

Research Article

Qi Li, Shuangjie Peng, and Wei Shuai*

On fractional logarithmic Schrödinger equations

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Abstract: We study the following fractional logarithmic Schrödinger equation:

$$(-\Delta)^s u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N,$$

where $N \geq 1$, $(-\Delta)^s$ denotes the fractional Laplace operator, $0 < s < 1$ and $V(x) \in C(\mathbb{R}^N)$. Under different assumptions on the potential $V(x)$, we prove the existence of positive ground state solution and least energy sign-changing solution for the equation. It is known that the corresponding variational functional is not well defined in $H^s(\mathbb{R}^N)$, and inspired by Cazenave (*Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal. **7** (1983), 1127–1140), we first prove that the variational functional is well defined in a subspace of $H^s(\mathbb{R}^N)$. Then, by using minimization method and Lions' concentration-compactness principle, we prove that the existence results.

Keywords: fractional logarithmic Schrödinger equations, positive ground state solutions, sign-changing solutions, variational methods

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1 Introduction

In this article, we study the following fractional logarithmic Schrödinger equation:

$$(-\Delta)^s u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $N \geq 1$, $0 < s < 1$, $V(x) \in C(\mathbb{R}^N)$ and $V_0 := \inf_x V(x) > 0$. The fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where P.V. stands for the principle value and $C_{N,s}$ is a normalization constant, see for instance [1] and references therein.

Equation (1.1) arises from looking for standing waves $\Phi(t, x) = e^{-i\omega t}u(x)$ of the following fractional logarithmic Schrödinger equation:

* **Corresponding author: Wei Shuai**, School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, P. R. China, e-mail: wshuai@mail.ccnu.edu.cn

Qi Li: College of Science and Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan University of Science and Technology, Wuhan 430065, P. R. China, e-mail: qili@mails.ccnu.edu.cn

Shuangjie Peng: School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, P. R. China, e-mail: sjpeng@mail.ccnu.edu.cn

$$i\frac{\partial\Phi}{\partial t} = (-\Delta)^s\Phi + (V(x) + \omega)\Phi - \Phi\log|\Phi|^2, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \quad (1.2)$$

which is a generalization of the following logarithmic Schrödinger equation:

$$i\frac{\partial\Phi}{\partial t} = -\Delta\Phi + (V(x) + \lambda)\Phi - \Phi\log|\Phi|^2, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \quad (1.3)$$

For physical and mathematical background of equation (1.3), we refer the reader to [2–8] and references therein.

There are few results on fractional logarithmic Schrödinger equation. For example, Ardila [9] recently studied the existence and stability of standing waves for nonlinear fractional Schrödinger equation (1.2). In [10], d'Avenia et al. employed non-smooth critical point theory and obtained infinitely many standing wave solutions to equation (1.2).

Equation (1.1) formally associated with the energy functional $I : H^s(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} (V(x) + 1)u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx,$$

where

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\},$$

with the norm

$$\|u\|_{H^s} := ([u]_s^2 + \|u\|_{L^2(\mathbb{R}^N)}^2)^{\frac{1}{2}}, \quad [u]_s := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

By the fractional logarithmic Sobolev inequality (see [11]), for each $u \in H^s(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} u^2 \log \left(\frac{u^2}{\|u\|_{L^2}^2} \right) dx + \left(N + \frac{N}{s} \log \alpha + \log \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right) \|u\|_{L^2}^2 \leq \frac{\alpha^2}{\pi^s} [u]_s^2, \quad \text{for any } \alpha > 0. \quad (1.4)$$

It is easy to see that $\int_{\mathbb{R}^N} u^2 \log u^2 < +\infty$, but the functional fails to be finite since the logarithm is singular at origin. Indeed, let u be a smooth function that satisfies

$$u(x) = \begin{cases} (|x|^{N/2} \log|x|)^{-1}, & |x| \geq 3, \\ 0, & |x| \leq 2. \end{cases} \quad (1.5)$$

One can verify directly that $u \in H^s(\mathbb{R}^N)$ but $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$. Thus, I fails to be C^1 on $H^s(\mathbb{R}^N)$.

Due to loss of smoothness, the classical critical point theory cannot be applied for I . The same difficulty also occurs for $s = 1$. In order to investigate the following logarithmic Schrödinger equation:

$$-\Delta u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N, \quad (1.6)$$

several approaches developed so far in the literature. Cazenave [5] worked in a suitable Banach space endowed with a Luxemburg-type norm, in which the corresponding energy functional is well defined. Squassina and Szulkin [12] applied non-smooth critical point theory for lower semi-continuous functionals, see also [13–15]. Tanaka and Zhang [16] used penalization technique. Recently, via direction derivative and constrained minimization method, Shuai [17] proved the existence of ground state solutions and sign-changing solutions for equation (1.6). Recently, Wang and Zhang [18] proved that the positive ground state solution of the power-law equations

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0 \end{cases} \quad (1.7)$$

converges to the Gaussian

$$U(x) := e^{\frac{N}{2}} e^{-\frac{\lambda}{4}|x|},$$

up to translations, which is the unique positive solution of the logarithmic equation

$$\begin{cases} -\Delta u = \lambda u \log u^2, & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases}$$

They also proved that the same result holds for bound state solutions.

Inspired by [5], we first find a suitable Banach space, in which the energy functional I is well defined. Then, we study the existence of positive ground state solution and least energy sign-changing solution for equation (1.1).

Throughout this article, we assume $V(x) \in C^{0,\gamma}(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$ and bounded from below. The following different types of potential are considered:

- (V₁) $\lim_{|x| \rightarrow \infty} V(x) = +\infty$.
- (V₂) $V(x)$ is radially symmetric, i.e., $V(x) = V(|x|)$.
- (V₃) $V(x)$ is 1-periodic in each variable of x_1, x_2, \dots, x_N .
- (V₄) For almost every $x \in \mathbb{R}^N$, $V(x) \leq \liminf_{|x| \rightarrow \infty} V(x) := V_\infty < +\infty$, and the inequality is strict in a subset of positive Lebesgue measure.
- (V₅) There exist positive constants M , C_0 and $m \in (0, 2s)$ such that

$$V(x) \leq V_\infty - \frac{C_0}{1 + |x|^m}, \quad \forall |x| \geq M.$$

Next, we denote

$$H := \left\{ u \in H^s(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\}$$

with the norm

$$\|u\|_H := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x) u^2 dx \right)^{\frac{1}{2}}.$$

The Orlicz space $L^A(\mathbb{R}^N)$ is defined by

$$L^A(\mathbb{R}^N) = \{u \in L^1_{\text{loc}}(\mathbb{R}^N) \mid A(|u|) \in L^1(\mathbb{R}^N)\}$$

which is equipped with the Luxemburg norm

$$\|u\|_{L^A} := \inf \left\{ k > 0 \mid \int_{\mathbb{R}^N} A(k^{-1}|u(x)|) dx \leq 1 \right\},$$

where A is given by

$$A(s) := \begin{cases} -s^2 \log s^2, & 0 \leq s \leq e^{-3}, \\ 3s^2 + 4e^{-3}s - e^{-6}, & s \geq e^{-3}. \end{cases}$$

We also denote

$$F(s) = s^2 \log s^2, \quad B(s) = F(s) + A(s), \quad \forall s \geq 0.$$

It is easy to check that A, B are nonnegative convex and increasing function on $[0, +\infty)$, and $A \in C^1([0, +\infty)) \cap C^2((0, +\infty))$.

Now, define

$$W^s(\mathbb{R}^N) := H \cap L^A(\mathbb{R}^N),$$

which is equipped with the norm

$$\|u\| = \|u\|_H + \|u\|_{L^A},$$

By Lemma 2.2 in [9], we know that

$$W^s(\mathbb{R}^N) = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 |\log u^2| < +\infty \right\}.$$

Now, we give the definition of weak solutions for equation (1.1).

