Research Article

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Large solutions of a class of degenerate equations associated with infinity Laplacian

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Abstract: In this article, we investigate the boundary blow-up problem

$$\begin{cases}\Delta_\infty^h u = f(x,u), & \text{in } \Omega,\\ u = \infty, & \text{on } \partial\Omega,\end{cases}$$

where $\Delta_\infty^h u = |Du|^{h-3}\langle D^2uDu, Du\rangle$ is the highly degenerate operator related to infinity Laplacian which comes from the absolutely minimizing Lipschitz extension and has a close relationship with the random game named tug-of-war. When the function f satisfies the Keller-Osserman-type condition, we establish the existence of the boundary blow-up viscosity solution. Moreover, for the separable case $f(x,u) = b(x)g(u)$, we establish the asymptotic estimate of the blow-up solution near the boundary under some regular conditions of the domain. Based on the asymptotic estimate and comparison principle, we obtain the uniqueness of the large viscosity solution. During this procedure, we also study the non-existence of the large solution. For the separable case, we show that the Keller-Osserman-type condition is sufficient and necessary for the existence of the boundary blow-up viscosity solution.

Keywords: infinity Laplacian, blow-up solution, comparison principle, existence, boundary asymptotic estimate

MSC 2020: 35J60, 35J70, 35B40

1 Introduction

In this article, we are interested in the following problem:

$$\begin{cases}\Delta_\infty^h u = f(x,u), & \text{in } \Omega,\\ u = \infty, & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where

$$\Delta_\infty^h u(x) := |Du(x)|^{h-3}\langle D^2uDu, Du\rangle = |Du(x)|^{h-3}\sum_{i,j=1}^{n} D_iuD_juD_{ij}u, \quad h > 1$$

denotes the degenerate elliptic operator and $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain.

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For $h = 3$, the operator $\Delta_\infty^h u$ is the infinity Laplacian operator $\Delta_\infty u = \sum_{i,j=1}^n D_i u D_j u D_{ij} u$, which is closely related to the absolutely minimizing Lipschitz extension (AMLE) that was first introduced by Aronsson [1–4] in 1960's. A function $u \in C(\Omega)$ is said to be an absolutely minimizing Lipschitz function in Ω if and only if for any $\Omega_0 \subset\subset \Omega$ and any $v \in C(\overline{\Omega}_0)$ with $v = u$ on $\partial\Omega_0$, there holds

$$\mathrm{Lip}(u, \Omega_0) \leq \mathrm{Lip}(v, \Omega_0).$$

And a function $u \in C(\Omega)$ is an absolutely minimizing Lipschitz function in Ω if and only if u is a viscosity solution to the infinity Laplacian equation in Ω,

$$\Delta_\infty u = 0, \tag{1.2}$$

and the viscosity solution u is also called an infinity harmonic function. It was only in 1993 that Jensen [5] showed the equivalence of the AMLE and viscosity solutions of the homogeneous infinity Laplace equation and the uniqueness of AMLE. Armstrong and Smart [6] gave an easy proof for the uniqueness of infinity harmonic functions. Crandall et al. [7] studied the uniqueness of the viscosity solution of the homogeneous infinity Laplacian equation in an unbounded domain. Crandall et al. [8] proved that an infinity harmonic function $u \in C(\Omega)$ is equivalent to it enjoys the so-called comparison property with cones in Ω. Aronsson et al. [9] showed a systematic treatment of the theory of AMLEs. For the inhomogeneous case

$$\Delta_\infty u = f(x),$$

Lu and Wang [10] proved the existence and uniqueness of a viscosity solution of the Dirichlet problem when the inhomogeneous term f does not change its sign. They established the fact that the viscosity solutions of the perturbed equations converge uniformly to the unique viscosity solution of the homogeneous equation when its right-hand side and boundary data are perturbed simultaneously. They also showed the comparison with standard function property for the viscosity solutions to the inhomogeneous equation with constant right-hand side, which extended the result of Crandall et al. [8]. In [11], Bhattacharya and Mohammed established the existence or nonexistence of viscosity solutions to the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f(x, u), & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for f with the sign and the monotonicity restrictions and $g \in C(\partial\Omega)$. In [12], they further removed the sign and the monotonicity restrictions and gave the existence result from the general structure condition on f. For more results of absolute minimizers, one can see Barron et al. [13], Barles and Busca [14], Yu [15] etc.

For $h = 1$, the operator $\Delta_\infty^h u$ is the normalized ∞-Laplacian operator (also called game ∞-Laplacian)

$$\Delta_\infty^G u = \frac{1}{|Du|^2} \sum_{i,j=1}^n D_i u D_j u D_{ij} u,$$

which has a "tug-of-war" stochastic game approach first introduced by Peres et al. [16]. Roughly speaking, the game is in a set Ω, a running payoff function f in Ω and a terminal payoff function $g(x)$ defined on the boundary of Ω. The game is played by two players who take turns depending on the outcome of a coin toss. A token is initially placed at a point $x_0 \in \Omega$. During each turn, the player can move the token to any point in an open ball of size ε around the current position. If the player's move takes the token to a point $x_g \in \partial\Omega$, then the game is over and the players are rewarded or penalized through the payoff functions f and g. For $\varepsilon \to 0$, the continuum value function of this game is shown to solve the following game infinity Laplace equation:

$$\begin{cases} \Delta_\infty^G u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Lu and Wang [17] gave a different proof for the Dirichlet problem (1.4) based on partial differential equation (PDE) methods. They established the well-posedness and the comparison with polar quadratic polynomials property for the normalized infinity Laplace equation. Note that the uniqueness is valid when the inhomogeneous term $f(x)$ is positive or negative for equations of Δ_∞^G [17,16] and Δ_∞ [10]. A counter-example in [16] showed that the uniqueness of a viscosity solution of the Dirichlet problem for the inhomogeneous equation is invalid if the inhomogeneous term $f(x)$ changes its sign. When f is non-positive or non-negative, it is still open whether the uniqueness holds. Lu and Wang [18] constructed a least and a greatest viscosity solution of equation (1.4) with sign-changing inhomogeneous term f in a bounded domain. Furthermore, they proved that these extremal solutions are absolutely extremal solutions. Lu and Wang [19] studied further the uniqueness for a general degenerate elliptic PDE including both normalized infinity Laplacian Δ_∞^G and non-normalized infinity Laplacian Δ_∞. For more game theory related to infinity Laplacian, we direct the reader to the papers of Barron et al. [20], Evans [21] and Rossi [22]. For generalized tug-of-war games, one can see [15,23–25].

Since the operator Δ_∞^h has no divergent structure, we understand a function $u \in C(\Omega)$ verifying the equation in the viscosity sense. And due to $u(x) \to \infty$ as $x \to \partial\Omega$, the viscosity solution to (1.1) is called large solution or boundary blow-up solution.

The infinity Laplacian has received a lot of attention in recent years, notably due to its many applications in image processing [26,27] and optimal mass transportation problems [28,29]. Note that equations (1.1) involved in Δ_∞^h, besides their wide applications, are not only degenerate, singular for $1 < h < 3$, but also not in divergence form and have no variational structure. They constitute a class of operators with particular properties.

In this article, we consider problem (1.1) involved in the family of the infinity Laplacian Δ_∞^h for all $h > 1$. The main purpose of this article is to investigate the comparison principle, uniqueness, existence, and non-existence of viscosity solutions. We also investigate the boundary behavior of the blow-up viscosity solutions. The general function $f(x, u)$ on the right-hand satisfies the following conditions:

(**f1**) $f \in C((\Omega, [0,\infty)), [0,\infty))$, $f(x,t)$ is non-decreasing in t for each $x \in \Omega$ and $f(x,0) = 0$ for all $x \in \Omega$ while $f(x,t) > 0$ for all $(x,t) \in \Omega \times (0,\infty)$;

(**f2**) For each $x \in \Omega_\delta = \{x \in \Omega : d(x) < \delta\}$, where $\delta > 0$ and $d(x) = \mathrm{dist}(x, \partial\Omega)$, the Keller-Osserman-type condition is as follows:

$$\int_1^\infty \frac{1}{F_*(t;\, x, r)^{\frac{1}{h+1}}} \mathrm{d}t < \infty, \quad t > 0, \tag{1.5}$$

where $r > 0$ and

$$F_*(t;\, x, r) := \int_0^t f_*(s;\, x, r)\mathrm{d}s, \quad f_*(t;\, x, r) := \min\{f(z,t) : z \in \overline{B}(x,r) \cap \overline{\Omega}\},$$

for each $(x, r) \in \overline{\Omega} \times (0, \infty)$.

