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Research Article

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Weighted critical exponents of Sobolev-type embeddings for radial functions

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Abstract: In this article, we prove the upper weighted critical exponents for some embeddings from weighted Sobolev spaces of radial functions into weighted Lebesgue spaces. We also consider the lower critical exponent for certain embedding.

Keywords: embeddings, critical exponents, compactness

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1 Introduction

We assume that the real functions A, V, and Q satisfy the following assumptions:

(A) $A \in C(0,\infty)$, $A(r) > 0$, and there exist $\ell, \ell_0 \in \mathbb{R}$ such that

$$\lim_{r\to\infty} \frac{A(r)}{r^{\ell}} > 0, \quad \lim_{r\to 0} \frac{A(r)}{r^{\ell_0}} > 0.$$

(V) $V \in C(0,\infty)$, $V(r) > 0$, and there exist $a, a_0 \in \mathbb{R}$ such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^{a}} > 0, \quad \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$, and there exist $b, b_0 \in \mathbb{R}$ such that

$$\limsup_{r\to\infty} \frac{Q(r)}{r^{b}} < \infty, \quad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

We assume $N \in \mathbb{N}$ and $1 < p < N + \min\{\ell, \ell_0\}$ and then define two numbers q_* and q^* in the following way.

$$q_* = \begin{cases} p + \dfrac{p^2(b-a)}{p(N-1+a) + (\ell - a)}, & b \geqslant a > \ell - p, \\ \dfrac{p(N+b)}{N+\ell-p}, & b \geqslant \ell - p,\ a \leqslant \ell - p, \\ p, & b \leqslant \max\{a, \ell - p\}, \end{cases}$$

$$q^* = \begin{cases} \dfrac{p(N+b_0)}{N+\ell_0-p}, & b_0 \geqslant \ell_0 - p, \quad a_0 \geqslant \ell_0 - p, \\ p + \dfrac{p^2(b_0-a_0)}{p(N-1+a_0) + (\ell_0 - a_0)}, & \ell_0 - p \geqslant a_0 > -\dfrac{p(N-1)+\ell_0}{p-1}, \quad b_0 \geqslant a_0, \\ \infty, & a_0 \leqslant -\dfrac{p(N-1)+\ell_0}{p-1}, \quad b_0 \geqslant a_0. \end{cases}$$

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Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) | u \text{ is radial}\}.$$

Let $D_r^{1,p}(\mathbb{R}^N; A)$ (resp., $D^{1,p}(\mathbb{R}^N; A)$) be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ (resp., $C_0^\infty(\mathbb{R}^N)$) under

$$\|u\|_A = \left(\int_{\mathbb{R}^N} A(x)|\nabla u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Define for $p > 1$

$$L^p(\mathbb{R}^N; V) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x < \infty \right\}.$$

Then define

$$W_r^{1,p}(\mathbb{R}^N; A, V) = D_r^{1,p}(\mathbb{R}^N; A) \cap L^p(\mathbb{R}^N; V), \quad W^{1,p}(\mathbb{R}^N; A, V) = D^{1,p}(\mathbb{R}^N; A) \cap L^p(\mathbb{R}^N; V).$$

They are, under conditions (A) and (V), reflexive Banach spaces with the norm

$$\|u\|_{A,V} = \left(\int_{\mathbb{R}^N} A(x)|\nabla u|^p + V(x)|u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Define for $q \geqslant 1$

$$L^q(\mathbb{R}^N; Q) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} Q(x)|u|^q \mathrm{d}x < \infty \right\}.$$

In [17], Su and Wang proved the following embedding theorem.

Theorem I. Assume $(A), (V)$, and (Q) with $1 < p < N + \min\{\ell_0, \ell\}$ and q_*, q^* being well defined. The embedding

$$W_r^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^q(\mathbb{R}^N; Q) \tag{1.1}$$

is continuous for $q_* \leqslant q \leqslant q^*$ when $q^* < \infty$ and for $q_* \leqslant q < \infty$ when $q^* = \infty$. The embedding (1.1) is compact for $q_* < q < q^*$.

If $b < \max\{a, \ell - p\}$ and $b_0 > \min\{a_0, \ell_0 - p\}$, then the embedding

$$W_r^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^p(\mathbb{R}^N; Q) \tag{1.2}$$

is compact.

We first obtain a new observation that (1.2) will be compact on $W^{1,p}(\mathbb{R}^N; A, V)$.

Theorem 1.1. *Assume (A), (V), and (Q) with $1 < p < N + \min\{\ell_0, \ell\}$, $b < \max\{a, \ell - p\}$, and $b_0 > \min\{a_0, \ell_0 - p\}$. The embedding*

$$W^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^p(\mathbb{R}^N; Q) \tag{1.3}$$

is compact.

One of the main results in this article is the following theorem on the upper weighted critical exponent of the embedding (1.1).

Theorem 1.2. *Assume* (A), (V), *and* (Q) *with* $1 < p < N + \min\{\ell_0, \ell\}$, $b_0 \geqslant \ell_0 - p$, *and* $a_0 \geqslant \ell_0 - p$. *There is no embedding from* $W_r^{1,p}(\mathbb{R}^N;\, A, V)$ *into* $L^q(\mathbb{R}^N;\, Q)$ *for any* $q > q^*$. *The embedding*

$$W_r^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^{q^*}(\mathbb{R}^N;\, Q) \tag{1.4}$$

is not compact.

Now we investigate the lower critical exponent of the above embedding. When (A), (V), and (Q) are assumed, we have that $q_* = p$ is exactly the lower critical exponent of the embedding (1.1) in some case.

Theorem 1.3. *Assume* (A), (V), *and* (Q) *with* $1 < p < N + \min\{\ell_0, \ell\}$, $a_0 > -N$, $b_0 > -N$, *and* $b = a \geqslant \sigma - p$. *There is no embedding from* $W_r^{1,p}(\mathbb{R}^N;\, A, V)$ *into* $L^q(\mathbb{R}^N;\, Q)$ *for any* $q < p$. *The embedding*

$$W_r^{1,p}(\mathbb{R}^N;\, A, V) \hookrightarrow L^p(\mathbb{R}^N;\, Q) \tag{1.5}$$

is not compact.

Let $A(x) = |x|^\sigma$, $V(x) = |x|^\beta$, and $Q(x) = |x|^\alpha$, set

$$q_*(\sigma, \alpha, \beta) = \begin{cases} p, & \text{if } \alpha \leqslant \beta, \\ p + \dfrac{p^2(\alpha - \beta)}{p(N - 1 + \beta) + \sigma - \beta}, & \text{if } \quad \alpha > \beta, \end{cases} \tag{1.6}$$

$$q^*(\sigma, \alpha, \beta) = \frac{p(N + \alpha)}{N + \sigma - p}. \tag{1.7}$$

We consider the following embeddings.

$$W_r^{1,p}(\mathbb{R}^N;\, |x|^\sigma, |x|^\beta) \hookrightarrow L^q(\mathbb{R}^N;\, |x|^\alpha), \tag{1.8}$$

$$W^{1,p}(\mathbb{R}^N;\, |x|^\sigma, |x|^\beta) \hookrightarrow L^q(\mathbb{R}^N;\, |x|^\alpha). \tag{1.9}$$

From Theorem I, Theorems 1.1, 1.2, and 1.3, we can obtain the following conclusions (Table 1).

Furthermore, taking the special numbers, we can obtain the classical critical exponents (Tables 2 and 3).

