_{\Sigma} \log f \mathrm{d}\mu \right), \quad q = 0.$$

We then define a functional $\widetilde{\mathcal{F}}_q : C^+(\Sigma) \to \mathbb{R}$ by

$$\widetilde{\mathcal{F}}_q(f) = \frac{\mathcal{E}(\langle f \rangle^*)}{\int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*} + \log \|f : \mu\|_{-q}.$$

For $\rho \in C^+(\Sigma)$, using $\rho_{\langle \rho \rangle} \geq \rho$, it is clear to have

$$\|\rho_{\langle \rho \rangle} : \mu\|_{-q} \geq \|\rho : \mu\|_{-q}. \tag{4.1}$$

Thus, combining (4.1) with $\mathcal{E}(\langle \rho_{\langle \rho \rangle} \rangle^*) = \mathcal{E}(\langle \rho \rangle^*)$ we have

$$\widetilde{\mathcal{F}}_q(\rho) \leq \widetilde{\mathcal{F}}_q(\langle \rho \rangle^*).$$

Next we shall focus on searching a maximizer to the set of all radial functions of Σ-reflectors, i.e., ρ_{R_0} is a maximizer to the maximization problem

$$\sup\{\widetilde{\mathcal{F}}_q(\rho) : \rho \in C^+(\Sigma)\}$$

if and only if ρ_{R_0} is a maximizer to the following maximization problem:

$$\sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}_o^n\},$$

where the functional

$$\widetilde{\mathcal{F}}_q(R) = \frac{\mathcal{E}(R^*)}{\int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*} + \log \|\rho_R : \mu\|_{-q},$$

for $R \in \mathcal{R}^n$.

In our setting, the functional $\widetilde{\mathcal{F}}_q$ is continuous and homogeneous of degree 0. Suppose $R_0 \in \mathcal{R}^n$ is a maximizer to $\sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}_o^n\}$, or equivalently ρ_{R_0} is a maximizer to $\sup\{\widetilde{\mathcal{F}}_q(\rho) : \rho \in C^+(\Sigma)\}$. Then we have:

Lemma 4.1. *For a real number q, assume μ is a Borel measure on Σ. Let $R_0 \in \mathcal{R}^n$. If*

$$\int_{\Sigma} \rho_R^{-q} \mathrm{d}\mu = \int_{\Sigma} g(y)\mathrm{d}y = \int_{\Sigma^*} g^*(m^*)\mathrm{d}m^* \tag{4.2}$$

and

$$\widetilde{\mathcal{F}}_q(R_0) = \sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}^n\}, \tag{4.3}$$

then

$$\mu = \widetilde{G}_q(R_0, \cdot). \tag{4.4}$$

Proof. Suppose $u \in C(\Sigma)$ is arbitrary but fixed. Let $\delta > 0$ and $t \in (-\delta, \delta)$. Consider the family $\rho(x, t) \in C^{+}(\Sigma)$, where the function $\rho(\cdot, t) : \Sigma \times (-\delta, \delta) \to \mathbb{R}$ is defined by

$$\log\rho(x, t) = \log\rho_{R_0}(x) + tu(x),$$

and let R_t denote the boundary of the Σ-convex hulls associated with $\rho(x, t)$.

For the proof of the case $q \neq 0$, we will use the inequality

$$|e^s - 1 - s| \leq es^2,$$

which is easily checked for every $s \in (-1, 1)$. If t is chosen by $|t| < 1/(|q|\max_{x\in\Sigma}u(x))$, then we can ensure that for every $x \in \Sigma$,

$$\left|\frac{e^{-tqu} - 1}{t} + qu(x)\right| \leq eq^2u(x)^2|t|.$$

This guarantees that

$$\frac{\rho_t^{-q} - \rho_0^{-q}}{t} = \frac{e^{-tqu} - 1}{t}\rho_{R_0}^{-q} \longrightarrow -qu\rho_{R_0}^{-q}, \text{ uniformly on } \Sigma,$$

as $t \to 0$. Hence, we prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\rho_{R_t} : \mu\|_{-q}\bigg|_{t=0} = \frac{1}{\int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*}\int_{\Sigma}\rho_{R_0}(x)^{-q}u(x)\mathrm{d}\mu. \tag{4.5}$$

It is easily pointed out that the above statement (4.5) also holds if $q = 0$, in which the hypothesis (4.2) turns to $|\mu| = \int_{\Sigma} g(y)\mathrm{d}y = \int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*$.