Our main results are summarized as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If f satisfies (**f1**) and (**f2**), then problem* (1.1) *admits a non-negative viscosity solution.*

In order to prove Theorem 1.1, we first establish the comparison principle and then combine Perron's method with compactness arguments. For the comparison result, the method we use is the classical perturbation argument for viscosity solutions. Due to the difficulty of the strong degeneracy we must construct suitable double variable function. As for the existence of blow-up solutions, due to the high degeneracy and singularity of (1.1), we adopt Perron's method. The key point is to construct suitable barrier functions and calculate carefully.

We also establish the non-existence result for problem (1.1).

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If f satisfies* (**f1**) *and the following condition at some $x_0 \in \partial\Omega$,*

$$\int_1^\infty \frac{1}{f^*(t;x_0,r)^{\frac{1}{h}}}\mathrm{d}t=\infty, \quad \textit{for some } r>0, \tag{1.6}$$

where

$$f^*(t;x,r):=\max\{f(z,t): x\in\overline{\Omega}, z\in\overline{B}(x,r)\cap\overline{\Omega}\},$$

then the problem (1.1) *has no viscosity solution.*

Due to the high degeneracy of Δ_∞^h, we prove the non-existence of the viscosity solution of (1.1) by the perturbation method. Note that Theorem 1.2 only gives a sufficient condition for the non-existence of the viscosity solution to (1.1), not a necessary condition. In fact, we can give the necessary and sufficient conditions for the non-existence of the viscosity solution in the separable case $f(x,u)=b(x)g(u)$. Now we first give the condition for the functions $b(\cdot)$ and $g(\cdot)$.

(**b1**) $b\in C(\overline{\Omega},(0,\infty))$;

(**b2**) There exist a function $k\in\Lambda$ and a positive constant b_0 such that

$$\lim_{d(x)\to 0}\frac{b(x)}{k^{h+1}(d(x))}=b_0, \tag{1.7}$$

where Λ represents the set of all positive, monotonic functions $k\in C^1$ which satisfy

$$\lim_{t\to 0^+}\left(\frac{K(t)}{k(t)}\right)'=\beta<\infty, \quad \text{where } K(t)=\int_0^t k(s)\mathrm{d}s. \tag{1.8}$$

(**g1**) $g\in C([0,\infty),[0,\infty))$, $g(0)=0$, $g(t)>0$ for $t>0$, and g is a non-decreasing function;

(**g2**) the Keller-Osserman-type condition,

$$\int_1^\infty \frac{1}{\sqrt[h+1]{G(t)}}\mathrm{d}t<\infty, \quad \text{where } G(t)=\int_0^t g(s)\mathrm{d}s. \tag{1.9}$$

Remark 1.1. If b satisfies (**b1**) and g satisfies (**g1**) and (**g2**), it is easy to obtain the existence of the viscosity solution to (1.1) for $f(x,u)=b(x)g(u)$ by Theorem 1.1.

Theorem 1.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Suppose that b satisfies (**b1**) and g satisfies (**g1**). If g fails to satisfy (**g2**), then the problem*

$$\begin{cases}\Delta_\infty^h u=b(x)g(u), & \textit{in } \Omega,\\ u=\infty, & \textit{on } \partial\Omega,\end{cases} \tag{1.10}$$

has no viscosity solution.

Remark that Theorem 1.3 shows that the Keller-Osserman-type condition is necessary for the existence of the blow-up solution. The non-existence of the viscosity solution of (1.10) is analyzed by the method of arguing by contradiction.

When the domain Ω possesses the additional C^1 regularity, we can establish the following asymptotic estimate near the boundary and then the uniqueness of the blow-up solution of (1.10) follows immediately based on the comparison principle.

Theorem 1.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded C^1 domain. Suppose that b satisfies (**b1**) and (**b2**), g satisfies (**g1**) and*

$$\lim_{t\to\infty}\frac{g(t)}{(h+1)G(t)^{h/(h+1)}}\int_t^\infty G(s)^{-1/(h+1)}\mathrm{d}s=\alpha\geq 1.$$

If $\alpha + \beta > 1$, where β is as in (1.8), *then the viscosity solution u to* (1.10) *satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\vartheta(K(d(x)))} = \left(\frac{\alpha + \beta - 1}{\alpha b_0}\right)^{\frac{\alpha-1}{h+1}}, \tag{1.11}$$

where ϑ satisfies

$$\int_{\vartheta(t)}^{\infty} \frac{1}{[(h+1)G(s)]^{\frac{1}{h+1}}} \mathrm{d}s = t.$$

Furthermore, if $\frac{g(t)}{t^h}$ is non-decreasing on $(0, \infty)$, *then problem* (1.10) *has a unique viscosity solution.*

One should note that if the regularity assumption of Theorem 1.4 holds, then the distance function is a solution of $\Delta_\infty^h v = 0$ near the boundary. Thus, we can perturb the distance function to analyze the boundary asymptotic behavior of the blow-up solutions based on the comparison principle and Karamata's regular variation theory. And then, by the comparison principle, the uniqueness result of the viscosity solution follows immediately.

In [30], for the particular case $h = 1$ and $f(u) = u^q$, Juutinen and Rossi showed that the Keller-Osserman-type condition

$$\int_a^{\infty} \frac{1}{\sqrt[2]{F(t)}} \mathrm{d}t < \infty, \quad \text{where } F(t) = \int_0^t f(s)\mathrm{d}s, \ a > 0 \tag{1.12}$$

is necessary and sufficient for the existence of the viscosity solutions to the boundary blow-up problem

$$\begin{cases} \Delta_\infty^G u = u^q, & \text{in } \Omega, \\ u = \infty, & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

They also established the following boundary asymptotic behavior of the blow-up solution,

$$u(x) \sim \left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}} d(x)^{-\frac{2}{q-1}}, \quad d(x) \to 0.$$

For $h = 3$, Mohammed and Mohammed [31,32] studied the large solutions to

$$\begin{cases} \Delta_\infty u = b(x)f(u), & \text{in } \Omega, \\ u = \infty, & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

where $b \in C(\overline{\Omega})$ is nonnegative, $f \in C[0, \infty) \cap C^1(0, \infty)$, $f(0) = 0$, $f(s) > 0$, $s > 0$, and $f(s)$ is nondecreasing on $[0, \infty)$. They proved that (1.14) has a non-negative viscosity solution if the following Keller-Osserman-type condition holds:

$$\int_a^{\infty} \frac{1}{\sqrt[4]{4F(t)}} \mathrm{d}t < \infty, \quad \text{where } F(t) = \int_0^t f(s)\mathrm{d}s, \ a > 0. \tag{1.15}$$

This means that if the weight function $b(x)$ is bounded, the presence of $b(x)$ is not really important for the existence of the large solutions. In [33], Wang et al. studied the exact boundary blow-up estimates with the first expansion when f is regularly varying at infinity with index $p > 3$ and the weighted function $b(x)$ is controlled on the boundary in some manner. Furthermore, for the case of $f(s) = sp(1 + cg(s))$, with $c \in \mathbb{R}$ and the function g normalized regularly varying with index $-q < 0$, they obtained the second expansion of solutions near the boundary based on Karamata regular variation theory. In [34], under appropriate structure conditions on the inhomogeneous term f, Zhang established the following the boundary estimate of large solutions to problem (1.14)

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(K(d(x)))} = 1,$$

where ψ satisfies $\int_{\psi(t)}^{\infty} \frac{1}{\sqrt[4]{4F(s)}} ds = t$ and $F(t) = \int_0^t f(s) ds$. In [35], the boundary behavior of the blow-up solutions to problem (1.14) is studied under different conditions on weight function $b(x)$ and the nonlinear term f.

We direct the reader to papers [36–38]. for more results about boundary behavior of the blow-up solutions.

The rest of the article will be organized as follows. In Section 2, we give the definition of the viscosity solution of the equation $\Delta_\infty^h u = f(x, u)$. And by the perturbation method of viscosity solutions, we prove the comparison principle to $\Delta_\infty^h u = f(x, u)$. In Section 3, by combining Perron's method and compactness arguments, we establish the existence of viscosity solutions to (1.1). In Section 4, we give the non-existence of viscosity solutions to (1.1) and (1.10). Finally, in Section 5, under some regular assumption of the domain, by the Karamata's regular variation theory and the comparison principle, we give the characteristic of the boundary blow-up solution near the boundary, and then we obtain the uniqueness of viscosity solutions to (1.10) based on the comparison principle.

