Research Article

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Existence and asymptotic behavior of solitary waves for a weakly coupled Schrödinger system

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Abstract: This paper deals with the following weakly coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u_1 + a_1(x)u_1 = |u_1|^{2p-2}u_1 + b|u_1|^{p-2}|u_2|^p u_1, & x \in \mathbb{R}^N, \\ -\Delta u_2 + a_2(x)u_2 = |u_2|^{2p-2}u_2 + b|u_2|^{p-2}|u_1|^p u_2, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 1$, $b \in \mathbb{R}$ is a coupling constant, $2p \in (2, 2^*)$, $2^* = 2N/(N-2)$ if $N \geq 3$ and $+\infty$ if $N = 1, 2$, $a_1(x)$ and $a_2(x)$ are two positive functions. Assuming that $a_i(x)$ $(i = 1, 2)$ satisfies some suitable conditions, by constructing creatively two types of two-dimensional mountain-pass geometries, we obtain a positive synchronized solution for $|b| > 0$ small and a positive segregated solution for $b < 0$, respectively. We also show that when $1 < p < \min\{2, 2^*/2\}$, the positive solutions are not unique if $b > 0$ is small. The asymptotic behavior of the solutions when $b \to 0$ and $b \to -\infty$ is also studied.

Keywords: Bose-Einstein condensates, nonlinear optics, nonunique, nontrivial constraints, Schrödinger systems, segregated, synchronized, weakly coupled

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1 Introduction and main results

In this paper, we study the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u_1 + a_1(x)u_1 = |u_1|^{2p-2}u_1 + b|u_1|^{p-2}|u_2|^p u_1, & x \in \mathbb{R}^N, \\ -\Delta u_2 + a_2(x)u_2 = |u_2|^{2p-2}u_2 + b|u_2|^{p-2}|u_1|^p u_2, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $a_1(x)$, $a_2(x)$ are two positive functions, and $b \in \mathbb{R}$ is a coupling constant. This type of systems arise when one considers standing waves for time-dependent k-coupled Schrödinger systems of the form

$$\begin{cases} i\dfrac{\psi_j}{\partial t} = \Delta\psi_j - c_j(x)\psi_j + |\psi_j|^{2p-2}\psi_j + |\psi_j|^{p-2}\psi_j \displaystyle\sum_{l=1, s\neq j}^{k} \beta_{js}|\psi_l|^p, & \text{in } \mathbb{R}^N, \\ \psi_j = \psi_j(x, t) \in \mathbb{C}, \quad t > 0, \quad j = 1, \dots, k, \end{cases} \tag{1.2}$$

where $c_j(x) > 0$ are positive functions, i denotes the imaginary part, and $\beta_{js} = \beta_{sj}$ are coupling constants.

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System (1.2) is applied to study the nonlinear optics in isotropic materials, for instance, the propagation pulses in a single-mode fiber. With the effects of birefringence, one pulse ψ tends to be spilt into two pulses in the two polarization directions, but Menyuk [21] proved that the two components ψ_1, ψ_2 in a birefringence optical fiber were governed by the two coupled nonlinear Schrödinger system in (1.2) ($k = 2$).

System (1.2) also has applications in Bose–Einstein condensates theory. For example, when $k = 2$ in (1.2), ψ_1 and ψ_2 are the wave functions of the corresponding condensates and b is the interspecies scattering length. If $b > 0$, then the components of states tend to obtain along with each other leading to synchronization, but if $b < 0$, the components tend to segregate each other, leading to phase separation.

In recent years, a lot of works have been done on the existence of nontrivial solutions for (1.1). For the existence of ground states, we refer to [4,5,9,15,16,18–20,22–24,28–31]; for the other existence results, we refer to [2,17]. See also [17,18] for some nonexistence results.

When $p = 2$, there are two ways of looking for nontrivial solutions to (1.1). One way is to look for minimizers on some candidate critical points sets with radial restriction or some nontrivial constraints. For example, in [2], nontrivial solutions to (1.1) were obtained by finding a (P.S.) sequence $\{w_n = (u_n^1, u_n^2)\}$ satisfying $\int_{\mathbb{R}^N} |u_n^i|^2 dx \equiv \lambda_i > 0$. In [3], the L^2-constraints in [2] had been used to obtain positive radial (both components are positive and radially symmetric) solutions of segregated type ($b < 0$). In [29], by minimizing on a nontrivial Nehari-type set, Sirakov proved that (1.1) had positive radial solutions if $-\infty < b < b_0$ for some positive constant b_0, see also [9,35]. In [33], assuming that $a_i(x) \equiv 1$ ($i = 1, 2$) and b is negative enough, Terracini and Verzini proved the existence of positive and radial solutions for (1.1). They showed that as $b \to -\infty$, the profile of each component u_i separates, in many pulses, from the others. The other way is mathematical reduction method. See [27], for instance, where assuming that $a_i(x) = 1 + O(|x|^{-m})$ ($m > 1$) as $|x| \to \infty$, Peng and Wang used the finite-dimensional reduction method to construct an unbounded sequence of nonradial positive vector solutions of segregated or synchronized type respectively.

However, system (1.1) is not well studied for a general exponent $2p \in (2, 2^*)$ or general potentials $a_i(x)(i = 1, 2)$. First, the finite reduction method in [26,27] does not work since the general assumptions on p and $a_i(x)$ make it hard to find a nondegenerate solution to the limit system of (1.1). Second, it is impossible to use the L^2-constraints like [2,3] for the nonconstant potentials $a_i(x)(i = 1, 2)$, since the nonvariant properties do not hold anymore. Moreover, the L^2-constraint argument just gives some minimizers of a nontrivial Nehari type of sets, which exactly solve (1.1) with $a_i(x)u_i$ being replaced by $a_i(x)u_i + \lambda_i u_i$ ($i = 1, 2$), where λ_i is some Lagrange multiplier. Third, when $p \neq 2$, it is impossible to obtain lineal real symmetric matrix when minimizing on Nehari type manifold; see [9,35] for example.

The aims of the present paper include the following two aspects. On the one hand, we want to obtain rid of nontrivial constraints that are usually used to look for nontrivial solutions on the candidate critical points added in [2,29] and the references therein. On the other hand, we will try to remove the radial restriction on the solutions in the full repulsive case $b < 0$ and look for segregated type solutions for system (1.1) with a general Sobolev exponent $2p \in (2, 2^*)$ and potentials $a_i(x)$ ($i = 1, 2$). Hereafter, we say a solution $w = (u_1, u_2)$ of (1.1) is nontrivial if $u_i \not\equiv 0$ and positive if $u_i > 0$, $i = 1, 2$.

To state our main results, we first recall some preliminaries and give some notations. We set the Hilbert space $\mathcal{H}$ as follows:

$$\mathcal{H} = \{w = (u_1, u_2) : u_1, u_2 \in H^1(\mathbb{R}^N)\},$$

with inner product

$$\langle \tilde{w}, w \rangle_{\mathcal{H}} = \sum_{i=1}^{2} \int_{\mathbb{R}^N} (\nabla u_i \nabla \tilde{u}_i + a_i(x) u_i \tilde{u}_i)$$

and its reduced norm

$$\|w\|_{\mathcal{H}}^2 = \|u_1\|_{a_1}^2 + \|u_2\|_{a_2}^2$$

for all $\tilde{w} = (\tilde{u}_1, \tilde{u}_2)$, $w = (u_1, u_2) \in \mathcal{H}$, where $\|\cdot\|_{a_i} = \left(\int_{\mathbb{R}^N} |\nabla \cdot|^2 + a_i(x)|\cdot|^2\right)^{\frac{1}{2}}$. We also need the following closed subspace $\mathcal{H}_r$ of $\mathcal{H}$,

$$\mathcal{H}_r = \{w \in \mathcal{H} : w \text{ is radially symmetric}\},$$

endowed with the same inner product and norm as $\mathcal{H}$.

We set

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{for } N > 2, \\ +\infty, & \text{for } N \le 2 \end{cases} \quad \text{and} \quad \sigma_p = \frac{p}{p-1} - \frac{N}{2}.$$

For $1 \le q < +\infty$ and $f : \mathbb{R}^N \to \mathbb{R}$ being Lebesgue measurable, $|f|_q^q$ denotes the integration $\int_{\mathbb{R}^N} |u|^q \mathrm{d}x$.

In the sequel, we assume that the positive functions $a_1(x)$ and $a_2(x)$ satisfy the following conditions that were first introduced in [12]

($\mathcal{A}_1$) $a_i(x) = a_i(|x|) \in C^2(\mathbb{R}^N)$.

($\mathcal{A}_2$) $a_i'(r) \ge 0$ for $r \ge 0$, $0 < a_i(0) \le \lim_{r\to\infty} a_i(r) < +\infty$.

($\mathcal{A}_3$) When $N \ge 3$, we assume

$$\inf_{r>0}\{a_i''(r)r^2 + (3+\beta)a_i'(r)r + 2\beta a_i(r)\} > 0,$$

where

$$\beta := \frac{2(N-1)(2p-2)}{2p+2}.$$

Define the functional corresponding to (1.1) as follows:

$$J_{a_1,a_2,b}(w) := \frac{1}{2}\|w\|_{\mathcal{H}}^2 - \frac{1}{2p}|u_1|_{2p}^{2p} - \frac{1}{2p}|u_2|_{2p}^{2p} - \frac{b}{p}|u_1u_2|_p^p, \quad \forall w = (u_1, u_2) \in \mathcal{H}.$$

Denote $b^+ = \max\{b, 0\}$. For a positive bounded function $a(x)$, define the functional $L_{a,b^+} : H^1(\mathbb{R}^N) \to \mathbb{R}$ as follows:

$$L_{a,b^+}(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)|u|^2) - \frac{1+b^+}{2p}\int_{\mathbb{R}^N} |u|^{2p}.$$

It is easy to see that the Euler–Lagrange equation corresponding to L_{a,b^+} is

$$-\Delta u + a(x)u = (1+b^+)|u|^{2p-2}u, \quad u \in H^1(\mathbb{R}^N). \tag{$\mathcal{P}_{a,b^+}$}$$

Define $U_{1,0}$ to be the unique positive solution (up to a translation) to ($\mathcal{P}_{1,0}$). Then, if $a(x) \equiv a > 0$,

$$U_{a,b^+}(\cdot) := \left(\frac{a}{1+b^+}\right)^{\frac{1}{2p-2}} U_{1,0}(\sqrt{a}\ \cdot)$$

is the unique positive solution of ($\mathcal{P}_{a,b^+}$). It is well known (see ([1,7,8,14]), for example) that

$$\limsup_{|x|\to\infty} U_{a,0}(x)|x|^{\frac{N-1}{2}} e^{\sqrt{a}|x|} < +\infty.$$

Furthermore, setting $\mathcal{E}(a, b^+) = L_{a,b^+}(U_{a,b^+})$, we can see

$$\mathcal{E}(a, b^+) := \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} L_{a,b^+}(tu) = \inf_{\gamma \in \Gamma_{a,b^+}} \max_{t\in[0,1]} L_{a,b^+}(\gamma(t)),$$

where $\Gamma_{a,b^+} = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \quad L_{a,b^+}(\gamma(1)) < 0\}$. We can also easily check that

$$\mathcal{E}(a, b^+) = \frac{a^{\sigma_p}}{(1+b^+)^{\frac{1}{p-1}}}\mathcal{E}(1, 0).$$

As a consequence, $\mathcal{E}(a, b^+)$ is continuous, increases about a, and decreases about b when $b > 0$.

The following proposition is well known when $N \geq 2$, see [12] for example. For the case $N = 1$, maybe this result has been proved somewhere, but since we cannot find it in literature, we will give its proof in the Appendix for the readers' convenience.

Proposition 1.1. *Let $N \geq 1$ and $a(x)$ be a positive radial function that satisfies $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. Then, the positive solution of $(\mathcal{P}_{a,b^+})$ in $H^1(\mathbb{R}^N)$ is radially symmetric and unique. The mountain pass value*

$$C_{a,b^+} = \inf_{\gamma \in \Gamma_{a,b^+}} \max_{t \in [0,1]} L_{a,b^+}(\gamma(t))$$

can be achieved by a unique positive radial solution U_{a,b^+}, where

$$\Gamma_{a,b^+} := \Big\{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \quad \text{and} \quad L_{a,b^+}(\gamma(1)) < 0\Big\}.$$

Moreover, there holds

$$\lim_{|x| \to \infty} U_{a,b^+}(x) e^{\sqrt{a(0)}\,|x|} \leq d$$

for some constant $d > 0$.

Now we state our main results.

