Research Article

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Chemotaxis-Stokes interaction with very weak diffusion enhancement: Blow-up exclusion via detection of absorption-induced entropy structures involving multiplicative couplings

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Abstract: The chemotaxis–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c), \\ c_t + u\cdot\nabla c = \Delta c - nc, \\ u_t = \Delta u + \nabla P + n\nabla\Phi, \quad \nabla\cdot u = 0, \end{cases}$$

is considered in a smoothly bounded convex domain $\Omega \subset \mathbb{R}^3$, with given suitably regular functions $D : [0,\infty) \to [0,\infty)$, $S : \overline{\Omega} \times [0,\infty) \times (0,\infty) \to \mathbb{R}^{3\times 3}$ and $\Phi : \overline{\Omega} \to \mathbb{R}$ such that $D > 0$ on $(0,\infty)$. It is shown that if with some nondecreasing $S_0 : (0,\infty) \to (0,\infty)$ we have

$$|S(x,n,c)| \le \frac{S_0(c)}{c^{\frac{1}{2}}} \quad \text{for all } (x,n,c) \in \overline{\Omega} \times [0,\infty) \times (0,\infty),$$

then for all $M > 0$ there exists $L(M) > 0$ such that whenever

$$\liminf_{n\to\infty} D(n) > L(M) \quad \text{and} \quad \liminf_{n\searrow 0} \frac{D(n)}{n} > 0,$$

for all sufficiently regular initial data (n_0, c_0, u_0) fulfilling $\|c_0\|_{L^\infty(\Omega)} \le M$ an associated no-flux/no-flux/Dirichlet initial-boundary value problem admits a global bounded weak solution, classical if additionally $D(0) > 0$. When combined with previously known results, this particularly implies global existence of bounded solutions when $D(n) = n^{m-1}$, $n \ge 0$, with arbitrary $m > 1$, but beyond this asserts global boundedness also in the presence of diffusivities which exhibit arbitrarily slow divergence to $+\infty$ at large densities and of possibly singular chemotactic sensitivities.

Keywords: chemotaxis, stokes equation, degenerate diffusion, boundedness

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1 Introduction

This manuscript is concerned with the initial-boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c), & x\in\Omega, \quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega, \quad t>0,\\ u_t = \Delta u + \nabla P + n\nabla\Phi, \quad \nabla\cdot u = 0, & x\in\Omega, \quad t>0,\\ (D(n)\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nu = 0, \quad \nabla c\cdot\nu = 0, \ u = 0, & x\in\partial\Omega, \quad t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

which arises in the description of collective behavior in populations of aerobic bacteria swimming in the domain $\Omega\subset\mathbb{R}^N$: With the aim to adequately capture pattern formation phenomena observed in colonies of *Bacillus subtilis* suspended to sessile water drops, the authors in [1] proposed an evolution system of this form as a model for the respective population density n, for the concentration c of oxygen acting as an attractive nutrient, and for the fluid field (u,P) through which both these former components are transported, and which itself is influenced by cells via buoyant forces mediated by the given gravitational potential Φ.

While in the first step achieved in [1] the essential ingredients of (1.1) were kept in comparatively simple functional forms, accounting for various developments in refined modeling of cell migration has subsequently led to the inclusion of more general choices of the key constituents. A first example in this regard can be found in [2], where linear Brownian cell diffusion corresponding to the choice $D\equiv$ const. was replaced by migration operators of porous medium type; in line with further observations on detailed facets of diffusive and cross-diffusive bacterial motion, and with associated modeling approaches [3,4,5], the analytical literature of the past few years has addressed problems of the form (1.1), and also large classes of related models for chemotaxis-fluid interaction in various partially more complex frameworks, in noticeable generality with respect to the choices of the scalar parameter function D and the matrix-valued chemotactic sensitivity S (cf., e.g., [6–16] for a small selection of recent examples).

The challenge of excluding blow-up. An issue forming a natural core of considerable activities in this respect consists in the question how far the interplay of the dissipative mechanisms in (1.1) can be identified as suitably efficient so as to rule out the occurrence of blow-up phenomena, as known to constitute a central characteristic of chemotactic cross-diffusion in contexts already of simple two-component Keller-Segel systems [17–19]. A particular focus in this regard has been on the role of the absorptive contribution $-nc$ to the second equation in (1.1) which marks the possibly most crucial difference in comparison to the latter class of notoriously explosion-supporting Keller-Segel systems in which instead, namely, a positive summand $+n$ forms the essential part of the corresponding zero-order contribution.

A first approach toward making appropriate use of this dissipative feature of (1.1) can be successfully pursued in the simple case when the chemotactic sensitivity is assumed to be the essentially scalar function determined by the choice $S\equiv id$. Indeed, in the evolution of expressions of the form

$$\mathcal{F}_1 := \int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{1}{c}|\nabla c|^2 + b\int_\Omega |u|^2, \qquad b>0, \tag{1.2}$$

it is precisely the action of said consumptive mechanism which brings about a favorable cancellation of the respective cross-diffusive contribution, and which hence implies a Lyapunov-type role of $\mathcal{F}_1$ in such situations. Although this observation already dates back to quite early studies concerned with (1.1), even at present it appears unclear whether the full potential of its consequences has already been exhausted. In fact, the a priori information obtained from corresponding entropy-dissipation inequalities could either directly be used to construct global weak solutions or alternatively be utilized as a starting point for iterative regularity arguments finally yielding global solutions enjoying additional boundedness properties. In this situation when $S\equiv id$, strategies of this type have led to an essentially complete theory of global bounded solutions in planar domains both in the presence of linear diffusion with $D\equiv$ const., and in the case of porous medium type diffusion determined by the choice $D(n)=n^{m-1}$, $n\geq 1$, with arbitrary $m>1$ [2,20–24]. In the corresponding three-dimensional counterpart, however, global bounded solutions have been constructed on this basis only for $m>\frac{9}{8}$ [25,26]. For general $m>1$, for the linear borderline case $m=1$, and also for the fast diffusion range $m \in \left(\frac{2}{3}, 1\right)$; however, up to now global solvability could be asserted only in spaces of possibly unbounded functions [22,27,28], albeit partially even in the substantially more complex variant of (1.1) involving the full three-dimensional Navier-Stokes system [28,29]; after all, at least in the case $D \equiv 1$ an analysis involving $\mathcal{F}_1$ has revealed eventual regularity, and especially ultimate boundedness, of such weak solutions even in the latter Navier-Stokes extension of (1.1) [30]. Similar energy-based arguments have also been underlying studies concerned with corresponding Cauchy problems posed in $\Omega = \mathbb{R}^N$ (see [31–33], for instance).

To date, however, it seems unclear how far functionals of the form in (1.2) can be used to assert boundedness in three-dimensional versions of (1.1) when in the above setting we have $m \le \frac{9}{8}$, or when S fails to be precisely of the indicated form. In the literature concerned with such situations, exploiting the absorptive character of $-nc$ in (1.1) seems to essentially reduce to relying on the estimate $\|c\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)}$, as thereby trivially implied. For general bounded and suitably smooth matrices S, for instance, bounded weak solutions could be constructed for $D(n) = n^{m-1}$, $n \ge 0$, with any $m > 1$ when $N = 2$ [34], and with $m > \frac{7}{6}$ when $N = 3$ [35], while even in two-dimensional settings, the linear case $m = 1$ could up to now be addressed only in very weak frameworks of possibly unbounded solutions [36], with boundedness and further regularity features only available after suitably large waiting times [37]. This basic L^∞ estimate could moreover be used to establish results on global existence of bounded classical solutions under additional assumptions on appropriate smallness of the initial data in various settings involving linear diffusion [38–40]. For chemotactic sensitivities exhibiting singular behavior near $c = 0$, as forming a core ingredient of the celebrated Keller-Segel consumption system [41], none of the mentioned techniques seem applicable, not even in scalar cases. Accordingly, the literature up to now seems restricted to the construction of weak solutions with possibly quite poor regularity features in a two-dimensional version of (1.1) with $D \equiv 1$ and $S(x, n, c) = \frac{1}{c} id$, $(x, n, c) \in \Omega \times [0, \infty) \times (0, \infty)$, and to a statement on eventual regularity [42,43].

An alternative approach toward taking advantage of signal consumption. The main objective of the present manuscript now is to present a method, deviating from those described above already in its principal design, which will turn out to be capable of exploiting the dissipative nature of the signal consumption mechanism in (1.1) in quite an efficient manner. Specifically, concentrating on the three-dimensional version of (1.1) the core of our approach will be formed by the ambition to estimate, rather than precisely cancel, a taxis-related contribution to the evolution undergone by the functional

$$\int_\Omega D_2(n), \qquad \text{where } D_2(\hat{n}) := \int_0^{\hat{n}} \int_0^s D(\sigma) d\sigma ds, \quad \hat{n} \ge 0, \tag{1.3}$$

that is, an expression of the form

$$\int_\Omega \frac{n^2}{c} |\nabla c|^2$$

(cf. Lemma 3.1). Instead of attempting to decouple this quantity and separately estimate resulting integrals exclusively containing n and c, in Lemma 3.2 we shall identify this expression, *as a whole*, as the dissipated part in an inequality describing the evolution of

$$\int_\Omega \frac{n}{c} |\nabla c|^2. \tag{1.4}$$

In fact, we shall see that under a very mild assumption on the behavior of $D(n)$ for large values of n, and within a large class of matrix-valued functions S, possibly singular near $c = 0$, corresponding ill-signed terms obtained in the course of analyzing the coupled quantity (1.4) can be compensated by appropriate linear combination with the function in (1.3), with

$$\int_\Omega \frac{1}{c^3} |\nabla c|^4, \tag{1.5}$$

and with a further zero-order integral involving a bounded function of n (Lemmas 3.4 and 3.5).

Suitably utilizing the entropy-like structure hence discovered (Lemmas 3.7, 4.1, and 4.2), essentially straightforward regularity and compactness arguments (Lemmas 4.3–5.1) will thereafter complete the derivation of the following main result of this manuscript, asserting global existence and boundedness of solutions in the three-dimensional version of (1.1) under assumptions on D and S which appear to be more general than those underlying any precedent-related study, and which especially apply to arbitrarily weak diffusion enhancement at large population densities, in the sense of merely requiring (1.8). Here and below, given $\beta > 0$ and a smoothly bounded domain $\Omega \subset \mathbb{R}^3$ we let A and A^β denote the Stokes operator with domain $D(A) := W^{2,2}(\Omega;\mathbb{R}^3) \cap W^{1,2}_{0,\sigma}(\Omega)$ and its corresponding fractional power, respectively, where $W^{1,2}_{0,\sigma}(\Omega) := \{\varphi \in W^{1,2}_0(\Omega;\mathbb{R}^3) \mid \nabla\cdot\varphi = 0\}$ [44,45].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and suppose that*

$$\begin{cases} S \in C^2(\overline{\Omega}\times[0,\infty)\times(0,\infty);\mathbb{R}^{3\times 3}) \text{ is such that} \\ |S(x,n,c)| \le \dfrac{S_0(c)}{c^{\frac{1}{2}}} \quad \text{for all } (x,n,c)\in\Omega\times(0,\infty)^2 \\ \text{with some nondecreasing } S_0 : (0,\infty)\to(0,\infty). \end{cases} \tag{1.6}$$

Then for all $M > 0$ there exists $L = L(M) > 0$ with the property that whenever

$$D \in \bigcup_{\vartheta\in(0,1)} C^{\vartheta}_{\mathrm{loc}}([0,\infty)) \cap C^2((0,\infty)) \quad \text{is positive on } (0,\infty) \tag{1.7}$$

and such that

$$\liminf_{n\to\infty} D(n) > L \tag{1.8}$$

as well as

$$\liminf_{n\searrow 0} \frac{D(n)}{n} > 0, \tag{1.9}$$

given any initial data (n_0, c_0, u_0) which are such that

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega) \text{ with } n_0 \ge 0 \quad \text{and } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \text{ with } c_0 > 0 \quad \text{in } \overline{\Omega} \quad \text{and} \\ u_0 \in D(A^\beta) \text{ with some } \beta \in \left(\frac{3}{4}, 1\right), \end{cases} \tag{1.10}$$

and that

$$\|c_0\|_{L^\infty(\Omega)} \le M,$$

one can find a global weak solution (n, c, u) of (1.1), *according to* Definition 2.1, *which is bounded in the sense that*

$$\operatorname*{ess\,sup}_{t>0}\{\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,2}(\Omega)}\} < \infty. \tag{1.11}$$

If additionally $D(0) > 0$, then this solution furthermore satisfies

$$\begin{cases} n \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ c \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \quad \text{and} \\ u \in C^0(\overline{\Omega}\times[0,\infty);\mathbb{R}^3) \cap C^{2,1}(\overline{\Omega}\times(0,\infty);\mathbb{R}^3), \end{cases} \tag{1.12}$$

and there exists $P \in C^{1,0}(\Omega\times(0,\infty))$ such that (n, c, u, P) solves (1.1) *classically in $\Omega\times(0,\infty)$.*

The following immediate consequence of the latter emphasizes that arbitrarily slow divergent behavior of $D(n)$ near $n = \infty$ is sufficient for the above conclusion.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and assume that S and D satisfy* (1.6), (1.7), (1.9), *and*

$$D(n) \to \infty \quad \text{as } n \to \infty. \tag{1.13}$$

Then for any choice of (n_0, c_0, u_0) *complying with* (1.10), *one can find a global bounded weak solution of* (1.1) *in the sense specified in* Theorem 1.1. *If, moreover,* $D(0) > 0$, *then this solution additionally satisfies* (1.12) *and solves* (1.1) *actually in the classical sense with some* $P \in C^{1,0}(\Omega \times (0, \infty))$.