Definition 1.1. We say $u \in W^s(\mathbb{R}^N)$ is a weak solution to equation (1.1) if u satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + V(x) u \varphi dx = \int_{\mathbb{R}^N} u \varphi \log u^2 dx, \quad \text{for each } \varphi \in W^s(\mathbb{R}^N).$$

It follows from Proposition 2.5 in [9] that the energy functional I is well defined on $W^s(\mathbb{R}^N)$ and belongs to $C^1(W^s(\mathbb{R}^N), \mathbb{R})$. Hence, the critical points of I are corresponding to weak solutions of equation (1.1).

Remark 1.1. We remark that, for $\lambda \neq 0$, if v is a solution of

$$(-\Delta)^s v + (V(x) - \log \lambda^2) v = v \log v^2, \quad x \in \mathbb{R}^N,$$

then λv solves equation (1.1). Thus, one can choose $\lambda > 0$ small enough such that $V(x) - \log \lambda^2 > 0$ for each $x \in \mathbb{R}^N$. Since $V(x)$ is bounded from below, we always assume $V_0 := \inf_x V(x) > 0$ in the sequent.

Before stating our main results, we give some notations. Denote

$$\mathcal{N} := \{u \in W^s(\mathbb{R}^N) \setminus \{0\} \mid J(u) = 0\}, \quad \mathcal{M} := \{u^\pm \neq 0 \mid \langle I(u), u^+ \rangle = \langle I(u), u^- \rangle = 0\},$$

where $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$ and J is given by

$$J(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx.$$

By Lemma 2.1, we know that \mathcal{N}, \mathcal{M} are nonempty. Then we define

$$c := \inf_{\mathcal{N}} I(u), \quad m = \inf_{\mathcal{M}} I(u).$$

Our first result can be stated as follows.

Theorem 1.1. If $u \in \mathcal{N}$ with $I(u) = c$, then u is a positive solution of equation (1.1). If $u \in \mathcal{M}$ with $I(u) = m$, then u is a sign-changing solution to equation (1.1). Moreover, if c and m are achieved, then $m > 2c$.

If u is a sign-changing solution to equation (1.1), then $u \in \mathcal{N}$, and we can easily check that

$$I(u) = I(u^+) + I(u^-) - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N+2s}} dx dy, \quad (1.8)$$

$$\langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy, \quad (1.9)$$

$$\langle I'(u), u^- \rangle = \langle I'(u^-), u^- \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy. \quad (1.10)$$

Note that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy < 0.$$

Hence, by (1.8)–(1.10), one obtains

$$I(u) > I(u^+) + I(u^-), \quad \text{and } u^+, u^- \notin \mathcal{N},$$

which is totally different from the case $s = 1$, because equation (1.1) is a nonlocal problem. And we cannot directly infer that $I(u) > 2c$. This property is called *energy doubling* by Weth [19], which is crucial in overcoming the difficulty of lack of compactness.

Now, we focus on whether c or m is achieved.

Theorem 1.2. *If one of the following four conditions hold: $(V_1); (V_2)$ and $N \geq 2$; $(V_3); (V_4)$, then c is achieved. If one of the following three conditions hold: $(V_1); (V_2)$ and $N \geq 2$; (V_4) and (V_5) , then m is achieved. In particular, if m is not achieved provided (V_3) holds.*

The proof of Theorem 1.2 is based on the concentration-compactness principle [20]. However, as we will see, the nonlocal operator $(-\Delta)^s$ and the logarithmic nonlinearity cause some obstacles, which need some new technique and subtle analysis.

Remark 1.2. We point out that (i) the variational framework also works for equation (1.6), while the analogous results for equation (1.6) are obtained by using direction derivative and constrained minimization method; (ii) Theorems 1.1 and 1.2 also hold for the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad (1.11)$$

where $2 < p < 2_s^*$. Even for equation (1.11), the results on the existence of sign-changing solutions are new.

In the rest of the article, we shall first prove some preliminary results and prove Theorem 1.1 in Section 2, and then we prove Theorem 1.2 in Section 3. We will use C to denote different positive constants from line to line.

2 Proof of Theorem 1.1

In this section, we first prove some technical lemmas, which is crucial for proving our main results.

Lemma 2.1.

- (i) Let $u \in W^s(\mathbb{R}^N) \setminus \{0\}$, then there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$;
- (ii) Let $u \in W^s(\mathbb{R}^N)$ with $u^\pm \neq 0$, then there exists a unique pair $(\alpha_u, \beta_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$.

Proof. Note that, by (1.14) in [21], we know that $u^\pm \in H^s(\mathbb{R}^N)$ if $u \in H^s(\mathbb{R}^N)$, thus, $u^\pm \in W^s(\mathbb{R}^N)$ if $u \in W^s(\mathbb{R}^N)$. We only prove (ii), and (i) can be proved by a similar argument. First, we prove the existence of (α_u, β_u) . By a direct calculation, we have

$$\begin{aligned}
\langle I'(au^+ + \beta u^-), au^+ \rangle &= \alpha^2 [u^+]_s^2 - \alpha\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy \\
&\quad + \alpha^2 \int_{\mathbb{R}^N} V(x)(u^+)^2 dx - \alpha^2 \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx - \alpha^2 \log \alpha^2 \int_{\mathbb{R}^N} |u^+|^2 dx
\end{aligned} \quad (2.1)$$

and

$$\begin{aligned}
\langle I'(au^+ + \beta u^-), \beta u^- \rangle &= \beta^2 [u^-]_s^2 - \alpha\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy \\
&\quad + \beta^2 \int_{\mathbb{R}^N} V(x)|u^-|^2 dx - \beta^2 \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx - \beta^2 \log \beta^2 \int_{\mathbb{R}^N} |u^-|^2 dx.
\end{aligned} \quad (2.2)$$

Define

$$F(\alpha, \beta) = (\langle I'(au^+ + \beta u^-), au^+ \rangle, \langle I'(au^+ + \beta u^-), \beta u^- \rangle).$$

It follows from (2.1) and (2.2) that there exists $R > r > 0$ such that

$$\begin{aligned}
\langle I'(ru^+ + \beta u^-), ru^+ \rangle &> 0 \quad \text{and} \quad \langle I'(Ru^+ + \beta u^-), Ru^+ \rangle < 0, \quad \forall \beta \in [r, R], \\
\langle I'(au^+ + ru^-), ru^- \rangle &> 0 \quad \text{and} \quad \langle I'(au^+ + Ru^-), Ru^- \rangle < 0, \quad \forall \alpha \in [r, R].
\end{aligned}$$

By Miranda theorem, there exists $(\alpha_u, \beta_u) \in (r, R) \times (r, R)$ such that $F(\alpha_u, \beta_u) = 0$, that is, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$.

Next, we prove the uniqueness of (α_u, β_u) on $\mathbb{R}^+ \times \mathbb{R}^+$. If $u \in \mathcal{M}$, then

$$[u^+]_s^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)|u^+|^2 dx = \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx \quad (2.3)$$

and

$$[u^-]_s^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)|u^-|^2 dx = \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx. \quad (2.4)$$

We claim that $(\alpha_u, \beta_u) = (1, 1)$ is the unique pair of positive numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$.

Indeed, let (α_u, β_u) satisfy $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$. By direct computation, we have

$$\begin{aligned}
&\alpha_u^2 [u^+]_s^2 - \alpha_u \beta_u \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \alpha_u^2 \int_{\mathbb{R}^N} V(x)(u^+)^2 dx \\
&= \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx + \alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx
\end{aligned} \quad (2.5)$$

and

$$\begin{aligned}
&\beta_u^2 [u^-]_s^2 - \alpha_u \beta_u \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \beta_u^2 \int_{\mathbb{R}^N} V(x)(u^-)^2 dx \\
&= \beta_u^2 \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx + \beta_u^2 \log \beta_u^2 \int_{\mathbb{R}^N} |u^-|^2 dx.
\end{aligned} \quad (2.6)$$

Without loss of generality, we assume that $0 < \alpha_u \leq \beta_u$. Then by (2.5),

$$\begin{aligned}
&\alpha_u^2 [u^+]_s^2 - \alpha_u^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \alpha_u^2 \int_{\mathbb{R}^N} V(x)(u^+)^2 dx \\
&\leq \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx + \alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx.
\end{aligned} \quad (2.7)$$

Combining (2.3) and (2.7), we deduce that

$$\alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx \geq 0,$$

which implies that $\alpha_u \geq 1$. Thus, $1 \leq \alpha_u \leq \beta_u$. On the other hand, by using a similar argument to (2.4) and (2.6), we obtain that

$$\beta_u^2 \log \beta_u^2 \int_{\mathbb{R}^N} |u^-|^2 dx \leq 0,$$

which implies $\beta_u \leq 1$. Therefore, $\alpha_u = \beta_u = 1$.