For the attractive and partial repulsive case, we have the following:

Theorem 1.2. *Let $N \geq 1$, $2p \in (2, 2^*)$, $a_i(x) \equiv a_i > 0$ $(i = 1, 2)$, and $\omega = a_1/a_2 \leq 1$. There exist constants b_ω, $\hat{b}_\omega > 0$ such that if $-\hat{b}_\omega < b < b_\omega$, system (1.1) has a positive radial solution $w_b = (u_b^1, u_b^2) \in \mathcal{H}_r$, which satisfies that*

(i) $u_b^1 \to U_{a_1,0}$, $u_b^2 \to U_{a_2,0}$ uniformly on every compact set of $\mathbb{R}^N$ as $b \to 0$;

(ii) $u_b^1 \to U_{a_1,0}$, $u_b^2 \to U_{a_2,0}$ strongly in $H^1(\mathbb{R}^N)$ as $b \to 0$.

If $a_i(x) > 0$ $(i = 1, 2)$ are not constants, considering Proposition 1.1 and using the transformation $u = (1 + b^+)^{-\frac{1}{2p-2}} v$, we see

$$\mathcal{E}(a, b^+) = (1 + b^+)^{-\frac{1}{p-1}} \mathcal{E}(a, 0). \tag{1.3}$$

The following result is a more general form of Theorem 1.2.

Theorem 1.3. *Let $N \geq 1$. Assume that $2p \in (2, 2^*)$, $a_i(x) = a_i(|x|) > 0$ $(i = 1, 2)$ satisfy $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. There exist constants b'_{a_1/a_2}, $\hat{b}'_{a_1/a_2} > 0$ such that if $-\hat{b}'_{a_1/a_2} < b < b'_{a_1/a_2}$, system (1.1) has a positive radial solution $w'_b = (u'^1_b, u'^2_b) \in \mathcal{H}_r$, which satisfies that*

(i) $u'^1_b \to U_{a_1,0}$, $u'^2_b \to U_{a_2,0}$ uniformly on every compact set of $\mathbb{R}^N$ as $b \to 0$;

(ii) $u'^1_b \to U_{a_1,0}$, $u'^2_b \to U_{a_2,0}$ strongly in $H^1(\mathbb{R}^N)$ as $b \to 0$.

For the special case $N = 2, 3$ and $p = 2$, it was proved in [13] that positive solutions of (1.1) with $a_i(x) \equiv a_i > 0$ are unique when $b > 0$ is small enough. Hence, when $b > 0$ small enough and $a_i(x) \equiv a_i > 0$, the solution we construct here is exactly that obtained in [2]. Probably, the positive solutions should be unique when $p > 2$ and $b > 0$ is small enough. However, when $1 < p < 2$, the following result and Theorem 1.2 imply that system (1.1) has at least two positive solutions when $b > 0$ is close to 0, which is very surprising.

Theorem 1.4. *Let $N \geq 1$. Assume that $2p \in (2, \min\{2^*, 4\})$, $a_i(x) = a_i(|x|) > 0$ $(i = 1, 2)$ satisfy $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. Then, for $b > 0$, system (1.1) has a positive least energy solution $\hat{w}_b = (\hat{u}_b^1, \hat{u}_b^2) \in \mathcal{H}$, which satisfies that*

(i) $\hat{u}_b^1 \to \hat{u}_*^1$, $\hat{u}_b^2 \to \hat{u}_*^2$ uniformly on every compact set of $\mathbb{R}^N$ as $b \to 0$;

(ii) $\hat{u}_*^1 \to \hat{u}_*^1$, $\hat{u}_b^2 \to \hat{u}_*^2$ strongly in $H^1(\mathbb{R}^N)$ as $b \to 0$,

where either $\hat{u}_*^1 \equiv 0$ or $\hat{u}_*^2 \equiv 0$, *and if* $\hat{u}_*^i \not\equiv 0$, *then* $\hat{u}_*^i > 0$ *satisfies* $-\Delta \hat{u}_*^i + a_i(x)\hat{u}_*^i = (\hat{u}_*^i)^{2p-1}$, *and* $L_{a_i(x),0}(\hat{u}_*^i) = \min\{C_{a_1,0}, C_{a_2,0}\}$. *Moreover, the ground state solution of* (1.1) *is nontrivial if and only if* $b > 0$.

Remark 1.5. By minimizing over Nehari-manifold, the existence part has been proved in [9]. We give a new proof here, from which we can obtain the energy and the asymptotic behavior of the positive least energy solution that are different from [9].

Now we come to the full repulsive case. We require that $a_i(x)$ satisfies the following two additional assumptions:

$$(\mathcal{A}_4)\quad \limsup_{\theta\to+\infty}\sup_{y\in\mathbb{R}^N} \frac{|y|e^{-p\min\{\sqrt{a_1(0)},\ \sqrt{a_2^\infty}\}\theta|y|}}{a_2^\infty - a_2(\theta|y|/2)} = 0,$$

and

$$(\mathcal{A}_5)\quad \mathcal{E}(a_1^\infty, 0) - \mathcal{E}(a_2^\infty, 0) \ge C_{a_1,0} - C_{a_2,0} \ge 0 \quad \text{and} \quad \left(\frac{C_{a_1,0}}{\mathcal{E}(a_2^\infty, 0)}, 1 + \frac{C_{a_1,0} - C_{a_2,0}}{\mathcal{E}(a_2^\infty, 0)}\right) \cap \mathbb{N} = \varnothing,$$

where $a_i^\infty = \lim_{|x|\to\infty} a_i(x)$.

Theorem 1.6. *Let* $N \ge 1$. *Assume that* $b < 0$, $2p \in (2, 2^*)$, $a_i(x) = a_i(|x|) > 0$ $(i = 1, 2)$ *satisfy* $(\mathcal{A}_1)$–$(\mathcal{A}_5)$. *Then, system* (1.1) *has a positive solution* $\bar{w}_b = (\bar{u}_b^1, \bar{u}_b^2) \in \mathcal{H}$. *Moreover, the following segregated properties hold*:

(*i*) $\limsup_{b\to-\infty} |\bar{u}_b^1 \bar{u}_b^2|_p^p = 0$.

(*ii*) *There exist* $\bar{u}_*^1, \bar{u}_*^2 \in H^1(\mathbb{R}^N)$ *with* $u_*^1 u_*^2 \equiv 0$ *on* $\mathbb{R}^N$ *such that* $\bar{u}_b^i \to \bar{u}_*^i$ *strongly in* $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

(*iii*) *For every compact* $K \subset\subset \mathbb{R}^N$, $b\int_K |u_b^1|^p |u_b^2|^p \to 0$ *as* $b \to -\infty$.

(*iv*) $(\bar{u}_b^1)^{p-1}(\bar{u}_b^1)^p + (\bar{u}_b^2)^{p-1}(\bar{u}_b^1)^p \to 0$ *on each compact set of* $\mathbb{R}^N$ *as* $b \to -\infty$. *In particular, letting* $x_b^i \in \mathbb{R}^N$ *satisfying* $\bar{u}_b^i(x_b^i) = \max_{\mathbb{R}^N} \bar{u}_b^i(x)$, *then* $\inf_{-\infty<b<0} \bar{u}_b^i(x_b^i) > 0$ *and* $\limsup_{b\to-\infty} \bar{u}_b^i(x_b^j) = 0$, *for* $i \ne j$.

Remark 1.7. Item (*i*) follows easily from the fact that $\sup_{b<0}\int_{\mathbb{R}^N} |\bar{u}_b^1|^p |\bar{u}_b^2|^p < +\infty$, and items (*ii*)–(*iii*) follow from [32]. As far as we know, all the known results on the existence of nontrivial solutions to system (1.1) with $b < 0$ are for the case $p \ge 2$ $(N \le 3)$, see [18,27,29] and the references therein. As a supplement, Theorems 1.4 and 1.6 assert the existence of positive solutions for (1.1) when $1 < p < \min\{2, 2^*/2\}$.

Remark 1.8. The uniqueness result in Proposition 1.1 is the key for the construction of nontrivial solutions w_b' of Theorem 1.3. Assumption $(\mathcal{A}_4)$ is used to construct the special mountain-pass geometry that we use to find a (P.S.) sequence. $(\mathcal{A}_5)$ is used to ensure the compactness of the (P.S.) sequence.

We want to emphasize that a large class of functions $a_i(x)$, $i = 1, 2$, satisfy $(\mathcal{A}_1)$–$(\mathcal{A}_5)$. First, noting that if $f(t) \in C^2((0, +\infty), \mathbb{R})$ satisfies

$$f, f', f'' \in L^\infty((0, +\infty)) \tag{1.4}$$

then by [12], we know that $f(t) + m$ satisfies $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ for sufficiently large $m > 1$. Now, we choose $a(x)$ as a C^2 function satisfying $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ and

$$\limsup_{\theta\to+\infty}\sup_{y\in\mathbb{R}^N} \frac{|y|e^{-p\min\{\sqrt{la(0)},\ \sqrt{a^\infty}\}\theta|y|}}{a^\infty - a(\theta|y|/2)} = 0, \tag{1.5}$$

where $l \ge 1$ is an arbitrary constant such that

$$\frac{C_{la,0}}{\mathcal{E}(a^\infty, 0)} = k_l \in \mathbb{N}. \tag{1.6}$$

Noting that all functions with polynomial decay or exponent decay but with decay rate being less than $e^{-p\min\{\sqrt{la(0)},\ \sqrt{a^\infty}\}|y|}$ satisfy the equation (1.5). With (1.5) and (1.6) at hand, we let

$$a_2(x) = a(x) \quad \text{and} \quad a_1(x) = la(x) \quad \forall x \in \mathbb{R}^N.$$

Let $U(x)$ be the ground solution of the following equation:

$$-\Delta U + a(l^{-1/2}x)U = |U|^{2p-2}U, \quad x \in \mathbb{R}^N,$$

and since $\hat{U}(y) = l^{\frac{1}{2p-2}}U(\sqrt{l}y)$ solves

$$-\Delta \hat{U} + la(x)\hat{U} = |\hat{U}|^{2p-2}\hat{U}, \quad x \in \mathbb{R}^N$$

and $a(l^{-1/2}t) \leq a(t)$ for $t \geq 0$, we have

$$\begin{aligned} C_{a_1,0} &= C_{la,0} \\ &\leq \max_{t>0}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\hat{U}|^2 + la(x)|\hat{U}|^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}|\hat{U}|^{2p}\right) \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\mathbb{R}^N}|\hat{U}|^{2p} = l^{\sigma_p}C_{a(l^{-1/2}\cdot),0} \leq l^{\sigma_p}C_{a,0}. \end{aligned}$$

Then, by the aforementioned analysis, we find

$$\limsup_{\theta\to+\infty}\sup_{y\in\mathbb{R}^N}\frac{|y|e^{-p\min\{\sqrt{a_1(0)},\ \sqrt{a_2^\infty}\}\theta|y|}}{a_2^\infty - a_2(\theta|y|/2)} = \limsup_{\theta\to+\infty}\sup_{y\in\mathbb{R}^N}\frac{|y|e^{-p\min\{\sqrt{la(0)},\ \sqrt{a^\infty}\}\theta|y|}}{a^\infty - a(\theta|y|/2)} = 0,$$

$$\mathcal{E}(a_1^\infty, 0) - \mathcal{E}(a_2^\infty, 0) = \mathcal{E}(la^\infty, 0) - \mathcal{E}(a^\infty, 0) = (l^{\sigma_p} - 1)\mathcal{E}(a^\infty, 0) > \big(l^{\sigma_p} - 1\big)C_{a,0} \geq C_{a_1,0} - C_{a_2,0}$$

and

$$\left(\frac{C_{a_1,0}}{\mathcal{E}(a^\infty, 0)}, 1 + \frac{C_{a_1,0} - C_{a_2,0}}{\mathcal{E}(a^\infty, 0)}\right) \cap \mathbb{N} = \left(k_l, 1 + k_l - \frac{C_{a,0}}{\mathcal{E}(a^\infty, 0)}\right) \cap \mathbb{N} = \varnothing,$$

which imply $a_i(x)$, $i = 1, 2$, also satisfy the assumptions $(\mathcal{A}_4)$ and $(\mathcal{A}_5)$.

Finally, let functions $a(x)$ and $a_2(x)$ and the constant l as mentioned earlier, since

$$1 - \frac{C_{a,0}}{\mathcal{E}(a^\infty, 0)} \in (0, 1),$$

it is easy to check by continuity that the redefined function $a_1(x) = la(\beta x)$ with $\beta > 1$ is a small perturbation of 1 still satisfies the assumptions $(\mathcal{A}_4)$ and $(\mathcal{A}_5)$.