Table 1: $1 < p < N + \sigma$

q	β, α	Embedding
$q_*(\sigma, \alpha, \beta) \leqslant q \leqslant q^*(\sigma, \alpha, \beta)$	$\beta \geqslant \sigma - p$ $\alpha \geqslant \sigma - p$	(1.8) holds
$q_*(\sigma, \alpha, \beta) < q < q^*(\sigma, \alpha, \beta)$	$\beta \geqslant \sigma - p$ $\alpha \geqslant \sigma - p$	(1.8) is compact
$q = q^*(\sigma, \alpha, \beta)$	$\beta \geqslant \sigma - p$ $\alpha \geqslant \sigma - p$	$q^*(\sigma, \alpha, \beta)$ is upper critical exponent of (1.8), (1.8) is noncompact
$q = p$	$\beta = \alpha \geq \sigma - p$	p is lower critical exponent of (1.8), (1.8) is noncompact
$q = p$	$\sigma - p < \alpha < \beta$	(1.9) is compact

Table 2: The upper critical exponents

p, σ	β	α	$q^*(\sigma, \alpha, \beta)$
$1 < p < N, \sigma = 0$	$\beta = 0$	$\alpha = -p$	Upper Hardy critical exponent, see [7]
$1 < p < N, \sigma = 0$	$\beta = 0$	$-p < \alpha < 0$	Upper Hardy-Sobolev critical exponent, see [7]
$1 < p < N, \sigma = 0$	$\beta = 0$	$\alpha = 0$	Upper Sobolev critical exponent, see [6,14] for $p = 2$
$1 < p < N, \sigma = 0$	$\beta = 0$	$\alpha > 0$	Upper Hénon-Sobolev critical exponent, see [19]
$p - N < \sigma < \infty$ $p > 1$	$\beta \geq \sigma - p$	$\sigma - p \leqslant \alpha \leqslant \frac{\sigma N}{N-p}$ for $1 < p < N$; $\alpha \geqslant \sigma - p$ for $p \geqslant N, p > 1$	Upper CKN critical exponent, see [3]

Table 3: The lower critical exponents

p, σ	β	α	$q_*(\sigma, \alpha, \beta)$
$1 < p < N, \sigma = 0$	$\beta = 0$	$\alpha = 0$	p is lower Sobolev critical exponent, see [6,14] for $p = 2$

In the case that $A \equiv 1$, V and Q satisfy assumptions (V) and (Q), the conclusions of Theorem I have been established in [15,16]. For $\alpha > 0$, the upper Hénon-Sobolev critical exponent is raised in [19] due to a reason that Hénon [9] first established a model of semilinear elliptic equation

$$\begin{cases} -\Delta u = |x|^\alpha u^{q-1}, \ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.10}$$

on the unit ball $B := \{x \in \mathbb{R}^N : |x| < 1\}$ with $\alpha > 0$ in studying the rotating stellar structures in 1973. Smets et al. in [13] first revealed the symmetric breaking phenomenon of the ground state of (1.10) for $2 < q < 2^* = \frac{2N}{N-2}$. Sinzeff and Willem in [12, Theorem 2.3] investigated the ground state of the equation

$$\begin{cases} -\Delta u + |x|^\beta u = |x|^\alpha u^{q-1}, \ u > 0 & \text{in } \mathbb{R}^N, \\ u \in W_r^{1,2}(\mathbb{R}^N; |x|^\beta) \end{cases} \tag{1.11}$$

for $\beta \geqslant 0$, $\alpha \geqslant 0$ and $\max\left\{2, 2 + \frac{4(\alpha-\beta)}{2(N-1)+\beta}\right\} < q < \frac{2(N+\alpha)}{N-2}$. Indeed, the power q in (1.11) lies strictly between $q_*(\sigma, \alpha, \beta)$ and $q^*(\sigma, \alpha, \beta)$ given by (1.6) and (1.7) with $\sigma = 0$ and $p = 2$.

It follows from Theorem I and Theorem 1.2 that q^* is exactly the upper critical exponent of the embedding (1.1) in the case $a_0, b_0 \geqslant \ell_0 - p$. In particular, the number $q^*(\sigma, \alpha, \beta)$ defined by (1.7) is exactly the upper critical exponent of the embedding (1.8).

The well-known Caffarelli-Kohn-Nirenberg (CKN) inequality [3] (see also [11]) reads as

$$\left(\int_{\mathbb{R}^N} |x|^{-tq} |u|^q \mathrm{d}x \right)^{\frac{p}{q}} \leqslant C_{s,t} \int_{\mathbb{R}^N} |x|^{-ps} |\nabla u|^p \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.12}$$

where

$$-\infty < s < \frac{N-p}{p}, \quad 0 \leqslant t - s \leqslant 1, \quad q = \frac{pN}{N-p+p(t-s)}, \quad \text{for } 1 < p < N, \tag{1.13}$$

$$-\infty < s < \frac{N-p}{p}, \quad \frac{N-p}{p} < t - s \leqslant 1, \quad q = \frac{pN}{N-p+p(t-s)}, \quad \text{for } p \geqslant N, \quad p > 1. \tag{1.14}$$

By the density, (1.12) holds for all $u \in D^{1,p}(\mathbb{R}^N;\ |x|^{-ps})$.

Let us reset $\alpha = -tq$ and $\sigma = -ps$. Then (1.12) is reformulated as

$$\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{q^*(\sigma,\alpha)} \mathrm{d}x\right)^{\frac{p}{q^*(\sigma,\alpha)}} \leqslant C_{s,t} \int_{\mathbb{R}^N} |x|^\sigma |\nabla u|^p \mathrm{d}x, \quad \forall u \in D^{1,p}(\mathbb{R}^N;\ |x|^\sigma), \tag{1.15}$$

where

$$q^*(\sigma,\alpha) := \frac{p(N+\alpha)}{N+\sigma-p}, \tag{1.16}$$

and (1.13), (1.14) can be reformulated, respectively, as

$$\begin{cases} p-N<\sigma<\infty, \quad \sigma-p \leqslant \alpha \leqslant \dfrac{\sigma N}{N-p}, & \text{for } 1<p<N; \\ p-N<\sigma<\infty, \quad \alpha \geqslant \sigma - p, & \text{for } p \geqslant N, \quad p>1. \end{cases} \tag{1.17}$$

For $p-N<\sigma<\infty$, from the proof of Theorem 1.2 we see that $q^*(\sigma,\alpha)$ coincides the upper critical exponent, given by (1.7), of the embedding (1.8). We name the number $q^*(\sigma,\alpha)$ as the CKN critical exponent.

We remark here that in the case that $p-N<\sigma<\infty$ and $1<p<N$, the region of α for the CKN inequality is

$$\sigma - p \leqslant \alpha \leqslant \frac{\sigma N}{N-p}. \tag{1.18}$$

Su and Wang in [17] proved inequality (1.15) for radial functions which extended the region of α for the CKN inequality to the case $\sigma - p \leqslant \alpha < \infty$. The case $p=2$ was obtained in Catrina and Wang [5, Theorem 1.3].