In view of the assumption we have that $R_0 \in \mathcal{R}^n$ is a maximizer for $\widetilde{\mathcal{F}}_q$ and also trivially that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{F}}_q(R_t)\bigg|_{t=0} = 0.$$

By Lemma 3.1, differentiating in $\widetilde{\mathcal{F}}_q(R_t)$ gives

$$\int_{\Sigma}\rho_{R_0}(x)^{-q}u(x)\mathrm{d}\mu - \int_{\Sigma}u(x)\widetilde{G}(R_0, x) = 0.$$

Since $u(x) \in C(\Sigma)$ is arbitrary, from (2.14) we conclude (4.4). □

5 Existence of solutions to the L_q Σ-reflector problem

As shown in Lemma 4.1, it is essential to see that the problem of finding a solution for the L_q Σ-reflector problem transfers to finding a maximizer for the maximization problem:

$$\sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}^n\}.$$

In this section, we show that the maximization problem does have a solution. We first discuss this maximization problem in the case $q = 0$, which is revealed as

$$\widetilde{\Phi}(R) = \frac{\mathcal{E}(R^*)}{\int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*} + \frac{\int_{\Sigma}\log\rho_R(x)\mathrm{d}\mu}{|\mu|}.$$

The basic observation is that the functional $\widetilde{\Phi} : \mathcal{R}^n \to \mathbb{R}$ is continuous and homogeneous of degree 0. For $q = 0$, the main conclusion is the following:

Lemma 5.1. *For any Borel measure μ on the unit norm sphere Σ we can find a Σ-reflector $R_0 \in \mathcal{R}^n$ such that*

$$\widetilde{\Phi}(R_0) = \sup\{\widetilde{\Phi}(R) : R \in \mathcal{R}^n\}. \tag{5.1}$$

Proof. Let

$$\mathcal{R} = \left\{R \in \mathcal{R}^n : \int_{\Sigma^*} \log\rho_{R^*}(m^*)g^*(m^*)\mathrm{d}m^* = 0\right\}.$$

We start with proving $\mathcal{R}$ is bounded. For $R \in \mathcal{R}$, we set $\rho_R(x_R)x_R \in R$, $x_R \in \Sigma$, and

$$\rho_R(x_R) = \max_{x\in\Sigma}\rho_R(x).$$

Combining the definition of $\rho_{R^*}(m^*)$, we deduce that

$$\rho_R(x_R)(1 - m^*x) \le \frac{1}{\rho_{R^*}(m^*)},$$

for all $m^* \in \Sigma^*$. It is clear from the last inequality that

$$\int_{\Sigma^*} \log\rho_R(x_R)g^*(m^*)\mathrm{d}m^* + \int_{\Sigma^*} \log(1 - m^*x)g^*(m^*)\mathrm{d}m^* \le -\int_{\Sigma^*} \log\rho_{R^*}(m^*)g^*(m^*)\mathrm{d}m^* = 0,$$

i.e.,

$$\log\rho_R(x_R)\int_{\Sigma^*} g^*(m^*)\mathrm{d}m^* + \int_{\Sigma^*} \log(1 - m^*x)g^*(m^*)\mathrm{d}m^* \le 0.$$

Hence, $\rho_R(x_R)$ is bounded.

Since $\widetilde{\Phi}$ is homogeneous of degree 0 and $\mathcal{R}$ is bounded, we choose a maximizing sequence $R_j \in \mathcal{R}$ such that $\widetilde{\Phi}(R_j)$ converges to

$$\sup\{\widetilde{\Phi}(R) : R \in \mathcal{R}\} \tag{5.2}$$

and dilate R_j such that $R_j \subset c_0K$. By Blaschke's selection theorem, there exists a subsequence of R_j (we denote it again by R_j) that converges to a closed convex surface R_0 such that $x_j \to x \in c_0\Sigma$, $b_j \to b$, $m_j^* \to m^*$. Clearly, $b_j \le 2c_0$. If R_0 is star-shaped relative to the point O, then we are done.