We would like to point out that to prove Theorems 1.3 and 1.6, the argument used in [33], the two nontrivial Nehari constraint argument in [29], and the L^2-constraint method in [2,3] do not work here since p and $a_i(x)$ are more general. Actually, in [33], $a_i(x)$ ($i = 1, 2$) should be positive constants and b is negative enough. In [2,3], the argument relies heavily on $N = 3, 4$ and $p \geq 1 + \frac{2}{N}$, and it is impossible to regard $a_i(x)$ as a Lagrange multiplier when $a_1(x)$ or $a_2(x)$ is not a constant. In [29], the cubic assumption $p + 1 = 3$ is necessary in obtaining a nondegenerate 2×2 matrix, see also [9,35]. We should also mention here that when $1 < p < 2$ or $\lim_{|x|\to\infty}(a_1(x) - a_2(x)) \neq 0$, the finite-dimensional reduction method in [26,27] does not work well either since when $1 < p < 2$, it is not clear that positive solutions of the corresponding limit system are nondegenerate.

In the present paper, we will prove Theorem 1.4 by estimating the functional energy corresponding to the mountain pass solutions. Theorems 1.2, 1.3, and 1.6 will be proved by a two-dimensional mountain-pass theorem under the sketch of the variational method. Hence, we should construct a suitable two-dimensional mountain-pass geometry. To this end, the most important thing is to construct the separated assumption (see Theorem 2.8 in [36], for example). For the attractive and partial repulsive case, we construct the mountain-pass geometry $\Gamma_{a_1,a_2,b}$ by letting its boundary including the unique radially symmetric ground state of $-\Delta U + a_i U = U^{2p-1}$. About the full repulsive case $b < 0$, we need to choose the boundary of the mountain-pass geometry $\overline{\Gamma}_{a_1,a_2,b}$ more skillfully. We let its boundary consist of the two segregated functions, i.e., $U_{a_1,0}$ and $U^{y_b}_{a_2^\infty,0}(\cdot) = U_{a_2^\infty,0}(\cdot + y_b)$, where $y_b \in \mathbb{R}^N$ is chosen to make $|b|\int_{\mathbb{R}^N} U_{a_1,0}U^{y_b}_{a_2^\infty,0}$ be a small perturbation.

After finding a (P.S.) sequence $\{w_n = (u_n^1, u_n^2)\}$ from the two-dimensional mountain-pass geometry, to obtain a nontrivial solution, we need to verify that the weak limit (u^1, u^2) of w_n in $\mathcal{H}$ is nontrivial. To our best knowledge, except the finite-dimensional reduction method, which requires $p = 2$ and $\lim_{|x|\to\infty} a_1(x) = \lim_{|x|\to\infty} a_2(x)$, almost all positive solutions for (1.1) are established by assuming a special exponent p and potential $a_i(x)$ and finding minimizer or minimax points on a candidate critical points set that has some nontrivial restriction [2,29]. Those constraints are always used to prove that the (P.S.) sequence $\{w_n\}$ is nontrivial, i.e., $\liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n^i|^{2p} > 0$ $(i = 1, 2)$. In the present paper, we need to check the following two properties:

$$(\mathcal{N}_1)\quad \liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n^i|^{2p} > 0,\ \ i = 1, 2;\quad (\mathcal{N}_2)\quad \liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n^1|^p|u_n^2|^p > 0.$$

To this end, we should have a precise estimate on the mountain-pass value, which is also obtained by our special construction of the mountain-pass geometry. When $-\hat{b}'_{a_1/a_2} < b < b'_{a_1/a_2}$, we can check that the mountain pass value is less than or equal to $\mathcal{E}(a_1, 0) + \mathcal{E}(a_2, 0)$. But for the full repulsive case, because of no radial restriction on the solutions, the problem becomes more difficult. We will use a type of global compactness result (see [6], for example), the special $y_b \in \mathbb{R}^N$, and the decay assumption of $a_i(x)$ $(i = 1, 2)$ (condition $(\mathcal{A}_4)$) to obtain lower and upper bounds of the mountain-pass value.

Finally, the asymptotic behavior of the nontrivial solutions can also be verified by employing the precise estimate on the mountain-pass value and the results in [25,32].

This paper is organized as follows. In Section 2, we prove Theorems 1.2, 1.3, and 1.4. The positive solutions asserted in Theorems 1.2 and 1.3 are constructed by using the minimax method ([36]) on a creative mountain pass geometry $\Gamma_{a_1,a_2,b}$ (see Theorem 2.7). Their asymptotic behavior is also studied in this section. In Section 3, we consider the full repulsive case. A more skillful mountain pass geometry $\overline{\Gamma}_{a_1,a_2,b}$ is constructed, from which we obtain a precise estimate on the mountain pass value $\overline{C}_{a_1,a_2,b}$ and then prove that the (P.S.) sequence $\{\bar{w}_n = (\bar{u}_n^1, \bar{u}_n^2)\}$ is nontrivial when $n \to \infty$. We show that $\liminf_{n\to\infty}|\bar{u}_n^1\bar{u}_n^2|_p^p > 0$ (see Lemma 3.5), from which we can obtain a positive solution $\bar{w}_b$. Partial segregated properties of $\bar{w}_b$ are discussed at the end of Section 3, which are typical examples of [32].

2 Proofs of Theorems 1.2, 1.3, and 1.4

In this section, we first give a short proof to Theorem 1.4. We prove Theorem 1.3 rather than Theorem 1.2 since it is more general. For the sake of simplicity, we divide his section into three subsections. In subsection 2.1, we prove Theorem 1.4 by a one-dimensional mountain pass Lemma and energy analysis argument. In subsection 2.2, we construct a suitable mountain pass geometry $\Gamma_{a_1,a_2,b}$ on $[0, 1]^2$. We estimate the mountain pass value accurately by choosing skillfully the boundary elements of $\Gamma_{a_1,a_2,b}$ on $\partial[0, 1]^2$. Then, following the estimates and the critical symmetric principle [36], we prove the existence assertion and asymptotic behavior of Theorem 1.3 in subsection 2.3.

We emphasize that only the three assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ are needed in this section. For the sake of convenience, we assume without loss of generality that

$$C_{a_1,0} = \min\left\{C_{a_1,0}, C_{a_2,0}\right\}$$

in this section.

2.1 Proof of Theorem 1.4

Recalling the definition of $J_{a_1,a_2,b}$, we define

$$\widehat{C}_{a_1,a_2,b,p} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{a_1,a_2,b}(\gamma(t)), \tag{2.1}$$

where

$$\Gamma := \left\{\gamma \in C([0,1],\mathcal{H}) : \gamma(0) = 0, \quad J_{a_1,a_2,b}(\gamma(1)) < 0\right\}.$$

Obviously, $\widehat{C}_{a_1,a_2,b,p} \le C_{a_1,0}$. One can infer from Theorem 2.5 in [18] that

$$\widehat{C}_{a_1,a_2,b,p} = C_{a_1,0}, \quad \text{if } p \ge 2 \quad \text{and} \quad b \le 2^{p-1} - 1.$$

However, for $1 < p < 2$, we have

Lemma 2.1. *Let* $1 < p < 2$. *Then,* $\widehat{C}_{a_1,a_2,b,p} < C_{a_1,0}$. *Moreover,* $\widehat{C}_{a_1,a_2,b,p}$ *can be achieved by a positive radial solution if* $b > 0$.

Proof. Let $0 < \sigma < 1$ be a positive constant and $w_\sigma = (U_{a_1,0}, \sigma U_{a_1,0})$, where $U_{a_1,0}$ is given in Proposition 1.1. The function $f(t) = J_{a_1,a_2,b}(tw_\sigma)$ $(t \in (0,+\infty))$ will take the maximum at a unique $t^*_{b,\sigma} > 0$ with

$$t^*_{b,\sigma} := \left(\frac{1 + \sigma^2 + \sigma^2\int_{\mathbb{R}^N}(a_2(x) - a_1(x))|U_{a_1,0}|^2\mathrm{d}x \,/\, \|U_{a_1,0}\|^2_{a_1}}{1 + \sigma^{2p} + 2b\sigma^p}\right)^{\frac{1}{2p-2}} > \left(\frac{1}{2+2b}\right)^{\frac{1}{2p-2}}. \tag{2.2}$$

Then, we have

$$\begin{aligned}\widehat{C}_{a_1,a_2,b,p} &\le f(t^*_{b,\sigma})\\ &\le C_{a_1,0} + (t^*_{b,\sigma})^2\left(\frac{\sigma^2}{2}\frac{\|U_{a_1,0}\|^2_{a_2}}{\|U_{a_1,0}\|^2_{a_1}} - \frac{(t^*_{b,\sigma})^{2p-2}\sigma^{2p}}{2p} - \frac{b(t^*_{b,\sigma})^{2p-2}\sigma^p}{p}\right)\|U_{a_1,0}\|^2_{a_1}\\ &< C_{a_1,0},\end{aligned} \tag{2.3}$$

when letting

$$0 < \sigma \le \left(\frac{b}{p(1+b)}\frac{\|U_{a_1,0}\|^2_{a_1}}{\|U_{a_1,0}\|^2_{a_2}}\right)^{\frac{1}{2-p}}.$$

Using the compact embedding $\mathcal{H}_r \hookrightarrow L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, we deduce that $\widehat{C}_{a_1,a_2,b,p}$ can be achieved by a nonnegative function $\hat{w}_b = (\hat{u}^1_b, \hat{u}^2_b) \in \mathcal{H}_r$, which is a solution to (1.1). Observing that if $\hat{w}_b = (\hat{u}^1_b, \hat{u}^2_b)$ is semitrivial, then it must hold

$$\widehat{C}_{a_1,a_2,b,p} = C_{a_1,0},$$

which contradicts with (2.3). As a result, $\hat{w}_b = (\hat{u}^1_b, \hat{u}^2_b)$ is nontrivial and the maximum principle in [11] concludes that $\hat{w}_b = (\hat{u}^1_b, \hat{u}^2_b)$ is positive. □

Lemma 2.2. *Let* $b \le 0$. *Then, if* w *is a nontrivial solution of* (1.1), *it holds*

$$J_{a_1,a_2,b}(w) \ge C_{a_1,0} + C_{a_2,0}.$$

Proof. Since $w = (u_1, u_2)$ is a solution of (1.1), the function $J_{a_1,a_2,b}(tu_1, su_2) : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ satisfies

$$
\begin{aligned}
&\max_{t,s>0} J_{a_1,a_2,b}(tu_1, su_2)\\
&= \max_{t,s>0}\left(\left(\frac{t^2}{2}-\frac{t^{2p}}{2p}\right)\|u_1\|_{a_1}^2+\left(\frac{s^2}{2}-\frac{s^{2p}}{2p}\right)\|u_1\|_{a_2}^2+\frac{1}{2p}(t^p-s^p)^2 b|u_1u_2|_p^p\right)\\
&\le \max_{t,s>0}\left(\left(\frac{t^2}{2}-\frac{t^{2p}}{2p}\right)\|u_1\|_{a_1}^2+\left(\frac{s^2}{2}-\frac{s^{2p}}{2p}\right)\|u_1\|_{a_2}^2\right)\\
&= J_{a_1,a_2,b}(w).
\end{aligned}
$$

Then, by the definition of $C_{a,0}$ in Proposition 1.1, we conclude that

$$J_{a_1,a_2,b}(w) \ge \max_{t,s>0} J_{a_1,a_2,b}(tu_1, su_2) > \max_{t,s>0}(L_{a_1,0}(tu_1) + L_{a_2,0}(su_2)) \ge C_{a_1,0} + C_{a_2,0}.$$

This completes the proof. □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. The existence of the solutions can be derived from Lemma 2.1. From Lemma 2.2, we see that the ground state solution of (1.1) is nontrivial if and only if $b > 0$.