Beyond this, however, we note that also in the case when $D \equiv D_0 = \text{const.}$, Theorem 1.1 appears to provide some progress by, namely, asserting the existence of global bounded classical solutions whenever $D_0 \geq L$ with some suitably large L depending on an upper bound for $\|c_0\|_{L^\infty(\Omega)}$ only. The above statement thereby partially extends the solution theory for the corresponding linear diffusion version of (1.1), as developed in [39] and [22] for essentially arbitrary relationships between D_0 and the initial data but in frameworks of possibly unbounded weak solutions only, so as to assert classical solvability and boundedness at least under said restrictions.

When concretized in the framework of models involving diffusion operators precisely of porous medium type, Theorem 1.1 evidently admits any choice of the corresponding adiabatic exponent which has not already been addressed in previous studies:

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and assume that S satisfies* (1.6). *Then for each $m \in (1, 2]$ and any choice of initial data fulfilling* (1.10), *the problem*

$$\begin{cases} n_t + u \cdot \nabla n = \nabla\cdot(n^{m-1}\nabla n) - \nabla\cdot(nS(x, n, c)\cdot\nabla c), & x \in \Omega, \quad t > 0, \\ \tau c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, \quad t > 0, \\ u_t = \Delta u + \nabla P + n\nabla\Phi, \quad \nabla\cdot u = 0, & x \in \Omega, \quad t > 0, \\ (n^{m-1}\nabla n - nS(x, n, c)\cdot\nabla c)\cdot\nu = 0, \nabla c\cdot\nu = 0, \quad u = 0, & x \in \partial\Omega, \quad t > 0, \\ n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.14}$$

admits a global bounded weak solution in the style specified in Theorem 1.1.

In the presence non-singular chemotactic sensitivities, but yet possibly containing off-diagonal matrix entries, in view of the results obtained in [35] for $m > \frac{7}{6}$ this particularly completes the picture concerning the corresponding version of (1.14) for arbitrary $m > 1$:

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and suppose that*

$$\begin{aligned} &S \in C^2(\bar\Omega \times [0, \infty) \times (0, \infty); \mathbb{R}^{3\times3}) \text{ is such that } |S(x, n, c)| \\ &\quad \leq S_0(c) \text{ for all } (x, n, c) \in \Omega \times (0, \infty)^2 \text{ with some nondecreasing } S_0 : (0, \infty) \to (0, \infty). \end{aligned} \tag{1.15}$$

Then given an arbitrary $m > 1$ and any (n_0, c_0, u_0) which satisfies (1.10) *and moreover is such that $u_0 \in \bigcap_{r>1} W^{2,r}(\Omega; \mathbb{R}^3)$, one can find a global bounded weak solution of* (1.14) *in the sense specified in* Theorem 1.1.

2 Preliminaries. Global solutions to regularized problems

The following notion of weak solvability has been imported from [35].

Definition 2.1. Assume (1.6), (1.7), and (1.10), and suppose that

$$\begin{cases} n \in L^1_{loc}(\bar\Omega \times [0, \infty)), \\ c \in L^\infty_{loc}(\bar\Omega \times [0, \infty)) \cap L^1_{loc}([0, \infty); W^{1,1}(\Omega)) \quad \text{and} \\ u \in L^1_{loc}([0, \infty); W^{1,1}(\Omega)) \end{cases}$$

are such that $n \geq 0$ and $c > 0$ a.e. in $\Omega \times (0, \infty)$, and that

$$D_1(n), \quad n|S(x, n, c)|\cdot|\nabla c| \text{ and } n|u| \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty)), \tag{2.1}$$

where $D_1(\hat{n}) := \int_0^{\hat{n}} D(s)ds$ for $\hat{n} \geq 0$. Then we call (n, c, u) a *global weak solution* of (1.1) if $\nabla \cdot u = 0$ a.e. in $\Omega \times (0, \infty)$, if

$$-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) = \int_0^\infty \int_\Omega D_1(n)\Delta\varphi + \int_0^\infty \int_\Omega n(S(x, n, c)\cdot\nabla c)\cdot\nabla\varphi + \int_0^\infty \int_\Omega nu\cdot\nabla\varphi \tag{2.2}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, \infty)$, if

$$-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla c\cdot\nabla\varphi - \int_0^\infty \int_\Omega nc\varphi + \int_0^\infty \int_\Omega cu\cdot\nabla\varphi \tag{2.3}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$, and if moreover

$$-\int_0^\infty \int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty \int_\Omega n\nabla\Phi\cdot\varphi \tag{2.4}$$

for all $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^3)$ such that $\nabla\cdot\varphi \equiv 0$ in $\Omega \times (0, \infty)$.

In order to construct such a solution by means of appropriate approximation, following the approaches pursued in [35] and [36] we fix, given $\varepsilon \in (0, 1)$,

$$\begin{cases} D_\varepsilon \in C^2([0, \infty)) \quad \text{such that} \quad D_\varepsilon(n) \geq \varepsilon \quad \text{for all } n \geq 0 \quad \text{and} \\ D(n) \leq D_\varepsilon(n) \leq D(n) + 2\varepsilon \quad \text{for all } n \geq 0, \end{cases} \tag{2.5}$$

as well as

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{with} \quad 0 \leq \rho_\varepsilon \leq 1 \text{ in } \Omega$$

and

$$\chi_\varepsilon \in C_0^\infty([0, \infty)) \quad \text{satisfying} \quad 0 \leq \chi_\varepsilon \leq 1 \text{ in } [0, \infty),$$

in such a way that

$$\rho_\varepsilon \nearrow 1 \text{ in } \Omega \quad \text{and} \quad \chi_\varepsilon \nearrow 1 \text{ in } [0, \infty) \quad \text{as } \varepsilon \searrow 0.$$

For $\varepsilon \in (0, 1)$, we then define $S_\varepsilon \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{3\times 3})$ by letting

$$S_\varepsilon(x, n, c) := \rho_\varepsilon(x)\cdot\chi_\varepsilon(n)\cdot S(x, n, c + \varepsilon), \quad x \in \bar{\Omega}, \quad n \geq 0, \quad c \geq 0, \tag{2.6}$$

and consider the regularized variant of (1.1) given by

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon\cdot\nabla n_\varepsilon = \nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon), & x \in \Omega, \quad t > 0, \\ \partial_t c_\varepsilon + u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon c_\varepsilon, & x \in \Omega, \quad t > 0, \\ \partial_t u_\varepsilon + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon\nabla\phi, & x \in \Omega, \quad t > 0, \\ \nabla\cdot u_\varepsilon = 0, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial n_\varepsilon}{\partial\nu} = \dfrac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ n_\varepsilon(x, 0) = n_0(x), \quad c_\varepsilon(x, 0) = c_0(x), \quad u_\varepsilon(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

which is globally solvable in the classical sense:

Lemma 2.2. *Let $\varepsilon \in (0,1)$. Then there exist functions*

$$\begin{cases} n_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ c_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ u_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \quad \textit{and} \\ P_\varepsilon \in C^{1,0}(\bar\Omega \times (0,\infty)), \end{cases}$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.7) *classically in $\Omega \times (0,\infty)$, and such that n_ε and c_ε are positive in $\overline{\Omega} \times (0,\infty)$. Furthermore,*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \quad \textit{for all } t > 0 \tag{2.8}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \textit{for all } t > 0. \tag{2.9}$$

Proof. This has precisely been covered by [35, Lemmas 2.1 and 2.2]. □

Without any further comment, throughout the sequel we let n_ε, c_ε, u_ε, and P_ε be as found in Lemma 2.2.

3 A quasi-entropy structure involving multiplicative couplings

The plan for this key section is to arrange an efficient analysis related to the evolution of the functionals in (1.3), (1.4), and (1.5). Our first observation in this regard is quite straightforward.

Lemma 3.1. *Suppose that* (1.7), (1.6), *and* (1.10) *hold. Then writing*

$$D_{1,\varepsilon}(n) := \int_0^n D_\varepsilon(s)ds \quad \textit{and} \quad D_{2,\varepsilon}(n) := \int_0^n D_{1,\varepsilon}(s)ds, \qquad n \ge 0,\ \varepsilon \in (0,1), \tag{3.1}$$

we have

$$\frac{d}{dt}\int_\Omega D_{2,\varepsilon}(n_\varepsilon) + \frac12 \int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 \le \frac12 S_0^2\Big(\|c_0\|_{L^\infty(\Omega)} + 1\Big)\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 \quad \textit{for all} \quad t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.2}$$

Proof. Let $\varepsilon \in (0,1)$. Then in view of the identity $\nabla\cdot u_\varepsilon = 0$, according to the first equation in (2.7) we find that since $D''_{2,\varepsilon} = D_\varepsilon$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega D_{2,\varepsilon}(n_\varepsilon) &= -\int_\Omega D''_{2,\varepsilon}(n_\varepsilon)D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon D''_{2,\varepsilon}(n_\varepsilon)\nabla n_\varepsilon \cdot (S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\ &= -\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot (S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\ &\le -\frac12\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac12\int_\Omega n_\varepsilon^2 |S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon|^2 \\ &\le -\frac12\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac12 S_0^2\Big(\|c_\varepsilon+\varepsilon\|_{L^\infty(\Omega)}\Big)\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 \quad \text{for all } t > 0 \end{aligned}$$

because of Young's inequality, (2.6) and (1.6). Using (2.9) in estimating $\|c_\varepsilon+\varepsilon\|_{L^\infty(\Omega)} \le \|c_\varepsilon\|_{L^\infty(\Omega)} + \varepsilon \le \|c_0\|_{L^\infty(\Omega)} + 1$ for $t > 0$, we thereby obtain (3.2). □

By describing the evolution of the coupled quantity from (1.4), the next lemma may now be viewed as the core of our analysis, through which it will become possible to suitably compensate the rightmost summand in (3.2). Our overall assumption on convexity of Ω is explicitly made use of here in order to keep the presentation as simple as possible (cf. (3.12)), and we mention that at the cost of some additional but essentially straightforward modification along the lines presented in [46], the argument can indeed be extended so as to cover arbitrary smoothly bounded domains.

Lemma 3.2. *Assume* (1.6). *Then for all $M > 0$ and any $\eta > 0$ one can find $C(M,\eta) > 0$ with the property that whenever* (1.7) *and* (1.10) *hold with $\|c_0\|_{L^\infty(\Omega)} \le M$, we have*

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \frac12\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 &\le \eta\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + C(M,\eta)\int_\Omega |\nabla n_\varepsilon|^2 + C(M,\eta)\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + C(M,\eta)\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\ &\quad + C(M,\eta)\int_\Omega|\nabla u_\varepsilon|^3 + C(M,\eta)\int_\Omega |u_\varepsilon|^6 \quad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1).\end{aligned} \tag{3.3}$$

Proof. Let $\varepsilon \in (0,1)$. Assuming (1.6), (1.7), and (1.10) with $\|c_0\|_{L^\infty(\Omega)} \le M$, on the basis of (2.7) and the identities $\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon = \frac12\Delta|\nabla c_\varepsilon|^2 - |D^2c_\varepsilon|^2$ and $\nabla|\nabla c_\varepsilon|^2 = 2D^2c_\varepsilon\cdot\nabla c_\varepsilon$, integrating by parts several times we then compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 &= \int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2\cdot\{\nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) - u_\varepsilon\cdot\nabla n_\varepsilon\} \\ &\quad + 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot\{\nabla\Delta c_\varepsilon - n_\varepsilon\nabla c_\varepsilon - c_\varepsilon\nabla n_\varepsilon - \nabla u_\varepsilon\cdot\nabla c_\varepsilon - D^2c_\varepsilon\cdot u_\varepsilon\} \\ &\quad - \int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\cdot\{\Delta c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\} \\ &= -\int_\Omega\frac{1}{c_\varepsilon}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 + \int_\Omega\frac{1}{c_\varepsilon^2}D_\varepsilon(n_\varepsilon)|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\ &\quad + \int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla|\nabla c_\varepsilon|^2\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) - \int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\ &\quad - \int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla n_\varepsilon) - \int_\Omega\frac{1}{c_\varepsilon}\nabla n_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 + \int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 \\ &\quad + \int_{\partial\Omega}\frac{n_\varepsilon}{c_\varepsilon}\cdot\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2 - 2\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 - 2\int_\Omega n_\varepsilon\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\ &\quad - 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) - 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) + \int_\Omega\frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\ &\quad + \int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + \int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla c_\varepsilon) \\ &= -2\int_\Omega\frac{1}{c_\varepsilon}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) + \int_\Omega\frac{1}{c_\varepsilon^2}D_\varepsilon(n_\varepsilon)|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\ &\quad + 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) - \int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\ &\quad - \int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla n_\varepsilon) - 2\int_\Omega\frac{1}{c_\varepsilon}\nabla n_\varepsilon(D^2c_\varepsilon\cdot\nabla c_\varepsilon) + 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+ \int_{\partial\Omega} \frac{n_\varepsilon}{c_\varepsilon} \cdot \frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2 - \int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 - 2\int_\Omega n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&- 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) + \int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2 \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&+ 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^2}\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) - 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + \int_\Omega \frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2 (u_\varepsilon \cdot \nabla c_\varepsilon) \quad \text{for all } t > 0.
\end{aligned}
$$