Theorem II. [17] *Assume* $1<p<N+\sigma$ *and* $\sigma-p \leqslant \alpha < \infty$. *Then*

$$\widetilde{C}\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{q^*(\sigma,\alpha)} \mathrm{d}x\right)^{\frac{p}{q^*(\sigma,\alpha)}} \leqslant \int_{\mathbb{R}^N} |x|^\sigma |\nabla u|^p \mathrm{d}x, \quad \forall u \in D_r^{1,p}(\mathbb{R}^N;|x|^\sigma), \tag{1.19}$$

where

$$\widetilde{C} = \omega_N^{\frac{p+\alpha-\sigma}{N+\alpha}} \left(\frac{N+\sigma-p}{p-1}\right)^{\frac{(p-1)(p+\alpha-\sigma)}{N+\alpha}} \left(\frac{N+\sigma-p}{p}\right)^{\frac{p(N+\sigma-p)}{N+\alpha}},$$

and ω_N *is the surface area of the unit sphere in* $\mathbb{R}^N$.

For the special case $\sigma=0$ and $\alpha=-p$, $\widetilde{C} = \left(\frac{N-p}{p}\right)^p$ is exactly the best Hardy constant. In general case, the constant $\widetilde{C}$ in (1.19) may not be optimal. We investigate the best constant for (1.19) in Theorem II by an alternative way. We define

$$S_{\sigma,\alpha} = \inf_{u\in D_r^{1,p}(\mathbb{R}^N;|x|^\sigma)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |x|^\sigma |\nabla u|^p \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{q^*(\sigma,\alpha)} \mathrm{d}x\right)^{\frac{p}{q^*(\sigma,\alpha)}}}. \tag{1.20}$$

Theorem 1.4. *Assume* $1<p<N+\sigma$ *and* $\sigma-p<\alpha<\infty$. *Then*

$$S_{\sigma,\alpha} = (N+\alpha)\left(\frac{1}{p+\alpha-\sigma}\right)^{\frac{p+\alpha-\sigma}{N+\alpha}} \left(\frac{N+\sigma-p}{p-1}\right)^{p-1} \left((p-1)\omega_N B\left(\frac{N+\alpha}{p+\alpha-\sigma}, \frac{(p-1)(N+\alpha)}{p+\alpha-\sigma}\right)\right)^{\frac{p+\alpha-\sigma}{N+\alpha}}, \tag{1.21}$$

where $B(\cdot,\cdot)$ is the Beta function. Furthermore, $S_{\sigma,\alpha}$ can be uniquely (up to dilations) achieved by the extremal function

$$U_{\varepsilon,\sigma,\alpha}(x)=\frac{\varepsilon^{\frac{N+\sigma-p}{p(p-1)}}\left(\frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}}\right)^{\frac{N+\sigma-p}{p(p+\alpha-\sigma)}}}{\left(\varepsilon^{\frac{p+\alpha-\sigma}{p-1}}+|x|^{\frac{p+\alpha-\sigma}{p-1}}\right)^{\frac{N+\sigma-p}{p+\alpha-\sigma}}}\quad \text{with}\quad \varepsilon>0. \tag{1.22}$$

We remark here that the best constant (1.21) and the extremal function (1.22) with the CKN critical exponent (1.16) have been widely considered. In [4], the authors established a transformation in investigating the best constant, the existence or nonexistence, and the symmetry of the extremal function for $p=2$ in the setting (1.12)–(1.14). In [2], the symmetry-breaking phenomenon of the extremal functions for CKN inequality with general p was first revealed. The result of Theorem 1.4 extends the CKN critical exponent (1.16) in radial case to a large region of the power α of the weight $|x|^\alpha$. This means that for all $\alpha>\sigma-p$, the best constant (1.20) is achieved exactly by the function (1.22) and can also be written explicitly as (1.21). The function $U_{\varepsilon,\sigma,\alpha}$ given by (1.22) is a solution of the weighted critical equation

$$\begin{cases}-\operatorname{div}(|x|^\sigma|\nabla u|^{p-2}\nabla u)=|x|^\alpha u^{q^*(\sigma,\alpha)-1}, & u>0, \quad \text{in} \quad \mathbb{R}^N,\\ u\in D_r^{1,p}(\mathbb{R}^N;\ |x|^\sigma).\end{cases} \tag{1.23}$$

From (1.19) and (1.20) we have that $S_{\sigma,\alpha}\geqslant\widetilde{C}$ for $\alpha>\sigma-p$.

For the problems restricted to the unit ball B (or a bounded spherically symmetric domain in $\mathbb{R}^N$), there is only the upper critical exponent for the embeddings being concerned and we only need the conditions on the weights A and Q near the origin.

We make the following assumptions.

(A′) $A\in C(0,1)$, $A(r)>0$, there exists $\ell_0>p-N$ such that $\lim_{r\to0}\frac{A(r)}{r^{\ell_0}}>0$.

(Q′) $Q\in C(0,1)$, $Q(r)>0$, and there exists $b_0\in\mathbb{R}$ such that $\limsup_{r\to0}\frac{Q(r)}{r^{b_0}}<\infty$.

Let $W_{0,r}^{1,p}(B;\ A)$ be the completion of $C_{0,r}^\infty(B)$ with respect to $\|u\|=\left(\int_B A(x)|\nabla u|^p\mathrm{d}x\right)^{1/p}$. Define $q^\star=\frac{p(N+b_0)}{N+\ell_0-p}$. Following the arguments in the proof of [17, Theorem 3.2], we have

Theorem 1.5. *Assume* (A′) *and* (Q′) *with* $1<p<N+\ell_0$ *and* $b_0\geqslant\ell_0-p$. *The embedding*

$$W_{0,r}^{1,p}(B;\ A)\hookrightarrow L^q(B;\ Q) \tag{1.24}$$

is continuous for $1\leqslant q\leqslant q^\star$. *The embedding* (1.24) *is compact for* $1\leqslant q<q^\star$.

For the weighted critical exponent on the unit ball, we have a similar result.

Theorem 1.6. *Assume* (A′) *and* (Q′) *with* $1<p<N+\ell_0$ *and* $b_0\geqslant\ell_0-p$. *There is no embedding from* $W_{0,r}^{1,p}(B;\ A)$ *into* $L^q(B;\ Q)$ *for any* $q>q^\star$. *The embedding*

$$W_{0,r}^{1,p}(B;\ A)\hookrightarrow L^{q^*}(B;\ Q)$$

is not compact.

Now, let $A(x)=|x|^\sigma$ and $Q(x)=|x|^\alpha$, we consider the embedding,

$$W_{0,r}^{1,p}(B;|x|^\sigma)\hookrightarrow L^q(B;|x|^\alpha). \tag{1.25}$$

Based on Theorems 1.5 and 1.6, we obtain the special results (Table 4).

Table 4: $1 < p < N + \sigma$

q	α	Embedding
$1 \leqslant q \leqslant q^*(\alpha, \beta)$	$\alpha \geqslant \sigma - p$	(1.25) holds
$1 \leqslant q < q^*(\alpha, \beta)$	$\alpha \geqslant \sigma - p$	(1.25) is compact
$q = q^*(\alpha, \beta)$	$\alpha \geqslant \sigma - p$	$q^*(\alpha, \beta)$ is upper critical exponent of (1.25), (1.25) is noncompact

We remark here that the embedding theorems obtained above are very important and are basic tools for applying the variational methods in the quasilinear elliptic equations of different types. For example, the compact embedding (1.9) with $\sigma = \beta = 0$ has been applied to study a class of weighted critical quasilinear elliptic problems in [20].

There are a few of interesting open problems on the weighted critical exponents for the embeddings. Is the upper embedding exponent q^* for the second case critical and is the lower exponent q_* critical in Theorem I? In particular, is the lower exponent $q_*(\sigma, \alpha, \beta)$ defined by (1.6) for the case $\alpha > \beta$ critical? We will keep working on these problems.