It remains to show the asymptotic behavior. Since $\lim_{n\to\infty} J'_{a_1,a_2,b}(\hat{w}_b) = 0$ in $\mathcal{H}'$, we have

$$\|\hat{w}_b\|_{\mathcal{H}}^2 - |\hat{u}_b^1|_{2p}^{2p} - |\hat{u}_b^2|_{2p}^{2p} - 2b|\hat{u}_b^1\hat{u}_b^2|_p^p = 0.$$

It follows that

$$\frac{1}{2p}\|\hat{w}_b\|_{\mathcal{H}}^2 = \frac{1}{2p}(|\hat{u}_b^1|_{2p}^{2p} + |\hat{u}_b^2|_{2p}^{2p}) + \frac{b}{p}|\hat{u}_b^1\hat{u}_b^2|_p^p. \tag{2.4}$$

Considering the fact that

$$\frac{1}{2}\|\hat{w}_b^1\|_{\mathcal{H}}^2 - \frac{1}{2p}(|\hat{u}_b^1|_{2p}^{2p} + |\hat{u}_b^2|_{2p}^{2p}) - \frac{b}{p}|\hat{u}_b^1\hat{u}_b^2|_p^p = \widehat{C}_{a_1,a_2,b,p} < C_{a_1,0}, \tag{2.5}$$

we conclude that $\sup_{b<1}\|\hat{w}_b\|_{\mathcal{H}} < +\infty$, which and the radial symmetry of $\hat{w}_b$ imply $\hat{w}_b \to \hat{w}_* = (\hat{u}_*^1, \hat{u}_*^2) \in \mathcal{H}$ strongly in $L^{2p}(\mathbb{R}^N)$ as $b \to 0^+$. As a result, we have $\hat{w}_b \to \hat{w}_*$ strongly in $\mathcal{H}$. If $\hat{u}_*^1, \hat{u}_*^2 \not\equiv 0$, then by the definitions of $C_{a_i,0}$, we deduce

$$J_{a_1,a_2,0}(\hat{w}_*) \ge C_{a_1,0} + C_{a_2,0},$$

which is a contradiction to Lemma 2.1. Consequently, it must hold that either $\hat{u}_*^1 \equiv 0$ or $\hat{u}_*^2 \equiv 0$. Moreover, letting $(tTU_{a_1,0}, 0)$ with $T > 0$ large be a special path in Γ, by the strong convergence of $\hat{w}_b$ in $\mathcal{H}$, we have

$$J_{a_1,a_2,0}(\hat{w}_*) \le C_{a_1,0},$$

which implies that $L_{a_1,0}(\hat{u}_*^1) = C_{a_1,0}$ or $L_{a_2,0}(\hat{u}_*^2) = C_{a_1,0}$.

Finally, we can check by the same argument as that in [11] that

$$\hat{w}_b \to \hat{w}_* \quad \text{uniformly on every compact subset of} \quad \mathbb{R}^N.$$

As a result, we complete the proof of Theorem 1.4. □

Remark 2.3. If $\{x \in \mathbb{R}^N : a_2(x) - a_1(x) > 0\} \ne \emptyset$, then since $C_{a_2,0} > C_{a_1,0}$, it must hold $\hat{u}_2^* \equiv 0$.

In the following, we give the proof of Theorem 1.3.

2.2 The mountain-pass geometry $\Gamma_{a_1,a_2,b}$

Definition 2.4. We say that a continuous path $\gamma : [0,1]^2 \to \mathcal{H}_r$ belongs to $\Gamma_{a_1,a_2,b}$ if

$$\gamma(\tau) = \left(tT_\delta U_{a_1,b^+}, sT_\delta U_{a_2,b^+},\right), \quad \forall \tau = (t,s) \in \partial[0,1]^2,$$

where $T_\delta = p^{\frac{1}{2p-2}} + \delta$ with that $\delta > 0$ is a small parameter. Note that $L_{a_i,b^+}(T_\delta U_{a_i,b^+}) < 0$, $i = 1, 2$.

We define the mountain pass value of $J_{a_1,a_2,b}(w)$ corresponding to $\Gamma_{a_1,a_2,b}$ as follows:

$$C_{a_1,a_2,b} := = \inf_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)).$$

For simplicity, we denote

$$C_{a,b^+} = C_{a,0}, \quad \text{if } a > 0 \quad \text{and} \quad b \le 0,$$

where C_{a,b^+} is given in Proposition 1.1.

Lemma 2.5. *Let*

$$b'_{a_1/a_2} := \left(\frac{C_{a_1,0} + C_{a_2,0}}{C_{a_2,0}}\right)^{p-1} - 1 \tag{2.6}$$

and $\hat{b}'_{a_1/a_2}$ *be the positive constant such that*

$$\frac{\hat{b}'_{a_1/a_2} T_\delta^{2p}}{p} |U_{a_1,0}U_{a_2,0}|_p^p = C_{a_1,0}. \tag{2.7}$$

If $0 < b < b'_{a_1/a_2}$, *then it holds*

$$C_{a_1,a_2,b} > C_{a_2,0} \ge \sup_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)). \tag{2.8}$$

Moreover,

$$C_{a_1,a_2,b} \le C^*(b), \tag{2.9}$$

where

$$C^*(b) := \max_{t,s\in[0,1]^2} \left(L_{a_1,0}\left(tT_\delta U_{a_1,b}\right) + L_{a_2,0}\left(sT_\delta U_{a_2,b}\right) - \frac{bt^p s^p T_\delta^{2p}}{p} \int_{\mathbb{R}^N} |U_{a_1,b}U_{a_2,b}|^p \right).$$

If $-\hat{b}'_{a_1/a_2} < b < 0$, *then we have*

$$C_{a_1,0} + C_{a_2,0} \le C_{a_1,a_2,b} \le C_{a_1,0} + C_{a_2,0} + \frac{|b| T_\delta^{2p}}{p} |U_{a_1,0}U_{a_2,0}|_p^p \tag{2.10}$$

and

$$C_{a_1,a_2,b} > \sup_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)). \tag{2.11}$$

Proof. Case 1: $b > 0$.

Obviously, $\Gamma_{a_1,a_2,b}$ is not empty. By the choice of γ, we have

$$\sup_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)) \le \max_{i=1,2} C_{a_i,0}.$$

For each $\gamma \in \Gamma_{a_1,a_2,b}$, assuming that $\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau))$, by Hölder inequality, we have

$$J_{a_1,a_2,b}(\gamma(\tau)) \ge L_{a_1,b}(\gamma_1(\tau)) + L_{a_2,b}(\gamma_2(\tau)), \quad \forall \tau \in [0,1]^2.$$

Observing that for each continuous map $c : [0,1] \to [0,1]^2$ with $c(0) \in \{0\} \times [0,1]$ and $c(1) \in \{1\} \times [0,1]$, it holds

$$L_{a_1,b}(\gamma_1(c(0))) = 0 \quad \text{and} \quad L_{a_1,b}(\gamma_1(c(1))) < 0.$$

Hence, $\gamma_1(c(s)) \in \Gamma_{a_1,b}$, which implies

$$\max_{s\in[0,1]} L_{a_1,b}(\gamma_1(c(s))) \geq C_{a_1,b}.$$

Similarly, letting $c : [0,1] \to [0,1]^2$ be a continuous map with $c(0) \in [0,1] \times \{0\}$ and $c(1) \in [0,1] \times \{1\}$, it holds

$$\max_{t\in[0,1]} L_{a_1,b}(\gamma_2(c(t))) \geq C_{a_2,b}.$$

Now using the same argument as that of the proof of Proposition 3.4 in [10], we can find a $\hat{\tau} \in [0,1]^2$ such that

$$L_{a_1,b}(\gamma_1(\hat{\tau})) \geq C_{a_1,b}, \quad L_{a_2,b}(\gamma_2(\hat{\tau})) \geq C_{a_2,b}.$$

Thus, from (1.3), we have

$$\max_{\tau\in[0,1]^2} J(\gamma(\tau)) \geq C_{a_1,b} + C_{a_2,b} = \left(\frac{1}{1+b}\right)^{\frac{1}{p-1}} (C_{a_1,0} + C_{a_2,0}). \tag{2.12}$$

(2.8) follows by $b \in (0, b'_{a_1/a_2})$.

(2.9) also follows by choosing $\gamma(\tau) = (tT_\delta U_{a_1,b}, sT_\delta U_{a_1,b})$ as a special path.

Case 2: $b < 0$.

Noting that for each $\gamma = (\gamma_1, \gamma_2) \in \Gamma_{a_1,a_2,b}$, it holds that

$$J_{a_1,a_2,b}(\gamma(\tau)) \geq L_{a_1,0}(\gamma_1(\tau)) + L_{a_2,0}(\gamma_2(\tau)), \quad \forall \tau \in [0,1]^2.$$

Then, by the same argument mentioned earlier, we have

$$C_{a_1,a_2,b} \geq C_{a_1,0} + C_{a_2,0}. \tag{2.13}$$

On the boundary of $\partial[0,1]^2$, considering $|b| < \hat{b}'_{a_1/a_2}$, we have

$$\begin{aligned}
&\sup_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)) \\
&= \max\Bigg\{ \max_{t\in[0,1]}\left(L_{a_1,0}\big(tT_\delta U_{a_1,0}\big) + L_{a_2,0}\big(T_\delta U_{a_2,0}\big) + \frac{|b|t^p T_\delta^{2p}}{p} |U_{a_1,0}U_{a_2,0}|_p^p \right), \\
&\qquad \max_{s\in[0,1]}\left(L_{a_2,0}\big(sT_\delta U_{a_2,0}\big) + L_{a_1,0}\big(T_\delta U_{a_1,0}\big) + \frac{|b|s^p T_\delta^{2p}}{p} |U_{a_1,0}U_{a_2,0}|_p^p \right), \quad C_{a_1,0}, C_{a_2,0} \Bigg\} \\
&\leq C_{a_2,0} + \frac{|b|T_\delta^{2p}}{p} |U_{a_1,0}U_{a_2,0}|_p^p < C_{a_2,0} + C_{a_1,0},
\end{aligned}$$

which and (2.13) give (2.11).

The second part of (2.10) can be proved similarly. □

Remark 2.6.

(i) It is easy to check by (2.12) and (2.9) that $C_{a_1,a_2,b} \to C_{a_1,0} + C_{a_2,0}$ when $|b| \to 0$.

(ii) When $a_i(x) \equiv a_i (i = 1,2)$ are positive constants, then the constant b'_{a_1/a_2} in (2.6) satisfies

$$b'_{a_1/a_2} = \big(1 + \omega^{\sigma_p}\big)^{p-1} - 1,$$

which is the constant b_ω in Theorem 1.2.

(iii) The radial symmetry here when $b > 0$ is natural, since we can use symmetric-rearrangement as in [18].

2.3 Existence and asymptotic behavior of the solution

In this subsection, we use the estimates of $C_{a_1,a_2,b}$ in (2.9) and (2.10) to obtain a positive solution w_b.

Theorem 2.7. *Let* $-\hat{b}'_{a_1/a_2} < b < b'_{a_1/a_2}$. *Then, system* (1.1) *has a positive and radial solution* $w_b = (u_b^1, u_b^2)$.

Proof. It follows from Lemma 2.5 that there exists a $(P.S.)$ sequence $\{w_n = (u_n^1, u_n^2) : n \in \mathbb{N}\} \subset \mathcal{H}_r$ such that for every $-\hat{b}'_{a_1/a_2} < b < b'_{a_1/a_2}$, it holds

$$J_{a_1,a_2,b}(w_n) \to C_{a_1,a_2,b} \quad \text{and} \quad J'_{a_1,a_2,b}(w_n) \to 0 \quad \text{in } \mathcal{H}'. \tag{2.14}$$

Hence,

$$\|w_n\|_{\mathcal{H}}^2 - |u_n^1|_{2p}^{2p} - |u_n^2|_{2p}^{2p} - 2b|u_n^1 u_n^2|_p^p = o_n(1)\|w_n\|_{\mathcal{H}},$$

which implies

$$\frac{1}{2p}\|w_n\|_{\mathcal{H}}^2 + o_n(1)\|w_n\|_{\mathcal{H}} = \frac{1}{2p}(|u_n^1|_{2p}^{2p} + |u_n^2|_{2p}^{2p}) + \frac{b}{p}|u_n^1 u_n^2|_p^p. \tag{2.15}$$

Considering the fact that

$$\lim_{n\to\infty}\left(\frac{1}{2}\|w_n\|_{\mathcal{H}}^2 - \frac{1}{2p}(|u_n^1|_{2p}^{2p} + |u_n^2|_{2p}^{2p}) - \frac{b}{p}|u_n^1 u_n^2|_p^p\right) = C_{a_1,a_2,b} < +\infty, \tag{2.16}$$

we conclude that $\{w_n\}$ is bounded in $\mathcal{H}$. Noting that $\int_{\mathbb{R}^N}|\nabla|u||\mathrm{d}x \le \int_{\mathbb{R}^N}|\nabla u|\mathrm{d}x$ for all $u \in H^1(\mathbb{R}^N)$, we can assume that w_n is nonnegative.