2 Comparison principles

In this section, we first introduce the concept of the viscosity solutions to equation

$$\Delta_\infty^h u = f(x, u), \quad \text{in } \Omega \tag{2.1}$$

and then establish the comparison results by the double variables method based on the viscosity solutions theory. It should be pointed out that the operator Δ_∞^h has no divergence structure and in order to give a reasonable explanation when the gradient vanishes, we use the definition of viscosity solutions based on semi-continuous extension and we refer the reader to [17,39,40]. Note that the singularity is removable when $h > 1$. Hence, we can rewrite equation (2.1) as

$$F_h(D^2 u, Du) = f(x, u), \quad x \in \Omega,$$

where $F_h : \mathbb{S} \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}$ and $F_h(M, p) := |p|^{h-3}(Mp)\cdot p$. Here $\mathbb{S}$ denotes the set of $n \times n$ real symmetric matrices. Since $h > 1$, we have $\lim_{p\to 0} F_h(M, p) = 0$ for arbitrary $M \in \mathbb{S}$. That is, the operator Δ_∞^h is continuous for $h > 1$. Hence, we can define the continuous extension of F_h as follows:

$$\overline{F}_h(M, p) := \begin{cases} F_h(M, p), & \text{if } p \neq 0, \\ 0, & \text{if } p = 0. \end{cases}$$

Now let us define the viscosity solution of equation (2.1).

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. An upper semi-continuous function $u : \Omega \to \mathbb{R}$ is said to be a viscosity subsolution to equation (2.1) if and only if for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) < \varphi(x)$ for all $x \in \Omega$ near x_0 and $x \neq x_0$, there holds

$$\overline{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \geq f(x_0, \varphi(x_0)).$$

Similarly, a lower semi-continuous function $u : \Omega \to \mathbb{R}$ is said to be a viscosity supersolution to equation (2.1) if and only if for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$ for all $x \in \Omega$ near x_0 and $x \neq x_0$, there holds

$$\overline{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \leq f(x_0, \varphi(x_0)).$$

If a continuous function $u \in C(\Omega)$ is both a viscosity subsolution and a viscosity supersolution of equation (2.1), then we say that u is a viscosity solution of equation (2.1).

We can also use super-jets and sub-jets (see [39]) to give the definition of the viscosity subsolution and the viscosity supersolution, respectively. Now we first recall the definition of super-jets and sub-jets.

The second order super-jet of an upper semi-continuous function u at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,+}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0\},$$

and its closure is

$$\overline{\mathcal{J}}^{2,+}u(x_0) := \{(p, M) \in \mathbb{R}^n \times \mathbb{S} : \exists (x_i, p_i, M_i) \in \Omega \times \mathbb{R} \times \mathbb{S} \text{ such that } (p_i, M_i) \in \mathcal{J}^{2,+}u(x_i) \text{ and } (x_i, p_i, M_i) \to (x_0, p, M)\}.$$

Similarly, the second-order sub-jet of a lower semi-continuous function u at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,-}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0\},$$

and its closure is

$$\overline{\mathcal{J}}^{2,-}u(x_0) := \{(p, M) \in \mathbb{R}^n \times \mathbb{S} : \exists (x_i, p_i, M_i) \in \Omega \times \mathbb{R} \times \mathbb{S} \text{ such that } (p_i, M_i) \in \mathcal{J}^{2,-}u(x_i) \text{ and } (x_i, p_i, M_i) \to (x_0, p, M)\}.$$

Definition 2.2. We say $u \in C(\Omega)$ is a viscosity subsolution to (2.1) if

$$\overline{F}_h(M, p) \geq f(x_0, u(x_0)), \quad \forall (p, M) \in \overline{\mathcal{J}}^{2,+}u(x_0), \ \forall x_0 \in \Omega.$$

Similarly, we say $u \in C(\Omega)$ is a viscosity supersolution to (2.1) if

$$\overline{F}_h(M, p) \leq f(x_0, u(x_0)), \quad \forall (p, M) \in \overline{\mathcal{J}}^{2,-}u(x_0), \ \forall x_0 \in \Omega.$$

By the definition of the viscosity subsolution of (2.1), it is easy to check that if u and v are both viscosity subsolutions of (2.1), then $\max\{u, v\}$ is also a viscosity subsolution of (2.1). And a similar result is also valid for the viscosity supersolution.

Now we recall the following maximum principle for infinity harmonic functions which can be deduced from Harnack's inequality [9,41].

Lemma 2.1. (Maximum principle) *If $u \in C(\overline{\Omega})$ satisfies $\Delta_\infty u \geq 0$ in the viscosity sense, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Moreover, the supremum occurs only on the boundary of Ω unless u is a constant.

Since the degeneracy at the points where the gradient vanishes and the explosive boundary condition, the general comparison results stated in [39] cannot be applied to equation (2.1). Now we give the comparison result by the double variable method based on the viscosity solution theory.

Theorem 2.1. (Comparison principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $f(x, t) \in C((\overline{\Omega}, (0, \infty)), (0, \infty))$ is non-decreasing in t for each $x \in \Omega$. Suppose also that $u \in C(\overline{\Omega})$ and $v \in C(\overline{\Omega})$ satisfy*

$$\Delta_\infty^h u \geq f(x, u), \quad x \in \Omega$$

and

$$\Delta_\infty^h v \leq f(x, v), \quad x \in \Omega$$

in the viscosity sense. If

$$\liminf_{x\to z\in\partial\Omega} u(x) \le \limsup_{x\to z\in\partial\Omega} v(x),$$

then there holds $u \le v$ in Ω.

Proof. Define

$$u_\varepsilon := u + \varepsilon\Big(u - \sup_{\partial\Omega} u\Big), \quad \varepsilon > 0.$$

By Lemma 2.1, we have $u \le \sup_{\partial\Omega} u$ and $u_\varepsilon \le u$ in Ω. Since u is a viscosity subsolution of (2.1), we have

$$\Delta_\infty^h u_\varepsilon \ge (1+\varepsilon)^h f(x,u) \ge (1+\varepsilon)^h f(x,u_\varepsilon) \ge f(x,u_\varepsilon) \tag{2.2}$$

in the viscosity sense.

Now we claim that $u_\varepsilon \le v$ in Ω. We argue by contradiction. Suppose that $u_\varepsilon > v$ somewhere in Ω. Set

$$M = \sup_\Omega (u_\varepsilon - v) = u_\varepsilon(x_0) - v(x_0) > 0, \quad x_0 \in \overline{\Omega}.$$

Using the arguments in [39], we double the variables

$$w_j(x,y) = u_\varepsilon(x) - v(y) - \frac{j}{4}|x-y|^4, \quad (x,y) \in \Omega\times\Omega,\ j = 1,2,\dots,$$

and let $(x_j, y_j) \in \overline{\Omega}\times\overline{\Omega}$ such that $w_j(x_j,y_j) = \sup_{(x,y)\in\Omega\times\Omega} w_j(x,y)$. According to Proposition 3.7 in [39], we have

$$\lim_{j\to\infty} M_j = \lim_{j\to\infty}(u_\varepsilon(x_j) - v(y_j) - j|x_j-y_j|^4/4) = M$$

and

$$\lim_{j\to\infty} j|x_j - y_j|^4/4 = 0.$$

Furthermore, $x_j \to x_0$, $y_j \to x_0$ as $j\to\infty$. Due to $M > 0 \ge \sup_{\partial\Omega}(u_\varepsilon - v)$, there is an open set Ω_0 such that x_0, x_j, and $y_j \in \Omega_0 \subseteq \Omega$ for $j\to\infty$.

Let

$$\psi(x) = j|x-y_j|^4/4, \quad \phi(y) = -j|x_j - y|^4/4.$$

It is clear that the functions $u_\varepsilon - \psi$ and $v - \phi$ have a local maximum at x_j and a local minimum at y_j, respectively. We consider the two cases: either $x_j \ne y_j$ or $x_j = y_j$ for j large enough.

Case 1: If $x_j = y_j$, we have $D\psi(x_j) = 0$ and $D^2\psi(x_j) = 0$. Since u_ε is a viscosity subsolution to $\Delta_\infty^h u = (1+\varepsilon)^h f(x,u)$, we have

$$(1+\varepsilon)^h f(x_j, \psi(x_j)) = (1+\varepsilon)^h f(x_j, u_\varepsilon(x_j)) \le 0, \tag{2.3}$$

which contradicts to $f > 0$.