Here, given $\eta > 0$ we may utilize Young's inequality together with (2.9) to see that for all $t > 0$,

$$
\begin{aligned}
-2\int_\Omega \frac{1}{c_\varepsilon} D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) &\le \frac{\eta}{2}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{2}{\eta}\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2 \\
&\le \frac{\eta}{2}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{2M}{\eta}\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2,
\end{aligned}
\tag{3.5}
$$

that

$$
\begin{aligned}
\int_\Omega \frac{1}{c_\varepsilon^2} D_\varepsilon(n_\varepsilon)|\nabla c_\varepsilon|^2 \nabla n_\varepsilon \cdot \nabla c_\varepsilon &\le \frac{\eta}{2}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{1}{2\eta}\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^6 \\
&\le \frac{\eta}{2}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{M}{2\eta}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \quad \text{for all } t > 0,
\end{aligned}
\tag{3.6}
$$

and that thanks to (2.6) and (1.6),

$$
\begin{aligned}
&2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}(D^2 c_\varepsilon \cdot \nabla c_\varepsilon)\cdot(S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot \nabla c_\varepsilon) \\
&\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 16 S_0^2(M+1)\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2 \\
&\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 16 M S_0^2(M+1)\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2 \quad \text{for all } t > 0
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
-\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2 \nabla c_\varepsilon \cdot (S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot \nabla c_\varepsilon) &\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 4 S_0^2(M+1)\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^6 \\
&\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 4 M S_0^2(M+1)\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \quad \text{for all } t > 0.
\end{aligned}
\tag{3.8}
$$

Next, again by Young's inequality and (2.9),

$$
\begin{aligned}
-\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 (u_\varepsilon \cdot \nabla n_\varepsilon) &\le \int_\Omega |\nabla n_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^4 |u_\varepsilon|^2 \\
&\le \int_\Omega |\nabla n_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^6 + \int_\Omega |u_\varepsilon|^6 \\
&\le \int_\Omega |\nabla n_\varepsilon|^2 + M^2 \int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \int_\Omega |u_\varepsilon|^6 \quad \text{for all } t > 0
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
-2\int_\Omega \frac{1}{c_\varepsilon}\nabla n_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) &\le \int_\Omega|\nabla n_\varepsilon|^2+\int_\Omega\frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2\\
&\le \int_\Omega|\nabla n_\varepsilon|^2+M\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 \quad \text{for all } t>0
\end{aligned}
\tag{3.10}
$$

as well as

$$
2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon^2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\le\frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 \quad\text{for all } t>0,
\tag{3.11}
$$

while

$$
\int_{\partial\Omega}\frac{n_\varepsilon}{c_\varepsilon}\cdot\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\le 0 \quad\text{for all } t>0
\tag{3.12}
$$

due to the positivity of n_ε and c_ε and the fact that $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\le 0$ on $\partial\Omega$ by convexity of Ω [47]. Apart from that, once more relying on Young's inequality and (2.9) we see that

$$
\begin{aligned}
-2\int_\Omega n_\varepsilon\nabla n_\varepsilon\cdot\nabla c_\varepsilon &\le \frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16\int_\Omega c_\varepsilon|\nabla n_\varepsilon|^2\\
&\le\frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16M\int_\Omega|\nabla n_\varepsilon|^2 \quad\text{for all } t>0,
\end{aligned}
\tag{3.13}
$$

that

$$
\begin{aligned}
&-2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon)\\
&\le \frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16\int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|\nabla u_\varepsilon|^2\\
&\le \frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^6+16\int_\Omega|\nabla u_\varepsilon|^3\\
&\le \frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+16M^2\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6+16\int_\Omega|\nabla u_\varepsilon|^3 \quad\text{for all } t>0
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&-2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon)\\
&\le 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2+\frac{1}{2}\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2|u_\varepsilon|^2\\
&\le 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2+\frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+\int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|u_\varepsilon|^4\\
&\le 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2+\frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^6+\int_\Omega|u_\varepsilon|^6\\
&\le 2\int_\Omega\frac{n_\varepsilon}{c_\varepsilon}|D^2c_\varepsilon|^2+\frac{1}{16}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2+M^2\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6+\int_\Omega|u_\varepsilon|^6 \quad\text{for all } t>0,
\end{aligned}
\tag{3.15}
$$

and that

$$\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \le \int_\Omega |\nabla n_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^6 \le \int_\Omega |\nabla n_\varepsilon|^2 + M\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \quad \text{for all } t>0 \tag{3.16}$$

and, again,

$$2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) \le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 16\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 \quad \text{for all } t>0 \tag{3.17}$$

as well as

$$\begin{aligned}\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^2}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla c_\varepsilon) &\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4|u_\varepsilon|^2\\ &\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 4\int_\Omega \frac{1}{c_\varepsilon^{\frac{9}{2}}}|\nabla c_\varepsilon|^6 + 4\int_\Omega |u_\varepsilon|^6\\ &\le \frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 4M^{\frac{1}{2}}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + 4\int_\Omega |u_\varepsilon|^6 \quad \text{for all } t>0.\end{aligned} \tag{3.18}$$

Estimating

$$-\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + 8\cdot\frac{1}{16}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 - 2\int_\Omega \frac{n_\varepsilon}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 \le -\frac{1}{2}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 \quad \text{for all } t>0,$$

from (3.4)–(3.18) we obtain (3.3) upon an obvious selection of $C(M,\eta)$. □

Now in order to prepare our identification of the third and fourth summands in (3.3) as part of the dissipative contribution to the evolution of the functional in (1.5), let us briefly state the following essentially elementary properties of arbitrary smooth positive functions on $\overline{\Omega}$ with vanishing normal derivative on $\partial\Omega$ (cf. [22] for a related precedent).

Lemma 3.3. *Let $\varphi\in C^2(\overline{\Omega})$ be such that $\varphi>0$ in $\overline{\Omega}$ and $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$. Then*

$$|D^2\varphi|^2 = \varphi^2|D^2\ln\varphi|^2 + \frac{1}{\varphi}\nabla\varphi\cdot\nabla|\nabla\varphi|^2 - \frac{1}{\varphi^2}|\nabla\varphi|^4 \quad \text{in } \Omega \tag{3.19}$$

and

$$\int_\Omega \frac{1}{\varphi^5}|\nabla\varphi|^6 \le (19+2\sqrt{3})\int_\Omega \frac{1}{\varphi}|\nabla\varphi|^2|D^2\ln\varphi|^2 \tag{3.20}$$

as well as

$$\int_\Omega \frac{1}{\varphi^3}|\nabla\varphi|^2|D^2\varphi|^2 \le (40+4\sqrt{3})\int_\Omega \frac{1}{\varphi}|\nabla\varphi|^2|D^2\ln\varphi|^2. \tag{3.21}$$

Proof. We integrate by parts and use the fact that $|\Delta\ln\varphi|\le\sqrt{3}|D^2\ln\varphi|$ along with the Cauchy-Schwarz inequality to see that

$$\begin{aligned}\int_\Omega \frac{1}{\varphi^5}|\nabla\varphi|^6 &= \int_\Omega |\nabla\ln\varphi|^4\nabla\ln\varphi\cdot\nabla\varphi\\ &= -\int_\Omega \varphi\nabla\ln\varphi\cdot\nabla|\nabla\ln\varphi|^4 - \int_\Omega \varphi|\nabla\ln\varphi|^4\Delta\ln\varphi\\ &= -4\int_\Omega \varphi|\nabla\ln\varphi|^2\nabla\ln\varphi\cdot(D^2\ln\varphi\cdot\nabla\ln\varphi) - \int_\Omega \varphi|\nabla\ln\varphi|^4\Delta\ln\varphi\end{aligned}$$

$$
\begin{aligned}
&= -4 \int_\Omega \frac{1}{\varphi^3} |\nabla\varphi|^2 \nabla\varphi \cdot (D^2 \ln\varphi \cdot \nabla\varphi) - \int_\Omega \frac{1}{\varphi^3} |\nabla\varphi|^4 \Delta \ln\varphi \\
&\le (4 + \sqrt{3}) \int_\Omega \frac{1}{\varphi^3} |\nabla\varphi|^4 |D^2 \ln\varphi| \\
&\le (4 + \sqrt{3}) \cdot \left\{ \int_\Omega \frac{1}{\varphi^5} |\nabla\varphi|^6 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \frac{1}{\varphi} |\nabla\varphi|^2 |D^2 \ln\varphi|^2 \right\}^{\frac{1}{2}},
\end{aligned}
$$

from which (3.20) immediately follows.

Noting that (3.19) can be verified by direct computation and that accordingly

$$
\begin{aligned}
|D^2\varphi|^2 &= \varphi^2 |D^2 \ln\varphi|^2 + \frac{2}{\varphi} \nabla\varphi \cdot (D^2\varphi \cdot \nabla\varphi) - \frac{1}{\varphi^2} |\nabla\varphi|^4 \\
&\le \varphi^2 |D^2 \ln\varphi|^2 + \frac{1}{2} |D^2\varphi|^2 + \frac{1}{\varphi^2} |\nabla\varphi|^4 \quad \text{in } \Omega,
\end{aligned}
$$

that is,

$$
|D^2\varphi|^2 \le 2\varphi^2 |D^2 \ln\varphi|^2 + \frac{2}{\varphi^2} |\nabla\varphi|^4 \quad \text{in } \Omega
$$

by Young's inequality, from (3.20) we obtain that moreover

$$
\int_\Omega \frac{1}{\varphi^3} |\nabla\varphi|^2 |D^2\varphi|^2 \le 2 \int_\Omega \frac{1}{\varphi} |\nabla\varphi|^2 |D^2 \ln\varphi|^2 + 2 \int_\Omega \frac{1}{\varphi^5} |\nabla\varphi|^6 \le (2 + 2 \cdot (19 + 2\sqrt{3})) \int_\Omega \frac{1}{\varphi} |\nabla\varphi|^2 |D^2 \ln\varphi|^2,
$$

and that hence also (3.21) holds. □

In fact, suitably making use of this we can derive the following.

Lemma 3.4. *There exists $\gamma > 0$ with the property that if* (1.6) *is satisfied, then for all $M > 0$ one can choose $C(M) > 0$ such that whenever* (1.7) *and* (1.10) *hold with $\|c_0\|_{L^\infty(\Omega)} \le M$,*

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^4 + \gamma \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2 + \gamma \int_\Omega \frac{1}{c_\varepsilon^5} |\nabla c_\varepsilon|^6 \\
&\le C(M) \int_\Omega |\nabla n_\varepsilon|^2 + C(M) \int_\Omega |\nabla u_\varepsilon|^3 + C(M) \int_\Omega |u_\varepsilon|^6 \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0, 1).
\end{aligned}
\tag{3.22}
$$

Proof. Again since $\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2} \Delta |\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$ and $\frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu}|_{\partial\Omega} \le 0$ by convexity of Ω, assuming (1.7), (1.6), and (1.10) with $\|c_0\|_{L^\infty(\Omega)} \le M$ we integrate by parts in the second equation from (2.7) to find that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^4 &= 4 \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 \nabla c_\varepsilon \cdot \{\nabla \Delta c_\varepsilon - n_\varepsilon \nabla c_\varepsilon - c_\varepsilon \nabla n_\varepsilon - \nabla u_\varepsilon \cdot \nabla c_\varepsilon - D^2 c_\varepsilon \cdot u_\varepsilon\} \\
&\quad - 3 \int_\Omega \frac{1}{c_\varepsilon^4} |\nabla c_\varepsilon|^4 \cdot \{\Delta c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon\} \\
&= 2 \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 \Delta |\nabla c_\varepsilon|^2 - 4 \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 |D^2 c_\varepsilon|^2 - 4 \int_\Omega \frac{n_\varepsilon}{c_\varepsilon^3} |\nabla c_\varepsilon|^4 - 4 \int_\Omega \frac{1}{c_\varepsilon^2} |\nabla c_\varepsilon|^2 \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&\quad - 4 \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - 4 \int_\Omega \frac{1}{c_\varepsilon^3} |\nabla c_\varepsilon|^2 \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) - 3 \int_\Omega \frac{1}{c_\varepsilon^4} |\nabla c_\varepsilon|^4 \Delta c_\varepsilon \\
&\quad + 3 \int_\Omega \frac{n_\varepsilon}{c_\varepsilon^3} |\nabla c_\varepsilon|^4 + 3 \int_\Omega \frac{1}{c_\varepsilon^4} |\nabla c_\varepsilon|^4 (u_\varepsilon \cdot \nabla c_\varepsilon)
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\le -2\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla|\nabla c_\varepsilon|^2|^2 - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + 6\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 \\
&\quad - 3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4\Delta c_\varepsilon - \int_\Omega \frac{n_\varepsilon}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 - 4\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&\quad - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) \\
&\quad + 3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4(u_\varepsilon\cdot\nabla c_\varepsilon) \quad \text{for all } t>0.
\end{aligned}
$$