Since $\{w_n\}$ is bounded, there exists $w_b = (u_b^1, u_b^2) \in \mathcal{H}_r$, which is nonnegative such that $w_n \rightharpoonup w_b$ weakly in $\mathcal{H}_r$ as $n \to \infty$. By the radial restriction, we have $w_n \to w_b$ strongly in $L^q(\mathbb{R}^N)$ for every $2 < q < 2^*$. But the strong convergence in turn implies that $w_n \to w_b$ strongly in $\mathcal{H}_r$. Hence, w_b satisfies

$$J_{a_1,a_2,b}(w_b) = C_{a_1,a_2,b} \tag{2.17}$$

and

$$\begin{cases} -\Delta u_b^1 + a_1 u_b^1 = |u_b^1|^{2p-2}u_b^1 + b|u_b^1|^{p-2}u_b^1|u_b^2|^p, & x \in \mathbb{R}^N, \\ -\Delta u_b^2 + a_2 u_b^2 = |u_b^2|^{2p-2}u_b^2 + b|u_b^2|^{p-2}u_b^2|u_b^1|^p, & x \in \mathbb{R}^N. \end{cases} \qquad (\mathcal{P}_{a_1,a_2,b})$$

Now suppose without loss of generality that $u_b^1 \equiv 0$. Noting (2.17), we see

$$-\Delta u_b^2 + a_2 u_b^2 = (u_b^2)^{2p-1} \quad \text{and} \quad L_{a_2,0}(u_2^b) = C_{a_1,a_2,b}. \tag{2.18}$$

By the strong maximum principle in [11], we have $u_b^2 > 0$. Hence, $u_b^2 = U_{a_2,0}$, which contradicts with (2.8) and (2.10). □

Remark 2.8. We can infer from Remark 2.6 and Lemma 2.1 that when $1 < p < 2$ and $0 < b < b'_{a_1/a_2}$, it holds

$$\widehat{C}_{a_1,a_2,b,p} < C_{a_1,a_2,b},$$

from which we assert that the two solutions in Theorems 1.3 and 1.4 are different and system (1.1) satisfying $1 < p < 2$ and $b > 0$ is close to 0 has at least two positive solutions.

We now prove the asymptotic behavior of w_b when $|b| \to 0$:

Theorem 2.9. *It holds*

(*i*) $u_b^1 \to U_{a_1,0}$, $u_b^2 \to U_{a_2,0}$ *uniformly on every compact set* $K \subset \mathbb{R}^N$, *as* $b \to 0$;

(*ii*) $u_b^1 \to U_{a_1,0}$, $u_b^2 \to U_{a_2,0}$ *strongly in* $H^1(\mathbb{R}^N)$, *as* $b \to 0$.

Proof. It is easy to check

$$\sup_{-\hat{b}'_{a_1/a_2} < b < b'_{a_1/a_2}} \|w_b\|_{\mathcal{H}} < +\infty,$$

from which we can assume that

$$w_b \rightharpoonup w = (u_0^1, u_0^2) \quad \text{in } \mathcal{H}_r$$

as $b \to 0$. Then, for each $\Phi \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, letting $b \to 0$ in $\langle J'_{a_1,a_2,b}(w_b), \Phi\rangle = 0$, we find that

$$\begin{cases} -\Delta u_0^1 + a_1 u_0^1 = (u_0^1)^{2p-1}, & \text{in } \mathbb{R}^N, \\ -\Delta u_0^2 + a_2 u_0^2 = (u_0^2)^{2p-1}, & \text{in } \mathbb{R}^N. \end{cases}$$

By using the standard Moser iteration and the standard elliptic regularity argument [11] on system $(\mathcal{P}_{a_1,a_2,b})$, we have

$$\sup_{|b|<\delta} \|w_b\|_{C^{2,\alpha}(\mathbb{R}^N)} < +\infty, \tag{2.19}$$

where $\delta > 0$ is a small parameter, $\alpha \in (0, 1)$.

On the other hand, by the uniqueness of $U_{a_i,0}$ and the fact that $C_{a_1,a_2,b} \to \mathcal{E}(a_1, 0) + \mathcal{E}(a_2, 0)$ as $b \to 0$, we see

$$u_0^i = U_{a_i,0}, \ \ i = 1, 2,$$

which combining with (2.19) completes the proof. □

Remark 2.10. When $|b| > 0$ is small enough, we can regard the solution w_b as a synchronized solution.

3 Proof of Theorem 1.6

In this section, we will prove Theorem 1.6. Note that the mountain-pass geometry in Definition 2.4 is not suitable when $b < 0$ is negative enough. Indeed, direct analysis shows that there exists a constant $\hat{b}^*_{a_1,a_2} > 0$ such that

$$\sup_{\gamma\in\Gamma_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)) \geq \mathcal{E}(a_1, 0) + \mathcal{E}(a_2, 0) \ \text{ if } \ b \leq -\hat{b}^*_{a_1,a_2}. \tag{3.1}$$

Hence, when $|b|$ is large, the separate condition in (2.8) does not hold. To handle with all the repulsive cases, we need to construct a more delicate mountain-pass geometry.

For the repulsive effect, we do not add radial restriction hereafter. Hence, the compactness of (P.S.) sequence $\{\bar{w}_n = (\bar{u}_n^1, \bar{u}_n^2)\} \subset \mathcal{H}$ will lose. In general, it is easy to obtain a relative compact (P.S.) sequence for a single equation with the similar potential $a_i(x)$. However, it is hard to check that the relative compactness is still true for a system. Hence, it is not easy to check that the (P.S.) sequence is nontrivial. The global compactness result in [6] tells us that if $\lim_{n\to\infty}|\bar{u}_n^i|_{2p}^{2p} = 0$, then the energy $\overline{C}_{a_1,a_2,b}$ (see (3.2)) will be spilt into several parts. We will try to prove that this dichotomy phenomenon does not occur, which in turn needs more accurate estimates of $\overline{C}_{a_1,a_2,b}$. As one can see in Lemma 3.2, the better boundary condition of $\overline{\Gamma}_{a_1,a_2,b}$ on $\partial[0, 1]^2$ is, the better estimates for $\overline{C}_{a_1,a_2,b}$ will be obtained. Hence, we need to choose the boundary element skillfully.

After showing that $\lim_{n\to\infty}|\bar{u}_n^i|_{2p}^{2p} > 0$, $(i = 1, 2)$, we will argue by contradiction and complicated analysis to verify that $\lim_{n\to\infty}|\bar{u}_n^1\bar{u}_n^2|_p^p > 0$, which is one of the difficult parts of this paper.

3.1 The mountain-pass geometry $\overline{\Gamma}_{a_1,a_2,b}$

Definition 3.1. We say that a continuous path $\gamma : [0, 1]^2 \to \mathcal{H}$ belongs to $\overline{\Gamma}_{a_1,a_2,b}$ if,

$$\gamma(\tau) = \left(tT_\delta U_{a_1,0}, sT_\delta U^{y_b}_{a_2^\infty,0}\right), \quad \forall \tau = (t,s) \in \partial[0,1]^2,$$

where $y_b = f(b)\alpha$ for some $\alpha \in \partial B_1(0)$, $f : (-\infty, 0) \to \mathbb{R}^+$ is a function that will be determined later, T_δ, $U_{a_1,0}$ and $U_{a_2^\infty,0}$ are the same as mentioned earlier.

Define

$$\bar{C}_{a_1,a_2,b} = \inf_{\gamma\in\bar{\Gamma}_{a_1,a_2,b}} \max_{\tau\in[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)). \tag{3.2}$$

We have

Lemma 3.2. *For $b < 0$, there exists a $f(b) > 0$ such that*

$$\bar{C}_{a_1,a_2,b} \ge C_{a_1,0} + C_{a_2,0} > \sup_{\gamma\in\bar{\Gamma}_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)). \tag{3.3}$$

Moreover, there holds

$$\bar{C}_{a_1,a_2,b} \le C_{a_1,0} + \mathcal{E}(a_2^\infty, 0) + F_b, \tag{3.4}$$

where $F_b < 0$ and $F_b \to 0$ as $b \to -\infty$, $C_{a_i,0}$ $(i = 1, 2)$ are given in Proposition 1.1.

Proof. On the boundary $\partial[0,1]^2$, by the exponential decay of $U_{a_1,0}$ and $U_{a_2^\infty,0}$, we have

$$\sup_{\gamma\in\bar{\Gamma}_{a_1,a_2,b}} J_{a_1,a_2,b}(\gamma) \le C_{a_1,0} + C|b|e^{-c|y_b|} = C_{a_1,0} + C|b|e^{-cf(b)}$$

for some $c > 0$ independent of b. Since

$$\limsup_{\theta\to+\infty} \sup_{b\in\mathbb{R}^N} C|b|e^{-c\theta|b|} = 0,$$

there exists a constant $\theta^* > 0$ such that

$$C|b|e^{-c\theta^*|b|} < C_{a_2,0}$$

for $b \in (-\infty, 0)$. Hence, letting

$$f(b) = \theta^*|b|, \tag{3.5}$$

we obtain

$$C_{a_1,0} + C_{a_2,0} > \sup_{\gamma\in\bar{\Gamma}_{a_1,a_2,b}} \max_{\tau\in\partial[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)).$$

On the other hand, for any given $\gamma \in \bar{\Gamma}_{a_1,a_2,b}$, we can check

$$J_{a_1,a_2,b}(\gamma(\tau)) \ge L_{a_1,0}(\gamma_1(\tau)) + L_{a_2,0}(\gamma_2(\tau)).$$

By the similar argument as that in the proof of Lemma 2.5 and our choice of T_δ, there exists $\hat{\tau} \in [0,1]^2$ such that

$$\max_{\tau\in[0,1]^2} J_{a_1,a_2,b}(\gamma(\tau)) \ge L_{a_1,0}(\gamma_1(\hat{\tau})) + L_{a_2,0}(\gamma_2(\hat{\tau})) \ge C_{a_1,0} + C_{a_2,0}.$$

Then, (3.3) follows.

Now letting $\gamma(\tau) = (tT_\delta U_{a_1,0}, sT_\delta U^{y_b}_{a_2^\infty,0})$ be a special path, we have

$$\begin{aligned}
\bar{C}_{a_1,a_2,b} &\le \max_{t,s\in[0,T_\delta]} \left(L_{a_1,0}\left(tU_{a_1,0}\right) + L_{a_2,0}\left(sU^{y_b}_{a_2^\infty,0}\right) + \frac{|b|t^p s^p}{p} \int_{\mathbb{R}^N} |U_{a_1,0}U^{y_b}_{a_2^\infty,0}|^p \right) \\
&\le C_{a_1,0} + \frac{|b|T_\sigma^{2p}}{p} \int_{\mathbb{R}^N} |U_{a_1,0}U^{y_b}_{a_2^\infty,0}|^p + \max_{s\in[0,1]} L_{a_2,0}\left(sU^{y_b}_{a_2^\infty,0}\right) \\
&\le C_{a_1,0} + \mathcal{E}(a_2^\infty, 0) + \frac{|b|T_\sigma^{2p}}{p} \int_{\mathbb{R}^N} |U_{a_1,0}U^{y_b}_{a_2^\infty,0}|^p - c_0 \int_{\mathbb{R}^N} (a_2^\infty - a_2(x - y_b))\left(U_{a_2^\infty,0}\right)^2 \\
&\coloneqq C_{a_1,0} + \mathcal{E}(a_2^\infty, 0) + F_b,
\end{aligned}$$

where $c_0 > 0$ is a constant independent of b.