The article is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we give the proofs of Theorems 1.2, 1.3, 1.4, and 1.6.

2 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We follow the arguments in [17].

Proof of Theorem 1.1. To prove the compactness of the embedding (1.3), we only need to prove that for any sequence $\{u_k\} \subset W^{1,p}(\mathbb{R}^N; A, V)$,

$$u_k \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(\mathbb{R}^N; A, V) \tag{2.1}$$

implies $\{u_k\} \subset L^p(\mathbb{R}^N; Q)$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} Q(x)|u_k|^p \mathrm{d}x = 0. \tag{2.2}$$

By (2.1), there exists $M > 0$ such that

$$\|u_k\|^p_{A,V} \leqslant M, \quad \forall k \in \mathbb{N}. \tag{2.3}$$

By (A), (V), and (Q), there exist $R_0 > r_0 > 0$, for some $C_0 > 0$,

$$\begin{cases} A(x) \geqslant C_0|x|^{\ell}, \quad V(x) \geqslant C_0|x|^{a}, \quad Q(x) \leqslant C_0|x|^{b}, \quad \text{for } |x| \geqslant R_0, \\ A(x) \geqslant C_0|x|^{\ell_0}, \quad V(x) \geqslant C_0|x|^{a_0}, \quad Q(x) \leqslant C_0|x|^{b_0}, \quad \text{for } 0 < |x| \leqslant r_0. \end{cases} \tag{2.4}$$

We will prove that for any $\varepsilon > 0$ there exist $R > R_0$ and $0 < r < r_0$ such that the integrals

$$\int_{\mathbb{R}^N\setminus B_R} Q(x)|u_k|^p \mathrm{d}x < \frac{\varepsilon}{3}, \tag{2.5}$$

$$\int_{B_r} Q(x)|u_k|^p \mathrm{d}x < \frac{\varepsilon}{3}. \tag{2.6}$$

In the following we will use C to denote various constants.

We first consider the case out of a big ball. Since $b < \max\{a, \ell - p\}$, it holds that either $b < a$ or $b < \ell - p$.

(1) Let $b < a$. For $R > R_0$, we have by (2.3) and (2.4) that

$$\int_{\mathbb{R}^N\setminus B_R} Q(x)|u_k|^p\,dx \leqslant C_0 \int_{\mathbb{R}^N\setminus B_R} |x|^b|u_k|^p\,dx \leqslant R^{b-a}\int_{\mathbb{R}^N} V(x)|u_k|^p\,dx \leqslant MR^{b-a}. \tag{2.7}$$

(2) Let $b < \ell - p$. We choose a smooth cutoff function $\phi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\phi(x) = \begin{cases} 1, & \text{for } |x| \geqslant R_0 + 1,\\ 0, & \text{for } 0 \leqslant |x| \leqslant R_0, \end{cases} \quad 0 \leqslant \phi \leqslant 1, \quad |\nabla\phi(x)| \leqslant C. \tag{2.8}$$

Then for $R > R_0 + 1$, we have by (2.3), (2.4), (2.8), and the CKN inequality (1.15) that

$$\begin{aligned} \int_{\mathbb{R}^N\setminus B_R} Q(x)|u_k|^p\,dx &\leqslant C_0R^{b+p-\ell}\int_{\mathbb{R}^N} |x|^{\ell-p}|\phi u_k|^p\,dx\\ &\leqslant CR^{b+p-\ell}\int_{\mathbb{R}^N} |x|^{\ell}|\nabla(\phi u_k)|^p\,dx\\ &\leqslant CR^{b+p-\ell}\int_{\mathbb{R}^N} [A(x)|\nabla u_k|^p + V(x)|u_k|^p]\,dx\\ &\leqslant CMR^{b+p-\ell}. \end{aligned} \tag{2.9}$$

In above we have used a fact that $|x|^\ell|\nabla\phi|^p \leqslant CV(x)$ for $R_0 \leqslant |x| \leqslant R_0 + 1$.

Next we consider the case in a small ball centered at the origin. It follows from $b_0 > \min\{a_0, \ell_0 - p\}$ that either $b_0 > a_0$ or $b_0 > \ell_0 - p$.

(3) Let $b_0 > a_0$. For $0 < r < r_0$, we have by (2.3) and (2.4) that

$$\int_{B_r} Q(x)|u_k|^p\,dx \leqslant r^{b_0-a_0}\int_{\mathbb{R}^N} V(x)|u_k|^p\,dx \leqslant Mr^{b_0-a_0}. \tag{2.10}$$

(4) Let $b_0 > \ell_0 - p$. We choose a smooth cutoff function $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\psi(x) = \begin{cases} 0, & \text{for } |x| \geqslant r_0,\\ 1, & \text{for } 0 \leqslant |x| \leqslant \frac{r_0}{2}, \end{cases} \quad 0 \leqslant \psi \leqslant 1, \quad |\nabla\psi(x)| \leqslant C. \tag{2.11}$$

Then for $0 < r < \frac{r_0}{2}$, we have by (2.3), (2.4), (2.11), and (1.15) that

$$\begin{aligned} \int_{B_r} Q(x)|u_k|^p\,dx &\leqslant C_0r^{b_0+p-\ell_0}\int_{\mathbb{R}^N} |x|^{\ell_0-p}|\psi u_k|^p\,dx\\ &\leqslant Cr^{b_0+p-\ell_0}\int_{\mathbb{R}^N} |x|^{\ell_0}|\nabla(\psi u_k)|^p\,dx\\ &\leqslant Cr^{b_0+p-\ell_0}\int_{\mathbb{R}^N} [A(x)|\nabla u_k|^p + V(x)|u_k|^p]\,dx\\ &\leqslant CMr^{b_0+p-\ell_0}. \end{aligned} \tag{2.12}$$

Since $b < \max\{a, \ell - p\}$ and $b_0 > \min\{a_0, \ell_0 - p\}$, it follows from (2.7) or (2.9) and (2.10) or (2.12) that for any $\varepsilon > 0$, there exist $R > R_0$ and $r_0 > r > 0$ such that (2.5) and (2.6) hold.

For $R > r > 0$ given above, we consider the integral over $B_R\setminus B_r$. We have by (Q) that

$$\int_{B_R\setminus B_r} Q(x)|u_k|^p\,dx \leqslant C\int_{B_R\setminus B_r} |x|^{b_0}|u_k|^p\,dx \leqslant C\int_{B_R\setminus B_r} |u_k|^p\,dx. \tag{2.13}$$

By the classical Sobolev embedding theorem [1], we have that

$$W^{1,p}(B_R\setminus B_r; A, V) \hookrightarrow W^{1,p}(B_R\setminus B_r) \hookrightarrow\hookrightarrow L^p(B_R\setminus B_r), \tag{2.14}$$

where $\hookrightarrow\hookrightarrow$ indicates compact embedding. It follows from (2.13), (2.14), and (2.1) that

$$\lim_{k\to\infty}\int_{B_R\setminus B_r} Q(x)|u_k|^p\,\mathrm{d}x = 0. \tag{2.15}$$

Since $\varepsilon > 0$ is arbitrary, by (2.5), (2.6), and (2.15) we arrive at (2.2). The proof is complete. □

We remark that in Theorem 1.1, the functions A, V, and Q are not needed to be radial.