Here another integration by parts shows that

$$
\begin{aligned}
-3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4\Delta c_\varepsilon &= 3\int_\Omega \frac{1}{c_\varepsilon^4}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^4 - 12\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\
&= 6\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 12\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \quad \text{for all } t>0,
\end{aligned}
$$

whence using the identity in (3.19) we can estimate the first four summands on the right-hand side of the inequality in (3.23) according to

$$
\begin{aligned}
&-2\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla|\nabla c_\varepsilon|^2|^2 - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + 6\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4\Delta c_\varepsilon \\
&= -2\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla|\nabla c_\varepsilon|^2|^2 - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\cdot\left\{c_\varepsilon^2|D^2\ln c_\varepsilon|^2 + \frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^4\right\} \\
&\quad + 12\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 12\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\
&= -4\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|D^2\ln c_\varepsilon|^2 - 2\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla|\nabla c_\varepsilon|^2|^2 + 8\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 - 8\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\
&= -4\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|D^2\ln c_\varepsilon|^2 - 2\int_\Omega \frac{1}{c_\varepsilon^3}\left|\nabla|\nabla c_\varepsilon|^2 - \frac{2}{c_\varepsilon}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\right|^2 \\
&\le -4\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|D^2\ln c_\varepsilon|^2 \quad \text{for all } t>0.
\end{aligned}
$$

As (3.20) and (3.21) assert that

$$
4\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2|D^2\ln c_\varepsilon|^2 \ge \frac{2}{40+4\sqrt{3}}\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + \frac{2}{19+2\sqrt{3}}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \quad \text{for all } t>0,
$$

by abbreviating $\gamma := \frac{1}{2}\cdot\min\left\{\frac{2}{40+4\sqrt{3}}, \frac{2}{19+2\sqrt{3}}\right\} \equiv \frac{1}{40+4\sqrt{3}}$ we thus infer from (3.23) upon dropping a nonpositive summand on its right that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + 2\gamma\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + 2\gamma\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\
&\le -4\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) - 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) \\
&\quad + 3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4(u_\varepsilon\cdot\nabla c_\varepsilon) \quad \text{for all } t>0.
\end{aligned} \tag{3.24}
$$

Now several applications of Young's inequality reveal that due to (2.9) we have

$$-4\int_\Omega \frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\nabla n_\varepsilon\cdot\nabla c_\varepsilon \le \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{16}{\gamma}\int_\Omega c_\varepsilon|\nabla n_\varepsilon|^2$$
$$\le \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{16M}{\gamma}\int_\Omega c_\varepsilon|\nabla n_\varepsilon|^2 \quad \text{for all } t>0$$

and

$$-4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \le 4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4|\nabla u_\varepsilon|$$
$$= \int_\Omega \left\{\frac{\gamma}{4}\cdot\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\right\}^{\frac{2}{3}}\cdot\left\{4\cdot\left(\frac{4}{\gamma}\right)^{\frac{2}{3}}c_\varepsilon^{\frac{1}{3}}|\nabla u_\varepsilon|\right\}$$
$$\le \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{1024}{\gamma^2}\int_\Omega c_\varepsilon|\nabla u_\varepsilon|^3$$
$$\le \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{1024M}{\gamma^2}\int_\Omega |\nabla u_\varepsilon|^3 \quad \text{for all } t>0$$

as well as

$$-4\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon)$$
$$\le \gamma\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + \frac{4}{\gamma}\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4|u_\varepsilon|^2$$
$$= \gamma\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + \int_\Omega \left\{\frac{\gamma}{4}\cdot\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\right\}^{\frac{2}{3}}\cdot\left\{\frac{4}{\gamma}\cdot\left(\frac{4}{\gamma}\right)^{\frac{2}{3}}c_\varepsilon^{\frac{1}{3}}|u_\varepsilon|^2\right\}$$
$$\le \gamma\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{1024M}{\gamma^5}\int_\Omega |u_\varepsilon|^6 \quad \text{for all } t>0$$

and, finally,

$$3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^4(u_\varepsilon\cdot\nabla c_\varepsilon) \le 3\int_\Omega \frac{1}{c_\varepsilon^4}|\nabla c_\varepsilon|^5|u_\varepsilon|$$
$$= \int_\Omega \left\{\frac{\gamma}{4}\cdot\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\right\}^{\frac{5}{6}}\cdot\left\{3\cdot\left(\frac{4}{\gamma}\right)^{\frac{5}{6}}c_\varepsilon^{\frac{1}{6}}|u_\varepsilon|\right\}$$
$$\le \frac{\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{729\cdot 1024M}{\gamma^5}\int_\Omega |u_\varepsilon|^6 \quad \text{for all } t>0.$$

Therefore, (3.22) results from (3.24) if we choose $C(M)$ appropriately large. □

Now both (3.3) and (3.22) contain integrals of $|\nabla n_\varepsilon|^2$, which with regard to their part restricted to subdomains of the form $\{n_\varepsilon > s_0\}$ with adequately large $s_0 > 0$ will finally be controlled by the dissipated part $\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2$ from (3.2) (cf. Lemma 3.7 and especially (3.45)). To provide suitable estimates within corresponding complementary regions involving small population densities, in view of possible diffusion degeneracies at $n = 0$ we apply an additional testing procedure to the first equation in (2.7), this time exclusively focusing on regions of small densities, and crucially relying on our assumption (1.9) now.

Lemma 3.5. *Assume* (1.7), (1.9), (1.6), *and* (1.10), *and for* $s_0 > 0$ *and* $\varepsilon \in (0,1)$ *let*

$$\psi_{0,\varepsilon}^{(s_0)}(n) := \begin{cases} \frac{1}{D_\varepsilon(n)}, & n \in (0, s_0), \\ \frac{2s_0 - n}{s_0 D_\varepsilon(s_0)}, & n \in [s_0, 2s_0], \\ 0, & n > 2s_0, \end{cases} \tag{3.25}$$

as well as

$$\psi_{1,\varepsilon}^{(s_0)}(n) := \int_{2s_0}^{n} \psi_{0,\varepsilon}^{(s_0)}(s)\mathrm{d}s \quad \text{and} \quad \psi_{2,\varepsilon}^{(s_0)}(n) := \int_{2s_0}^{n} \psi_{1,\varepsilon}^{(s_0)}(s)\mathrm{d}s, \quad n > 0. \tag{3.26}$$

Then $\psi_{2,\varepsilon}^{(s_0)} \in C^2((0,\infty))$ *with*

$$0 \le \psi_{2,\varepsilon}^{(s_0)}(n) \le \frac{3s_0}{\kappa(s_0)} \quad \text{for all } n > 0, \tag{3.27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \psi_{2,\varepsilon}^{(s_0)}(n_\varepsilon) + \frac{1}{2} \int_{\{n_\varepsilon < s_0\}} |\nabla n_\varepsilon|^2 \le \frac{S_0^2(\|c_0\|_{L^\infty(\Omega)} + 1)}{\kappa^2(s_0)} \int_\Omega \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 \quad \text{for all } t > 0, \tag{3.28}$$

where

$$\kappa(s_0) := \inf_{n \in (0,s_0)} \frac{D(n)}{n}. \tag{3.29}$$

Proof. The claimed regularity feature is a direct consequence of (3.26) and the fact that $\psi_{0,\varepsilon}^{(s_0)}$ is continuous on $(0,\infty)$ according to (3.25). Moreover, using (2.5) and (3.29) to estimate

$$\frac{1}{D_\varepsilon(n)} \le \frac{1}{D(n)} \le \frac{1}{\kappa(s_0)n} \quad \text{for all } n \in (0, s_0) \text{ and } \varepsilon \in (0,1),$$

and

$$0 \le \psi_{0,\varepsilon}^{(s_0)}(n) \le \frac{s_0}{s_0 D_\varepsilon(s_0)} = \frac{1}{D_\varepsilon(s_0)} \le \frac{1}{s_0 \kappa(s_0)} \quad \text{for all } n \in [s_0, 2s_0] \text{ and } \varepsilon \in (0,1),$$

we see that $\psi_{1,\varepsilon}^{(s_0)} \le 0$ with $\psi_{1,\varepsilon}^{(s_0)} \equiv 0$ on $(2s_0, \infty)$,

$$|\psi_{1,\varepsilon}^{(s_0)}(n)| \le \int_{s_0}^{2s_0} \frac{\mathrm{d}s}{s_0 \kappa(s_0)} = \frac{1}{\kappa(s_0)} \quad \text{for all } n \in [s_0, 2s_0]$$

and

$$|\psi_{1,\varepsilon}^{(s_0)}(n)| \le |\psi_{1,\varepsilon}^{(s_0)}(s_0)| + \int_n^{s_0} \frac{\mathrm{d}s}{D_\varepsilon(s)} \le \frac{1}{\kappa(s_0)} + \frac{1}{\kappa(s_0)} \ln \frac{s_0}{n} \quad \text{for all } n \in (0, s_0)$$

whenever $\varepsilon \in (0,1)$. Accordingly, for any such ε we have $\psi_{2,\varepsilon}^{(s_0)} \equiv 0$ on $(2s_0, \infty)$ and

$$0 \le \psi_{2,\varepsilon}^{(s_0)}(n) \le \int_{s_0}^{2s_0} |\psi_{1,\varepsilon}^{(s_0)}(s)|\mathrm{d}s \le \frac{s_0}{\kappa(s_0)} \quad \text{for all } n \in [s_0, 2s_0]$$

as well as

$$0 \le \psi_{2,\varepsilon}^{(s_0)}(n) \le \psi_{2,\varepsilon}^{(s_0)}(s_0) + \int_0^{s_0} |\psi_{1,\varepsilon}^{(s_0)}(s)| ds$$
$$\le \frac{s_0}{\kappa(s_0)} + \int_0^{s_0} \left\{ \frac{1}{\kappa(s_0)} + \frac{1}{\kappa(s_0)} \ln \frac{s_0}{s} \right\} ds$$
$$= \frac{s_0}{\kappa(s_0)} + \frac{s_0}{\kappa(s_0)} + \frac{s_0 \ln s_0}{\kappa(s_0)} - \frac{1}{\kappa(s_0)} \cdot (s_0 \ln s_0 - s_0)$$
$$= \frac{3s_0}{\kappa(s_0)} \quad \text{for all } n \in (0, s_0),$$

so that (3.27) follows.