We estimate F_b next. Note that

$$\frac{|b|T_\sigma^{2p}}{p}\int_{\mathbb{R}^N}|U_{a_1,0}U^{y_b}_{a_2^\infty,0}|^p \leq C_1|b|e^{-p\min\{\sqrt{a_1(0)},\ \sqrt{a_2^\infty}\}|y_b|}.$$

Since

$$\int_{\mathbb{R}^N}(a_2^\infty - a_2(x-y_b))\big(U_{a_2^\infty,0}\big)^2dx \geq \int_{B_{|y_b|/2}(0)}(a_2^\infty - a_2(x-y_b))\big(U_{a_2^\infty,0}\big)^2 \geq C_2(a_2^\infty - a_2(y_b/2)),$$

letting the constant θ^* in (3.5) be large enough, we conclude by condition $(\mathcal{A}_4)$ that

$$\begin{aligned}F_b &\leq C_1|b|e^{-p\min\{\sqrt{a_1(0)},\sqrt{a_2^\infty}\}|y_b|} - C_2(a_2^\infty - a_2(y_b/2))\\ &= C_1|b|e^{-p\min\{\sqrt{a_1(0)},\ \sqrt{a_2^\infty}\}\theta^*|b|} - C_2(a_2^\infty - a_2(\theta^*|b|/2))\\ &< 0\end{aligned}\tag{3.6}$$

for $b \in (-\infty,\ 0)$. □

Remark 3.3. By Theorem 2.8 in [36] again, we can obtain a nonnegative sequence $\{\bar{w}_n = (\bar{u}_n^1, \bar{u}_n^2)\} \subset \mathcal{H}$ such that

$$\lim_{n\to\infty}J'_{a_1,a_2,b}(\bar{w}_n) = 0 \quad \text{in } \mathcal{H}' \quad \text{and} \quad \lim_{n\to\infty}J_{a_1,a_2,b}(\bar{w}_n) = \bar{C}_{a_1,a_2,b}.\tag{3.7}$$

Since in this section we do not add symmetric restriction, it is quite hard to obtain the following two properties for $\{\bar{w}_n : n \in \mathbb{N}\}$:

$$(i)\quad \liminf_{n\to\infty}|\bar{u}_n^i|_{2p} > 0\ (i = 1, 2);\quad (ii)\quad \liminf_{n\to\infty}|\bar{u}_n^1\bar{u}_n^2|_p > 0.$$

Thanks to the estimates of $\bar{C}_{a_1,a_2,b}$ in (3.3) and (3.4), we have the following lemma, which is essential for proving the existence assertion in Theorem 1.6.

Lemma 3.4. *We have*

$$\liminf_{n\to\infty}|\bar{u}_n^i|_{2p} > 0\ (i = 1, 2).$$

Proof. First, we suppose to the contrary that there exists a subsequence of $\{\bar{u}_n^2 : n \in \mathbb{N}\}$, still denoted by $\{\bar{u}_n^2 : n \in \mathbb{N}\}$, such that

$$\lim_{n\to\infty}|\bar{u}_n^2|_{2p} = 0.$$

Then, by the global compactness result in [6] (for its application in single equation, we refer the readers to [34]), there exists a $\bar{u}_1 \in H^1(\mathbb{R}^N)$, an integer $k \geq 0$ and functions $v_j^1 \in H^1(\mathbb{R}^N)$, $j = 1, \ldots, k$, such that

$$\bar{C}_{a_1,a_2,b} = L_{a_1,0}(\bar{u}_1) + \sum_{j=1}^k L_{a_1^\infty,0}(v_j^1),$$

where $\bar{u}_1 \not\equiv 0$ and v_j^1 are nonnegative solutions of $(\mathcal{P}_{a_1,0})$ and $(\mathcal{P}_{a_1^\infty,0})$, respectively.

By the strong maximum principle in [11], we conclude that $\bar{u}_1$ and v_j^1 $(j = 1, \ldots, k)$ are positive. Hence, it follows from the uniqueness of $U_{a_1^\infty,0}$ and $U_{a_1,0}$ that

$$\bar{C}_{a_1,a_2,b} = C_{a_1,0} + k\mathcal{E}(a_1^\infty, 0).\tag{3.8}$$

Combining with the lower and upper bounds of $\bar{C}_{a_1,a_2,b}$ in Lemma 3.2, we conclude that

$$C_{a_1,0} + C_{a_2,0} \leq C_{a_1,0} + k\mathcal{E}(a_1^\infty, 0) < C_{a_1,0} + \mathcal{E}(a_2^\infty, 0),\tag{3.9}$$

which contradicts with the condition $(\mathcal{A}_5)$.

Second, we assume that

$$\lim_{n\to\infty}|\bar{u}_n^1|_{2p} = 0.$$

Then, by the same reason mentioned earlier, we have

$$C_{a_1,0} + C_{a_2,0} < C_{a_2,0} + l\mathcal{E}(a_2^\infty, 0) < C_{a_1,0} + \mathcal{E}(a_2^\infty, 0),$$

i.e.,

$$\frac{C_{a_1,0}}{\mathcal{E}(a_2^\infty, 0)} < l < 1 + \frac{C_{a_1,0} - C_{a_2,0}}{\mathcal{E}(a_2^\infty, 0)}. \tag{3.10}$$

This is impossible by $(\mathcal{A}_5)$. The proof is complete. □

It is easy to check from the aforementioned lemma and Lemma 1.21 in [36] that there exists at least one $(x_n^i)_{n\in\mathbb{N}}$ $(i = 1, 2)$ such that

$$\liminf_{n\to\infty} \int_{B_1(x_n^i)} |\bar{u}_n^i|^{2p} > 0. \tag{3.11}$$

Accordingly, we define

$$\mathcal{R}_i = \inf\left\{R > 0 : \liminf_{n\to\infty} \int_{B_R(0)} |\bar{u}_n^i|^{2p}\mathrm{d}x > 0\right\}.$$

It is easy to see that $0 < \mathcal{R}_i \le +\infty$.

In the repulsive case $b < 0$, it may happen that

$$\liminf_{n\to\infty}|\bar{u}_n^1\bar{u}_n^2|_p^p = 0,$$

which makes it impossible to find nontrivial solutions to system (1.1) by letting $n \to \infty$ in $J'_{a_1,a_2,b}(\bar{w}_n)$. Note that if $\mathcal{R}_1 < +\infty$ and $\mathcal{R}_2 < +\infty$, we can prove easily by contradiction that the limit profile $\bar{w}_b = (\bar{u}_b^1, \bar{u}_b^2)$ will be nontrivial and $|\bar{u}_b^1\bar{u}_b^2|_p^p > 0$, see Lemma 3.6 later for details. Hence, what we need to do is to prove that either

$$\mathcal{R}_1 + \mathcal{R}_2 < +\infty, \tag{3.12}$$

or

$$\liminf_{n\to\infty}|\bar{u}_n^1\bar{u}_n^2|_p^p > 0. \tag{3.13}$$

Lemma 3.5. *If both* (3.12) *and* (3.13) *are false, then*

$$\overline{C}_{a_1,a_2,b} \ge C_{a_1,0} + \mathcal{E}(a_2^\infty, 0). \tag{3.14}$$

Proof. By (3.11), since $\mathcal{R}_1 + \mathcal{R}_2 = +\infty$, there exist $(x_n^1)_{n\in\mathbb{N}}$, $(x_n^2)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ with $\lim_{n\to\infty}(|x_n^1| + |x_n^2|) = +\infty$, such that

$$\liminf_{n\to\infty} \int_{B_1(x_n^i)} \bar{u}_n^i(x)\mathrm{d}x > 0.$$

First, assume that $\lim_{n\to\infty}|x_n^1| < +\infty$ and $\lim_{n\to\infty}|x_n^2| = +\infty$. Let $\tilde{u}_n^{ij}(x) = \bar{u}_n^i(x + x_n^j)$, $i, j = 1, 2$ and $\tilde{w}_n^j(x) = (\tilde{u}_n^{1j}(x), \tilde{u}_n^{2j}(x))$, $j = 1, 2$. By (3.4), we can check by using the similar argument in proving (2.15) and (2.16) that

$$\left(\frac{1}{2}-\frac{1}{2p}\right)\|\bar{w}_n\|_{\mathcal{H}}^2 + o_n(1)\|\bar{w}_n\|_{\mathcal{H}} = \overline{C}_{a_1,a_2,b} < C_{a_1,0} + \mathcal{E}(a_2^\infty, 0),$$

which means that $\sup_n\|\bar{w}_n\|_{\mathcal{H}} < +\infty$. Then, there exists $\tilde{w}^j = (\tilde{u}_{1j}, \tilde{u}_{2j})$ such that $\tilde{w}_n^j \rightharpoonup \tilde{w}^j$ weakly in $\mathcal{H}$.

We now estimate the energy of $J_{a_1,a_2,b}(\bar{w}_n)$. Letting $n \to \infty$ in the system satisfied by $\tilde{w}_n^j$ and considering that (3.13) is false, we have

$$\begin{cases}-\Delta\tilde{u}_{11} + a_1(x + x_*^1)\tilde{u}_{11} = |\tilde{u}_{11}|^{2p-2}\tilde{u}_{11}, & \text{in } \mathbb{R}^N,\\ -\Delta\tilde{u}_{21} + a_2(x + x_*^1)\tilde{u}_{21} = |\tilde{u}_{21}|^{2p-2}\tilde{u}_{21}, & \text{in } \mathbb{R}^N\end{cases} \tag{3.15}$$

and

$$\begin{cases}-\Delta\tilde{u}_{12} + a_1^\infty\tilde{u}_{12} = |\tilde{u}_{12}|^{2p-2}\tilde{u}_{12}, & \text{in } \mathbb{R}^N,\\ -\Delta\tilde{u}_{22} + a_2^\infty\tilde{u}_{22} = |\tilde{u}_{22}|^{2p-2}\tilde{u}_{22}, & \text{in } \mathbb{R}^N,\end{cases} \tag{3.16}$$

where we assume that $x_n^1 \to x_*^1$ as $n \to \infty$. It is easy to see that $\tilde{u}_{11}$ and $\tilde{u}_{22}$ are nontrivial. Hence, combining with (3.15) and (3.16) and the Sobolev embedding (see [36], for example), we have

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\sum_{i=1}^{2}\int\limits_{(B_R(x_n^1)\bigcup B_R(x_n^2))}\frac{|\nabla\bar{u}_n^i|^2 + a_i(x)|\bar{u}_n^i|^2}{2} - \frac{1}{2p}|\bar{u}_n^i|^{2p}\mathrm{d}x + \frac{|b|}{p}\int\limits_{(B_R(x_n^1)\bigcup B_R(x_n^2))}|\bar{u}_n^1|^p|\bar{u}_n^2|^p\mathrm{d}x\right)\\
&\ge \sum_{j=1}^{2}\int\limits_{\mathbb{R}^N}\frac{|\nabla\tilde{u}_{j1}|^2 + a_1(x + x_*^1)|\tilde{u}_{j1}|^2}{2} - \frac{1}{2p}|\tilde{u}_{j1}|^{2p}\mathrm{d}x + \sum_{j=1}^{2}\int\limits_{\mathbb{R}^N}\frac{|\nabla\tilde{u}_{j2}|^2 + a_2^\infty|\tilde{u}_{j2}|^2}{2} - \frac{1}{2p}|\tilde{u}_{j2}|^{2p}\mathrm{d}x + o_R(1)\\
&\ge \frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla\tilde{u}_{11}|^2 + a_1(x + x_*^1)|\tilde{u}_{11}|^2\mathrm{d}x - \frac{1}{2p}\int\limits_{\mathbb{R}^N}|\tilde{u}_{11}|^{2p}\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla\tilde{u}_{22}|^2 + a_2^\infty|\tilde{u}_{22}|^2\mathrm{d}x - \frac{1}{2p}\int\limits_{\mathbb{R}^N}|\tilde{u}_{22}|^{2p}\mathrm{d}x\\
&\quad + o_R(1)\\
&\ge \mathcal{E}(a_2^\infty, 0) + \frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla\tilde{u}_{11}^*|^2 + a_1(x)|\tilde{u}_{11}^*|^2\mathrm{d}x - \frac{1}{2p}\int\limits_{\mathbb{R}^N}|\tilde{u}_{11}^*|^{2p}\mathrm{d}x + o_R(1)\\
&\ge C_{a_1,0} + \mathcal{E}(a_2^\infty, 0) + o_R(1),
\end{aligned} \tag{3.17}$$

where we have used the fact that $\tilde{u}_{11}^*(\cdot) = \tilde{u}_{11}(\cdot - x_*^1)$ solves equation $-\Delta u + a_1(x)u = |u|^{2p-2}u$ in $\mathbb{R}^N$ by (3.15).