Case 2: If $x_j \ne y_j$, applying the maximum principle for semi-continuous functions, there exist symmetric matrices $X_j, Y_j \in \mathbb{S}$ such that

$$(j|x_j - y_j|^2(x_j - y_j), X_j) \in \bar{\mathcal{J}}^{2,+} u_\varepsilon(x_j),$$
$$(j|x_j - y_j|^2(x_j - y_j), Y_j) \in \bar{\mathcal{J}}^{2,-} v(y_j),$$

and

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le D^2\theta_j(x_j,y_j) + \frac{1}{j}(D^2\theta_j(x_j,y_j))^2,$$

where $\theta_j(x,y) := \frac{j}{4}|x-y|^4$.

For simplicity, we denote $p_j = j|x_j - y_j|^2(x_j - y_j)$ and $z_j = x_j - y_j$. Then

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le j(|z_j|^2 + 2|z_j|^4)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 16\begin{pmatrix} z_j \otimes z_j & -z_j \otimes z_j \\ -z_j \otimes z_j & z_j \otimes z_j \end{pmatrix}.$$

This means that

$$(X_j\xi)\cdot \xi \le (Y_j\xi)\cdot \xi, \quad \forall \xi \in \mathbb{R}^n.$$

That is $Y_j - X_j \ge 0$. Again since u_ε is a viscosity subsolution to $\Delta_\infty^h u = (1+\varepsilon)^h f(x,u)$ and v is a viscosity supersolution to (2.1), we can conclude that

$$\begin{aligned} 0 &\le |p_j|^{h-3}\langle X_j p_j, p_j\rangle - (1+\varepsilon)^h f(x_j, u_\varepsilon(x_j)) \\ &\le |p_j|^{h-3}\langle Y_j p_j, p_j\rangle - f(y_j, v(y_j)) + f(y_j, v(y_j)) - (1+\varepsilon)^h f(x_j, u_\varepsilon(x_j)) \\ &\le f(y_j, v(y_j)) - (1+\varepsilon)^h f(x_j, u_\varepsilon(x_j)), \end{aligned}$$

where we have used $Y_j - X_j \ge 0$. Letting $j \to \infty$, we obtain

$$f(x_0, v(x_0)) - (1+\varepsilon)^h f(x_0, u_\varepsilon(x_0)) \ge 0. \tag{2.4}$$

Since $f(x,t)$ is non-decreasing in t and $u_\varepsilon(x_0) > v(x_0)$, we have $f(x_0, u_\varepsilon(x_0)) \ge f(x_0, v(x_0))$, which contradicts to (2.4). Hence, we have proven $u_\varepsilon \le v$ in Ω. Letting $\varepsilon \to 0$, we have $u \le v$ in Ω. □

Remark 2.1. The comparison principle still holds for $f(x,t) < 0$ in Ω. In fact, we can take $v_\varepsilon := (1+\varepsilon)v$ with $\varepsilon > 0$. It is easy to verify that v_ε is a viscosity supersolution of (2.1) and then one can immediately obtain the symmetric result.

Based on Theorem 2.1, we can establish the comparison principle of the boundary blow-up solutions.

Theorem 2.2. (Comparison principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $f(x,t) \in C(\overline{\Omega}, (0,\infty))$ and $\frac{f(x,t)}{t^h}$ are both positive, non-decreasing in t for each $x \in \Omega$.*

Suppose also that $u \in C(\overline{\Omega})$ and $v \in C(\overline{\Omega})$ satisfy

$$\Delta_\infty^h u \ge f(x,u), \quad x \in \Omega$$

and

$$\Delta_\infty^h v \le f(x,v), \quad x \in \Omega$$

in the viscosity sense. If

$$\limsup_{x \to \partial\Omega} \frac{u(x)}{v(x)} \le 1, \tag{2.5}$$

then we have $u \le v$ in Ω.

Proof. By (2.5), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$u \le (1+\varepsilon)v, \quad \text{in } \Omega_\delta := \{x \in \Omega : d(x) < \delta\}. \tag{2.6}$$

Since $\frac{f(x,t)}{t^h}$ is positive, non-decreasing in t for each $x \in \Omega$, then we obtain

$$\Delta_\infty^h((1+\varepsilon)v) = (1+\varepsilon)^h \Delta_\infty^h v \le (1+\varepsilon)^h f(x,v) \le f(x,(1+\varepsilon)v).$$

That is, $(1+\varepsilon)v$ is a viscosity supersolution of (2.1). Since $u \le (1+\varepsilon)v$ on $\partial(\Omega\backslash\overline{\Omega}_\delta)$, by the comparison principle, Theorem 2.1, we have

$$u \le (1+\varepsilon)v, \quad \text{in } \Omega\backslash\overline{\Omega}_\delta. \tag{2.7}$$

By (2.6) and (2.7), we have

$$u \le (1+\varepsilon)v, \quad \text{in } \Omega.$$

Since $\varepsilon > 0$ is arbitrary, then we have $u \le v$ in Ω. ☐

Remark 2.2. If $f(x,u) = b(x)g(u)$ and $\frac{g(t)}{t^h}$ is non-decreasing on $(0,\infty)$, then the comparison principle is still valid.

3 Existence of boundary large solutions

In this section, we investigate conditions on the non-linearity f in problem (1.1) that would lead to existence of viscosity solutions. First, we find radial solutions to

$$\begin{cases} \Delta_\infty^h u = g(u), & \text{in } B_R(x_0), \\ u = \infty, & \text{on } \partial B_R(x_0), \end{cases} \tag{3.1}$$

for any $x_0 \in \mathbb{R}^n$ with $u(x_0) = a \in (0,\infty)$, and prove the existence of viscosity solutions to

$$\begin{cases} \Delta_\infty^h u = f(x,u), & \text{in } \Omega, \\ u = M, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $M \ge 1$. And then we establish the existence of blow-up solutions of (1.1) based on Perron's method and compactness arguments.

Now we concern the solvability of the boundary blow-up problem (3.1) in a ball $B_R(x_0)$.

In (3.1), we can take the radius $R \to \infty$, that is, $B = \mathbb{R}^n$, in which case the boundary condition means $u(x) \to \infty$ as $|x| \to \infty$. We look for viscosity solutions of the form $u(x) = \zeta(r)$, where $r = |x - x_0|$. By direct calculation, we obtain

$$Du(x) = \zeta'(r)\frac{x - x_0}{|x - x_0|}$$

and

$$D^2u(x) = \zeta''(r)\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2} + \zeta'(r)\frac{1}{|x-x_0|}I - \zeta'(r)\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^3}$$

for $x \ne x_0$, where $\otimes$ denotes the tensor product. Therefore, u satisfies

$$\Delta_\infty^h u = g(u) \tag{3.3}$$

if and only if

$$|\zeta'(r)|^{h-1}\zeta''(r) = g(\zeta),$$

which is a one-dimensional case for (3.3).

Theorem 3.1. *Assume g satisfies (**g1**) and (**g2**). Then there exists $0 < R^* \le \infty$ such that problem (3.1) admits a viscosity solution $u \in C(B_R(x_0))$ with $0 < R < R^*$.*

Proof. Step 1: Let us consider the initial value problem

$$\begin{cases} |\zeta'|^{h-1}\zeta'' = g(\zeta), \\ \zeta'(0) = 0, \\ \zeta(0) = a, \end{cases} \tag{3.4}$$

where $a > 0$. Multiplying by $\zeta'(r)$ in (3.4) and integrating on $(0, r)$, we obtain

$$\frac{1}{h+1}|\zeta'(r)|^{h+1} = \int_a^{\zeta(r)} g(t)\mathrm{d}t \geq g(a)(\zeta(r) - a). \tag{3.5}$$

Since $\zeta'(0) = 0$ and $\zeta'(r) > 0$ in $(0, R)$, there holds

$$\zeta'(r)(\zeta(r) - a)^{-\frac{1}{h+1}} \geq g(a)^{\frac{1}{h+1}}.$$

Integrating once again on $(0, r)$, we have

$$\zeta(r) - a \geq Cr^{\frac{h+1}{h}}, \quad r \in (0, R), \tag{3.6}$$

where C is a positive constant.