3 The weighted critical exponents

In this section, we give the proofs of those theorems about the weighted critical exponents. We first prove Theorem 1.2. Denote $B_R = \{x \in \mathbb{R}^N : |x| < R\}$.

Proof of Theorem 1.2. To prove the first conclusion of this theorem, we only need to construct three weight radial functions $\widetilde{A}$, $\widetilde{V}$, and $\widetilde{Q}$, satisfying the assumptions (A), (V), and (Q), respectively, and to construct a sequence $\{u_k\}$ such that for any $q > q^*$, $\{u_k\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N, \widetilde{A}, \widetilde{V})$ but $\{u_k\}$ is unbounded in $L^q(\mathbb{R}^N;\widetilde{Q})$. Define a smooth cutoff function $\varphi \in C_{0,r}^\infty(\mathbb{R}^N)$ satisfying $\varphi(x) = \begin{cases} 1, & \text{for} \quad |x| \leqslant 1 \\ 0, & \text{for} \quad |x| \geqslant 2 \end{cases}$ and $0 \leqslant \varphi \leqslant 1$. Define

$$\begin{aligned}\widetilde{A}(x) &= \varphi(x)|x|^{\ell_0} + (1-\varphi(x))|x|^{\ell},\\ \widetilde{V}(x) &= \varphi(x)|x|^{a_0} + (1-\varphi(x))|x|^{a},\\ \widetilde{Q}(x) &= \varphi(x)|x|^{b_0} + (1-\varphi(x))|x|^{b}.\end{aligned}$$

Then $\widetilde{A}$, $\widetilde{V}$, and $\widetilde{Q}$ satisfy assumptions (A), (V), and (Q), respectively. Construct a sequence $\{u_k\}_{k=1}^\infty$ as follows:

$$u_k(x) = k^{\frac{N+\ell_0-p}{p^2}} e^{-\frac{k|x|^p}{p\cdot q^*}}, \quad k \in \mathbb{N}. \tag{3.1}$$

For the integral term containing gradient, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \widetilde{A}(x)|\nabla u_k|^p\,\mathrm{d}x\\
&\leqslant \int_{B_2} |x|^{\ell_0}|\nabla u_k|^p\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_1} |x|^{\ell}|\nabla u_k|^p\,\mathrm{d}x\\
&= \frac{k^{\frac{N+\ell_0-p}{p}+p}}{(q^*)^p}\omega_N \int_0^2 r^{N+\ell_0+p(p-1)-1} e^{-\frac{kr^p}{q^*}}\,\mathrm{d}r + \frac{k^{\frac{N+\ell_0-p}{p}+p}}{(q^*)^p}\omega_N \int_1^\infty r^{N+\ell+p(p-1)-1} e^{-\frac{kr^p}{q^*}}\,\mathrm{d}r\\
&= (q^*)^{\frac{N+\ell_0-p}{p}} \frac{\omega_N}{p} \int_0^{\frac{k2^p}{q^*}} s^{\frac{N+\ell_0+p(p-1)}{p}-1} e^{-s}\,\mathrm{d}s \left(s = \frac{k}{q^*}r^p\right) + k^{\frac{\ell_0-\ell}{p}} (q^*)^{\frac{N+\ell-p}{p}} \frac{\omega_N}{p} \int_{\frac{k}{q^*}}^\infty s^{\frac{N+\ell+p(p-1)}{p}-1} e^{-s}\,\mathrm{d}s\\
&\leqslant (q^*)^{\frac{N+\ell_0-p}{p}} \frac{\omega_N}{p} \Gamma\left(\frac{N+\ell_0+p^2-p}{p}\right) + k^{\frac{\ell_0-\ell}{p}} (q^*)^{\frac{N+\ell-p}{p}} \frac{\omega_N}{p} \int_{\frac{k}{q^*}}^\infty s^{\frac{N+\ell+p(p-1)}{p}-1} e^{-s}\,\mathrm{d}s,
\end{aligned} \tag{3.2}$$

where Γ is the Gamma function. It is easily seen that

$$k^{\frac{\ell_0-\ell}{p}} \int_{\frac{k}{q^*}}^\infty s^{\frac{N+\ell+p(p-1)}{p}-1} e^{-s}\,\mathrm{d}s \to 0 \quad \text{as } k \to \infty. \tag{3.3}$$

It follows from (3.2) and (3.3) that for some $C_1 > 0$

$$\int_{\mathbb{R}^N} \widetilde{A}(x)|\nabla u_k|^p \mathrm{d}x \leqslant C_1, \quad \forall k \in \mathbb{N}. \tag{3.4}$$

Next we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \widetilde{V}(x)|u_k|^p \mathrm{d}x \\
&\leqslant \int_{B_2} |x|^{a_0}|u_k|^p \mathrm{d}x + \int_{\mathbb{R}^N \setminus B_1} |x|^a |u_k|^p \mathrm{d}x \\
&= k^{\frac{N+\ell_0-p}{p}} \omega_N \int_0^2 r^{N-1+a_0} e^{-\frac{kr^p}{q^*}} \mathrm{d}r + k^{\frac{N+\ell_0-p}{p}} \omega_N \int_1^\infty r^{N-1+a} e^{-\frac{kr^p}{q^*}} \mathrm{d}r \\
&\leqslant (q^*)^{\frac{N+a_0}{p}} k^{-\frac{p+a_0-\ell_0}{p}} \frac{\omega_N}{p} \Gamma\left(\frac{N+a_0}{p}\right) + (q^*)^{\frac{N+a}{p}} k^{-\frac{p+a-\ell_0}{p}} \frac{\omega_N}{p} \int_{\frac{k}{q^*}}^\infty s^{\frac{N+a}{p}-1} e^{-s} \mathrm{d}s.
\end{aligned} \tag{3.5}$$

Combining (3.5) with

$$k^{-\frac{p+a-\ell_0}{p}} \int_{\frac{k}{q^*}}^\infty s^{\frac{N+a}{p}-1} e^{-s} \mathrm{d}s \to 0, \quad \text{as } k \to \infty,$$

we have for some $C_2 > 0$

$$\int_{\mathbb{R}^N} \widetilde{V}(x)|u_k|^p \mathrm{d}x \leqslant C_2, \quad \forall k \in \mathbb{N}. \tag{3.6}$$

By (3.4) and (3.6), we see that the sequence $\{u_k\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N; \widetilde{A}, \widetilde{V})$.

Let $q \geqslant q^*$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \widetilde{Q}(x)|u_k|^q \mathrm{d}x &\geqslant \int_{B_1} |x|^{b_0}|u_k|^q \mathrm{d}x + \int_{\mathbb{R}^N \setminus B_2} |x|^b |u_k|^q \mathrm{d}x \\
&\geqslant k^{\frac{q(N+\ell_0-p)}{p^2}} \omega_N \int_{B_1} |x|^{b_0} e^{-\frac{kq|x|^p}{p\cdot q^*}} \mathrm{d}x \\
&= \frac{\omega_N}{p} \left(\frac{pq^*}{q}\right)^{\frac{N+b_0}{p}} k^{\frac{q(N+\ell_0-p)-p(N+b_0)}{p^2}} \int_0^{\frac{kq}{pq^*}} s^{\frac{N+b_0}{p}-1} e^{-s} \mathrm{d}s.
\end{aligned} \tag{3.7}$$

Since $q > q^*$ implies that $q(N+\ell_0-p) - p(N+b_0) > 0$ and

$$\lim_{k\to\infty} \int_0^{\frac{kq}{pq^*}} s^{\frac{N+b_0}{p}-1} e^{-s} \mathrm{d}s = \Gamma\left(\frac{N+b_0}{p}\right), \tag{3.8}$$

it follows from (3.7) that

$$\int_{\mathbb{R}^N} \widetilde{Q}(x)|u_k|^q \mathrm{d}x \to \infty \quad \text{as } k \to \infty. \tag{3.9}$$

Therefore, there is no embedding from $W_r^{1,p}(\mathbb{R}^N; A, V)$ into $L^q(\mathbb{R}^N; Q)$ for any $q > q^*$.