If $u \notin \mathcal{M}$, then there exists $(\alpha_u, \beta_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$. If there exists another pair $(\bar{\alpha}_u, \bar{\beta}_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\bar{\alpha}_u u^+ + \bar{\beta}_u u^- \in \mathcal{M}$. Denote $v = \alpha_u u^+ + \beta_u u^-$ and $\bar{v} = \bar{\alpha}_u u^+ + \bar{\beta}_u u^-$, then we have

$$\frac{\bar{\alpha}_u}{\alpha_u} v^+ + \frac{\bar{\beta}_u}{\beta_u} v^- = \bar{v} \in \mathcal{M}.$$

Hence, by the above analysis, we conclude that

$$\frac{\bar{\alpha}_u}{\alpha_u} = \frac{\bar{\beta}_u}{\beta_u} = 1.$$

Therefore, (α_u, β_u) is unique. \square

Lemma 2.2. Let $u \in W^s(\mathbb{R}^N)$ with $u^\pm \neq 0$ such that $\langle I'(u), u^+ \rangle \leq 0$ and $\langle I'(u), u^- \rangle \leq 0$. Then the unique pair (α_u, β_u) obtained in Lemma 2.1 satisfies $0 < \alpha_u \leq 1$ and $0 < \beta_u \leq 1$.

Proof. Without loss of generality, we assume that $0 < \beta_u \leq \alpha_u$. Since $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$, then

$$\begin{aligned} & \alpha_u^2 [u^+]_s^2 - \alpha_u \beta_u \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \alpha_u^2 \int_{\mathbb{R}^N} V(x)|u^+|^2 dx \\ &= \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx + \alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} & \beta_u^2 [u^-]_s^2 - \alpha_u \beta_u \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \beta_u^2 \int_{\mathbb{R}^N} V(x)|u^-|^2 dx \\ &= \beta_u^2 \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx + \beta_u^2 \log \beta_u^2 \int_{\mathbb{R}^N} |u^-|^2 dx. \end{aligned} \quad (2.9)$$

Note that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy < 0.$$

Thus, it follows from (2.8) that

$$\begin{aligned} & \alpha_u^2 [u^+]_s^2 - \alpha_u^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \alpha_u^2 \int_{\mathbb{R}^N} V(x)|u^+|^2 dx \\ & \geq \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx + \alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx. \end{aligned} \quad (2.10)$$

By using the fact $\langle I'(u), u^+ \rangle \leq 0$, we have

$$[u^+]_s^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)|u^+|^2 dx \leq \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx. \quad (2.11)$$

Combining (2.10) with (2.11), we deduce that

$$\alpha_u^2 \log \alpha_u^2 \int_{\mathbb{R}^N} |u^+|^2 dx \leq 0,$$

which implies that $0 < \alpha_u \leq 1$. Therefore, $0 < \beta_u \leq \alpha_u \leq 1$. \square

Lemma 2.3. For fixed $u \in W^s(\mathbb{R}^N)$ with $u^\pm \neq 0$, define $\Phi(\alpha, \beta) := I(\alpha u^+ + \beta u^-)$. Then (α_u, β_u) obtained in Lemma 2.1 is the unique maximum point of $\Phi(\alpha, \beta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$.

Proof. By the proof of Lemma 2.1, we deduce that (α_u, β_u) is the unique critical point of $\Phi(\alpha, \beta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$. By direct calculation, we conclude that $\Phi(\alpha, \beta) \rightarrow -\infty$ as $|\alpha, \beta| \rightarrow +\infty$. So it is sufficient to check that the maximum is not achieved on the boundary of $\mathbb{R}^+ \times \mathbb{R}^+$. Without loss of generality, we may suppose by contradiction that $(0, \beta_0)$ is a maximum point of Φ . Then we can deduce that this is impossible, since

$$\begin{aligned} h(\alpha) := & \frac{\alpha^2}{2} [u^+]_s^2 - \alpha \beta_0 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy + \frac{\beta_0^2}{2} [u^-]_s^2 \\ & + \frac{\alpha^2}{2} \int_{\mathbb{R}^N} (V(x) + 1)|u^+|^2 dx + \frac{\beta_0^2}{2} \int_{\mathbb{R}^N} (V(x) + 1)|u^-|^2 dx \\ & - \frac{\alpha^2}{2} \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx - \frac{1}{2} \alpha^2 \log \alpha^2 \int_{\mathbb{R}^N} |u^+|^2 dx \\ & - \frac{\beta_0^2}{2} \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx - \frac{1}{2} \beta_0^2 \log \beta_0^2 \int_{\mathbb{R}^N} |u^-|^2 dx \end{aligned} \quad (2.12)$$

is strictly increasing in α for $\alpha > 0$ small enough. \square

Lemma 2.4. Let $u \in W^s(\mathbb{R}^N)$ be a non-negative solution of (1.1), then $u \in L^\infty(\mathbb{R}^N)$.

Proof. The proof is inspired by Proposition 2.2 in [22]. Let us define, for $\beta \geq 1$ and $T > 0$ large enough,

$$\varphi(s) := \varphi_{\beta, T}(s) = \begin{cases} 0, & s \leq 0, \\ s^\beta, & 0 < s < T, \\ \beta T^{\beta-1}(s - T) + T^\beta, & s \geq T. \end{cases}$$

Obviously, φ is convex and differential function. Then, for $u \in W^s(\mathbb{R}^N)$, one has

$$(-\Delta)^s \varphi(u) \leq \varphi'(u)(-\Delta)^s u. \quad (2.13)$$

Since $\varphi(u)$ grows linearly and φ is Lipschitz with constant $K = \beta T^{\beta-1}$, therefore,

$$\|\varphi(u)\|_H := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)|\varphi(u(x))|^2 dx \right)^{\frac{1}{2}} \leq C \|u\|_H$$

and

$$\int_{\mathbb{R}^N} ||\varphi(u(x))|^2 \log |\varphi(u(x))|^2| dx \leq C_1 \int_{\mathbb{R}^N} u^2 dx + C_2 \int_{\mathbb{R}^N} |u^2 \log u^2| dx.$$

Thus, $\varphi(u(x)) \in W^s(\mathbb{R}^N)$ if $u \in W^s(\mathbb{R}^N)$. Observe that by the Sobolev embedding theorem, we have the following inequality:

$$[\varphi(u)]_s^2 = \|(-\Delta)^{\frac{s}{2}} \varphi(u)\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \varphi(u)(-\Delta)^s \varphi(u) dx \geq S(N, s) \|\varphi(u)\|_{L^{2_s^*}(\mathbb{R}^N)}^2, \quad (2.14)$$

where $S(N, s)$ is the best fractional Sobolev constant defined by

$$S(N, s) := \inf\{[u]_s^2 \mid u \in H^s(\mathbb{R}^N) \text{ and } \|u\|_{L^{2_s^*}(\mathbb{R}^N)} = 1\}.$$

On the other hand, since $\varphi(u)\varphi'(u) \in W^s(\mathbb{R}^N)$, thus

$$\int_{\mathbb{R}^N} \varphi(u)(-\Delta)^s \varphi(u) dx \leq \int_{\mathbb{R}^N} \varphi(u)\varphi'(u)(-\Delta)^s u dx \leq C \int_{\mathbb{R}^N} \varphi(u)\varphi'(u)|u|^{2_s^*-1} dx. \quad (2.15)$$

Since $u\varphi'(u) \leq \beta\varphi(u)$, the above estimate (2.14)–(2.15) becomes

$$\left(\int_{\mathbb{R}^N} (\varphi(u(x)))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq C\beta \int_{\mathbb{R}^N} (\varphi(u(x)))^2 |u|^{2_s^*-2} dx. \quad (2.16)$$

We point out that since $\varphi(u)$ grows linearly, both sides of (2.16) are finite.