Step 2: We claim that the function $u(x) := \zeta(|x - x_0|)$ is a viscosity solution of

$$\Delta_\infty^h u = g(u), \quad \text{in } B_R(x_0). \tag{3.7}$$

Indeed, we note that $u \in C^2(B_R(x_0)\backslash\{x_0\}) \cap C^1(B_R(x_0))$, which implies that u is a classical solution in $B_R(x_0)\backslash\{x_0\}$. Let $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$. Since $D\varphi(x_0) = Du(x_0)$, we have

$$u(x) - u(x_0) \leq \varphi(x) - \varphi(x_0) = \frac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2).$$

Taking $x = x_0 + t\vec{e}$, where $\vec{e}$ is a unit vector and $t > 0$, from (3.6), we have

$$Ct^{\frac{h+1}{h}} \leq \zeta(t) - a = u(x_0 + t\vec{e}) - u(x_0) \leq \frac{1}{2}\langle D^2\varphi(x_0)\vec{e}, \vec{e}\rangle t^2 + o(t^2),$$

which is impossible because φ is twice differentiable at the point x_0. Thus, there is no $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains its local maximum at $x_0 \in \Omega$. Therefore, u is a viscosity subsolution of (3.7). Since $\zeta'(0) = 0$, we have $Du(x_0) = 0$. Let $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, and then we have $\Delta_\infty^h \varphi(x_0) = 0 \leq g(u(x_0))$. That is, u is a viscosity supersolution of (3.7).

Step 3: Next we show that problem (3.4) admits an appropriate solution $\zeta \in C^2((0, R)) \cap C^1([0, R))$ such that

$$\zeta(r) \to \infty, \quad r \to R.$$

Let $0 < R_0 \leq \infty$ such that $[0, R_0)$ is the maximal interval for the existence of the viscosity solution. From (3.5), we have

$$\frac{1}{h+1}|\zeta'(r)|^{h+1} = \int_a^{\zeta(r)} g(t)\mathrm{d}t = G(\zeta(r)) - G(a), \quad G(t) = \int_0^t g(s)\mathrm{d}s.$$

Since $\zeta'(r) > 0$, then

$$\frac{\zeta'(r)}{((h+1)(G(\zeta(r)) - G(a)))^{\frac{1}{h+1}}} = 1, \quad 0 < r < R_0.$$

Integrating this on $(0, r)$, we obtain

$$\int_a^{\zeta(r)} \frac{1}{((h+1)(G(t) - G(a)))^{\frac{1}{h+1}}}\mathrm{d}t = r, \quad 0 < r < R_0. \tag{3.8}$$

By (3.5), we have

$$G(t) - G(a) \geq g(a)(t - a), \quad t \geq a$$

and

$$G(t) - G(a) \geq ag(a) \geq G(a) - G(0) = G(a), \quad t \geq 2a$$

such that

$$G(t) - G(a) \geq \frac{1}{2}G(t), \quad t \geq 2a.$$

And then we obtain

$$\frac{1}{\sqrt[h+1]{h+1}}\int_{2a}^{\zeta(r)} \frac{1}{G(t)^{\frac{1}{h+1}}}\mathrm{d}t \leq \int_{2a}^{\zeta(r)} \frac{1}{((h+1)(G(t)-G(a)))^{\frac{1}{h+1}}}\mathrm{d}t \leq \frac{1}{\sqrt[h+1]{(h+1)/2}}\int_{2a}^{\zeta(r)} \frac{1}{G(t)^{\frac{1}{h+1}}}\mathrm{d}t. \tag{3.9}$$

If g satisfies (**g2**), let

$$\psi(a) := \int_{a}^{\infty} \frac{1}{((h+1)(G(t)-G(a)))^{\frac{1}{h+1}}}\mathrm{d}t, \quad a > 0.$$

We have $\lim_{a\to\infty}\psi(a) = 0$. Making the change of variable $s = G(t) - G(a)$, we obtain

$$\psi(a) = \int_{0}^{\infty} \frac{1}{g(G^{-1}(s+G(a)))}\frac{1}{((h+1)s)^{\frac{1}{h+1}}}\mathrm{d}s. \tag{3.10}$$

It is easy to check that $\psi \in C((0,\infty))$ is decreasing. From (3.10) and the Monotone convergence theorem, we have

$$\lim_{a\to 0}\psi(a) = \int_{0}^{\infty} \frac{1}{g(G^{-1}(s))}\frac{1}{((h+1)s)^{\frac{1}{h+1}}}\mathrm{d}s$$

and $0 < \psi(0) \leq \infty$. Let $R^* = \psi(0)$. Given $0 < R < R^*$, we choose $a > 0$ such that $\psi(a) = R$, that is,

$$\int_{a}^{\infty} \frac{1}{((h+1)(G(t)-G(a)))^{\frac{1}{h+1}}}\mathrm{d}t = R.$$

And we note that $\zeta(R) = \infty$ and $\zeta(0) = a$. Hence, $\zeta(\cdot)$ is the desired solution to problem (3.4). □

Remark 3.1. When g satisfies (**g2**), problem (3.1) has a viscosity solution in a bounded domain Ω.

Remark 3.2. If g fails to satisfy (**g2**), from (3.8) and the first inequality in (3.9), we have $R_0 = \infty$, for each $a > 0$. And then problem (3.1) admits a viscosity solution $u \in C(\mathbb{R}^n)$ with $u(x_0) = a$, but there are no viscosity solutions in a bounded domain Ω.

Now we are in the position to prove the existence of viscosity solutions to the Dirichlet problem (3.2).

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and f satisfy (**f1**). For each $M \geq 1$, there exists a unique, non-negative viscosity solution $u_M \in C(\overline{\Omega})$ to problem* (3.2).

Proof. The uniqueness follows immediately by the comparison principle, Theorem 2.1. The proof of the existence relies on Perron's method applied to viscosity solutions. It is obvious that $\bar{u} = M$ is a viscosity supersolution of problem (3.2).

Next we want to construct a viscosity subsolution with appropriate boundary value. Let

$$v_z(x) = M - C|x-z|^{\alpha},$$

where $\alpha \in (0,1)$, $C \geq 1$, and $z \in \partial\Omega$. Then there is a sufficiently small $\delta > 0$, independent of C and z, such that

$$\Delta_{\infty}^{h} v_z(x) = (1-\alpha)(\alpha C)^h|x-z|^{\alpha h-h-1} \geq f(x,M) \geq f(x,v_z), \quad x \in B_{\delta}(z).$$

Choose C so large that $v_z \le 0$ outside $B_{\delta/2}(z)$, and let

$$\underline{u}(x) = \max\left\{0, \sup_{z\in\partial\Omega} v_z(x)\right\}.$$

Then $\underline{u}(x)$ is the desired viscosity subsolution. Hence, using the standard Perron's method, we can obtain the existence of a non-negative viscosity solution u_M to problem (3.2). □

We now prove the existence result of problem (1.1).

Proof of Theorem 1.1. For each positive integer M, by Lemma 3.1, we can obtain a unique viscosity solution u_M to the following problem:

$$\begin{cases} \Delta_\infty^h u = f(x,u), & \text{in } \Omega, \\ u = M, & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle, Theorem 2.1, we have $0 \le u_M \le u_{M+1}$ for each $M \ge 1$.

Let $0 < \varepsilon < \delta/2$ and $D_\varepsilon = \Omega\backslash\overline{\Omega}_\varepsilon$. Since $\partial D_\varepsilon \subseteq \Omega_\delta$, we note that f satisfies (**f2**) at every point of ∂D_ε. Thus, given $z \in \partial D_\varepsilon$, let $0 < r_z \le \varepsilon$ such that condition (**f2**) holds. Then, by Theorem 3.1, the problem

$$\begin{cases} \Delta_\infty^h v = f_*(v; z, r_z), & \text{in } B(z, r_z), \\ u = \infty, & \text{on } \partial B(z, r_z) \end{cases}$$

has a viscosity solution v_z. Since

$$\Delta_\infty^h u_M = f(x, u_M) \ge f_*(u_M; z, r_z), \quad \text{in } B(z, r_z),$$

by the comparison principle, Theorem 2.1, we have $u_M \le v_z$ in $B(z, r_z)$ for all M. Let

$$K_z := \max\left\{v_z(x) : x \in B\left(z, \frac{r_z}{2}\right)\right\}.$$

We have $u_M \le K_z$ in $B\left(z, \frac{r_z}{2}\right)$ for all $M \ge 1$. From the open cover $\mathcal{U} := \left\{B\left(z, \frac{r_z}{2}\right) : z \in \partial D_\varepsilon\right\}$ of ∂D_ε, we pick a finite subcover $\left\{B\left(z_j, \frac{r_{z_j}}{2}\right) : j = 1, 2, \ldots, k\right\}$. Let

$$K := \max\{K_{z_j} : j = 1, 2, \ldots, k\},$$

and then, we have

$$u_M \le K, \quad \text{on } \partial D_\varepsilon.$$

Since K is a viscosity supersolution of $\Delta_\infty^h u = f(x, u)$, by Theorem 2.1 again, we have $u_1 \le u_M \le K$ in D_ε. And by Lemma 2.9 of [9], $\{u_M\}$ is locally uniformly bounded and equicontinuous. Therefore, the sequence $\{u_M\}$ converges to a viscosity solution $u \in C(\Omega)$ locally uniformly as $M \to \infty$. On observing that $u_M \le u$ in Ω for all M, it follows that $u(x) \to \infty$ as $x \to \partial\Omega$. Hence, u is a viscosity solution of problem (1.1). □

By Theorem 1.1, we can immediately obtain the existence of the viscosity solution for the separable case $f(x, u) = b(x)g(u)$.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that b satisfies (**b1**) and g satisfies (**g1**) and (**g2**). Then problem* (1.10) *admits a viscosity solution $u \in C(\Omega)$.*

Remark 3.3. When $f(x, u) = u^q$, $q > h > 1$, the existence follows immediately by Theorem 1.1. For more results in this case, one can see [42].