Now we prove the second conclusion of this theorem. Let $q = q^*$ in (3.7). Then we have

$$\int_{\mathbb{R}^N} \widetilde{Q}(x)|u_k|^{q^*}dx \geqslant p^{\frac{N+b_0-p}{p}}\omega_N \int_0^{\frac{k}{p}} s^{\frac{N+b_0}{p}-1}e^{-s}ds \to p^{\frac{N+b_0-p}{p}}\omega_N\Gamma\left(\frac{N+b_0}{p}\right) > 0, \tag{3.10}$$

as $k \to \infty$. From the definition in (3.1) we see that

$$u_k(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } k \to \infty. \tag{3.11}$$

Therefore, $\{u_k\}$ has no convergent subsequence in $L^{q^*}(\mathbb{R}^N;\widetilde{Q})$. □

Remark 3.1. From the arguments in the proof of Theorem 1.2 we have some remarks. Since $W_r^{1,p}(\mathbb{R}^N; A, V)$ is a subspace of $W^{1,p}(\mathbb{R}^N; A, V)$, under the assumptions of Theorem 1.2, there is no embedding from $W^{1,p}(\mathbb{R}^N; A, V)$ into $L^q(\mathbb{R}^N; Q)$ for any $q > q^*$. In particular, there is no embedding from $W^{1,p}(\mathbb{R}^N; |x|^\sigma, |x|^\beta)$ into $L^q(\mathbb{R}^N; |x|^\alpha)$ for any $q > q^*(\sigma, \alpha, \beta)$ when $1 < p < N + \sigma$, $\beta \geqslant \sigma - p$, and $\alpha \geqslant \sigma - p$. However, we do not know whether or not there is a continuous embedding from $W^{1,p}(\mathbb{R}^N; A, V)$ into $L^{q^*}(\mathbb{R}^N; Q)$.

Now we turn our attention to the lower critical exponent and give the proof of Theorem 1.3.

Proof of Theorem 1.3. Define

$$\widehat{A}(x) = \psi(x)|x|^{\ell_0} + (1 - \psi(x))|x|^{\ell},$$
$$\widehat{V}(x) = \psi(x)|x|^{a_0} + (1 - \psi(x))|x|^{a},$$
$$\widehat{Q}(x) = \psi(x)|x|^{b_0} + (1 - \psi(x))|x|^{b},$$

where $\psi \in C_{0,r}^\infty(\mathbb{R}^N)$ satisfying $\psi(x) = \begin{cases} 1, & \text{for } |x| \leqslant 1 \\ 0, & \text{for } |x| \geqslant 2 \end{cases}$ and $0 \leqslant \psi \leqslant 1$. Then $\widehat{A}$, $\widehat{V}$, and $\widehat{Q}$ satisfy assumptions (A), (V), and (Q), respectively. Define a sequence $\{u_k\}_{k=1}^\infty$ as follows:

$$u_k(x) = k^{-\frac{N+b}{p^2}} e^{-\frac{|x|^p}{pk}}, \quad k \in \mathbb{N}. \tag{3.12}$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^N} \widehat{A}(x)|\nabla u_k|^p dx &\leqslant \int_{B_2} |x|^{\ell_0}|\nabla u_k|^p dx + \int_{\mathbb{R}^N\setminus B_1} |x|^{\ell}|\nabla u_k|^p dx \\ &= \omega_N k^{-\frac{N+b}{p}-p} \int_0^2 r^{N+\ell_0+p(p-1)-1} e^{-\frac{r^p}{k}} dr + \omega_N k^{-\frac{N+b}{p}-p} \int_1^\infty r^{N+\ell+p(p-1)-1} e^{-\frac{r^p}{k}} dr \\ &\leqslant \omega_N k^{-\frac{N+b}{p}-p} \int_0^2 r^{N+\ell_0+p(p-1)-1} dr + k^{\frac{\ell-b-p}{p}} \frac{\omega_N}{p} \int_{\frac{1}{k}}^\infty s^{\frac{N+\ell+p(p-1)}{p}-1} e^{-s} ds \\ &\leqslant \frac{2^{N+\ell_0+p(p-1)}\omega_N}{N+\ell_0+p(p-1)} + \frac{\omega_N k^{\frac{\ell-b-p}{p}}}{p}\Gamma\left(\frac{N+\ell+p(p-1)}{p}\right),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\int_{\mathbb{R}^N} \widehat{V}(x)|u_k|^p dx &\leqslant \int_{B_2} |x|^{a_0}|u_k|^p dx + \int_{\mathbb{R}^N\setminus B_1} |x|^{a}|u_k|^p dx \\ &= k^{-\frac{N+b}{p}}\omega_N \int_0^2 r^{N-1+a_0} e^{-\frac{r^p}{k}} dr + k^{-\frac{N+b}{p}}\omega_N \int_1^\infty r^{N-1+a} e^{-\frac{r^p}{k}} dr \\ &\leqslant k^{-\frac{N+b}{p}}\omega_N \int_0^2 r^{N-1+a_0} dr + \frac{\omega_N}{p} \int_{\frac{1}{k}}^\infty s^{\frac{N+a}{p}-1} e^{-s} ds \\ &\leqslant \omega_N \frac{2^{N+a_0}}{N+a_0} + \frac{\omega_N}{p}\Gamma\left(\frac{N+a}{p}\right).\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that the sequence $\{u_k\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N; \widehat{A}, \widehat{V})$.

For $q < p$, we have

$$\int_{\mathbb{R}^N} \widehat{Q}(x)|u_k|^q \mathrm{d}x \geqslant \int_{\mathbb{R}^N\setminus B_2} |x|^b|u_k|^q \mathrm{d}x = \frac{1}{p}\left(\frac{p}{q}\right)^{\frac{N+b}{p}} k^{\frac{(p-q)(N+b)}{p^2}} \omega_N \int_{\frac{q2^p}{pk}}^{\infty} s^{\frac{N+b}{p}-1}e^{-s}\mathrm{d}s. \tag{3.15}$$

Since $N + b > 0$, we have that

$$\lim_{k\to\infty} \int_{\frac{q2^p}{pk}}^{\infty} s^{\frac{N+b}{p}-1}e^{-s}\mathrm{d}s = \Gamma\left(\frac{N+b}{p}\right). \tag{3.16}$$

It follows from (3.15), (3.16), and $q < p$ that

$$\int_{\mathbb{R}^N} \widehat{Q}(x)|u_k|^q \mathrm{d}x \to \infty \quad \text{as } k \to \infty. \tag{3.17}$$

Therefore, there are no embeddings from $W_r^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^q(\mathbb{R}^N; Q)$ for any $q < p$.