To verify (3.28) for fixed $\varepsilon \in (0,1)$, we once more use the solenoidality of u_ε to see that due to the first equation in (2.7), Young's inequality, (2.6), (1.6), and (2.9),

$$\begin{aligned} \frac{d}{dt} \int_\Omega \psi_{2,\varepsilon}^{(s_0)}(n_\varepsilon) &= - \int_\Omega \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) \nabla n_\varepsilon \cdot (S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) \\ &\le -\frac{1}{2} \int_\Omega \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 + \frac{1}{2} \int_\Omega \frac{n_\varepsilon^2 \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon)}{D_\varepsilon(n_\varepsilon)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon|^2 \\ &\le -\frac{1}{2} \int_\Omega \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 + \frac{1}{2} S_0^2 \left(\|c_0\|_{L^\infty(\Omega)} + 1 \right) \int_\Omega \frac{n_\varepsilon^2 \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon)}{D_\varepsilon(n_\varepsilon)} \cdot \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 \quad \text{for all } t \\ &> 0, \end{aligned} \tag{3.30}$$

because $(\psi_{2,\varepsilon}^{(s_0)})'' \equiv \psi_{0,\varepsilon}^{(s_0)} \ge 0$ and $D_\varepsilon > 0$ on $(0, \infty)$. Here, (3.25) asserts that

$$\int_\Omega \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 \ge \int_{\{n_\varepsilon < s_0\}} \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon) D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 = \int_{\{n_\varepsilon < s_0\}} |\nabla n_\varepsilon|^2 \quad \text{for all } t > 0,$$

while by (2.5) and (3.29),

$$\frac{n^2 \cdot (2s_0 - n)}{s_0 D_\varepsilon(s_0) D_\varepsilon(n)} \le \frac{n^2 s_0}{s_0 D_\varepsilon(s_0) D_\varepsilon(n)} = \frac{n}{D_\varepsilon(s_0)} \cdot \frac{n}{D_\varepsilon(n)} \le \frac{2s_0}{D_\varepsilon(s_0)} \cdot \frac{n}{D_\varepsilon(n)} \le \frac{2}{\kappa^2(s_0)} \quad \text{for all } n \in [s_0, 2s_0],$$

so that

$$\begin{aligned} \int_\Omega \frac{n_\varepsilon^2 \psi_{0,\varepsilon}^{(s_0)}(n_\varepsilon)}{D_\varepsilon(n_\varepsilon)} \cdot \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 &= \int_{\{n_\varepsilon < s_0\}} \frac{n_\varepsilon^2}{D_\varepsilon^2(n_\varepsilon)} \cdot \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 + \int_{\{s_0 \le n_\varepsilon \le 2s_0\}} \frac{n_\varepsilon^2 \cdot (2s_0 - n_\varepsilon)}{s_0 D_\varepsilon(s_0) D_\varepsilon(n_\varepsilon)} \cdot \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 \\ &\le \frac{1}{\kappa^2(s_0)} \int_{\{n_\varepsilon < s_0\}} \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 + \frac{2}{\kappa^2(s_0)} \int_{\{s_0 \le n_\varepsilon \le 2s_0\}} \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 \\ &\le \frac{2}{\kappa^2(s_0)} \int_\Omega \frac{1}{c_\varepsilon} |\nabla c_\varepsilon|^2 \quad \text{for all } t > 0. \end{aligned}$$

Therefore, (3.28) results from (3.30). □

As a final preparation for our construction of an entropy-like functional for (2.7), we employ a Poincaré inequality in estimating the zero-order integral differentiated in (3.2) in terms of the associated dissipation rate appearing therein:

Lemma 3.6. *Suppose that* (1.6), (1.7), (1.8), *and* (1.10) *hold with some* $L > 0$. *Then there exists* $C = C(D, n_0, c_0, u_0) > 0$ *such that with* $(D_{2,\varepsilon})_{\varepsilon \in (0,1)}$ *as in* (3.1), *we have*

$$\int_\Omega D_{2,\varepsilon}(n_\varepsilon) \le C \int_\Omega D_\varepsilon^2(n_\varepsilon) |\nabla n_\varepsilon|^2 + C \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.31}$$

Proof. According to a Poincaré inequality ([48, Corollary 9.1.4]), we can fix $C_1 > 0$ fulfilling

$$\int_\Omega \varphi^2 \le C_1 \int_\Omega |\nabla\varphi|^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ such that } |\{\varphi = 0\}| \ge \frac{|\Omega|}{2}, \tag{3.32}$$

and to derive (3.31) from this, assuming (1.6), (1.7), (1.8), and (1.10) to hold with some $L > 0$ we let

$$\delta_\varepsilon \equiv \delta_\varepsilon(D, n_0, c_0, u_0) := D_{1,\varepsilon}\left(\frac{2}{|\Omega|}\int_\Omega n_0\right), \quad \varepsilon \in (0,1), \tag{3.33}$$

with $(D_{1,\varepsilon})_{\varepsilon\in(0,1)}$ taken from (3.1), observing that since

$$\int_\Omega n_0 = \int_\Omega n_\varepsilon \ge a|\{n_\varepsilon \ge a\}| \quad \text{for all } t > 0, \varepsilon \in (0,1) \text{ and } a > 0$$

due to (2.8) and the Chebyshev inequality, (3.33) together with the fact that $D'_{1,\varepsilon} > 0$ for all $\varepsilon \in (0,1)$ ensures that

$$\begin{aligned} |\{D_{1,\varepsilon}(n_\varepsilon) < \delta_\varepsilon\}| &= \left|\left\{n_\varepsilon < \frac{2}{|\Omega|}\int_\Omega n_0\right\}\right| \\ &= |\Omega| - \left|\left\{n_\varepsilon \ge \frac{2}{|\Omega|}\int_\Omega n_0\right\}\right| \\ &\ge |\Omega| - \frac{|\Omega|}{2} = \frac{|\Omega|}{2} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{3.34}$$

Now the upward monotonicity of $D_{1,\varepsilon}$ furthermore implies that

$$D_{2,\varepsilon}(n) = \int_0^n D_{1,\varepsilon}(s)ds \le nD_{1,\varepsilon}(n) \quad \text{for all } n \ge 0 \text{ and } \varepsilon \in (0,1),$$

and that thus

$$\int_\Omega D_{2,\varepsilon}(n_\varepsilon) \le \int_\Omega n_\varepsilon^2 + \frac{1}{4}\int_\Omega D_{1,\varepsilon}^2(n_\varepsilon) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1) \tag{3.35}$$

by Young's inequality, where the first summand on the right can essentially be controlled in terms of the second one, because our assumption (1.8) on D guarantees that with some $s_1 = s_1(D) > 0$ we have $D_\varepsilon(n) \ge L$ for all $n \ge s_1$ and each $\varepsilon \in (0,1)$. By definition of $(D_{1,\varepsilon})_{\varepsilon\in(0,1)}$, namely, this implies that

$$D_{1,\varepsilon}(n) \ge \int_{s_1}^n L\, ds = L \cdot (n - s_1) \quad \text{for all } n \ge s_1 \text{ and } \varepsilon \in (0,1),$$

so that

$$\begin{aligned} \int_\Omega n_\varepsilon^2 &= \int_{\{n_\varepsilon \le s_1\}} n_\varepsilon^2 + \int_{\{n_\varepsilon > s_1\}} n_\varepsilon^2 \\ &\le s_1^2|\Omega| + \int_\Omega \left\{s_1 + \frac{D_{1,\varepsilon}(n_\varepsilon)}{L}\right\}^2 \\ &\le 3s_1^2|\Omega| + \frac{2}{L^2}\int_\Omega D_{1,\varepsilon}^2(n_\varepsilon) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Therefore, a combination of (3.35) with (3.32), the latter being applicable here thanks to (3.34), shows that abbreviating $C_2 \equiv C_2(D) := \frac{2}{L^2} + \frac{1}{4}$ and $C_3 \equiv C_3(D) := 3s_1^2|\Omega|$ we have

$$\begin{aligned}\int_\Omega D_{2,\varepsilon}(n_\varepsilon) &\le C_2 \int_\Omega D_{1,\varepsilon}^2(n_\varepsilon) + C_3 \\ &\le 2C_2 \int_\Omega (D_{1,\varepsilon}(n_\varepsilon) - \delta_\varepsilon)_+^2 + 2C_2\delta_\varepsilon^2|\Omega| + C_3 \\ &\le 2C_1C_2 \int_\Omega |\nabla(D_{1,\varepsilon}(n_\varepsilon) - \delta_\varepsilon)_+|^2 + 2C_2\delta_\varepsilon^2|\Omega| + C_3 \\ &\le 2C_1C_2 \int_\Omega |\nabla D_{1,\varepsilon}(n_\varepsilon)|^2 + 2C_2\delta_\varepsilon^2|\Omega| + C_3 \\ &= 2C_1C_2 \int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + 2C_2\delta_\varepsilon^2|\Omega| + C_3 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),\end{aligned}$$

because $D_{1,\varepsilon}' \equiv D_\varepsilon$ for all $\varepsilon \in (0,1)$. □

The main result of this section can now be established by appropriately arranging a combination of Lemmas 3.1, 3.2, 3.4, and 3.5 with Lemma 3.6, and by making suitable use of the hypothesis (1.8) with some carefully chosen $L = L(M)$:

Lemma 3.7. *Assume* (1.6). *Then for all $M > 0$ one can fix $L = L(M) > 0$, $b_1 = b_1(M) > 0$, $b_2 = b_2(M) > 0$ and $b_3 = b_3(M) > 0$ in such a way that whenever* (1.7), (1.8), (1.9), *and* (1.10) *hold with $\|c_0\|_{L^\infty(\Omega)} \le M$, there exist $s_0 = s_0(D, n_0, c_0, u_0) > 0$, $\mu = \mu(D, n_0, c_0, u_0) > 0$ and $\Gamma = \Gamma(D, n_0, c_0, u_0) > 0$ such that for any choice of $\varepsilon \in (0,1)$,*

$$\mathcal{F}_\varepsilon(t) := \int_\Omega D_{2,\varepsilon}(n_\varepsilon) + b_1 \int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 + b_2 \int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + b_3 \int_\Omega \psi_{2,\varepsilon}^{(s_0)}(n_\varepsilon), \quad t \ge 0, \tag{3.36}$$

with $D_{2,\varepsilon}$ and $\psi_{2,\varepsilon}^{(s_0)}$ taken from (3.1) *and* (3.26), *satisfies*

$$\mathcal{F}_\varepsilon'(t) + \mu\mathcal{F}_\varepsilon(t) + \int_\Omega |\nabla n_\varepsilon|^2 \le \Gamma \cdot \left\{ \int_\Omega |\nabla u_\varepsilon|^3 + \int_\Omega |u_\varepsilon|^6 + 1 \right\} \quad \text{for all } t > 0. \tag{3.37}$$

Proof. Assuming (1.6), given $M > 0$ we let

$$b_1 \equiv b_1(M) := 2S_0^2(M+1) \tag{3.38}$$

and apply Lemma 3.2 with $\eta \equiv \eta(M) := \frac{1}{4b_1}$ as well as Lemma 3.4 to find $C_1 = C_1(M) > 0$ and $C_2 = C_2(M) > 0$ with the property that if (1.7) and (1.10) hold with $\|c_0\|_{L^\infty(\Omega)} \le M$, then for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \frac{1}{2}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 \\ &\quad\le \frac{1}{4b_1}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + C_1\int_\Omega |\nabla n_\varepsilon|^2 \\ &\qquad + C_1\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + C_1\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + C_1\int_\Omega |\nabla u_\varepsilon|^3 + C_1\int_\Omega |u_\varepsilon|^6,\end{aligned} \tag{3.39}$$

and that with $\gamma > 0$ as introduced in Lemma 3.4 we moreover have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + \gamma\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + \gamma\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 \\ &\quad\le C_2\int_\Omega |\nabla n_\varepsilon|^2 + C_2\int_\Omega |\nabla u_\varepsilon|^3 + C_2\int_\Omega |u_\varepsilon|^6 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).\end{aligned} \tag{3.40}$$

We now define

$$b_2 \equiv b_2(M) := \frac{2b_1C_1}{\gamma} \quad \text{and} \quad b_3 \equiv b_3(M) := 2(b_1C_1 + b_2C_2 + 1) \tag{3.41}$$

as well as

$$L \equiv L(M) := \sqrt{8(b_1C_1 + b_2C_2 + 1)}, \tag{3.42}$$

and henceforth fixing functions D and (n_0, c_0, u_0) fulfilling (1.7), (1.8), (1.9), and (1.10) with $\|c_0\|_{L^\infty(\Omega)} \le M$, according to (1.8) and (2.5) we can pick $s_0 = s_0(D, n_0, c_0, u_0) > 0$ such that

$$D_\varepsilon(n) \ge L \quad \text{for all } n > s_0 \text{ and any } \varepsilon \in (0,1). \tag{3.43}$$