It remains to estimate the energy outside $B_R(x_n^2) \cup B_R(x_n^1)$. Let $\eta_R \in C^\infty(\mathbb{R}^N)$ satisfy $0 \le \eta_R \le 1$, $x \in \mathbb{R}^N$, $\eta_R = 0$ on B_R and $\eta_R = 1$ on $\mathbb{R}^N \backslash B_{2R}$. Define

$$\Psi_R(x) = \eta_R(x - x_n^2)\eta_R(x - x_n^1), \quad x \in \mathbb{R}^N.$$

Since

$$\langle J'_{a_1,a_2,b}(\bar{w}_n), \Psi_R\bar{w}_n\rangle = o_n(1)\|\Psi_R\bar{w}_n\|_{\mathcal{H}},$$

by Sobolev embedding and direct computation, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\sum_{i=1}^{2}\int\limits_{(B_R(x_n^2)\bigcup B_R(x_n^1))^c}\left(\frac{|\nabla\bar{u}_n^i|^2 + a_i(x)|\bar{u}_n^i|^2}{2} - \frac{1}{2p}|\bar{u}_n^i|^{2p}\right)\mathrm{d}x - \frac{b}{p}\int\limits_{(B_R(x_n^1)\bigcup B_R(x_n^2))^c}|\bar{u}_n^1|^p|\bar{u}_n^2|^p\mathrm{d}x\right)\\
&\ge -\limsup_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}|\nabla\Psi_R|(|\bar{u}_n^1\nabla\bar{u}_n^1| + |\bar{u}_n^2\nabla\bar{u}_n^2|)\mathrm{d}x\right.\\
&\quad\left. + \left(\frac{1}{2}-\frac{1}{2p}\right)\int\limits_{(B_{2R}(0)\backslash B_R(0))}(|\tilde{u}_n^{11}|^{2p} + |\tilde{u}_n^{21}|^{2p}) + (|\tilde{u}_n^{12}|^{2p} + |\tilde{u}_n^{22}|^{2p})\right) = o_R(1).
\end{aligned} \tag{3.18}$$

Letting $R \to \infty$, it follows from (3.17) and (3.18) that

$$\liminf_{n\to\infty} J_{a_1,a_2,b}(\bar{w}_n) \geq C_{a_1,0} + \mathcal{E}(a_2^\infty, 0). \tag{3.19}$$

Second, assume that $\lim_{n\to\infty}|x_n^1| = +\infty$ and $\lim_{n\to\infty}|x_n^2| < +\infty$. Proceeding as we have just done to prove (3.19) and using $(\mathcal{A}_5)$, we obtain

$$\liminf_{n\to\infty} J_{a_1,a_2,b}(\bar{w}_n) \geq \mathcal{E}(a_1^\infty, 0) + C_{a_2,0} \geq C_{a_1,0} + \mathcal{E}(a_2^\infty, 0). \tag{3.20}$$

Third, assume that $\lim_{n\to\infty}|x_n^1| = +\infty$ and $\lim_{n\to\infty}|x_n^2| = +\infty$. Similarly, we have

$$\liminf_{n\to\infty} J_{a_1,a_2,b}(\bar{w}_n) \geq \mathcal{E}(a_1^\infty, 0) + \mathcal{E}(a_2^\infty, 0) > C_{a_1,0} + \mathcal{E}(a_2^\infty, 0). \tag{3.21}$$

As a result, we complete the proof. □

We can see that (3.14) is a contradiction to the upper bound of $\overline{C}_{a_1,a_2,b}$ in (3.4). Hence, one of (3.12) and (3.13) must be true. If (3.13) is true, then we are done. But if (3.12) is true, we have

Lemma 3.6. *If*

$$\mathcal{R}_1 + \mathcal{R}_2 < +\infty,$$

then

$$\liminf_{n\to\infty} |\bar{u}_n^1 \bar{u}_n^2|_p^p > 0.$$

Proof. From (3.12), there exists a $R > 0$ such that

$$\liminf_{n\to\infty} \int\limits_{B_R(0)} |\bar{u}_n^i|^{2p} dx > 0, \quad i = 1, 2.$$

Then, going if necessary to a subsequence, we assume that $\bar{w}_n = (\bar{u}_n^1, \bar{u}_n^2) \rightharpoonup \bar{w} = (\bar{u}_1, \bar{u}_2)$ weakly in $\mathcal{H}$.

Letting $n \to \infty$ in the system satisfied by $\bar{w}_n$, we find

$$\begin{cases} -\Delta \bar{u}_1 + a_1(x)\bar{u}_1 = |\bar{u}_1|^{2p-2}\bar{u}_1 + b|\bar{u}_1|^{p-2}\bar{u}_1|\bar{u}_2|^p, & x \in \mathbb{R}^N, \\ -\Delta \bar{u}_2 + a_2(x)\bar{u}_2 = |\bar{u}_2|^{2p-2}\bar{u}_2 + b|\bar{u}_2|^{p-2}\bar{u}_2|\bar{u}_1|^p, & x \in \mathbb{R}^N. \end{cases}$$

Note that $\bar{u}_i \geq 0$ and $\bar{u}_i \not\equiv 0$. Suppose that $\bar{u}_1\bar{u}_2 = 0$. Then, we see

$$-\Delta \bar{u}_1 + a_1(x)\bar{u}_1 = (\bar{u}_1)^{2p-1} \text{in } \mathbb{R}^N. \tag{3.22}$$

It follows from the strong maximum principle that $\bar{u}_1 > 0$, which means that $\bar{u}_2 \equiv 0$ in $\mathbb{R}^N$. Hence, we obtain a contradiction. Therefore,

$$\bar{u}_1(x)\bar{u}_2(x) > 0, \quad \forall x \in \mathbb{R}^N. \tag{3.23}$$

The conclusion then follows from Fatou's lemma. □

3.2 Existence and asymptotic behavior in Theorem 1.6

Theorem 3.7. *Let $a_i(x)$ $(i = 1, 2)$ satisfy $(\mathcal{A}_1) - (\mathcal{A}_5)$. Then, system* (1.1) *has a positive solution $\bar{w}_b = (\bar{u}_b^1, \bar{u}_b^2)$ for $b < 0$.*

Proof. Let $\{\bar{w}_n : n \in \mathbb{N}\}$ be the sequence given in Remark 3.3. Proceeding as we have done in the proof of Lemma 3.5, we deduce that $\{\bar{w}_n\}$ is bounded in $\mathcal{H}$. Hence, there exists a $\bar{w}_b = (\bar{u}_b^1, \bar{u}_b^2) \in \mathcal{H}$ such that $\bar{w}_n \rightharpoonup \bar{w}_b$ weakly in $\mathcal{H}$. Letting $n \to \infty$ in $\langle J'_{a_1,a_2,b}(\bar{w}_n), \Psi\rangle$ for $\Psi = (\phi_1, \phi_2) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, we conclude that $\bar{w}_b$ solves system (1.1). Moreover, by Lemmas 3.4, 3.5, and 3.6, $\bar{w}_b$ is nontrivial. Hence, as we prove (3.23), we conclude that $\bar{w}_b$ is positive. □

We end this subsection by showing the asymptotic behavior of $\bar{w}_b$ as $b \to -\infty$.

Theorem 3.8. *Let $\bar{w}_b$ be the solution of system* (1.1) *found in* Theorem 3.7. *Then,*

(*i*) $\limsup_{b\to-\infty}|\bar{u}_b^1\bar{u}_b^2|_p^p = 0$.

(*ii*) *There exists $\bar{u}_*^1, \bar{u}_*^2 \in H^1(\mathbb{R}^N)$ with $u_*^1u_*^2 \equiv 0$ on $\mathbb{R}^N$ such that $\bar{u}_b^i \to \bar{u}_*^i$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.*

(*iii*) *For every compact $K \subset\subset \mathbb{R}^N$, it holds $b\int_K |u_b^1|^p|u_b^2|^p \to 0$ as $b \to -\infty$.*

(*iv*) $(\bar{u}_b^1)^{p-1}(\bar{u}_b^1)^p + (\bar{u}_b^2)^{p-1}(\bar{u}_b^1)^p \to 0$ *on each compact set of* $\mathbb{R}^N$ *as* $b \to -\infty$.

Proof. Items (*ii*) and (*iii*) follow from [32]. Since $\bar{w}_b$ solves system (1.1), we find

$$\overline{C}_{a_1,a_2,b} = \left(\frac{1}{2} - \frac{1}{2p}\right)\|\bar{w}_b\|^2_{\mathcal{H}} = \left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\mathbb{R}^N} (|\bar{u}_b^1|^{2p} + |\bar{u}_b^2|^{2p} + 2b|\bar{u}_b^1\bar{u}_b^2|^p).$$

Noting the estimate of $\overline{C}_{a_1,a_2,b}$ in Lemma 3.2, we have

$$\sup_{-\infty<b<0} \|\bar{w}_b\|^2_{\mathcal{H}} = \sup_{-\infty<b<0} \frac{2p}{p-1}\overline{C}_{a_1,a_2,b} \le \frac{2p}{p-1}(\mathcal{E}(a_1^\infty, 0) + \mathcal{E}(a_2^\infty, 0)).$$

Then, by Sobolev embedding theorem, we conclude that

$$\limsup_{n\to\infty}|b||\bar{u}_b^1\bar{u}_b^2|_p^p < +\infty,$$

which implies (*i*).

It is easy to see that $\bar{w}_b$ satisfies

$$\begin{cases} -\Delta\bar{u}_b^1 + a_1(x)\bar{u}_b^1 = (\bar{u}_b^1)^{2p-1} + b(\bar{u}_b^1)^{p-1}(\bar{u}_b^2)^p, & \text{in } \mathbb{R}^N, \\ -\Delta\bar{u}_b^2 + a_2(x)\bar{u}_b^2 = (\bar{u}_b^2)^{2p-1} + b(\bar{u}_b^2)^{p-1}(\bar{u}_b^1)^p, & \text{in } \mathbb{R}^N. \end{cases} \tag{3.24}$$

By the standard Moser iteration and the standard regularity argument in [11], we conclude that $\bar{u}_b^i \in C^{2,\alpha}(\mathbb{R}^N)$ and

$$\sup_{-\infty<b<0} \|\bar{u}_b^i\|_{C^{2,\alpha}(K)} < +\infty$$

for $i = 1, 2$ and any compact set $K \subset \mathbb{R}^N$. Then, (*ii*) follows by the fact that

$$[|\bar{u}_b^1|^{p-1}|\bar{u}_b^2|^p + |\bar{u}_b^2|^{p-1}|\bar{u}_b^1|^p] = \frac{1}{b}\sum_{i=1}^{2}(-\Delta\bar{u}_b^i + a_i(x)\bar{u}_b^i - (\bar{u}_b^i)^{2p-1}).$$

Now for $i = 1, 2$, we claim that

$$\inf_{-\infty<b<0} \|\bar{u}_b^i\|_{a_i} > 0.$$

Actually, from $b < 0$ and the Sobolev embedding, we can check

$$\|\bar{u}_b^i\|^2_{a_i} \le C\|\bar{u}_b^i\|^{2p}_{a_i}.$$

Moreover, by Hölder inequality, we see

$$\|\bar{u}_b^i\|^2_{a_i} \le \|\bar{u}_b^i\|^{2p-2}_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\bar{u}_b^i|^2 \mathrm{d}x.$$

Hence, it holds

$$\inf_{-\infty<b<0} \|\bar{u}_b^i\|_{L^\infty(\mathbb{R}^N)} \geq \sigma_0 > 0.$$

So there exists $x_b^i \in \mathbb{R}^N$, $i = 1, 2$ such that

$$\max_{\mathbb{R}^N} \bar{u}_b^i(x) = \bar{u}_b^i(x_b^i) \geq \sigma_0.$$

Therefore, using the fact that $\bar{u}_b^1\bar{u}_b^2 \to 0$ strongly in $L^p(\mathbb{R}^N)$ as $b \to -\infty$, we conclude (iv). □

Remark 3.9. (i) We conjecture that the solution $\bar{w}_b$ is nonradial when $|b|$ is large enough. In particular, observing the estimate of mountain pass value $\overline{C}_{a_1,a_2,b}$ in Lemma 3.2, we guess the following conclusions hold:

(1) $|x_b^1 - x_b^2| \to +\infty$;

(2) $\bar{u}_b^1(x + x_b^1) \to U_{a_1,0}$ and $\bar{u}_b^2(x + x_b^2) \to U_{a_2^\infty,0}$;

(3) $x_b^1 \to 0$, $x_b^2 \to +\infty$.

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Appendix A

Part of the proof for Proposition 1.1

In this section, we will prove the uniqueness assertion in Proposition 1.1 when $N = 1$.