Now on one hand, we have

$$\int_{\mathbb{R}^N} \widehat{Q}(x)|u_k|^p \mathrm{d}x \geqslant \int_{\mathbb{R}^N\setminus B_2} |x|^b|u_k|^p \mathrm{d}x = \frac{\omega_N}{p} \int_{\frac{2^p}{k}}^{\infty} s^{\frac{N+b}{p}-1}e^{-s}\mathrm{d}s \to \frac{\omega_N}{p}\Gamma\left(\frac{N+b}{p}\right) > 0, \quad \text{as } k \to \infty. \tag{3.18}$$

It follows that $\{\|u_k\|_{L^p(\mathbb{R}^N;\widehat{Q})}\}$ is bounded from below by a positive constant. On the other hand, by the definition of (3.12), we see easily that

$$u_k(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } k \to \infty. \tag{3.19}$$

Thus, $\{u_k\}$ has no convergent subsequence in $L^p(\mathbb{R}^N; \widehat{Q})$. □

Remark 3.2. From the arguments in the proof of Theorem 1.3, we have a new observation. The assumptions of Theorem 1.3 plus $Q(x) \leqslant V(x)$ for all $x \in \mathbb{R}^N$ imply that there is no embedding from $W^{1,p}(\mathbb{R}^N; A, V)$ into $L^p(\mathbb{R}^N; Q)$ for any $q < p$ and the embedding $W^{1,p}(\mathbb{R}^N; A, V) \hookrightarrow L^p(\mathbb{R}^N; Q)$ is not compact.

Next we give the proof of Theorem 1.4. We follow some ideas from [8] by Gladiali et al. After finishing this article we found a paper by Horiuchi [10]. By a careful comparison we saw that (1.21) and (1.22) had been given in [10] without a detailed argument. We would like to keep the arguments of the proof for the readers' convenience.

Proof of Theorem 1.4. For $u \in D_r^{1,p}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |x|^\sigma |\nabla u|^p \mathrm{d}x = \omega_N \int_0^{\infty} r^{N+\sigma-1}|u'(r)|^p \mathrm{d}r. \tag{3.20}$$

Set

$$r = s^{\frac{p}{p+\alpha-\sigma}}, \quad v(s) = u(s^{\frac{p}{p+\alpha-\sigma}}) = u(r). \tag{3.21}$$

Then

$$\int_0^{\infty} r^{N+\sigma-1}|u'(r)|^p \mathrm{d}r = \left(\frac{p+\alpha-\sigma}{p}\right)^{p-1} \int_0^{\infty} s^{\kappa-1}|v'(s)|^p \mathrm{d}s, \tag{3.22}$$

where

$$\kappa = \frac{p(N+\alpha)}{p+\alpha-\sigma} > p. \tag{3.23}$$

For $1 < p < \kappa$ and $\tau = \frac{\kappa p}{\kappa - p}$, we obtain by applying [18, Lemma 2] that

$$\int_0^\infty s^{\kappa-1}|v'(s)|^p \mathrm{d}s \geqslant S_\kappa(\mathbb{R}_+)\left(\int_0^\infty s^{\kappa-1}|v(s)|^\tau \mathrm{d}s\right)^{\frac{p}{\tau}}, \tag{3.24}$$

where

$$S_\kappa(\mathbb{R}_+) = \kappa\left(\frac{\kappa-p}{p-1}\right)^{p-1}\left[\left(\frac{p-1}{p}\right)B\left(\frac{\kappa}{p}, \frac{\kappa(p-1)}{p}\right)\right]^{\frac{p}{\kappa}}. \tag{3.25}$$

Now it follows from (3.20) to (3.25) that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\sigma |\nabla u|^p \mathrm{d}x &\geqslant \omega_N\left(\frac{p+\alpha-\sigma}{p}\right)^{p-1} S_\kappa(\mathbb{R}_+)\left(\int_0^\infty s^{\kappa-1}|v(s)|^\tau ds\right)^{\frac{p}{\tau}} \\ &= \omega_N^{\frac{p}{\kappa}}\left(\frac{p+\alpha-\sigma}{p}\right)^{\frac{p(\kappa-1)}{\kappa}} S_\kappa(\mathbb{R}_+)\left(\omega_N\int_0^\infty r^{N+\alpha-1}|u(r)|^{\frac{p\kappa}{\kappa-p}}\mathrm{d}r\right)^{\frac{\kappa-p}{\kappa}} \\ &= S_{\sigma,\alpha}\left(\int_{\mathbb{R}^N}|x|^\alpha|u|^{q^*(\sigma,\alpha)}\mathrm{d}x\right)^{\frac{p}{q^*(\sigma,\alpha)}},\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}S_{\sigma,\alpha} &= \omega_N^{\frac{p}{\kappa}}\left(\frac{p+\alpha-\sigma}{p}\right)^{\frac{p(\kappa-1)}{\kappa}} S_\kappa(\mathbb{R}_+) \\ &= \kappa\left(\frac{p+\alpha-\sigma}{p}\right)^{\frac{p(\kappa-1)}{\kappa}}\left(\frac{\kappa-p}{p-1}\right)^{p-1}\left[\omega_N\left(\frac{p-1}{p}\right)B\left(\frac{\kappa}{p}, \frac{\kappa(p-1)}{p}\right)\right]^{\frac{p}{\kappa}} \\ &= (N+\alpha)\left(\frac{1}{p+\alpha-\sigma}\right)^{\frac{p+\alpha-\sigma}{N+\alpha}}\left(\frac{N+\sigma-p}{p-1}\right)^{p-1}\left[(p-1)\omega_N B\left(\frac{N+\alpha}{p+\alpha-\sigma}, \frac{(p-1)(N+\alpha)}{p+\alpha-\sigma}\right)\right]^{\frac{p+\alpha-\sigma}{N+\alpha}}.\end{aligned} \tag{3.27}$$

By the density, we arrive at (1.20) and (1.21).

It follows from (3.24) and (3.21) that an extremal function u for $S_{\sigma,\alpha}$ is exactly an extremal function v for $S_\kappa(\mathbb{R}_+)$, i.e., the function v satisfies

$$\int_0^\infty s^{\kappa-1}|v'(s)|^p \mathrm{d}s = S_\kappa(\mathbb{R}_+)\left(\int_0^\infty s^{\kappa-1}|v(s)|^{\frac{\kappa p}{\kappa-p}}\mathrm{d}s\right)^{\frac{\kappa-p}{\kappa}}. \tag{3.28}$$

By [18, Lemma 2], there is a unique extremal function for $S_\kappa(\mathbb{R}_+)$ taking the form

$$\tilde{v}(s) = \left(m + ns^{\frac{p}{p-1}}\right)^{\frac{p-\kappa}{p}}, \quad m, n > 0. \tag{3.29}$$

Changing variable back to $|x|$ by (3.21), we see that the extremal function for $S_{\sigma,\alpha}(\mathbb{R}^N)$ takes the form

$$\tilde{u}(x) = \left(m + n|x|^{\frac{p+\alpha-\beta}{p-1}}\right)^{-\frac{N+\sigma-p}{p+\alpha-\sigma}}. \tag{3.30}$$

Precisely, the function $\tilde{u}$ satisfies

$$\int_{\mathbb{R}^N} |x|^\sigma |\nabla \tilde{u}|^p \mathrm{d}x = S_{\sigma,\alpha}\left(\int_{\mathbb{R}^N} |x|^\alpha |\tilde{u}|^{q^*(\sigma,\alpha)} \mathrm{d}x\right)^{\frac{p}{q^*(\sigma,\alpha)}}. \tag{3.31}$$