We then let $(\mathcal{F}_\varepsilon)_{\varepsilon\in(0,1)}$ be as correspondingly defined through (3.36), and combining the outcome of Lemma 3.1 with (3.39), (3.40), and Lemma 3.5 we obtain that with κ as in (3.29),

$$\begin{aligned}
\mathcal{F}_\varepsilon'(t) \le{}& \left\{-\frac{1}{2}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \frac{1}{2}S_0^2(M+1)\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2\right\} + b_1\cdot\left\{-\frac{1}{2}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \frac{1}{4b_1}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2\right.\\
&\left. + C_1\int_\Omega|\nabla n_\varepsilon|^2 + C_1\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + C_1\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + C_1\int_\Omega|\nabla u_\varepsilon|^3 + C_1\int_\Omega|u_\varepsilon|^6\right\}\\
&+ b_2\cdot\left\{-\gamma\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 - \gamma\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + C_2\int_\Omega|\nabla n_\varepsilon|^2 + C_2\int_\Omega|\nabla u_\varepsilon|^3 + C_2\int_\Omega|u_\varepsilon|^6\right\}\\
&+ b_3\cdot\left\{-\frac{1}{2}\int_{\{n_\varepsilon<s_0\}}|\nabla n_\varepsilon|^2 + \frac{S_0^2(M+1)}{\kappa^2(s_0)}\int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2\right\}\\
={}& \left(-\frac{1}{2}+\frac{b_1}{4b_1}\right)\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + (b_1C_1+b_2C_2)\int_\Omega|\nabla n_\varepsilon|^2 - \frac{b_3}{2}\int_{\{n_\varepsilon<s_0\}}|\nabla n_\varepsilon|^2\\
&+\left(\frac{1}{2}S_0^2(M+1)-\frac{b_1}{2}\right)\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + (b_1C_1-b_2\gamma)\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 + (b_1C_1-b_2\gamma)\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\\
&+(b_1C_1+b_2C_2)\int_\Omega|\nabla u_\varepsilon|^3 + (b_1C_1+b_2C_2)\int_\Omega|u_\varepsilon|^6 + \frac{b_3S_0^2(M+1)}{\kappa^2(s_0)}\int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2\\
={}& -\frac{1}{4}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + (b_1C_1+b_2C_2)\int_\Omega|\nabla n_\varepsilon|^2 - \frac{b_3}{2}\int_{\{n_\varepsilon<s_0\}}|\nabla n_\varepsilon|^2 - \frac{b_1}{4}\int_\Omega\frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2\\
&-\frac{b_2\gamma}{2}\int_\Omega\frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^2|D^2c_\varepsilon|^2 - \frac{b_2\gamma}{2}\int_\Omega\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + (b_1C_1+b_2C_2)\int_\Omega|\nabla u_\varepsilon|^3 + (b_1C_1+b_2C_2)\int_\Omega|u_\varepsilon|^6\\
&+\frac{b_3S_0^2(M+1)}{\kappa^2(s_0)}\int_\Omega\frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 \quad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned} \tag{3.44}$$

because $\frac{1}{2}S_0^2(M+1) - \frac{b_1}{2} = -\frac{b_1}{4}$ by (3.38) and $b_1C_1 - b_2\gamma = -\frac{b_2\gamma}{2}$ by (3.41). Here our selections of b_3 and L in (3.41) and (3.42) enable us to infer from (3.43) that

$$\begin{aligned}
&(b_1C_1+b_2C_2+1)\int_\Omega|\nabla n_\varepsilon|^2 - \frac{b_3}{2}\int_{\{n_\varepsilon<s_0\}}|\nabla n_\varepsilon|^2\\
&= \left(b_1C_1+b_2C_2+1-\frac{b_3}{2}\right)\int_{\{n_\varepsilon<s_0\}}|\nabla n_\varepsilon|^2 + (b_1C_1+b_2C_2+1)\int_{\{n_\varepsilon\ge s_0\}}\frac{1}{D_\varepsilon^2(n_\varepsilon)}\cdot D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2
\end{aligned} \tag{3.45}$$

$$\leq \frac{b_1C_1 + b_2C_2 + 1}{L^2} \int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2$$
$$= \frac{1}{8} \int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

whence dropping a favorably signed summand we infer from (3.44) that

$$\mathcal{F}_\varepsilon'(t) + \frac{1}{8}\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + \int_\Omega |\nabla n_\varepsilon|^2 + \frac{b_1}{4}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \frac{b_2\gamma}{2}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6$$
$$\leq (b_1C_1 + b_2C_2)\int_\Omega |\nabla u_\varepsilon|^3 + (b_1C_1 + b_2C_2)\int_\Omega |u_\varepsilon|^6 + \frac{b_3S_0^2(M+1)}{\kappa^2(s_0)}\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 \tag{3.46}$$
$$\text{for all } t > 0 \text{ and } \varepsilon \in (0,1).$$

To create an absorptive term on the left-hand side in the style of the claim concerning (3.37), we now employ Lemma 3.6 to pick $C_3 = C_3(D, n_0, c_0, u_0) > 0$ such that

$$\int_\Omega D_{2,\varepsilon}(n_\varepsilon) \leq C_3\int_\Omega D_\varepsilon^2(n_\varepsilon)|\nabla n_\varepsilon|^2 + C_3 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

and use Young's inequality together with (2.9) in estimating

$$\int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 = \int_\Omega \left\{\frac{b_2\gamma}{4}\cdot\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\right\}^{\frac{2}{3}} \cdot \left\{\left(\frac{4}{b_2\gamma}\right)^{\frac{2}{3}} c_\varepsilon^{\frac{1}{3}}\right\}$$
$$\leq \frac{b_2\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{16}{b_2^2\gamma^2}\int_\Omega c_\varepsilon$$
$$\leq \frac{b_2\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{16M|\Omega|}{b_2^2\gamma^2} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).$$

As, similarly,

$$\left(\frac{b_3S_0^2(M+1)}{\kappa^2(s_0)} + 1\right)\cdot\int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 = \int_\Omega \left\{\frac{b_2\gamma}{4}\cdot\frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6\right\}^{\frac{1}{3}} \cdot \left\{\left(\frac{4}{b_2\gamma}\right)^{\frac{1}{3}}\left(\frac{b_3S_0^2(M+1)}{\kappa^2(s_0)} + 1\right)c_\varepsilon^{\frac{2}{3}}\right\}$$
$$\leq \frac{b_2\gamma}{4}\int_\Omega \frac{1}{c_\varepsilon^5}|\nabla c_\varepsilon|^6 + \frac{2}{b_2^{\frac{1}{2}}\gamma^{\frac{1}{2}}}\left(\frac{b_3S_0^2(M+1)}{\kappa^2(s_0)} + 1\right)^{\frac{3}{2}} M|\Omega|$$

for all $t > 0$ and $\varepsilon \in (0,1)$, from (3.46) we therefore obtain that

$$\mathcal{F}_\varepsilon'(t) + \frac{1}{8C_3}\int_\Omega D_{2,\varepsilon}(n_\varepsilon) + \frac{b_1}{4}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon^3}|\nabla c_\varepsilon|^4 + \int_\Omega |\nabla n_\varepsilon|^2$$
$$\leq (b_1C_1 + b_2C_2)\int_\Omega |\nabla u_\varepsilon|^3 + (b_1C_1 + b_2C_2)\int_\Omega |u_\varepsilon|^6$$
$$+ \frac{1}{8} + \frac{16M|\Omega|}{b_2^2\gamma^2} + \frac{2}{b_2^{\frac{1}{2}}\gamma^{\frac{1}{2}}}\left(\frac{b_3S_0^2(M+1)}{\kappa^2(s_0)} + 1\right)^{\frac{3}{2}} M|\Omega| \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

and that thus (3.37) holds with suitably chosen μ and Γ, because by Young's inequality,

$$\frac{b_1}{4}\int_\Omega \frac{n_\varepsilon^2}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \int_\Omega \frac{1}{c_\varepsilon}|\nabla c_\varepsilon|^2 \geq \sqrt{b_1}\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

and because

$$\int_\Omega \psi_{2,\varepsilon}^{(s_0)}(n_\varepsilon) \le \frac{3s_0|\Omega|}{\kappa(s_0)} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

according to (3.27). □

4 Fluid regularity and entropy-based further estimates

In this section we now perform a series of arguments from the regularity theories of parabolic equations and the Stokes evolution system to adequately exploit the structural feature discovered in Lemma 3.7. Our first step in this direction draws on a standard result on maximal Sobolev regularity to provide some control for the integrals on the right of (3.37) in relationship to the correspondingly dissipated quantity on the left-hand side therein.

Lemma 4.1. *Assume* (1.7), (1.6), *and* (1.10). *Then for all $\lambda > 0$ and each $\eta > 0$ there exists $C = C(\lambda, \eta, D, n_0, c_0, u_0) > 0$ such that*

$$\int_0^T \int_\Omega e^{\lambda t} \cdot \{|\nabla u_\varepsilon|^3 + |u_\varepsilon|^6\} \le \eta \int_0^T \int_\Omega e^{\lambda t} |\nabla n_\varepsilon|^2 + Ce^{\lambda T} \quad \textit{for all } T > 0 \textit{ and } \varepsilon \in (0,1). \tag{4.1}$$

Proof. We fix any $q \in \left(2, \frac{18}{7}\right)$ and recall a well-known result on maximal Sobolev regularity in the Stokes evolution system [49] to fix $C_1 > 0$ with the property that whenever $T > 0$ and $\varphi \in \bigcup_{\vartheta > \frac{3}{4}} C^0([0,T]; D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega} \times (0,T); \mathbb{R}^3)$ is such that $\varphi(\cdot, 0) = 0$ in Ω, we have

$$\int_0^T \|A\varphi(\cdot,t)\|_{L^2(\Omega)}^q \mathrm{d}t \le C_1 \int_0^T \|\varphi_t(\cdot,t) + A\varphi(\cdot,t)\|_{L^2(\Omega)}^q \mathrm{d}t. \tag{4.2}$$

We furthermore let $r_1 := \frac{18-6q}{q}$ and $r_2 := \frac{36-6q}{6+q}$ and note that since $q > 2$, we then have $r_1 < \frac{18-6\cdot 2}{2} = 3$ and $r_2 < \frac{36-6\cdot 2}{6+2} = 3$, and that our restriction $q < \frac{18}{7}$ ensures that $r_1 > \frac{18-6\cdot\frac{18}{7}}{\frac{18}{7}} = 1$ and $r_2 > \frac{36-6\cdot\frac{18}{7}}{6+\frac{18}{7}} = \frac{12}{5} > 1$. Therefore, by using that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ we may rely on a Gagliardo-Nirenberg interpolation to pick $C_2 > 0$ and $C_3 > 0$ fulfilling

$$\|\nabla\varphi\|_{L^3(\Omega)}^3 \le C_2 \|A\varphi\|_{L^2(\Omega)}^q \|\varphi\|_{L^{r_1}(\Omega)}^{\frac{3r_1}{6+r_1}} \quad \text{for all } \varphi \in D(A) \tag{4.3}$$

and

$$\|\varphi\|_{L^6(\Omega)}^6 \le C_3 \|A\varphi\|_{L^2(\Omega)}^q \|\varphi\|_{L^{r_2}(\Omega)}^{\frac{12r_2}{6+r_2}} \quad \text{for all } \varphi \in D(A), \tag{4.4}$$

and similarly we find $C_4 > 0$ such that

$$\|\varphi\|_{L^2(\Omega)}^q \le C_4 \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{3q}{5}} \|\varphi\|_{L^1(\Omega)}^{\frac{2q}{5}} + C_4 \|\varphi\|_{L^1(\Omega)}^q \quad \text{for all } \varphi \in W^{1,2}(\Omega).$$

We now combine (2.8) with a standard argument based on known smoothing estimates for the Dirichlet Stokes semigroup $(e^{-tA})_{t\ge 0}$ on Ω (cf., e.g., [50, Lemma 2.5] and [35, Corollary 3.4] for detailed arguments in closely related cases) to obtain $C_i = C_i(n_0, c_0, u_0) > 0$, $i \in \{5,6,7\}$, such that $\hat{u}_\varepsilon(\cdot,t) := u_\varepsilon(\cdot,t) - e^{-tA}u_0$, $t \ge 0$, $\varepsilon \in (0,1)$, satisfies

$$\|\hat{u}_\varepsilon(\cdot,t)\|_{L^{r_1}(\Omega)} \le C_5, \quad \|\hat{u}_\varepsilon(\cdot,t)\|_{L^{r_2}(\Omega)} \le C_6 \quad \text{and} \quad \|\hat{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_7 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

because $r_1 \in (1,3)$ and $r_2 \in (1,3)$. Observing that for each $\varepsilon \in (0,1)$ we have $\hat{u}_\varepsilon|_{t=0} = 0$ and

$$\partial_t\Big\{e^{\frac{\lambda}{q}t}\hat{u}_\varepsilon\Big\} + e^{\frac{\lambda}{q}t}A\hat{u}_\varepsilon = e^{\frac{\lambda}{q}t}\mathcal{P}[n_\varepsilon\nabla\Phi] + \frac{\lambda}{q}e^{\frac{\lambda}{q}t}\hat{u}_\varepsilon \quad \text{in } \Omega\times(0,\infty)$$

according to (2.7), writing $C_8 \equiv C_8(n_0,c_0,u_0) := C_2C_5^{\frac{3r_1}{6+r_1}} + C_3C_6^{\frac{12r_2}{6+r_2}}$ we therefore conclude from (4.2) and (2.7) that

$$\begin{aligned}
&\int_0^T\int_\Omega e^{\lambda t}\{|\nabla\hat{u}_\varepsilon|^3 + |\hat{u}_\varepsilon|^6\}\\
&\le C_2\int_0^T e^{\lambda t}\|A\hat{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^q\|\hat{u}_\varepsilon(\cdot,t)\|_{L^{r_1}(\Omega)}^{\frac{3r_1}{6+r_1}}dt + C_3\int_0^T e^{\lambda t}\|A\hat{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^q\|\hat{u}_\varepsilon(\cdot,t)\|_{L^{r_2}(\Omega)}^{\frac{12r_2}{6+r_2}}dt\\
&\le C_8\int_0^T \|e^{\frac{\lambda}{q}t}A\hat{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^q dt\\
&\le C_1C_8\int_0^T \Big\|e^{\frac{\lambda}{q}t}\mathcal{P}[n_\varepsilon(\cdot,t)\nabla\Phi] + \frac{\lambda}{q}e^{\frac{\lambda}{q}t}\hat{u}_\varepsilon(\cdot,t)\Big\|_{L^2(\Omega)}^q dt\\
&\le 2^qC_1C_8\|\nabla\Phi\|_{L^\infty(\Omega)}^q\int_0^T e^{\lambda t}\|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^q dt + 2^q\cdot\Big(\frac{\lambda}{q}\Big)^q C_1C_8\int_0^T e^{\lambda t}\|\hat{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^q dt\\
&\le C_9\int_0^T e^{\lambda t}\cdot\Big\{\|\nabla n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{3q}{5}}\|n_\varepsilon\|_{L^1(\Omega)}^{\frac{2q}{5}} + \|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)}^q\Big\}dt + C_{10}\int_0^T e^{\lambda t}dt\\
&\qquad\text{for all } T>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}\tag{4.5}$$