Claim: Let β_1 be such that $2\beta_1 = 2_s^*$, then $u \in L^{\beta_1 2_s^*}(\mathbb{R}^N)$.

To prove this, we take R large to be determined later. Then, Hölder's inequality gives

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi(u(x)))^2 |u|^{2_s^*-2} dx &= \int_{\{u \leq R\}} (\varphi(u(x)))^2 |u|^{2_s^*-2} dx + \int_{\{u > R\}} (\varphi(u(x)))^2 |u|^{2_s^*-2} dx \\ &\leq \int_{\{u \leq R\}} |\varphi(u(x))|^2 R^{2_s^*-2} dx + \left(\int_{\mathbb{R}^N} |\varphi(u(x))|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \left(\int_{\{u > R\}} |u(x)|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}}. \end{aligned}$$

By the Monotone convergence theorem, we may take R so that

$$\left(\int_{\{u > R\}} |u(x)|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta_1}.$$

In this way, the second term above is absorbed by the left-hand side of (2.16) to obtain

$$\left(\int_{\mathbb{R}^N} (\varphi(u(x)))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq 2C\beta_1 \int_{\{u \leq R\}} |\varphi(u(x))|^2 R^{2_s^*-2} dx. \quad (2.17)$$

Using that $\varphi_{\beta_1, T}(u) \leq u^{\beta_1}$ in the right-hand side of (2.17) and letting $T \rightarrow +\infty$ in the left-hand side, since $2\beta_1 = 2_s^*$, we obtain

$$\left(\int_{\mathbb{R}^N} |u(x)|^{2_s^* \beta_1} dx \right)^{\frac{2}{2_s^*}} \leq 2C\beta_1 \int_{\mathbb{R}^N} |u(x)|^{2_s^*} R^{2_s^*-2} dx < +\infty.$$

This proves the claim.

We now go back to inequality (2.16) and we use as before $\varphi_{\beta_1, T}(u) \leq u^{\beta_1}$ in the right-hand side and then we take $T \rightarrow +\infty$ in the left-hand side

$$\left(\int_{\mathbb{R}^N} |u(x)|^{2^* \beta} dx \right)^{\frac{2}{2^*}} \leq C\beta \int_{\mathbb{R}^N} |u(x)|^{2\beta+2^*-2} dx. \quad (2.18)$$

Then, by standard bootstrap technique, one can prove that $u \in L^\infty(\mathbb{R}^N)$. We omit the details here. \square

Proposition 2.5. *Let $u \in W^s(\mathbb{R}^N)$ be a non-negative solution of equation (1.1), then u is a classical solution of equation (1.1), i.e., $(-\Delta)^s u$ can be written as*

$$(-\Delta)^s u = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

and equation (1.1) is satisfied pointwise in all \mathbb{R}^N .

Proof. Let $u \in W^s(\mathbb{R}^N)$ be a non-negative solution of (1.1), we prove the conclusion by two cases.

Case 1. If $V(x)$ is bounded, by Theorem 3.4 in [23], we can deduce that $u \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0, 1)$. The function $h(x) := u(x) - V(x)u(x) + |u(x)|^2 \log|u(x)|^2$ belongs to $C^{0,\sigma}(\mathbb{R}^N)$ for certain $\sigma > 0$. Let η_1 be a non-negative, smooth function with support in $B_1(0)$ such that $\eta_1 = 1$ in $B_{1/2}(0)$. Let $u_1 \in H^s(\mathbb{R}^N)$ be the solution of the equation

$$(-\Delta)^s u_1 + u_1 = \eta_1 h(x), \quad x \in \mathbb{R}^N.$$

It follows from Lemma 2.4 that $\eta_1 h(x) \in L^q(\mathbb{R}^N)$ for all $q > 2$, then $u_1 \in W^{2s,q}(\mathbb{R}^N)$ and thus $u_1 \in C^{0,\sigma_0}(\mathbb{R}^N)$ for some $\sigma_0 \in (0, \sigma)$.

Now we look at the equation

$$-\Delta w = -u_1 + \eta_1 h \in C^{0,\sigma_0}(\mathbb{R}^N).$$

By Hölder's regularity theory for the Laplacian, we find $w \in C^{2,\sigma_0}(\mathbb{R}^N)$, so that if $2s + \sigma_0 > 1$, then $(-\Delta)^{1-s} w \in C^{1,2s+\sigma_0-1}(\mathbb{R}^N)$, while if $2s + \sigma_0 \leq 1$, then $(-\Delta)^{1-s} w \in C^{0,2s+\sigma_0}(\mathbb{R}^N)$. Then, since

$$(-\Delta)^s(u_1 - (-\Delta)^{1-s} w) = 0,$$

the function $u_1 - (-\Delta)^{1-s} w$ is s -harmonic, we find that u_1 has the same regularity as $(-\Delta)^{1-s} w$. By the similar arguments as the proof of Theorems 1.3 and 3.4 in [23], one obtains that u has the same regularity as u_1 . Thus, we conclude that $u \in C^{1,2s+\sigma_0-1}(\mathbb{R}^N)$ if $2s + \sigma_0 > 1$, while $u \in C^{0,2s+\sigma_0}(\mathbb{R}^N)$ if $2s + \sigma_0 \leq 1$. Note that the conclusion holds locally, but the corresponding Hölder norms depend only on η_1 , s , N , $\|u\|_{H^s}$ and $\|V(x)\|_{C^{0,\sigma_0}(\mathbb{R}^N)}$, so these estimates are global in \mathbb{R}^N .

Case 2. If $V(x)$ is coercive, i.e., $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$. Similar to the proof of Case 1, we can deduce that $u \in C_{\text{loc}}^{1,2s+\sigma_0-1}(\mathbb{R}^N)$ if $2s + \sigma_0 > 1$, while $u \in C_{\text{loc}}^{0,2s+\sigma_0}(\mathbb{R}^N)$ if $2s + \sigma_0 \leq 1$.

For each point $x \in \mathbb{R}^N$, since $u \in L^\infty(\mathbb{R}^N)$ and $u \in C^{1,2s+\sigma_0-1}(B_1(x))$ if $2s + \sigma_0 > 1$, while $u \in C^{0,2s+\sigma_0}(B_1(x))$ if $2s + \sigma_0 \leq 1$, thus

$$\int_{B_1(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy < +\infty, \quad \int_{\mathbb{R}^N \setminus B_1(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy < +\infty.$$

Therefore,

$$(-\Delta)^s u = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

and thus equation (1.1) is satisfied pointwise in all \mathbb{R}^N . \square

Now, we prove the minimizers of c or m are indeed solutions to equation (1.1) by quantitative deformation lemma and degree theory.

Proof of Theorem 1.1. (i) Suppose that $u \in \mathcal{N}$ and $I(u) = c$, by Lagrange multiplier theorem, there exists $\mu \in \mathbb{R}$ such that

$$I'(u) = \mu J'(u).$$

Thus, we have

$$0 = \langle I'(u), u \rangle = \mu \langle J'(u), u \rangle.$$

Noting that

$$\langle J'(u), u \rangle = -2 \int_{\mathbb{R}^N} u^2 dx < 0.$$

Hence, $\mu = 0$, that is, $I'(u) = 0$. By standard arguments, $u \geq 0$ or $u \leq 0$. Without loss of generality, assume $u \geq 0$. If there exists $x_0 \in \mathbb{R}^N$ such that $u(x_0) = 0$, then

$$(-\Delta)^s u(x_0) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x - y|^{N+2s}} dy = -C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)}{|x - y|^{N+2s}} dy < 0. \quad (2.19)$$

On the other hand, it holds

$$(-\Delta)^s u(x_0) = -V_{\infty} u(x_0) + u(x_0) \log |u(x_0)|^2 = 0,$$

which contradicts with (2.19). Therefore, $u(x) > 0$ for each $x \in \mathbb{R}^N$. Consequently, u is a positive ground state solution to equation (1.1).