4 Non-existence of boundary large solutions

In this section, we show that non-existence results of problem (1.1) and (1.10). Since f has less information, we give a sufficient condition that the viscosity solution of (1.1) does not exist. When $f(x,u) = b(x)g(u)$, we obtain a necessary and sufficient condition for the non-existence of viscosity solutions.

Proof of Theorem 1.2. We argue by contradiction. Assume that there is a viscosity solution $u \in C(\Omega)$. For $x_0 \in \partial\Omega$, there exist $r > 0$ and $\varepsilon > 0$ such that $u > \varepsilon > 0$ in $\Omega_0 := B(x_0, r) \cap \Omega$. Let

$$\beta(t) = \frac{1}{t}\int_t^{2t} f^*(s; x, r)\mathrm{d}s, \quad t > 0.$$

One can verify that $\beta(\cdot)$ is a non-decreasing C^1 function on $(0,\infty)$ with $\beta(0) = 0$, and $\beta(t) > 0$ for $t > 0$. As a consequence of the monotonicity of f^*, we note that

$$f^*(t) \le \beta(t) \le f^*(2t), \quad t > 0. \tag{4.1}$$

By inequality (4.1) and condition (1.6), we have

$$\int_1^{\infty} \frac{1}{\beta(t)^{\frac{1}{h}}}\mathrm{d}t = \infty. \tag{4.2}$$

We consider the following smooth increasing function $\eta : [\varepsilon,\infty) \to [0,\infty)$,

$$\eta(t) := \int_{\varepsilon}^{t} \frac{1}{\beta(s)^{\frac{1}{h}}}\mathrm{d}s, \quad t \ge \varepsilon.$$

Setting

$$w(x) := \eta(u(x)), \quad x \in \Omega_0,$$

we have $\Delta_\infty^h w \le 1$ in Ω_0. Indeed, let $\varphi \in C^2(\Omega_0)$ and suppose $w - \varphi$ has a local minimum at $z \in \Omega_0$. Then for some small $\delta > 0$,

$$w(x) \ge \varphi(x), \quad x \in B_\delta(z) \subseteq \Omega_0.$$

Since $w(z) > 0$, we have $\varphi > 0$ in $B_\delta(z)$ when δ is small enough. Since η is increasing, we have

$$u(z) = \eta^{-1}(\varphi(z)), \quad u(x) \ge \eta^{-1}(\varphi(x)),$$

for $x \in B_\delta(z)$. Let $\psi := \eta^{-1}(\varphi)$. We have $\psi \in C^2(B_\delta(z))$ and $u - \psi$ has a local minimum at z. Since u is a viscosity solution of (1.1), we see that

$$\Delta_\infty^h \psi(z) \le f(z, u(z)). \tag{4.3}$$

In $B_\delta(z)$, by direct calculations, we have

$$\begin{aligned}
D\psi &= (\eta^{-1}(\varphi))' D\varphi, \\
D^2\psi &= (\eta^{-1}(\varphi))'' D\varphi \otimes D\varphi + (\eta^{-1}(\varphi))' D^2\varphi, \\
(\eta^{-1}(\varphi))' &= (\beta(\eta^{-1}(\varphi)))^{\frac{1}{h}}, \\
(\eta^{-1}(\varphi))'' &= \frac{1}{h}(\beta(\eta^{-1}(\varphi)))^{\frac{2-h}{h}}\beta'(\eta^{-1}(\varphi)),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\infty^h \psi &= \left|(\beta(\eta^{-1}(\varphi)))^{\frac{1}{h}} D\varphi\right|^{h-1} \left[\frac{1}{h}(\beta(\eta^{-1}(\varphi)))^{\frac{2-h}{h}}\beta'(\eta^{-1}(\varphi)) D\varphi \otimes D\varphi + (\beta(\eta^{-1}(\varphi)))^{\frac{1}{h}} D^2\varphi\right] \\
&= \frac{1}{h}(\beta(\eta^{-1}(\varphi)))^{\frac{1}{h}}\beta'(\eta^{-1}(\varphi))|D\varphi|^{h+1} + \beta(\eta^{-1}(\varphi))\Delta_\infty^h \varphi,
\end{aligned}$$

where ⊗ denotes the tensor product. Since β is nonnegative and nondecreasing, we obtain

$$\Delta_\infty^h \psi \geq \beta(\eta^{-1}(\varphi))\Delta_\infty^h \varphi, \quad x \in B_\delta(z). \tag{4.4}$$

By (4.3) and (4.4), we have

$$f(z, u(z)) \geq \Delta_\infty^h \psi(z) \geq \beta(\psi(z))\Delta_\infty^h \varphi.$$

Recalling (4.1), we obtain

$$\Delta_\infty^h \varphi(z) \leq \frac{f(z, u(z))}{\beta(u(z))} \leq 1.$$

Thus, we have

$$\Delta_\infty^h w \leq 1, \quad \text{in } \Omega_0. \tag{4.5}$$

By (4.2) and $u = \infty$ on $\partial\Omega$, we obtain $w = \infty$ on $B(x_0, r) \cap \partial\Omega$.

We now proceed to show that the conclusion in (4.5) leads to a contradiction. Let α be a continuous function supported on $\partial\Omega_0$ and

$$\alpha(x) = \begin{cases} 1, & x \in B(x_0, r/3) \cap \partial\Omega, \\ 0, & x \in \partial\Omega_0 \backslash (B(x_0, r/2) \cap \partial\Omega). \end{cases} \tag{4.6}$$

From [19,43], we obtain the existence and uniqueness of a viscosity solution τ of the following problem:

$$\begin{cases} \Delta_\infty^h \tau = 1, & \text{in } \Omega_0, \\ \tau = \alpha, & \text{on } \partial\Omega_0. \end{cases}$$

And then for any constant $k \geq 1$, the function $\tau_k(x) := k\tau(x)$ satisfies

$$\begin{cases} \Delta_\infty^h \tau_k = k^h \geq 1, & \text{in } \Omega_0, \\ \tau_k = k\alpha, & \text{on } \partial\Omega_0 \end{cases}$$

in the viscosity sense. By the comparison principle, Theorem 2.2, there holds $\tau_k \leq w$ in Ω_0 for any $k \geq 1$. Due to $\Delta_\infty^h \tau = |D\tau|^{h-3}\Delta_\infty \tau = 1 > 0$, by Lemma 2.1, there exists $x_1 \in \Omega_0$ such that $\tau(x_1) > 0$. Then we have $\tau_k(x_1) = k\tau(x_1) \leq w(x_1)$ for all $k \geq 1$, which leads to a contradiction. □

For the separable case $f(x, t) = b(x)g(t)$, we can also obtain the non-existence result, which is an improvement of Theorem 1.2.

Proof of Theorem 1.3. We argue by contradiction. Let $g_0(t) := Mg(t)$ and $M := \max_\Omega b$. If u is a viscosity solution to (1.10) in Ω, then

$$\Delta_\infty^h u \leq g_0(u).$$

Since g fails to satisfy (**g2**), given

$$a > u(x_0), \quad x_0 \in \Omega,$$

we apply Remark 3.2 to find a viscosity solution $v \in C(\mathbb{R}^n)$ of $\Delta_\infty^h u = g_0(u)$ in $\mathbb{R}^n$ with $v(x_0) = a$. Since Ω is bounded, we see that $v \leq u$ in Ω, by the comparison principle, Theorem 2.1. In particular,

$$a = v(x_0) \leq u(x_0),$$

which is impossible. □

Theorem 1.3 together with Corollary 3.1, provides a necessary and sufficient condition for problem (1.10) to have a viscosity solution.