where $C_9 \equiv C_9(n_0,c_0,u_0) := 2^qC_1C_4C_8\|\nabla\Phi\|_{L^\infty(\Omega)}^q$ and $C_{10} \equiv C_{10}(n_0,c_0,u_0) := 2^q\cdot(\frac{\lambda}{q})^qC_1C_7^qC_8$. As $\frac{3q}{5} < \frac{54}{35} < 2$, given $\eta > 0$ we may here use Young's inequality along with (2.8) to see abbreviating $C_{11} \equiv C_{11}(n_0,c_0,u_0) := \|n_0\|_{L^1(\Omega)}$ that for all $T > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
&C_9\int_0^T e^{\lambda t}\cdot\Big\{\|\nabla n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{3q}{5}}\|n_\varepsilon\|_{L^1(\Omega)}^{\frac{2q}{5}} + \|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)}^q\Big\}dt\\
&\le C_9C_{11}^{\frac{2q}{5}}\int_0^T e^{\lambda t}\|\nabla n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{3q}{5}}dt + C_9C_{11}^q\int_0^T e^{\lambda t}dt\\
&= \int_0^T e^{\lambda t}\cdot\Big\{\frac{\eta}{64}\|\nabla n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2\Big\}^{\frac{3q}{10}}\cdot\Big\{\Big(\frac{64}{\eta}\Big)^{\frac{3q}{10}}C_9C_{11}^{\frac{2q}{5}}\Big\}dt + C_9C_{11}^q\int_0^T e^{\lambda t}dt\\
&\le \frac{\eta}{64}\int_0^T e^{\lambda t}\|\nabla n_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2dt + \Big\{\Big(\frac{64}{\eta}\Big)^{\frac{3q}{10-3q}}C_9^{\frac{10}{10-3q}}C_{11}^{\frac{4q}{10-3q}} + C_9C_{11}^q\Big\}\cdot\int_0^T e^{\lambda t}dt.
\end{aligned}\tag{4.6}$$

Since moreover $\int_0^T e^{\lambda t}dt \le \frac{1}{\lambda}e^{\lambda T}$ for all $T > 0$, (4.6) together with (4.5) readily implies (4.1) according to the rough pointwise estimate

$$|\nabla u_\varepsilon|^3 + |u_\varepsilon|^6 \le 64\cdot\{|\nabla\hat{u}_\varepsilon|^3 + |\hat{u}_\varepsilon|^6 + |\nabla e^{-tA}u_0|^3 + |e^{-tA}u_0|^6\}, \quad \varepsilon\in(0,1),$$

because both $\sup_{t>0}\|\nabla e^{-tA}u_0\|_{L^3(\Omega)}$ and $\sup_{t>0}\|e^{-tA}u_0\|_{L^6(\Omega)}$ are finite in view of (1.10) and classical regularization features of $(e^{-tA})_{t\ge0}$. □

An application of the latter to suitably small $\eta > 0$, and to $\lambda = \mu$ with μ as in Lemma 3.7, enables us to derive the following estimates as particular consequences obtained upon an integration of (3.37) in time.

Lemma 4.2. *Let $M > 0$, and let* (1.7), (1.8), (1.9), *and* (1.6) *be satisfied with $L = L(M)$ as provided by* Lemma 3.7. *Then for any (n_0, c_0, u_0) fulfilling* (1.10) *with $\|c_0\|_{L^\infty(\Omega)} \le M$, there exists $C = C(D, n_0, c_0, u_0) > 0$ such that*

$$\int_\Omega \frac{1}{c_\varepsilon^3(\cdot, t)} |\nabla c_\varepsilon(\cdot, t)|^4 \le C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \tag{4.7}$$

as well as

$$\int_0^T \int_\Omega e^{-\mu(T-t)} |\nabla n_\varepsilon(x, t)|^2 \mathrm{d}x\mathrm{d}t \le C \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1), \tag{4.8}$$

where $\mu = \mu(D, n_0, c_0, u_0)$ is taken from Lemma 3.7.

Proof. With $\Gamma = \Gamma(D, n_0, c_0, u_0)$ as in Lemma 3.7, we employ Lemma 4.1 to choose $C_1 = C_1(D, n_0, c_0, u_0) > 0$ such that

$$\Gamma \int_0^T \int_\Omega e^{\mu t} \cdot \{|\nabla u_\varepsilon|^3 + |u_\varepsilon|^6\} \le \frac{1}{2} \int_0^T \int_\Omega e^{\mu t} |\nabla n_\varepsilon|^2 + C_1 e^{\mu T} \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1). \tag{4.9}$$

As (3.37) says that the functions $\mathcal{F}_\varepsilon$ defined in (3.36) satisfy

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\{e^{\mu t}\mathcal{F}_\varepsilon(t)\} &= e^{\mu t} \cdot \{\mathcal{F}_\varepsilon'(t) + \mu \mathcal{F}_\varepsilon(t)\} \\ &\le -e^{\mu t} \int_\Omega |\nabla n_\varepsilon|^2 + \Gamma e^{\mu t} \int_\Omega \{|\nabla u_\varepsilon|^3 + |u_\varepsilon|^6\} + \Gamma e^{\mu t} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

upon an integration we thus infer that (4.9) implies the inequality

$$\begin{aligned} e^{\mu T}\mathcal{F}_\varepsilon(T) &\le \mathcal{F}_\varepsilon(0) - \int_0^T \int_\Omega e^{\mu t} |\nabla n_\varepsilon|^2 + \Gamma \int_0^T \int_\Omega e^{\mu t} \cdot \{|\nabla u_\varepsilon|^3 + |u_\varepsilon|^6\} + \Gamma \int_0^T e^{\mu t} \mathrm{d}t \\ &\le \mathcal{F}_\varepsilon(0) - \frac{1}{2} \int_0^T \int_\Omega e^{\mu t} |\nabla n_\varepsilon|^2 + C_1 e^{\mu T} + \Gamma \int_0^T e^{\mu t} \mathrm{d}t \\ &\le \mathcal{F}_\varepsilon(0) + \left(C_1 + \frac{\Gamma}{\mu}\right) e^{\mu T} \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1). \end{aligned}$$

Since from (3.36), (3.1), (2.5), and (3.27) we know that

$$\begin{aligned} \mathcal{F}_\varepsilon(0) &\le |\Omega| \cdot \sup_{\varepsilon \in (0,1)} \int_0^{\|n_0\|_{L^\infty(\Omega)}} \int_0^s D_\varepsilon(\sigma) \mathrm{d}\sigma \mathrm{d}s + b_1 \int_\Omega \frac{n_0}{c_0} |\nabla c_0|^2 + b_2 \int_\Omega \frac{1}{c_0^3} |\nabla c_0|^4 + \frac{3 b_3 s_0 |\Omega|}{\kappa(s_0)} \\ &\le C_2 \equiv C_2(D, n_0, c_0, u_0) := \frac{|\Omega|}{2} \|n_0\|_{L^\infty(\Omega)}^2 \cdot (\|D\|_{L^\infty((0, \|n_0\|_{L^\infty(\Omega)}))} + 2) \\ &\quad + b_1 \int_\Omega \frac{n_0}{c_0} |\nabla c_0|^2 + b_2 \int_\Omega \frac{1}{c_0^3} |\nabla c_0|^4 + \frac{3 b_3 s_0 |\Omega|}{\kappa(s_0)} \quad \text{for all } \varepsilon \in (0, 1), \end{aligned}$$

by nonnegativity of $D_{2,\varepsilon}$ and $\psi_{2,\varepsilon}^{(s_0)}$ for $\varepsilon \in (0, 1)$ this particularly implies that for all $T > 0$ and $\varepsilon \in (0, 1)$,

$$b_2 \int_\Omega \frac{1}{c_\varepsilon^3(\cdot, T)} |\nabla c_\varepsilon(\cdot, T)|^4 + \frac{1}{2} \int_0^T \int_\Omega e^{-\mu(T-t)} |\nabla n_\varepsilon(x, t)|^2 \mathrm{d}x\mathrm{d}t \le C_2 + C_1 + \frac{\Gamma}{\mu},$$

and hence establishes the claim. □

Essentially since the summability power 4 in the estimate (4.7) for the taxis gradient exceeds the considered spatial dimension, the above can be used, in the course of a standard L^p testing argument applied to the first equation in (2.7), to assert bounds on n_ε in L^p spaces for arbitrarily large finite p.

Lemma 4.3. *Let $M > 0$, and suppose that* (1.7), (1.8), (1.9), *and* (1.6) *hold with $L = L(M)$ as in* Lemma 3.7. *Then, assuming* (1.10) *with $\|c_0\|_{L^\infty(\Omega)} \le M$, for each $p > 2$ one can find $C = C(p, D, n_0, c_0, u_0) > 0$ such that*

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le C \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{4.10}$$

Proof. Since $0 \le n \mapsto (n-1)_+^p$ belongs to $C^2([0,\infty))$, we may use (2.7) along with the fact that $C_1 \equiv C_1(D) := \inf_{n\ge1} D(n)$ is positive by (1.7) and (1.8) to see that thanks to (2.5), Young's inequality, (2.6), (1.6), and (2.9),

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n_\varepsilon-1)_+^p + \frac{2(p-1)C_1}{p}\int_\Omega \left|\nabla(n_\varepsilon-1)_+^{\frac{p}{2}}\right|^2 + \int_\Omega (n_\varepsilon-1)_+^p \\
&= p\int_\Omega (n_\varepsilon-1)_+^{p-1}\nabla\cdot\{D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon - n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\} + \frac{p(p-1)C_1}{2}\int_\Omega (n_\varepsilon-1)_+^{p-2}|\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon-1)_+^p \\
&= -p(p-1)\int_\Omega (n_\varepsilon-1)_+^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega n_\varepsilon(n_\varepsilon-1)_+^{p-2}\nabla n_\varepsilon\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\
&\quad + \frac{p(p-1)C_1}{2}\int_\Omega (n_\varepsilon-1)_+^{p-2}|\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon-1)_+^p \\
&\le -\frac{p(p-1)C_1}{2}\int_\Omega (n_\varepsilon-1)_+^{p-2}|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega n_\varepsilon(n_\varepsilon-1)_+^{p-2}\nabla n_\varepsilon\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) + \int_\Omega (n_\varepsilon-1)_+^p \\
&\le \frac{p(p-1)}{2C_1}\int_\Omega n_\varepsilon^2(n_\varepsilon-1)_+^{p-2}|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon|^2 + \int_\Omega (n_\varepsilon-1)_+^p \\
&\le C_2\int_\Omega n_\varepsilon^2(n_\varepsilon-1)_+^{p-2}\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^{\frac{3}{2}}} + \int_\Omega (n_\varepsilon-1)_+^p \quad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned} \tag{4.11}$$

with $C_2 \equiv C_2(p, D, n_0, c_0, u_0) := \frac{p(p-1)\,\|c_0\|_{L^\infty(\Omega)}^{\frac{1}{2}}\,S_0^2(\|c_0\|_{L^\infty(\Omega)}+1)}{2C_1}$. Noting that

$$C_3 \equiv C_3(D, n_0, c_0, u_0) := \sup_{\varepsilon\in(0,1)}\sup_{t>0}\left\{\int_\Omega \frac{1}{c_\varepsilon^3(\cdot,t)}|\nabla c_\varepsilon(\cdot,t)|^4\right\}^{\frac{1}{2}}$$

is finite by Lemma 4.2, and that

$$n^4(n-1)_+^{2p-4} \le 8(n-1)_+^{2p} + 8(n-1)_+^{2p-4} \le 16(n-1)_+^{2p} + 8 \quad \text{for all } n \ge 0$$

by Young's inequality, using the Cauchy-Schwarz inequality we can here estimate

$$\begin{aligned}
\int_\Omega n_\varepsilon^2(n_\varepsilon-1)_+^{p-2}\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^{\frac{3}{2}}} &\le C_3\cdot\left\{\int_\Omega n_\varepsilon^4(n_\varepsilon-1)_+^{2p-4}\right\}^{\frac{1}{2}} \\
&\le C_3\cdot\left\{16\int_\Omega (n_\varepsilon-1)_+^{2p} + 8|\Omega|\right\}^{\frac{1}{2}} \\
&\le 4C_3\cdot\left\{\int_\Omega (n_\varepsilon-1)_+^{2p}\right\}^{\frac{1}{2}} + (8|\Omega|)^{\frac{1}{2}}C_3 \quad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

As furthermore, thanks to the same token,

$$\int_\Omega (n_\varepsilon - 1)_+^p \le |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_\Omega (n_\varepsilon - 1)_+^{2p} \right\}^{\frac{1}{2}} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

by means of an Ehrling inequality associated with the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ we can therefore utilize (2.8) to find $C_i = C_i(p, D, n_0, c_0, u_0) > 0$, $i \in \{4, 5, 6\}$, such that

$$\begin{aligned} C_2 \int_\Omega n_\varepsilon^2 (n_\varepsilon - 1)_+^{p-2} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^{\frac{3}{2}}} + \int_\Omega (n_\varepsilon - 1)_+^p &\le C_4 \left\| (n_\varepsilon - 1)_+^{\frac{p}{2}} \right\|_{L^4(\Omega)}^2 + C_4 \\ &\le \frac{2(p-1)C_1}{p} \left\| \nabla (n_\varepsilon - 1)_+^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + C_5 \left\| (n_\varepsilon - 1)_+^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 + C_4 \\ &\le \frac{2(p-1)C_1}{p} \left\| \nabla (n_\varepsilon - 1)_+^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + C_6 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Consequently, (4.11) implies that

$$\frac{d}{dt} \int_\Omega (n_\varepsilon - 1)_+^p + \int_\Omega (n_\varepsilon - 1)_+^p \le C_6 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

and that thus

$$\int_\Omega (n_\varepsilon - 1)_+^p \le \max \left\{ C_6, \int_\Omega (n_0 - 1)_+^p \right\} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

from which (4.10) immediately follows. □

According to a fairly well-established bootstrap-like series of arguments, this information implies the following.