(ii) Now, suppose that $u \in \mathcal{M}$ and $I(u) = m$, then $u^+, u^- \neq 0$ and

$$\langle I'(u), u^+ \rangle = \langle I'(u), u^- \rangle = 0.$$

It follows from Lemma 2.3 that, for any $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus (1, 1)$,

$$I(\alpha u^+ + \beta u^-) < I(u^+ + u^-) = m.$$

If $I'(u) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|I'(v)\| \geq \theta, \quad \text{for all } \|v - u\| \leq 3\delta.$$

Define $D := \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$ and $g(\alpha, \beta) := \alpha u^+ + \beta u^-$. Then

$$\tilde{m} := \max_{\partial D} I(g(\alpha, \beta)) < m.$$

For $S_\delta := B_\delta(u)$ and $\varepsilon = \min\left\{\frac{m-\tilde{m}}{2}, \frac{\theta\delta}{8}\right\}$, it follows from deformation lemma (see Lemma 2.3, [24]) that there exists a deformation $\eta \in C([0, 1] \times W^s(\mathbb{R}^N), W^s(\mathbb{R}^N))$ such that

- (a) $\eta(1, u) = u$ if $u \notin I^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$,
- (b) $\eta(1, I^{m+\varepsilon} \cap S_\delta) \subset I^{m-\varepsilon}$,
- (c) $I(\eta(1, u)) \leq I(u)$ for all $u \in W^s(\mathbb{R}^N)$,

where $S_\delta = \{w : \|w - v\| \leq 2\delta, v \in S\}$ and $I^c = \{u : I(u) \leq c\}$.

It is easy to check that

$$\max_{(\alpha, \beta) \in \overline{D}} I(\eta(1, g(\alpha, \beta))) < m. \quad (2.20)$$

Now, we prove that

$$\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset,$$

which is contradiction to the definition of m . Setting $h(\alpha, \beta) = \eta(1, g(\alpha, \beta))$ and

$$\begin{aligned} \Psi_0(\alpha, \beta) &= (\langle I'(g(\alpha, \beta), u^+), \langle I'(g(\alpha, \beta), u^-) \rangle, \\ \Psi_1(\alpha, \beta) &= \left(\frac{1}{\alpha} \langle I'(h(\alpha, \beta), h^+(\alpha, \beta)) \rangle, \frac{1}{\beta} \langle I'(h(\alpha, \beta), h^-(\alpha, \beta)) \rangle \right). \end{aligned}$$

By Lemma 2.1 and topological degree theory, we know that $\deg(\Psi_0, D, 0) = 1$. Moreover, (2.20) and (a) imply that $h = g$ on ∂D . It follows from the property of homotopy invariance of topological degree that

$$\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1.$$

Therefore, there exists $(\alpha_0, \beta_0) \in D$ such that

$$\eta(1, g(\alpha_0, \beta_0)) = h(\alpha_0, \beta_0) \in \mathcal{M},$$

which yields a contradiction with (2.20). Hence, $I'(u) = 0$. As a direct result, u is a least energy sign-changing solution to equation (1.1).

(iii) Suppose $u \in \mathcal{M}$ such that $I(u) = m$, then $u^+, u^- \neq 0$, similar to the proof of Lemmas 2.1–2.2, we conclude that there exists a unique $\alpha_{u^+} \in (0, 1)$ such that $\alpha_{u^+}u^+ \in \mathcal{N}$, and a unique $\beta_{u^-} \in (0, 1)$ such that $\beta_{u^-}u^- \in \mathcal{N}$.

On the other hand, one has

$$I(\alpha_{u^+}u^+ + \beta_{u^-}u^-) = I(\alpha_{u^+}u^+) + I(\beta_{u^-}u^-) - \alpha_{u^+}\beta_{u^-} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy \quad (2.21)$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy < 0.$$

Thus, by Lemma 2.3, we conclude that

$$m = I(u^+ + u^-) > I(\alpha_{u^+}u^+ + \beta_{u^-}u^-) > I(\alpha_{u^+}u^+) + I(\beta_{u^-}u^-) \geq 2c. \quad \square$$

3 Existence and nonexistence of minimizer

In this section, we study whether c or m is achieved under different types of potential. The following Brézis-Lieb-type lemma for $u^2 \log u^2$ is crucial.

Lemma 3.1. (Lemma 2.3, [9]) *Let $\{u_n\}$ be a bounded sequence in $W^s(\mathbb{R}^N)$ such that $u_n \rightarrow u$ a.e. in \mathbb{R}^N . Then $u \in W^s(\mathbb{R}^N)$ and*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [u_n^2 \log u_n^2 - (u_n - u)^2 \log(u_n - u)^2] dx = \int_{\mathbb{R}^N} u^2 \log u^2 dx.$$

3.1 Compact potential

Lemma 3.2. *Assume (V_1) or (V_2) , then both $c > 0$ and $m > 0$ are achieved.*

Proof. We only prove the case for m , one can argue similarly to verify that $c > 0$ is achieved. Noting that $\mathcal{M} \subset \mathcal{N}$, we have $J(v) = 0$ for any $v \in \mathcal{M}$. Take $\alpha = \pi^{\frac{s}{2}}$ in the logarithm Sobolev inequality (1.4), then

$$\int_{\mathbb{R}^N} v^2 \log \frac{v^2}{\|v\|_{L^2}^2} dx + \left(N + N \log \sqrt{\pi} + \log \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right) \|v\|_{L^2}^2 \leq [v]_s^2. \quad (3.1)$$

Combining $J(v) = 0$ with (3.1), we deduce that

$$\int_{\mathbb{R}^N} V(x)v^2 dx + \left(N + N \log \sqrt{\pi} + \log \frac{s\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2s}\right)} \right) \|v\|_{L^2}^2 \leq \|v\|_{L^2}^2 \log \|v\|_{L^2}^2,$$

which implies that

$$\|v\|_{L^2}^2 \geq e^N \pi^{\frac{N}{2}} \frac{s\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2s}\right)} > 0.$$

Therefore, we can derive

$$I(v) = I(v) - \frac{1}{2}J(v) = \frac{1}{2}\|v\|_{L^2}^2 \geq \frac{1}{2}e^N \pi^{\frac{N}{2}} \frac{s\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2s}\right)} > 0,$$

which implies that $m > 0$.

Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of m , then

$$\lim_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} \left(I(u_n) - \frac{1}{2}J(u_n) \right) = \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx = m.$$

Hence, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$. By (1.4) and $J(u_n) = 0$, we can derive

$$\left(1 - \frac{\alpha^2}{\pi^s} \right) [u_n]_s^2 + \int_{\mathbb{R}^N} V(x)u_n^2 dx \leq \|u_n\|_{L^2}^2 \log \left[\|u_n\|_{L^2}^2 \left(\frac{e^{-N}\Gamma\left(\frac{N}{2s}\right)}{s\alpha^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} \right) \right]. \quad (3.2)$$

Taking $\alpha > 0$ is small enough in (3.2), we deduce that $\{u_n\}$ is bounded in H . Thus, up to a subsequence, there exists $u \in H$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H, \\ u_n \rightarrow u & \text{in } L^p(\mathbb{R}^N), \quad 2 < p < 2_s^*, \\ u_n \rightarrow u & \text{a.e. on } \mathbb{R}^N. \end{cases} \quad (3.3)$$

For any $2 < p < 2_s^*$, we have

$$0 < C \leq \|u^+\|_H^2 \leq \int_{\mathbb{R}^N} (|u_n^+|^2 \log |u_n^+|^2)^+ dx \leq C_p \int_{\mathbb{R}^N} |u_n^+|^p dx$$

and

$$0 < C \leq \|u^-\|_H^2 \leq \int_{\mathbb{R}^N} (|u_n^-|^2 \log |u_n^-|^2)^+ dx \leq C_p \int_{\mathbb{R}^N} |u_n^-|^p dx.$$