Lemma A.1. *Assume that* $\delta > 0$ *and* $u_1(r), u_2(r) \in C^1([0,\delta),\mathbb{R}) \cap C^2((0,\delta),\mathbb{R})$ *are solutions of*

$$\begin{cases} u'' + u^{2p-1} - V(r)u = 0, & \text{in } (0,\delta), \\ u'(0) = 0, \quad u(r) > 0, & \text{in } (0,\delta). \end{cases}$$

Moreover, suppose

$$0 < u_1(r) < u_2(r) \quad \text{in } [0,\delta).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_1(r)}{u_2(r)}\right) > 0 \quad \text{in } (0,\delta).$$

Proof. Let $f(r) = u_1'(r)u_2(r) - u_1(r)u_2'(r)$. By direct computations, we have

$$f'(r) = u_1''(r)u_2(r) - u_2''(r)u_1(r) = u_1(r)u_2(r)(u_2^{2p-2}(r) - u_1^{2p-2}(r)) > 0 \quad \text{in } [0,\delta).$$

Hence, $f(r) > f(0) = 0$ in $(0,\delta)$. Then, the conclusion follows. □

In the sequel, we suppose that

$$\begin{cases} u'' + u^{2p-1} - V(r)u = 0, & \text{in } (0,+\infty), \\ u(r) > 0, & \text{in } [0,+\infty) \end{cases} \tag{A.1}$$

has two distinct positive even solutions $u_1(r)$, $u_2(r)$. Denote

$$\sharp\{r \in (0,+\infty);\ u_1(r) = u_2(r)\}$$

as the number of intersections of $u_1(r)$ and $u_1(r)$.

Then, we have the following lemma.

Lemma A.2. *Assume that equation* (A.1) *has two distinct positive even solutions* $u_1(r)$, $u_2(r)$ *such that* $u_1(0) < u_2(0)$. *Then, there exists a positive even solution* $u_3(r)$ *of* (A.1) *such that*

$$u_3(0) \geq u_2(0), \quad \sharp\{r \in (0,+\infty);\ u_1(r) = u_3(r)\} \leq 1. \tag{A.2}$$

Proof. We assume that $u_1(r)$, $u_2(r)$ satisfy

$$\sharp\{r \in (0,+\infty);\ u_1(r) = u_2(r)\} \geq 2.$$

If not, $u_3(r) \equiv u_2(r)$ is the desired solution. We set $\alpha_1 = u_1(0)$, $\alpha_2 = u_2(0)$ $(0 < \alpha_1 < \alpha_2)$. Now we study solutions $u(r;\alpha)$ of the equation for $\alpha \geq \alpha_2$:

$$\begin{cases} u'' + u^{2p-1} - V(r)u = 0, \\ u'(0) = 0, \quad u(0) = \alpha. \end{cases} \tag{A.3}$$

Let $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ be the first and second intersection points of $u(r;\alpha)$ and $u_1(r) = u(r;\alpha_1)$. $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ exist at least α close to α_2. We begin with $\alpha = \alpha_2$ and increase α progressively and track $\sigma_2(\alpha)$. We divide our argument into three steps.

Step 1: Existence of $\bar{\alpha} > \alpha_2$ such that

$$(i)\ u(r, \bar{\alpha}) \text{ hits zero at some } \quad r_0 \in (0, +\infty); \tag{A.4}$$

$$(ii)\ \sharp\{r \in (0, r_0);\ u(r; \bar{\alpha}) = u_1(r)\} = 1. \tag{A.5}$$

Step 2: Existence of $\alpha_3 \in (\alpha_2, \bar{\alpha})$ such that $\sigma_2(\alpha) \to \infty$ as $\alpha \to \alpha_3^-$.
Step 3: $u_3(r) = u(r, \alpha_3)$ has the desired property (A.2).

Step 1: Existence of $\bar{\alpha} > \alpha_2$ satisfying (4.4)–(4.5).

First, we note the following:

Claim 1. Assume $u(r; \alpha)$ is a solution to (A.3) and set

$$v(s, \alpha) = \alpha^{-1} u(\alpha^{-(p-1)} s;\ \alpha).$$

Then, $v(s; \alpha) \to w(s)$ in $C^1_{\mathrm{loc}}([0, +\infty), \mathbb{R})$ as $\alpha \to +\infty$, where $w(s)$ is a solution of

$$\begin{cases} w'' + w^{2p-1} = 0, \\ w'(0) = 0,\ \ w(0) = 1. \end{cases} \tag{A.6}$$

Proof of Claim 1. It is not difficult to see that $v(s) = v(s, \alpha)$ satisfies

$$\begin{cases} v_{ss} + v^{2p-1} - \alpha^{-(2p-2)} V(\alpha^{-(p-1)} s) v = 0, \\ v_s(0) = 0,\ \ v(0) = 1. \end{cases}$$

Thus, we can find that

$$v(s; \alpha) \to w(s) \quad \text{in} \quad C^1_{\mathrm{loc}}([0, +\infty), \mathbb{R}) \quad \text{as } \alpha \to +\infty,$$

where $w(s)$ is a solution of (A.6).

For the solution $w(s)$ of (A.6), we have

Claim 2. The solution $w(s)$ hits zero in finite time, that is, there is $s_0 \in (0, +\infty)$ such that

$$w(s_0) = 0, \quad w(s) > 0 \quad \text{in } (0, s_0). \tag{A.7}$$

Proof of Claim 2. It is well known that

$$\begin{cases} -u'' = u^{2p-1}, & |x| < 1, \\ u > 0, & |x| < 1, \\ u = 0, & |x| = 1, \end{cases}$$

has a positive even solution $u_0(x) = u_0(|x|)$ for $2p \in (2, +\infty)$. Then, we can see

$$w(s) = \alpha_0^{-1} u_0(\alpha_0^{-(p-1)/2} s), \quad \alpha_0 = u_0(0)$$

is the solution of (A.6) and $w(s)$ satisfies (A.7) with $s_0 = u_0(0)^{\frac{1}{p-1}}$.

Conclusion of Step 1. By Claims 1 and 2, we can see easily that (4.4)–(4.5) holds for sufficiently large $\bar{\alpha} > \alpha_2$.

Step 2: Existence of $\alpha_3 \in (\alpha_2, \bar{\alpha})$ such that $\sigma_2(\alpha) \to +\infty$ as $\alpha \to \alpha_3^-$.

Let $\alpha > \alpha_2$. Since

$$u_2'(\sigma_j(\alpha)) \neq u'(\sigma_j(\alpha);\ \alpha) \quad \text{for } j = 1, 2,$$

(by the uniqueness of solutions of the initial value problem at $r = \sigma_j(\alpha)$), we can see that $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ vary continuously as α moves.

Let $[\alpha_2, \alpha^*)$ be the maximal interval in which both of $\sigma_1(\alpha), \sigma_2(\alpha)$ exist.

We **claim** for $\alpha \in [\alpha_2, \alpha^*)$

$$u(r;\, \alpha) > 0 \quad \text{in } [0, \sigma_2(\alpha)]. \tag{A.8}$$

Proof of (A.8). First, we remark that (A.8) holds for $\alpha = \alpha_2$. By a contradiction argument, we assume that for some $\alpha' \in (\alpha_2, \alpha^*)$

$$u(r';\, \alpha') \le 0 \quad \text{for some } r' \in [0, \sigma_2(\alpha)).$$

Denote

$$\alpha_0 = \inf\Big\{\alpha \in [\alpha_2, \alpha^*);\ \min_{r\in[0,\sigma_2(\alpha)]} u(r, \alpha) \le 0\Big\}.$$

Then, $\alpha_0 \in (\alpha_2, \alpha']$ and $\min_{r\in[0,\sigma_2(\alpha)]}u(r;\, \alpha_0) = 0$. Thus, for some $r_0 \in [0, \sigma_2(\alpha_0)]$,

$$u(r_0;\, \alpha_0) = 0, \quad u'(r_0, \alpha_0) = 0. \tag{A.9}$$

Since (A.3), we conclude that $u(\cdot;\, \alpha_0) \equiv 0$, which contradicts with the uniqueness of solutions of the initial value problem for (A.3) at $r = r_0$. Therefore, (A.8) holds for all $\alpha \in (\alpha_2, \alpha^*)$.

Conclusion of Step 2. By (A.7) and Step 1, we can see $\alpha^* < \bar{\alpha}$ and $\sigma_2(\alpha^*) = +\infty$. Setting $\alpha_3 = \alpha^*$, the conclusion of Step 2 holds.

Step 3: $u_3(r) = u(r;\, \alpha_3)$ has the desired property.

We discuss the following two cases:

$$\begin{aligned} &\text{Case 1:} \quad \sigma_1(\alpha) \to +\infty \quad \text{as} \quad \alpha \to \alpha_3^-;\\ &\text{Case 2:} \quad \sigma_0 = \lim_{\alpha\to\alpha_3^-}\sigma_1(\alpha) \in (0, +\infty) \quad \text{exists}. \end{aligned}$$

Case 1: $\sigma_1(\alpha) \to +\infty$ as $\alpha \to \alpha_3^-$. It follows from the definition of $\sigma_1(\alpha)$ that

$$u(r, \alpha) > u_1(r) \quad \text{in } [0, \sigma_1(\alpha)].$$

From Lemma A.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u(r, \alpha)}{u_1(r)}\right) < 0 \quad \text{in } [0, \sigma_1(\alpha)].$$

Therefore,

$$0 < u(r, \alpha) < \frac{\alpha}{\alpha_1}u_1(r) \quad \text{in } [0, \sigma_1(\alpha)].$$

Letting $\alpha \to \alpha_3^-$, we have

$$0 < u_3(r, \alpha_3) < \frac{\alpha_3}{\alpha_1}u_1(r) \quad \text{in } (0, +\infty).$$

Since $u_1(r) \to 0$ exponentially as $r \to \infty$, so is $u_3(r)$. Thus, we can know that $u_3(r)$ satisfies (A.1). Obviously, $\sharp\{r > 0;\ u_3(r) = u_1(r)\} = 0$ in this case.

Case 2: $\sigma_0 = \lim_{\alpha\to\alpha_3^-}\sigma_1(\alpha) \in (0, +\infty)$ exists. Let $\alpha \in (\alpha_2, \alpha_3)$. Then, by the definition of $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$, we have

$$\begin{aligned} &0 < u_1(r) < u(r;\, \alpha) \quad \text{in } (0, \sigma_1(\alpha)),\\ &0 < u(r;\, \alpha) < u_1(r) \quad \text{in } (\sigma_1(\alpha), \sigma_2(\alpha)). \end{aligned}$$

Noting that $\sigma_1(\alpha) \to \sigma_0(0, \infty)$, $\sigma_2(\alpha) \to \infty$ as $\alpha \to \alpha_3^-$, we have

$$\begin{aligned} &0 < u_1(r) < u(r;\, \alpha_3) \quad \text{in } (0, \sigma_0),\\ &0 < u(r;\, \alpha_3) < u_1(r) \quad \text{in } (\sigma_0, +\infty). \end{aligned}$$

Thus, we can know that $u_3(r)$ satisfies (A.1). Moreover, we have

$$\sharp\{r \in (0, +\infty);\ u_3(r) = u_1(r)\} = 1.$$

Thus, we obtain the desired results in both cases. □

With Lemma A.2, we are now in a position to prove the uniqueness assertion of Proposition 1.1 when $N = 1$.

Proof of Proposition 1.1 when $N = 1$. We argue by contradiction that (A.1) has two distinct positive radial solutions $u_1(r)$, $u_2(r)$. By Lemma A.2, we can assume that

$$0 < u_1(r) < u_2(r) \quad \text{in } [0, \sigma), \tag{A.10}$$

$$0 < u_2(r) < u_1(r) \quad \text{in } (\sigma, +\infty) \tag{A.11}$$

for some $\sigma \in (0, +\infty]$ (When $\sigma = +\infty$, we disregard the second condition).

We set for $j = 1, 2$

$$E(r;\ u_j) = \frac{1}{2}u_j'(r)^2 + \frac{1}{2p}u_j^{2p}(r) - \frac{1}{2}V(r)u_j(r)^2.$$

Let $F(r) = E(r;\ u_2) - \left(\frac{u_2}{u_1}\right)^2 E(r, u_1)$. We can check that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_1(r)}{u_2(r)}\right) > 0 \quad \text{in } (0, \infty), \tag{A.12}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_2(r)}{u_1(r)}\right)^2 = 2\frac{u_2(r)}{u_1(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_2(r)}{u_1(r)}\right) < 0 \quad \text{in } (0, \infty). \tag{A.13}$$

By direct computations, it is easy to check that

$$F'(r) = -\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\left(\frac{u_2}{u_1}\right)^2\right)\right\}E(r, u_1).$$

Hence,

$$F(+\infty) - F(0) = \int_0^\infty -\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\left(\frac{u_2}{u_1}\right)^2\right)\right\}E(r, u_1) > 0, \tag{A.14}$$

where we have used the fact that $E(r, u_1) > 0$.

Applying the exponential decay of u_j, we see that

$$F(+\infty) = \lim_{r\to\infty} F(r) = 0.$$

But, on the other hand, we have

$$F(0) = \frac{1}{2p}(u_2^{2p}(0) - u_1^{2p}(0)) > 0,$$

which is a contradiction to (A.14). The proof is complete. □