5 Boundary asymptotic estimates

In this section, we investigate the asymptotic behavior near the boundary of the blow-up solutions to problem (1.10). Our approach relies on Karamata's regular variation theory.

Next we recall the definitions and properties of the regularly varying functions which we will use in the sequel. See for example [44,45].

Definition 5.1. A positive measurable function $f : [a, \infty) \to (0, \infty)$, for some $a > 0$, is called regularly varying at infinity with index $\rho \in \mathbb{R}$, written $f \in \mathrm{RV}_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho.$$

In particular, when $\rho = 0$, f is called slowly varying at infinity.

Clearly, if $f \in \mathrm{RV}_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity. And a positive measurable function g defined on $(0, a)$ for some $a > 0$ is regularly varying at zero with index $\rho \in \mathbb{R}$ and write $g \in \mathrm{RV\,Z}_\rho$, if $g(1/s) \in \mathrm{RV}_{-\rho}$.

Definition 5.2. A positive measurable function $f : [A, \infty) \to (0, \infty)$, for some $A > 0$, is called rapidly varying at infinity, if for each $\rho > 1$, there holds

$$\lim_{t\to\infty} \frac{f(t)}{t^\rho} = \infty.$$

Proposition 5.1. (Representation theorem) *A function L is slowly varying at infinity if and only if it can be written in the form*:

$$L(s) = c(s)\exp\left(\int_{a_1}^{s} \frac{y(t)}{t}\mathrm{d}t\right), \quad s \ge a_1,$$

for some $a_1 \ge a$, *where the functions c and y are measurable and for* $s \to \infty$, $y(s) \to 0$, *and* $c(s) \to c_0$, $c_0 > 0$.

Particularly, if $c(s) \equiv c_0$, then

$$\hat{L}(s) := c_0 \exp\left(\int_{a_1}^{s} \frac{y(t)}{t}\mathrm{d}t\right), \quad s \ge a_1,$$

is normalized slowly varying at infinity and

$$f(s) = s^\rho \hat{L}(s), \quad s \ge a_1,$$

is normalized regularly varying at infinity with index ρ and write $f \in \mathrm{NRV}_\rho$. Similarly, g is called normalized regularly varying at zero with index ρ, written $g \in \mathrm{NRV\,Z}_\rho$, if $g(1/s) \in \mathrm{NRV}_{-\rho}$.

If g satisfies (**g2**) and ϑ satisfies

$$\int_{\vartheta(t)}^{\infty} \frac{1}{[(h+1)G(s)]^{\frac{1}{h+1}}}\mathrm{d}s = t, \tag{5.1}$$

then we observe that ϑ is a decreasing function with $\vartheta(t) \to \infty$ as $t \to 0^+$. Furthermore, a direct computation shows that

$$|\vartheta'(t)|^{h-1}\vartheta''(t) = g(\vartheta(t)), \quad 0 < t < \psi(0), \tag{5.2}$$

and

$$-\vartheta'(t) = \sqrt[h+1]{(h+1)G(\vartheta(t))}. \tag{5.3}$$

Now we recall some properties of the normalized regularly varying functions, which can be found in [46].

Lemma 5.1. *Let $g \in C([0,\infty))$ with $g(0) = 0$ and $g(t) > 0$ for $t > 0$.*

(*i*) *Suppose that the following condition holds*:

$$\lim_{t\to\infty} \frac{g(t)}{(h+1)G(t)^{h/(h+1)}} \int_t^\infty G(s)^{-1/(h+1)} ds = \alpha \geq 1. \tag{5.4}$$

Then $G \in \mathrm{NRV}_{\frac{\alpha(h+1)}{\alpha-1}}$ and $g \in \mathrm{NRV}_{\frac{\alpha(h+1)}{\alpha-1}-1}$ if $\alpha > 1$, whereas G is rapidly varying at infinity if $\alpha = 1$.

(*ii*) *If $\alpha > 1$ and $G \in \mathrm{NRV}_{\frac{\alpha(h+1)}{\alpha-1}}$, then g satisfies (**g2**) and* (5.4).

(*iii*) *If G is rapidly varying at infinity, then g satisfies (**g2**) too.*

(*iv*) *Suppose that the function ϑ is the inverse of the decreasing function* (5.1) *and the condition* (5.4) *holds, then*

$$-\vartheta' \in \mathrm{NRV\,Z}_{-\alpha}, \quad \vartheta \in \mathrm{NRV\,Z}_{1-\alpha}, \tag{5.5}$$

and

$$\lim_{t\to 0} \frac{[(h+1)G(\vartheta(t))]^{h/(h+1)}}{tg(\vartheta(t))} = \frac{1}{\alpha}. \tag{5.6}$$

Now, we are ready to prove the boundary estimate for the viscosity solution $u \in C(\Omega)$ of the problem (1.10) based on the Karamata's regular variation theory.

Proof of Theorem 1.4. Denote $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$ for some $\delta > 0$. Since Ω is a C^1 bounded domain, we have $d(x) \in C^1(\Omega_\delta)$. Moreover, $|Dd(x)| = 1$ in Ω_δ. Then it is obvious that $\Delta_\infty^h d(x) = 0$ in the viscosity sense in Ω_δ. For $k \in \Lambda$ on $(0,\delta)$ and any $\varepsilon \in (0, b_0/2) > 0$, by (1.7), we have

$$b_0 - \varepsilon \leq \frac{b(x)}{k^{h+1}(d(x))} \leq b_0 + \varepsilon, \quad d(x) < \delta. \tag{5.7}$$

Now we construct the comparison functions. We divide it into two cases.

Case (i): First we will show that if $k \in \Lambda$ is non-increasing and $\delta_0 = \delta_0(\varepsilon) \in (0, \delta/2)$, then

$$u^*(x) = \vartheta(c_1[K(d(x)) - K(\delta_0)]), \quad x \in \Omega_\delta \setminus \overline{\Omega}_{\delta_0} =: \Omega^-,$$

and

$$u_*(x) = \vartheta(c_2[K(d(x)) + K(\delta_0)]), \quad x \in \Omega_{\delta-\delta_0} =: \Omega^+,$$

are a viscosity supersolution and a viscosity subsolution of (1.10) in Ω^- and in Ω^+, respectively, where

$$c_1 = \left(\frac{\alpha(b_0-\varepsilon)}{\alpha+\beta-1}\right)^{\frac{1}{h+1}} \quad \text{and} \quad c_2 = \left(\frac{\alpha(b_0+\varepsilon)}{\alpha+\beta-1}\right)^{\frac{1}{h+1}} \tag{5.8}$$

are positive constants.

Now we prove that u^* is a viscosity supersolution of (1.10) in Ω^-. The proof that u_* is a viscosity subsolution is similar and we omit it. For any $x_0 \in \Omega^-$ and $\varphi \in C^2(\Omega^-)$, $u^* - \varphi$ attains local minimum at x_0. We set

$$w(t) := \vartheta(c_1[K(t) - K(\delta_0)]), \quad t \in (\delta_0, \delta),$$

and ξ is the inverse of w. Since w is decreasing, ξ is decreasing in (δ_0, δ). Let

$$\psi := \xi(\varphi) \in C^2(\Omega^-).$$

It is easy to verify that

$$\Delta_\infty^h \psi = |\xi'(\varphi)|^{h-1}|D\varphi|^{h+1}\xi''(\varphi) + |\xi'(\varphi)|^{h-1}\xi'(\varphi)\Delta_\infty^h \varphi. \tag{5.9}$$

Since $\Delta_\infty^h d(x) = 0$, we have $\Delta_\infty^h \psi(x_0) \geq 0$ by the definition of viscosity solution. Then by (5.9) and $\xi' < 0$, we have

$$\Delta_\infty^h \varphi(x_0) \leq -(\xi'(\varphi(x_0)))^{-1}\xi''(\varphi(x_0))|D\varphi(x_0)|^{h+1}.$$

Note that $|Dd(x)| = 1$ in Ω^- and $d - \psi$ has a local maximum at x_0, we have

$$1 = |Dd(x_0)| = |\xi'(\varphi(x_0))D\varphi(x_0)|.$$

Then,

$$\Delta_\infty^h \varphi(x_0) \leq -|\xi'(\varphi(x_0))|^{-h-1}\xi'(\varphi(x_0))\xi''(\varphi(x_0)) = |\xi'(\varphi(x_0))|^{-h-2}\xi''(\varphi(x_0)). \tag{5.10}$$