Lemma 4.4. *Let $M > 0$, assume* (1.7), (1.8), (1.9), *and* (1.6) *with $L = L(M)$ as in* Lemma 3.7, *and suppose that* (1.10) *holds with $\|c_0\|_{L^\infty(\Omega)} \le M$. Then there exists $C = C(D, n_0, c_0, u_0) > 0$ such that*

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1) \tag{4.12}$$

and

$$\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1) \tag{4.13}$$

and that, with β taken from (1.10),

$$\|A^\beta u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \le C \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{4.14}$$

Proof. In view of a standard argument based on L^p–L^q estimates for the Dirichlet Stokes semigroup [51], an application of Lemma 4.3 to any fixed $p > \max\left\{2, \frac{6}{7-4\beta}\right\}$ yields (4.14). Thereupon, (4.13) results by combining known smoothing features of the Neumann heat semigroup on Ω [52] with the fact that according to Lemma 4.3, (4.14), Lemma 4.2 and the continuity of the embedding $D(A_2^\beta) \hookrightarrow L^\infty(\Omega; \mathbb{R}^3)$ [44,53], $(n_\varepsilon c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^\infty((0,\infty); L^q(\Omega))$ with $q := 4$ exceeding the considered spatial dimension. Having (4.13) at hand, however, we can readily derive (4.12) by means of a Moser-type iterative argument following [54, Lemma A.1], based on (1.7), (1.9) and the boundedness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty((0,\infty); L^p(\Omega))$ for all $p > 2$, and of $(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon) + n_\varepsilon u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty((0,\infty); L^q(\Omega; \mathbb{R}^3))$ for all $q < 6$, and hence especially for some $q > 5$, as asserted by Lemma 4.2, (4.13), (4.14), (2.6), and (1.6). □

Thanks to the L^∞ bound in (4.12), a simple application of the maximum principle enables us to derive a pointwise lower estimate for the second solution component, hence especially asserting some favorable distance of solutions to the point $c = 0$ of a possible singularity of S throughout any fixed finite time interval.

Lemma 4.5. *Suppose that $M > 0$, that* (1.7), (1.8), (1.9), *and* (1.6) *hold with $L = L(M)$ as in* Lemma 3.7, *and that* (1.10) *is satisfied with $\|c_0\|_{L^\infty(\Omega)} \le M$. Then for all $T > 0$ there exists $C = C(T, D, n_0, c_0, u_0) > 0$ such that*

$$c_\varepsilon(x,t) \ge C \quad \textit{for all } x \in \Omega, t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{4.15}$$

Proof. According to Lemma 4.4, there exists $C_1 = C_1(D, n_0, c_0, u_0) > 0$ such that $n_\varepsilon \le C_1$ in $\Omega \times (0,\infty)$ for all $\varepsilon \in (0,1)$, so that

$$c_{\varepsilon t} \ge \Delta c_\varepsilon - C_1 c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \quad \text{in } \Omega \times (0,\infty) \quad \text{for all } \varepsilon \in (0,1).$$

From a straightforward argument based on the comparison principle it hence follows that $c_\varepsilon \ge \underline{c}$ in $\Omega \times (0,\infty)$ for all $\varepsilon \in (0,1)$, where $\underline{c}(x,t) := C_2 e^{C_1 t}$, $(x,t) \in \overline{\Omega} \times [0,\infty)$, with $C_2 \equiv C_2(n_0, c_0, u_0) := \inf_{x\in\Omega} c_0(x)$ being positive by (1.10). The claim therefore results if we let $C \equiv C(T, D, n_0, c_0, u_0) := C_2 e^{-C_1 T}$ for $T > 0$. □

5 Global bounded solutions. Proofs of the main results

The last step in our construction of global bounded solutions to (1.1) is now quite straightforward:

Lemma 5.1. *Let $M > 0$, assume* (1.7), (1.8), (1.9), *and* (1.6) *with $L = L(M)$ as in* Lemma 3.7, *and suppose that* (1.10) *holds with $\|c_0\|_{L^\infty(\Omega)} \le M$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as well as functions*

$$\begin{cases} n \in L^\infty(\Omega \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \\ c \in L^\infty((0,\infty); W^{1,\infty}(\Omega)) \quad \textit{and} \\ u \in L^\infty((0,\infty); W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{5.1}$$

such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $n \ge 0$ and $c > 0$ a.e. in $\Omega \times (0,\infty)$, that (n, c, u) forms a global weak solution of (1.1) *in the sense of* Definition 2.1, *and that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$n_\varepsilon \to n \quad \textit{a.e. in } \Omega \times (0,\infty) \textit{ and in } L^p_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ for all } p > 1, \tag{5.2}$$

$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{5.3}$$

$$c_\varepsilon \to c \quad \textit{a.e. in } \Omega \times (0,\infty) \textit{ and in } L^p_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ for all } p > 1, \tag{5.4}$$

$$\nabla c_\varepsilon \overset{\star}{\rightharpoonup} \nabla c \quad \textit{in } L^\infty(\Omega \times (0,\infty)), \tag{5.5}$$

$$u_\varepsilon \rightharpoonup u \quad \textit{in } L^p_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ for all } p > 1, \quad \textit{and} \tag{5.6}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)). \tag{5.7}$$

If moreover $D(0) > 0$, then even (1.12) *holds, and then there exists $P \in C^{1,0}(\Omega \times (0,\infty))$ such that (n, c, u, P) solves* (1.1) *in the classical sense.*

Proof. Abbreviating $C_1 \equiv C_1(D, n_0, c_0, u_0) := \sup_{\varepsilon\in(0,1)} \|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$ and noting that C_1 is finite by Lemma 4.4, for each fixed $\varphi \in C^\infty(\overline{\Omega})$ we can estimate

$$\left|\int_\Omega n_{\varepsilon t}\varphi\right| = \left| -\int_\Omega D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi + \int_\Omega n_\varepsilon(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\cdot\nabla\varphi + \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi\right|$$
$$\le \left\{\sup_{\varepsilon\in(0,1)} \|D_\varepsilon\|_{L^\infty((0,C_1))}\right\}\cdot\|\nabla n_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + C_1 S_0(\|c_0\|_{L^\infty(\Omega)}+1)\left\|\frac{\nabla c_\varepsilon}{c_\varepsilon^{\frac12}}\right\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}$$
$$+ C_1\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \quad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

As $(\nabla n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ due to Lemma 3.7, from (2.5), (1.7), and Lemma 4.4 we hence obtain that $(n_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^2_{loc}([0,\infty);(W^{1,2}(\Omega))^\star)$. Furthermore, the boundedness of $(n_\varepsilon c_\varepsilon + u_\varepsilon\cdot\nabla c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$, as implied by Lemma 4.4, clearly ensures boundedness of $(c_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$. Therefore, the existence of a sequence $(\varepsilon_j)_{j\in(0,1)}\subset(0,1)$ fulfilling $\varepsilon_j\searrow 0$ as $j\to\infty$, and of functions n, c, and u satisfying (5.1)–(5.7) as $\varepsilon=\varepsilon_j\searrow 0$, can be obtained by means of a standard extraction procedure involving Lemma 4.4, an Aubin–Lions lemma and the Vitali convergence theorem, whereupon the verification of (2.2), (2.3), and (2.4) can be achieved in a straightforward manner by taking $\varepsilon=\varepsilon_j\searrow 0$ in corresponding weak formulations of (2.7) and by making use of Lemma 4.5 and the approximation properties expressed in (2.6) and (2.5).

In the case when additionally $D(0)>0$ and hence D is uniformly positive throughout $[0,\infty)$, we may first draw on the regularity features from Lemma (4.4) again to see that thanks to a known result on Hölder regularity in scalar parabolic equations [55], for each $T>0$ there exists $\theta_1=\theta_1(T,D,n_0,c_0,u_0)\in(0,1)$ such that $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])$. Thereupon, standard regularity arguments from the analysis of the Stokes evolution system (see, e.g., [56, Lemma 4.4 and Corollary 4.5] for details in a related setting) show that for each $T>0$ there exists $\theta_2=\theta_2(T,D,n_0,c_0,u_0)\in(0,1)$ such that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\theta_2,\frac{\theta_2}{2}}(\overline{\Omega}\times[0,T];\mathbb{R}^3)$, and that given any $\tau>0$ and $T>\tau$ one can find $\theta_3=\theta_3(\tau,T,D,n_0,c_0,u_0)\in(0,1)$ in such a way that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{2+\theta_3,1+\frac{\theta_3}{2}}(\overline{\Omega}\times[\tau,T];\mathbb{R}^3)$. This in turn enables us to apply Hölder estimates and parabolic Schauder theory to the second equation in (2.7) to see that for all $T>0$ there exists $\theta_4=\theta_4(T,D,n_0,c_0,u_0)\in(0,1)$ such that $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\theta_4,\frac{\theta_4}{2}}(\overline{\Omega}\times[0,T])$, and that whenever $0<\tau<T$, the family $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{2+\theta_5,1+\frac{\theta_5}{2}}(\overline{\Omega}\times[\tau,T])$ for some $\theta_5=\theta_5(\tau,T,D,n_0,c_0,u_0)\in(0,1)$. From (5.2), (5.4), (5.6), and the Arzelà-Ascoli theorem, we thus infer that $n_\varepsilon\to n$ in $C^0_{loc}(\overline{\Omega}\times[0,\infty))$, that $c_\varepsilon\to c$ in $C^0_{loc}(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$, and that $u_\varepsilon\to u$ in $C^0_{loc}(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$, and that according to a standard pressure construction [45], with some $P\in C^{1,0}(\Omega\times(0,\infty))$, the second and third sub-problems in (1.1)) are satisfied in the classical sense. But being a bounded generalized solution of $n_t=\nabla\cdot(D(n)\nabla n - nS(x,n,c)\cdot\nabla c - nu)$ in $\Omega\times(0,\infty)$ with $(D(n)\nabla n - nS(x,n,c)\cdot\nabla c - nu)\cdot\nu|_{\partial\Omega}=0$ and $n|_{t=0}=n_0\in W^{1,\infty}(\Omega)$ in the usual weak sense underlying [57] and [58], in line with a standard argument based on well-known regularity theory for non-degenerate parabolic equations [57,58] and the regularity features of c and u just asserted, the function n must actually belong to $C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$ and satisfy the said problem in the classical sense (cf. also [38, Lemma 5.7] for further details in a similar problem). □

All of our goals have thereby been accomplished:

Proof of Theorem 1.1. We only need to take $L=L(M)>0$ be as introduced in Lemma 3.7, and apply Lemma 5.1. □

Proof of Corollary 1.2. The statement has fully been covered by Theorem 1.1, as (1.13) implies (1.8) for any $L>0$. □

Proof of Corollary 1.3. This immediately results from the first part of Theorem 1.1 due to the fact that $D(n):=n^{m-1}$, $n\ge 0$, satisfies (1.7) and is such that $D(n)\to+\infty$ as $n\to\infty$ and $\frac{D(n)}{n}=n^{m-2}\ge 1$ for all $n\in(0,1)$ due to the assumption that $m\le 2$. □

Proof of Corollary 1.4. Since in the case when $m > \frac{7}{6}$ our additional hypothesis that u_0 belong to $\bigcap_{r>1} W^{2,r}(\Omega; \mathbb{R}^3)$ enables us to directly infer the existence of such a solution from [35], the statement in fact reduces to a by-product of Corollary 1.3. □

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