Hence, by using (3.3), we obtain

$$\int_{\mathbb{R}^N} (|u^+|^2 \log |u^+|^2)^+ dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|u_n^+|^2 \log |u_n^+|^2)^+ dx \geq C > 0 \quad (3.4)$$

and

$$\int_{\mathbb{R}^N} (|u^-|^2 \log |u^-|^2)^+ dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|u_n^-|^2 \log |u_n^-|^2)^+ dx \geq C > 0. \quad (3.5)$$

Now, by the fact that $\langle I'(u_n), u_n^+ \rangle = 0$, (3.4) and Fatou lemma, one has

$$\begin{aligned} \|u^+\|_H^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} (|u^+|^2 \log |u^+|^2)^- dx \\ \leq \liminf_{n \rightarrow \infty} \left[\|u_n^+\|_H^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n^+(x)u_n^-(y) + u_n^-(x)u_n^+(y)}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} (|u_n^+|^2 \log |u_n^+|^2)^- dx \right] \\ = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|u_n^+|^2 \log |u_n^+|^2)^+ dx = \int_{\mathbb{R}^N} (|u^+|^2 \log |u^+|^2)^+ dx, \end{aligned}$$

which implies

$$\|u^+\|_H^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx \leq 0. \quad (3.6)$$

Similarly, one can prove

$$\|u^-\|_H^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx \leq 0. \quad (3.7)$$

Thus, Lemma 2.1 yields a unique $(\alpha_u, \beta_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\bar{u} := \alpha_u u^+ + \beta_u u^- \in \mathcal{M}$. It follows from Lemma 2.2, (3.6) and (3.7) that $0 < \alpha_u \leq 1$ and $0 < \beta_u \leq 1$. Thus, by direct computation, we have

$$\begin{aligned} m \leq I(\bar{u}) &= I(\bar{u}) - \frac{1}{2}J(\bar{u}) = \frac{1}{2} \int_{\mathbb{R}^N} \bar{u}^2 dx \\ &\leq \frac{\alpha_u}{2} \int_{\mathbb{R}^N} |u^+|^2 dx + \frac{\beta_u}{2} \int_{\mathbb{R}^N} |u^-|^2 dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 dx = m, \end{aligned}$$

which implies $\alpha_u = \beta_u = 1$. As a result, $\bar{u} = u$ and $I(u) = m$. \square

3.2 Periodic potential

Lemma 3.3. *If potential $V(x)$ satisfies (V_3) , then c can be achieved.*

Proof. Let $\{u_n\}$ be a minimizer sequence of c , then

$$\lim_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} \left[I(u_n) - \frac{1}{2}J(u_n) \right] = \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx = c,$$

which implies that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$. By the logarithm Sobolev inequality (1.4), we can obtain that $\{u_n\}$ is bounded in H . One can check that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $B(|u|) \leq C_\varepsilon |u|^{2+\varepsilon}$. Using $u_n \in \mathcal{N}$, we have

$$\int_{\mathbb{R}^N} A(|u_n|) dx = \int_{\mathbb{R}^N} B(|u_n|) dx - \|u_n\|_s^2 \leq C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{2+\varepsilon} dx \leq C.$$

It follows from Lemma 2.1 in [5] that $\{u_n\}$ is bounded in $L^A(\mathbb{R}^N)$. Therefore, $\{u_n\}$ is bounded in $W^s(\mathbb{R}^N)$.

We claim that there exists $\{y_n\}$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_1(y_n)} u_n^2 dx > 0.$$

If not, then we have

$$\limsup_{n \rightarrow \infty} \int_{B_1(y_n)} u_n^2 dx = 0,$$

which implies that $u_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$. Using $u_n \in \mathcal{N}$ again, we have

$$\|u_n\|_s^2 = \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx \leq \int_{\mathbb{R}^N} (u_n^2 \log u_n^2)^+ dx \leq C_p \int_{\mathbb{R}^N} |u_n|^p dx,$$

which implies that $\|u_n\|_{L^p} \geq C > 0$. This is a contradiction.

Let $v_n(x) = u_n(x + y_n)$. Noting that for any $y_n \in \mathbb{R}^N$, $\{v_n\}$ is still a bounded minimizing sequence of c , we may assume that, up to a subsequence, there exists $v \in H \setminus \{0\}$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H, \\ v_n \rightarrow v & \text{in } L_{\text{loc}}^p(\mathbb{R}^N), \quad 2 \leq p < 2_s^*, \\ v_n \rightarrow v & \text{a.e. on } \mathbb{R}^N. \end{cases}$$

By Lemma 3.1, we can know that $v \in W^s(\mathbb{R}^N)$.

Now, we intend to show that $J(v) = 0$ and $I(v) = c$. If $J(v) < 0$, then there exists $0 < \lambda < 1$ such that $J(\lambda v) = 0$. Direct computation yields that

$$c \leq I(\lambda v) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} v^2 dx < \lim_{m \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} v_m^2 dx = c,$$

which leads to a contradiction. If $J(v) > 0$, then we claim that $J(v_n - v) < 0$ for sufficiently large n .

Indeed, by Lemma 3.1, we have

$$\lim_{n \rightarrow \infty} J(v_n - v) = \lim_{n \rightarrow \infty} J(v_n) - J(v) = -J(v) < 0.$$

Thus, by a similar argument above, we also have

$$c \leq \lim_{m \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} |v_n - v|^2 dx = c - \frac{1}{2} \int_{\mathbb{R}^N} v^2 dx < c,$$

which is a contradiction.

As a result, we have $J(v) = 0$. Consequently, we have

$$c \leq J(v) \leq \lim_{m \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} v_m^2 dx = c.$$

Thus, we complete the proof. \square

Lemma 3.4. *If potential $V(x)$ satisfies (V_3) , then m cannot be achieved.*

Proof. If $u \in \mathcal{N}$, then $J(|u|) \leq J(u) = 0$. Thus, there exists $t \in (0, 1]$ such that $J(t|u|) = 0$ and $I(t|u|) \leq I(u)$. Hence, we have

$$c = \inf\{I(u) : u \in \mathcal{N} \text{ and } u \geq 0 \text{ in } \mathbb{R}^N\}.$$

By density argument, one obtains

$$c = \inf\{I(u) : u \in \mathcal{N} \cap C_0^\infty(\mathbb{R}^N) \text{ and } u \geq 0 \text{ in } \mathbb{R}^N\}.$$

Therefore, for any $\varepsilon > 0$, we can choose a nonnegative function $u \in \mathcal{N}$ such that

$$I(u) < c + \varepsilon.$$

Without loss of generality, we may assume that $\text{supp}(u) \subset B_R(0)$. Define

$$\bar{u} := u(x + ke_1) - u(x - ke_1),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$. By Lemma 2.1, there exists $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\alpha u(x + ke_1) - \beta u(x - ke_1) = \alpha \bar{u}^+ + \beta \bar{u}^- \in \mathcal{M}.$$

Combining this with $u \in \mathcal{N}$, direct computation yields that

$$\alpha^2 \log \alpha^2 \int_{\mathbb{R}^N} u^2 dx = -\alpha \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\bar{u}^+(x) \bar{u}^-(y) + \bar{u}^-(x) \bar{u}^+(y)}{|x - y|^{N+2s}} dx dy > 0 \quad (3.8)$$

and

$$\beta^2 \log \beta^2 \int_{\mathbb{R}^N} u^2 dx = -\alpha \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\bar{u}^+(x) \bar{u}^-(y) + \bar{u}^-(x) \bar{u}^+(y)}{|x - y|^{N+2s}} dx dy > 0. \quad (3.9)$$