A complicated calculation shows that

$$S_{\sigma,\alpha}\left(\int_{\mathbb{R}^N} |x|^\alpha |\tilde{u}|^{q^*(\sigma,\alpha)} \mathrm{d}x\right)^{\frac{p-q^*(\sigma,\alpha)}{q^*(\sigma,\alpha)}} = \frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}} mn^{p-1}. \tag{3.32}$$

By the Lagrange multiplier theorem and (3.32) we have

$$-\mathrm{div}(|x|^\sigma |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}) = \frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}} mn^{p-1}|x|^\alpha \tilde{u}^{q^*(\sigma,\alpha)-1}. \tag{3.33}$$

After a first scaling, we have that the function

$$U(x) = \left(\frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}}\right)^{\frac{N+\sigma-p}{p(p+\alpha-\sigma)}} \frac{\left(\frac{n}{m}\right)^{\frac{(p-1)(N+\sigma-p)}{p(p+\alpha-\sigma)}}}{\left(1+\frac{n}{m}|x|^{\frac{p+\alpha-\sigma}{p-1}}\right)^{\frac{N+\sigma-p}{p+\alpha-\sigma}}} \tag{3.34}$$

is the extremal function for $S_{\sigma,\alpha}$. By scaling invariance, for $\lambda > 0$, we obtain $U_\lambda(x) := \lambda^{\frac{N+\sigma-p}{p}} U(\lambda x)$ is the extremal function of $S_{\sigma,\alpha}$ and a solution of equation (1.23). Taking $\lambda_0 = \left(\frac{m}{n}\right)^{\frac{p-1}{p+\alpha-\sigma}}$, then

$$U_{\lambda_0}(x) = \frac{\left(\frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}}\right)^{\frac{N+\sigma-p}{p(p+\alpha-\sigma)}}}{\left(1+|x|^{\frac{p+\alpha-\sigma}{p-1}}\right)^{\frac{N+\sigma-p}{p+\alpha-\sigma}}}. \tag{3.35}$$

By a second scaling, we obtain the functions

$$U_{\varepsilon,\sigma,\alpha}(x) = \frac{\varepsilon^{\frac{N+\sigma-p}{p(p-1)}}\left(\frac{(N+\sigma-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}}\right)^{\frac{N+\sigma-p}{p(p+\alpha-\sigma)}}}{\left(\varepsilon^{\frac{p+\alpha-\sigma}{p-1}}+|x|^{\frac{p+\alpha-\sigma}{p-1}}\right)^{\frac{N+\sigma-p}{p+\alpha-\sigma}}},$$

which are the extremal functions for $S_{\sigma,\alpha}$ and the solutions of (1.23). □

Proof of Theorem 1.6. The proof is similar to Theorem 1.2. Define $\overline{A}(x) = |x|^{\ell_0}$, $\overline{Q}(x) = |x|^{b_0}$. They satisfy (A') and (Q'), respectively. Define a sequence $\{u_k\}_{k=1}^\infty$ as follows

$$u_k(|x|) = k^{\frac{N+\ell_0-p}{p^2}}\left(e^{-\frac{k|x|^p}{p\cdot q^*}} - e^{-\frac{k}{p\cdot q^*}}\right), \quad |x| < 1, \quad k \in \mathbb{N}. \tag{3.36}$$

Since

$$\int_B \bar{A}(x)|\nabla u_k|^p \mathrm{d}x = \frac{k^{\frac{N+\ell_0-p}{p}+p}}{(q^*)^p}\int_B |x|^{\ell_0+p(p-1)} e^{-\frac{k|x|^p}{q^*}} \mathrm{d}x \leqslant (q^*)^{\frac{N+\ell_0-p}{p}} \frac{\omega_N}{p}\Gamma\left(\frac{N+\ell_0+p(p-1)}{p}\right). \tag{3.37}$$

It follows that $\{u_k\}$ is bounded in $W_{0,r}^{1,p}(B, \overline{A})$. Let $q > q^*$. By a direct calculation, we have

$$
\begin{aligned}
\int_B \bar{Q}(x)|u_k|^q \mathrm{d}x &= \frac{\omega_N}{p} k^{\frac{q(N+\ell_0-p)}{p^2}-\frac{N+b_0}{p}} \int_0^k s^{\frac{N+b_0}{p}-1} \left| e^{-\frac{s}{p\cdot q^*}} - e^{-\frac{k}{p\cdot q^*}} \right|^q \mathrm{d}s \\
&\geqslant \frac{1}{p} k^{\frac{q(N+\ell_0-p)}{p^2}-\frac{N+b_0}{p}} \left| e^{-\frac{1}{2p\cdot q^*}} - e^{-\frac{1}{p\cdot q^*}} \right|^q \omega_N \int_0^{\frac{1}{2}} s^{\frac{N+b_0}{p}-1} \mathrm{d}s \\
&= \frac{\omega_N (N+b_0) 2^{-\frac{N+b_0}{p}}}{p^2} k^{\frac{q(N+\ell_0-p)}{p^2}-\frac{N+b_0}{p}} \left| e^{-\frac{1}{2p\cdot q^*}} - e^{-\frac{1}{p\cdot q^*}} \right|^q .
\end{aligned} \tag{3.38}
$$

Since $q > q^*$ implies $q(N + \ell_0 - p) - p(N + b_0) > 0$, we have by (3.38) that

$$
\int_B \overline{Q}(|x|)|u_k|^q \mathrm{d}x \to \infty \quad \text{as} \quad k \to \infty. \tag{3.39}
$$

Therefore, there is no embedding from $W_r^{1,p}(B;\overline{A})$ into $L^q(B;\overline{Q})$ for any $q > q^*$. On one hand, we have

$$
\begin{aligned}
\int_B \overline{Q}(x)|u_k|^{q^*} \mathrm{d}x &= \frac{\omega_N}{p} \int_0^k s^{\frac{N+b_0}{p}-1} \left| e^{-\frac{s}{p\cdot q^*}} - e^{-\frac{k}{p\cdot q^*}} \right|^{q^*} \mathrm{d}s \\
&\geqslant \frac{1}{p} \left| e^{-\frac{1}{2p\cdot q^*}} - e^{-\frac{1}{p\cdot q^*}} \right|^{q^*} \omega_N \int_0^{\frac{1}{2}} s^{\frac{N+b_0}{p}-1} \mathrm{d}s \\
&= \frac{\omega_N (N+b_0)}{2^{\frac{N+b_0}{p}} p^2} \left| e^{-\frac{1}{2p\cdot q^*}} - e^{-\frac{1}{p\cdot q^*}} \right|^{q^*} > 0.
\end{aligned} \tag{3.40}
$$

On the other hand, by the definition of (3.36), we have

$$
u_k(x) \to 0 \quad \text{a.e. in} \quad B \quad \text{as} \quad k \to \infty. \tag{3.41}
$$

Thus, $\{u_k\}$ has no convergent subsequence in $L^{q^*}(B;\overline{Q})$. □

Remark 3.3. We finally remark that under the assumptions of Theorem 1.6 there is no embedding from $W_0^{1,p}(B;\, A, V)$ into $L^q(B;\, Q)$ for any $q > q^*$ and there is no embedding from $W_0^{1,p}(B;\, |x|^\sigma)$ into $L^q(B;\, |x|^\alpha)$ for any $q > q^*(\sigma, \alpha)$ as $1 < p < N + \sigma, \alpha \geqslant \sigma - p$.

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