Otherwise, we assume that $\rho_j = \min\{\rho_{R_j}(x) : x \in \Sigma\} \to 0$. By our assumptions, if

$$\Omega_j = \left\{x \in \Sigma : 1 - m_j^*x > \frac{1}{\log\frac{1}{\rho_j}}\right\},$$

then for $x \in \Omega_j$, we have

$$\rho_{R_j}(x) \le \frac{b_j}{1 - m_j^*x} < b_j \log\frac{1}{\rho_j} \le 2c_0 \log\frac{1}{\rho_j}$$

and

$$\log\rho_{R_j}(x) \le \log 2c_0 + \log\log\frac{1}{\rho_j}.$$

Therefore, we obtain the inequality

$$\liminf_{j\to\infty} \frac{\int_{\Omega_j} \log\rho_{R_j}(x)^{-1}\mathrm{d}\mu}{\log\frac{1}{\rho_j}} \ge \liminf_{j\to\infty} \frac{\int_{\Omega_j}\left(\log\frac{1}{2c_0} - \log\log\frac{1}{\rho_j}\right)\mathrm{d}\mu}{\log\frac{1}{\rho_j}} \ge \liminf_{j\to\infty} \mu(\Omega_j).$$

As μ is a Borel measure on Σ, there exists a constant $C > 0$ such that

$$\int_{\Sigma} (1 - m_j^* x) \mathrm{d}\mu \geq C.$$

Turning to Ω_j, by the monotone convergence theorem it follows that

$$\lim_{j\to\infty} \mu(\Omega_j) = \mu\left(\left\{x \in \Sigma | 1 - m_j^* x > \frac{1}{\log \frac{1}{\rho_j}}\right\}\right) > 0.$$

Certainly, we can pick a positive number c_1 so that

$$\mu(\Omega_j) \geq c_1.$$

Finally, those arguments lead to

$$\begin{aligned}\widetilde{\Phi}(R_j) &= \frac{\int_{\Sigma} \log \rho_{R_j}(x) \mathrm{d}\mu}{|\mu|} + \frac{\int_{\Sigma^*} \log \rho_{R_j^*}(m^*) g^*(m^*) \mathrm{d}m^*}{\int_{\Sigma^*} g^*(m^*) \mathrm{d}m^*} \\ &= -\frac{\int_{\Sigma} \log \rho_{R_j}(x)^{-1} \mathrm{d}\mu}{|\mu|} \\ &\leq -\frac{\int_{\Omega_j} \log \rho_{R_j}(x)^{-1} \mathrm{d}\mu}{|\mu|} \\ &\leq -c_1 \frac{\log \frac{1}{\rho_j}}{|\mu|},\end{aligned}$$

which yields $\widetilde{\Phi}(R_j) \to -\infty$ by $\rho_j \to 0$. This contradicts (5.2), and the proof of the Lemma is completed. □

For positive q, we have the following lemma.

Lemma 5.2. *Suppose $q \in (0, \infty)$. If μ is a Borel measure on Σ, then there exists a Σ-reflector $R_0 \in \mathcal{R}^n$ such that*

$$\sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}^n\} = \widetilde{\mathcal{F}}_q(R_0). \tag{5.3}$$

Proof. Consider the set

$$\mathcal{R} = \left\{R \in \mathcal{R}^n : \int_{\Sigma^*} \log \rho_{R^*}(m^*) g^*(m^*) \mathrm{d}m^* = 0\right\}.$$

Indeed, with an argument similar to the one used in the proof of Lemma 5.1, we conclude that $\mathcal{R}$ is bounded to $c_0 K$, where $c_0 > 0$.

Since $\widetilde{\mathcal{F}}_q(R)$ is homogeneous of degree 0, it is surely possible to find a maximizing sequence $\{R_j\}$ which satisfies the conditions

$$R_j \in \mathcal{R}, \quad \lim_{j\to\infty} \widetilde{\mathcal{F}}_q(R_j) = \sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}\}.$$

In addition, the Blaschke's selection theorem ensures that there exists a subsequence (we denote it again by R_j) converges to a closed convex surface R_0, i.e.,

$$R_j \to R_0.$$

We will then show that the origin O contained in the interior of R_0. We assume, on the contrary, that $\rho_j = \min\{\rho_{R_j}(x) : x \in \Sigma\} \to 0$, and seek a contradiction. As R_j is bounded to c_0K and $q > 0$, one immediately checks that

$$\liminf_{j\to\infty} \frac{\log \int_\Sigma \frac{1}{\rho_{R_j}(x)}^q d\mu}{\log \frac{1}{\rho_j}} \geq \liminf_{j\to\infty} \frac{\log \int_{\Omega_j} \frac{1}{\rho_{R_j}(x)}^q d\mu}{\log \frac{1}{\rho_j}} \geq \liminf_{j\to\infty} \frac{q \log \frac{1}{2c_0} - q \log\log \frac{1}{\rho_j} + \log \mu(\Omega_j)}{\log \frac{1}{\rho_j}} \geq q. \tag{5.4}$$