Hence, $\alpha = \beta > 1$. On the other hand, it follows from $\text{supp}(u) \subset B_R(0)$ that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\bar{u}^+(x) \bar{u}^-(y)}{|x - y|^{N+2s}} dx dy = \int_{B_R(-2ke_1)} \int_{B_R(2ke_1)} \frac{\bar{u}(x) \bar{u}(y)}{|x - y|^{N+2s}} dx dy = O\left(\frac{1}{|k - R|^{N+2s}}\right).$$

Let $k \rightarrow \infty$, by (3.8) and (3.9), we have

$$\log \alpha^2 \int_{\mathbb{R}^N} u^2 dx = o_k(1) = \log \beta^2 \int_{\mathbb{R}^N} u^2 dx,$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow +\infty$. Therefore, $\alpha \rightarrow 1$ and $\beta \rightarrow 1$ as $k \rightarrow \infty$. It follows from Lemma 2.3 that

$$m \leq I(\alpha \bar{u}^+ + \beta \bar{u}^-) = 2I(u) + o_k(1) \leq 2c + \varepsilon + o_k(1).$$

Since ε is arbitrary, we deduce that

$$m \leq 2c,$$

which is contradiction to Theorem 1.1. Thus, we complete the proof. \square

3.3 $(V)_\alpha$ potential

In this subsection, we prove c and m are achieved under conditions (V_4) and (V_5) . In order to overcome the difficulties caused by the loss of compactness, we first study the following problem in bounded domain, the solutions will be used as minimizing sequences.

$$\begin{cases} (-\Delta)^s u + V(x)u = u \log u^2, & x \in B_R, \\ u = 0, & x \in \mathbb{R}^N \setminus B_R, \end{cases} \quad (3.10)$$

where $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$. We now define

$$c_R = \inf_{u \in \mathcal{N}_R} I(u), \quad m_R = \inf_{u \in \mathcal{M}_R} I(u),$$

where

$$\mathcal{N}_R = \mathcal{N} \cap X_0^s(B_R), \quad \mathcal{M}_R = \mathcal{M} \cap X_0^s(B_R)$$

and

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) \mid u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}.$$

Obviously, I is well defined on $X_0^s(B_R)$ and $I \in C^1(X_0^s(B_R), \mathbb{R})$. Therefore, one can easily verify the following result.

Lemma 3.5. c_R, m_R are achieved. Assume that $u \in \mathcal{N}_R$ and $I(u) = c_R$, or $u \in \mathcal{M}_R$ and $I(u) = m_R$, then u is a solution of equation (3.10).

Lemma 3.6. Both c_R and m_R are decreasing with respect to R and converge to c, m , respectively, as $R \rightarrow \infty$.

Proof. Obviously, both c_R and m_R are decreasing with respect to R , and $c_R \geq c, m_R \geq m$. Now, we check that $m_R \rightarrow m$ as $R \rightarrow \infty$. By the definition of m , for any $\varepsilon > 0$, there exists $u \in \mathcal{M}$ such that $I(u) \leq m + \varepsilon$. Choose a cutoff function ξ_R satisfying

$$\begin{cases} \xi_R \equiv 1 & \text{on } B_{\frac{R}{2}}, \quad \text{supp}(\xi_R) \subset B_R, \\ 0 \leq \xi_R \leq 1, |\nabla \xi_R| \leq \frac{C}{R}. \end{cases}$$

Define $u_R := \xi_R u$, then we have

$$\int_{\mathbb{R}^N} |u_R^+|^2 \log |u_R^+|^2 dx \rightarrow \int_{\mathbb{R}^N} |u^+|^2 \log |u^+|^2 dx, \quad \text{as } n \rightarrow \infty \quad (3.11)$$

and

$$\int_{\mathbb{R}^N} |u_R^-|^2 \log |u_R^-|^2 dx \rightarrow \int_{\mathbb{R}^N} |u^-|^2 \log |u^-|^2 dx, \quad \text{as } n \rightarrow \infty. \quad (3.12)$$

Now, we calculate that

$$\begin{aligned} [u_R]_s^2 &= \int_{|x| \leq \frac{R}{2}} \int_{|y| \leq \frac{R}{2}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + 2 \int_{|x| \leq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|u_R(x) - u_R(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\quad + \int_{|x| \geq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|u_R(x) - u_R(y)|^2}{|x - y|^{N+2s}} dx dy \\ &:= I_1 + I_2 + I_3. \end{aligned} \quad (3.13)$$

By a direct computation, we have

$$\begin{aligned} I_2 &\leq 4 \left(\int_{|x| \leq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|\xi_R(x) - \xi_R(y)|^2 |u(x)|^2}{|x - y|^{N+2s}} dx dy + \int_{|x| \leq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|u(x) - u(y)|^2 |\xi_R(y)|^2}{|x - y|^{N+2s}} dx dy \right) \\ &\leq C \left(\int_{|x| \leq \frac{R}{2}} |u(x)|^2 \int_{|y| \geq \frac{R}{2}} \frac{\min(1, |x - y|^2/R^2)}{|x - y|^{N+2s}} dx dy + \int_{|y| \geq \frac{R}{2}} \int_{|x| \leq \frac{R}{2}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right) \\ &\leq C \left(O(R^{-2s}) + \int_{|y| \geq \frac{R}{2}} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right) \\ &= O(R^{-2s}) + o_R(1), \end{aligned} \quad (3.14)$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$. The last equality is because of

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty.$$

Similarly, we can deduce that

$$I_3 = o_R(1), \quad \text{as } R \rightarrow +\infty. \quad (3.15)$$

It follows from (3.13)–(3.15) that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x) - u_R(y)|^2}{|x - y|^{N+2s}} dx dy \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad \text{as } R \rightarrow +\infty. \quad (3.16)$$

Moreover, it also holds that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_R^+(x)u_R^-(y) + u_R^-(x)u_R^+(y)}{|x - y|^{N+2s}} dx dy \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N+2s}} dx dy. \quad (3.17)$$

Therefore,

$$\langle I'(u_R), u_R^+ \rangle \rightarrow \langle I'(u), u^+ \rangle = 0, \quad \langle I'(u_R), u_R^- \rangle \rightarrow \langle I'(u), u^- \rangle = 0, \quad \text{as } R \rightarrow +\infty.$$

By using Lemma 2.1, there exists $\alpha_{u_R}, \beta_{u_R} > 0$ such that $\alpha_{u_R}u_R^+ + \beta_{u_R}u_R^- \in \mathcal{M}_R$ and $\alpha_{u_R} \rightarrow 1, \beta_{u_R} \rightarrow 1$ as $R \rightarrow +\infty$. Thus,

$$\begin{aligned} m_R &\leq I(\alpha_{u_R}u_R^+ + \beta_{u_R}u_R^-) \\ &\leq I(\alpha_{u_R}u^+ + \beta_{u_R}u^-) + C \left(\int_{|x| \leq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|u_R(x) - u_R(y)|^2}{|x - y|^{N+2s}} dx dy \right. \\ &\quad \left. + \int_{|x| \geq \frac{R}{2}} \int_{|y| \geq \frac{R}{2}} \frac{|u_R(x) - u_R(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{|x| \geq \frac{R}{2}} V(x)u^2 + |u^2 \log u^2| dx \right) \\ &\leq I(u^+ + u^-) + o_R(1) \\ &\leq m + \varepsilon + o_R(1), \end{aligned}$$

which implies $\lim_{R \rightarrow +\infty} m_R \leq m$. Considering $m_R \geq m$, we then conclude that $\lim_{R \rightarrow +\infty} m_R = m$.

Similarly, we can prove $\lim_{R \rightarrow +\infty} c_R = c$. □

Next, we need to study the following limiting functional:

$$I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} (V_\infty + 1)u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx, \quad u \in W^s(\mathbb{R}^N).$$

Define

$$J^\infty(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_\infty u^2 dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx, \quad u \in W^s(\mathbb{R}^N),$$

and

$$\mathcal{N}^\infty := \{u \in W^s(\mathbb{R}^N) \setminus \{0\} \mid J^\infty(u) = 0\}, \quad c^\infty := \inf_{u \in \mathcal{N}^\infty} I^\infty(u).$$

It follows from Lemma 3.3 that $c^\infty > 0$ is achieved.