For convenience, let $\phi(t) = c_1[K(\xi(t)) - K(\delta_0)]$. A direct calculation shows that

$$\begin{aligned} \xi'(t) &= [c_1\vartheta'(\phi(t))k(\xi(t))]^{-1}, \\ \xi''(t) &= -[c_1\vartheta'(\phi(t))k(\xi(t))]^{-3}[\vartheta''(\phi(t))(c_1k(\xi(t)))^2 + c_1k'(\xi(t))\vartheta'(\phi(t))]. \end{aligned}$$

From (5.2), (5.3), and (5.10), we have

$$\begin{aligned} \Delta_\infty^h \varphi(x_0) &\leq |\xi'(\varphi(x_0))|^{-h-2}\xi''(\varphi(x_0)) \\ &= c_1|\vartheta'(\phi_0(x_0))k(d(x_0))|^{h-1}[\vartheta''(\phi_0(x_0))(c_1k(d(x_0)))^2 + c_1\vartheta'(\phi_0(x_0))k'(d(x_0))] \\ &= k(d(x_0))^{h+1}g(u^*(x_0))\left[\frac{c_1^{h+1}|\vartheta'(\phi_0(x_0))|^{h-1}\vartheta''(\phi_0(x_0))}{g(\vartheta(\phi_0(x_0)))} + \frac{c_1^h k'(d(x_0))|\vartheta'(\phi_0(x_0))|^{h-1}\vartheta'(\phi_0(x_0))}{k^2(d(x_0))g(\vartheta(\phi_0(x_0)))}\right] \\ &\leq k(d(x_0))^{h+1}g(u^*(x_0))\left[c_1^{h+1} + c_1^h\frac{k'(d(x_0))K(d(x_0))}{k^2(d(x_0))}\frac{|\vartheta'(\phi_0(x_0))|^{h-1}\vartheta'(\phi_0(x_0))}{K(d(x_0))g(\vartheta(\phi_0(x_0)))}\right] \\ &\leq k(d(x_0))^{h+1}g(u^*(x_0))\left[c_1^{h+1} - c_1^{h+1}\frac{k'(d(x_0))K(d(x_0))}{k^2(d(x_0))}\frac{[(h+1)G(\vartheta(\phi_0(x_0)))]^{\frac{h}{h+1}}}{\phi_0(x_0)g(\vartheta(\phi_0(x_0)))}\right], \end{aligned} \tag{5.11}$$

where $\phi_0(x_0) := c_1[K(d(x_0)) - K(\delta_0)]$. And when $d(x_0) \to 0$, $\vartheta(d(x_0)) \to \infty$. For each $k \in \Lambda$, it is easy to obtain

$$\lim_{t\to 0^+}\frac{k'(t)K(t)}{k^2(t)} = 1 - \beta. \tag{5.12}$$

Then using (5.6) and (5.12), we obtain

$$c_1^{h+1} - c_1^{h+1}\frac{k'(d(x_0))K(d(x_0))}{k^2(d(x_0))}\frac{[(h+1)G(\vartheta(\phi_0(x_0)))]^{\frac{h}{h+1}}}{\phi_0(x_0)g(\vartheta(\phi_0(x_0)))} \to c_1^{h+1}\frac{\alpha + \beta - 1}{\alpha}, \quad d(x_0) \to 0.$$

By (5.7), we have

$$b(x_0)g(u^*(x_0)) \geq (\ell - \varepsilon)k^{h+1}(d(x_0))g(u^*(x_0)). \tag{5.13}$$

Combining with (5.8), (5.11), and (5.13), we obtain

$$\Delta_\infty^h \varphi(x_0) \leq b(x_0)g(u^*(x_0)).$$

That is, u^* is a viscosity supersolution.

Case (ii): If $k \in \Lambda$ is non-decreasing and $\delta_0 = \delta_0(\varepsilon) \in (0, \delta/2)$, we define

$$\begin{aligned} u^*(x) &= \vartheta(c_1K(d(x) - \delta_0)) \quad x \in \Omega^-, \\ u_*(x) &= \vartheta(c_2K(d(x) + \delta_0)), \quad x \in \Omega^+, \end{aligned}$$

where c_1, c_2 are as in (5.8). Similarly, we can obtain that u^* is a viscosity supersolution of (1.10) in Ω^- and u_* is a viscosity subsolution of (1.10) in Ω^+.

Next, we want to establish the boundary asymptotic estimate of the blow-up solutions.

Let u be a positive viscosity solution of problem (1.10) in Ω. Note that the existence of u is guaranteed by Corollary 3.1 and Lemma 5.1. Now we will give the asymptotic estimate of u by the viscosity subsolution u_* and supersolution u^* based on the comparison principle. We note that

$$u^*(x) \to \infty, \; d(x) \to \delta_0, \quad \text{and} \quad u > u_* \text{ on } \partial\Omega.$$

Since ϑ and k are non-increasing (equality holds when k is non-decreasing), we can take a large constant $M > 0$ such that

$$u \le u^* + M, \quad x \in \{x \in \Omega \mid d(x) = \delta\}$$

and

$$u_* \le u + M, \quad x \in \{x \in \Omega \mid d(x) = \delta - \delta_0\}.$$

Therefore, $u \le u^* + M$ on $\partial\Omega^-$, and $u_* \le u + M$ on $\partial\Omega^+$. By the comparison principle, Theorem 2.1, we obtain

$$u \le u^* + M, \quad \text{in } \Omega^- \quad \text{and} \quad u_* - M \le u, \quad \text{in } \Omega^+.$$

Letting $\delta_0 \to 0$, we obtain

$$u_* - M \le u \le u^* + M \quad \text{in } \Omega^+ \cap \Omega^-.$$

That is,

$$\vartheta(c_2 K(d(x))) - M \le u \le \vartheta(c_1 K(d(x))) + M \quad \text{in } \Omega^+ \cap \Omega^-.$$

Since $\vartheta(K(d(x))) \to \infty$ as $d(x) \to 0$, by (5.5), we have

$$\begin{aligned} c_2^{1-\alpha} = \lim_{d(x)\to 0} \frac{\vartheta(c_2 K(d(x)))}{\vartheta(K(d(x)))} &\le \liminf_{d(x)\to 0} \frac{u(x)}{\vartheta(K(d(x)))} \\ &\le \limsup_{d(x)\to 0} \frac{u(x)}{\vartheta(K(d(x)))} \le \lim_{d(x)\to 0} \frac{\vartheta(c_1(K(d(x))))}{\vartheta(K(d(x)))} = c_1^{1-\alpha}. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, we have

$$\lim_{d(x)\to 0} \frac{u(x)}{\vartheta(K(d(x)))} = \left(\frac{\alpha + \beta - 1}{\alpha b_0}\right)^{\frac{\alpha-1}{h+1}}.$$

Now, we turn to prove the uniqueness of the viscosity solution when $\frac{g(t)}{t^h}$ is non-decreasing on $(0, \infty)$. We argue by contradiction. Suppose that u and v are two viscosity solutions of (1.10). Taking small $\delta > 0$, we can assume that $u > 0$ and $v > 0$ in Ω_δ. And since $\vartheta(d(x)) \to \infty$ for $d(x) \to 0$, we can assume that $u_* > 0$ in Ω_δ. Then we obtain

$$\frac{u(x)}{v(x)} \le \frac{u^*(x)}{u_*(x)}, \quad x \in \Omega_\delta,$$

which implies that

$$\limsup_{x\to\partial\Omega} \frac{u(x)}{v(x)} \le 1.$$

By Theorem 2.2, we have $u \le v$ in Ω. Similarly, by swapping u and v, we can obtain $v \le u$ in Ω. The uniqueness result is proven. ☐

Remark 5.1. When $\alpha = 1$, g is rapidly varying at infinity and (1.11) reduces to

$$\lim_{d(x)\to 0} \frac{u(x)}{\vartheta(K(d(x)))} = 1. \tag{5.14}$$

Therefore, relations (1.11) and (5.14) reveal the difference between the blow-up rate of the boundary blow-up solutions near the boundary of (1.10) when g is regularly varying at infinity ($\alpha > 1$) and rapidly varying at infinity ($\alpha = 1$). Furthermore, if g is rapidly varying at infinity, the blow-up rate does not depend on b_0 or β.

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