From (5.4) we infer that

$$\begin{aligned} \widetilde{\mathcal{F}}_q(R_j) &= -\frac{1}{q} \log \int_\Sigma \rho_{R_j}(x)^{-q} d\mu(x) + \frac{1}{q} \log|\mu| \\ &\leq -\frac{1}{q} \log \int_{\Omega_j} \frac{1}{\rho_{R_j}(x)}^{-q} d\mu(x) + \frac{1}{q} \log|\mu| \\ &\leq -\log \frac{1}{\rho_j} + \frac{1}{q} \log|\mu|. \end{aligned} \tag{5.5}$$

From (5.5), passing to the limit as $j \to \infty$, we obtain

$$\widetilde{\mathcal{F}}_q(R_j) \to -\infty, \tag{5.6}$$

which clearly contradicts to $\{R_j\}$ being a maximizer sequence. This finishes the proof of Lemma 5.2. □

For $q < 0$, we also have Lemma 5.3.

Lemma 5.3. *Suppose $q \in (-\infty, 0)$. If μ is a Borel measure on Σ, then there exists Σ-reflector $R_0 \in \mathcal{R}^n$ such that*

$$\sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}^n\} = \widetilde{\mathcal{F}}_q(R_0). \tag{5.7}$$

Proof. We just begin by considering the set

$$\mathcal{R} = \left\{ R \in \mathcal{R}^n : \int_{\Sigma^*} \log \rho_{R^*}(m^*) g^*(m^*) dm^* = 0 \right\}$$

and we note that $\mathcal{R}$ is bounded to c_0K for $c_0 > 0$.

On the other hand, since $\mathcal{R}$ is bounded, we may write a maximizer sequence $\{R_j : R_j \in \mathcal{R}\}$ such that

$$\lim_{j\to\infty} \widetilde{\mathcal{F}}_q(R_j) = \sup\{\widetilde{\mathcal{F}}_q(R) : R \in \mathcal{R}^n\}.$$

By the Blaschke's selection theorem, the sequence $\{R_j\}$ with dilated has a convergent subsequence, denoted again by $\{R_j\}$, such that

$$R_j \longrightarrow R_0,$$

for a Σ-reflector R_0.

We next show that $O \in \partial R_0 = R_0$. Otherwise, there exists $\rho_j = \min\{\rho_{R_j}(x) : x \in \Sigma\} \to 0$.

For $j = 1, 2, \dots,$ define

$$\varepsilon_j = \max\left\{ |\bar{m}^* - \bar{m}_j^*|, \rho_j^{\frac{1}{\log \frac{1}{\rho_j}}} \right\}$$

and

$$\Omega_j' = \{x \in \Sigma : 1 - \bar{m}_j^* x > \varepsilon_j\}.$$

Since $m_j^* \to m^*$ and $\rho_j \to 0$, there holds

$$\bar{m}_j^* \to \bar{m}^*, \quad \lim_{j\to\infty} \varepsilon_j = 0.$$

Recalling the definition of ε_j, we see that for $x \in \Omega_j'$,

$$1 - \bar{m}^* x = 1 - \bar{m}_j^* x - x \cdot (\bar{m}^* - \bar{m}_j^*) > \varepsilon_j - |\bar{m}^* - \bar{m}_j^*| \geq 0,$$

and we obtain

$$\Omega_j' \subset \Sigma.$$

Moreover, since $q < 0$, for $x \in \Omega_j'$ we deduce that

$$0 \geq \lim_{j\to\infty} \frac{\int_{\Omega_j'} \rho_{R_j}(x)^{-q} \mathrm{d}\mu}{\log \rho_j} \geq \lim_{j\to\infty} \mu(\Omega_j') \frac{(2c_0)^{-q} \rho_j^{\frac{q}{\log \frac{1}{\rho_j}}}}{\log \frac{1}{\rho_j}} \geq 0.$$

Let

$$\Omega_{\delta_k}' = \{x \in \Sigma : 1 - \bar{m}_j^* x > \delta_k\}.$$

We shall choose a sequence $1 > \delta_1 > \delta_2 > \cdots > \delta_k \to 0$ satisfying the following conditions:

$$\Sigma \backslash \Omega_{\delta_1}' \supset \Sigma \backslash \Omega_{\delta_2}' \supset \cdots, \quad \bigcap_{k=1}^{\infty} \left(\Sigma \backslash \Omega_{\delta_k}'\right) = \Sigma \bigcap \{x \in \mathbb{R}^n : 1 - m_0^* x \leq 0\}.$$

Since we have supposed that μ is a non-zero finite Borel measure on Σ, it is easy to obtain

$$\lim_{k\to\infty} \mu\left(\Sigma \backslash \Omega_{\delta_k}'\right) = \mu(\Sigma \bigcap \{x \in \mathbb{R}^n : 1 - m_0^* x \leq 0\}) = 0. \tag{5.8}$$

According to the definition of Ω_j', we have

$$0 < 1 - x \cdot \bar{m}^* = 1 - x \cdot \bar{m}_j^* + x \cdot (\bar{m}_j^* - \bar{m}^*) \leq \varepsilon_j + |\bar{m}^* - \bar{m}_j^*|$$

for every $x \in \Sigma \backslash \Omega_j'$.

One also finds, by elementary computations, that

$$\lim_{j\to\infty} (\varepsilon_j + |\bar{m}^* - \bar{m}_j^*|) = 0,$$

it suffices to observe that for fixed k, if j is sufficiently large, we have $\varepsilon_j - |\bar{m}^* - \bar{m}_j^*| < \delta_k$. Then there holds

$$1 - x \cdot \bar{m}^* < \delta_k$$

and

$$x \in \Sigma \backslash \Omega_{\delta_k}'.$$

Hence, for fixed k, when j is sufficiently large,

$$\Sigma \backslash \Omega_j' \subset \Sigma \backslash \Omega_{\delta_k}'.$$

From (5.8), it is evident that

$$\lim_{j\to\infty} \mu(\Sigma \backslash \Omega_j') = 0.$$

Furthermore, we point out that

$$0 \geq \lim_{j\to\infty} \frac{\int_{\Sigma \backslash \Omega_j'} \rho_{R_j}(x)^{-q} \mathrm{d}\mu}{\log \rho_j} \geq \lim_{j\to\infty} \mu(\Sigma \backslash \Omega_j') = 0.$$

Therefore,

$$\begin{aligned}\widetilde{\mathcal{F}}_q(R_j) &= -\frac{1}{q}\log\int_{\Sigma}\varrho_{R_j}(x)^{-q}\mathrm{d}\mu(x) + \frac{1}{q}\log|\mu| \\ &= -\frac{1}{q}\log\int_{\Omega_j'}\rho_{R_j}(x)^{-q}\mathrm{d}\mu(x) - \frac{1}{q}\int_{\Sigma\setminus\Omega_j'}\rho_{R_j}(x)^{-q}\mathrm{d}\mu(x) + \frac{1}{q}\log|\mu| \\ &\le -\frac{1}{q}\log\rho_j + \frac{1}{q}\log|\mu| \to -\infty,\end{aligned}$$

as $j \to \infty$, which clearly contradicts to the fact that $\{R_j\}$ is a maximizer sequence. Finally, this contradiction tells that R_0 is the reflector we desired. □

The above considerations raise the following two answers to the L_q Σ–reflector problem.

Theorem 5.1. *If $q = 0$, there exists Σ-reflector $R_0 \in \mathcal{R}^n$ such that $\mu = \widetilde{G}(R_0, \cdot)$ if and only if μ is a non-zero finite Borel measure on the unit norm sphere Σ and $|\mu| = \int_\Sigma g(y)\mathrm{d}y = \int_{\Sigma^*} g^*(m^*)\mathrm{d}m^*$.*

Proof. The necessity part is clear. Lemmas 4.1 and 5.1 lead to the sufficiency part. □

When $q \neq 0$, Lemmas 4.1, 5.2, and 5.3 give a complete solution to the L_q Σ-reflector problem.

Theorem 5.2. *Let $q \neq 0$. There exists Σ-reflector $R_0 \in \mathcal{R}^n$ such that $\mu = \widetilde{G}_q(R_0, \cdot)$ if and only if μ is a non-zero finite Borel measure on the unit norm sphere Σ.*

Proof. Only the if part needs a proof. This is a consequence of Lemmas 4.1, 5.2, and 5.3. □

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