Lemma 3.7. Let u_R be a solution of equation (3.10). Suppose that u_R is bounded in $W^s(\mathbb{R}^N)$ with respect to R , and up to a subsequence $R_n \rightarrow \infty$, that

$$\int_{B_{R_n}} |u_n|^p dx \rightarrow \lambda \in (0, \infty),$$

where $u_n := u_{R_n}$ and $2 < p < 2_s^*$. Then there exist $\beta \in (0, 1]$ and $\{x_n\}$ satisfying that for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that, for any $r' \geq r \geq r_\varepsilon > 0$,

$$\liminf_{n \rightarrow \infty} \int_{B_r(x_n)} |u_n|^p dx \geq \beta \lambda - \varepsilon \quad (3.18)$$

and

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_{r'}(x_n)} |u_n|^p dx \geq (1 - \beta)\lambda - \varepsilon. \quad (3.19)$$

Moreover, if $\beta < 1$, then $\liminf_{n \rightarrow \infty} I(u_n) \geq c + c^\infty$.

Proof. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and $\int_{B_{R_n}} |u_n|^p dx$ is bounded away from zero, then the existence of such a number $\beta \in (0, 1]$ follows from a result of P.-L. Lions (see Lemma I.1. [20] or Lemma 4.3 [25]).

Now, we assume that $\beta < 1$. Choose $\varepsilon_n \rightarrow 0$ and $r'_n \geq r_n \rightarrow +\infty$ such that, up to a subsequence, we may assume

$$\liminf_{n \rightarrow \infty} \int_{B_{r_n}(x_n)} |u_n|^p dx \geq \beta \lambda - \varepsilon_n, \quad \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_{r'_n}(x_n)} |u_n|^p dx \geq (1 - \beta)\lambda - \varepsilon_n. \quad (3.20)$$

Choose a cutoff function $\xi(x)$ satisfying

$$\begin{cases} \xi \equiv 1 & \text{on } B_3(0) \setminus B_2(0), \quad \text{supp}(\xi) \subset B_4(0) \setminus B_1(0), \\ 0 \leq \xi \leq 1, |\nabla \xi| \leq 2. \end{cases}$$

Define $\xi_n(x) := \xi(|x - x_n|/r_n)$, then $\xi_n u_n \in W^s(\mathbb{R}^N)$. Direct computation yields

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2 \xi_n(x)}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x) u_n^2 \xi_n dx - \int_{\mathbb{R}^N} \xi_n (u_n^2 \log u_n^2)^- dx \\ &= \int_{\mathbb{R}^N} \xi_n (u_n^2 \log u_n^2)^+ dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\xi_n(x) - \xi_n(y))u_n(y)}{|x - y|^{N+2s}} dx dy \\ &\leq C \int_{\mathbb{R}^N} \xi_n |u_n|^p dx + [u_n]_s \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\xi_n(x) - \xi_n(y)|^2 |u_n(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\leq C \int_{\mathbb{R}^N} \xi_n |u_n|^p dx + C [u_n]_s \left(\int_{\mathbb{R}^N} |u_n(y)|^2 \int_{|x-y| \leq r_n} \frac{|x - y|^2 / r_n^2}{|x - y|^{N+2s}} dx dy \right. \\ &\quad \left. + \int_{\mathbb{R}^N} |u_n(y)|^2 \int_{|x-y| \geq r_n} \frac{1}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\leq C \int_{\mathbb{R}^N} \xi_n |u_n|^p dx + O(r_n^{-s}). \end{aligned} \quad (3.21)$$

Thus,

$$\int_{2r_n \leq |x-x_n| \leq 3r_n} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{2r_n \leq |x-x_n| \leq 3r_n} V(x) u_n^2 \xi_n + |u_n^2 \log u_n^2| dx = o_n(1). \quad (3.22)$$

Take another cutoff function $\eta(x)$ satisfying

$$\begin{cases} \eta \equiv 1 & \text{on } B_2(0), \\ 0 \leq \eta \leq 1, |\nabla \eta| \leq 2. \end{cases} \quad \text{supp}(\eta) \subset B_3(0),$$

Define

$$w_n(x) = \eta\left(\frac{|x - x_n|}{r_n}\right) u_n(x), \quad v_n(x) = \left[1 - \eta\left(\frac{|x - x_n|}{r_n}\right)\right] u_n(x).$$

Using (3.20), we can infer

$$\int_{\mathbb{R}^N} |w_n|^p dx \geq \int_{B_{2r_n}(x_n)} |u_n|^p dx \geq \beta \lambda - \varepsilon_n \quad (3.23)$$

and

$$\int_{\mathbb{R}^N} |v_n|^p dx \geq \int_{\mathbb{R}^N \setminus B_{3r_n}(x_n)} |u_n|^p dx \geq (1 - \beta) \lambda - \varepsilon_n. \quad (3.24)$$

On the other hand, since $\text{supp}(v_n) \subset \mathbb{R}^N \setminus B_{2r_n}(x_n)$, one can deduce that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w_n(x) - w_n(y))(v_n(x) - v_n(y))}{|x-y|^{N+2s}} dx dy \\ &= \int_{|x-x_n| \geq 2r_n} \int_{|y-x_n| \geq 2r_n} \frac{(w_n(x) - w_n(y))(v_n(x) - v_n(y))}{|x-y|^{N+2s}} dx dy \\ &+ \int_{|x-x_n| \geq 2r_n} \int_{|y-x_n| < 2r_n} \frac{(w_n(x) - w_n(y))(v_n(x) - v_n(y))}{|x-y|^{N+2s}} dx dy \\ &+ \int_{|x-x_n| < 2r_n} \int_{|y-x_n| \geq 2r_n} \frac{(w_n(x) - w_n(y))(v_n(x) - v_n(y))}{|x-y|^{N+2s}} dx dy \\ &= o_n(1). \end{aligned} \quad (3.25)$$

It follows from (3.22)–(3.25) that

$$I(u_n) = I(w_n) + I(v_n) + o_n(1). \quad (3.26)$$

Moreover, by using (3.25) again, we derive that

$$J(w_n) = \langle I'(u_n), w_n \rangle + o_n(1) = o_n(1)$$

and

$$J(v_n) = \langle I'(u_n), v_n \rangle + o_n(1) = o_n(1).$$

Therefore, there exist two sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n w_n \in \mathcal{N}$, $\beta_n v_n \in \mathcal{N}$, and $\alpha_n \rightarrow 1, \beta_n \rightarrow 1$ as $n \rightarrow \infty$.

If $\{x_n\}$ is bounded, then

$$\liminf_{n \rightarrow \infty} I(\beta_n v_n) = \liminf_{n \rightarrow \infty} I^\infty(\beta_n v_n) \geq c^\infty. \quad (3.27)$$

Furthermore, it follows from (3.26) and (3.27) that

$$\liminf_{n \rightarrow \infty} I(u_n) = \liminf_{n \rightarrow \infty} I(\alpha_n w_n) + \liminf_{n \rightarrow \infty} I(\beta_n v_n) \geq c + c^\infty.$$

If $\{\chi_n\}$ is unbounded, then

$$\liminf_{n \rightarrow \infty} I(\alpha_n w_n) = \liminf_{n \rightarrow \infty} I^\infty(\alpha_n w_n) \geq c^\infty, \quad (3.28)$$

which upon combining with (3.26) also yields $\liminf_{n \rightarrow \infty} I(u_n) \geq c + c^\infty$. \square

Lemma 3.8. Assume (V_4) holds, then c is achieved.

Proof. First, we claim that $c < c^\infty$. It follows from Lemma 3.3 that there exists $w \in \mathcal{N}^\infty$ such that $I^\infty(w) = c^\infty$ and $J^\infty(w) = 0$. Moreover, by a similar argument of the proof of Theorem 1.1, we conclude that $w(x) > 0$ for each $x \in \mathbb{R}^N$.

It follows from (V_4) that

$$c \leq \max_{\alpha \geq 0} I(\alpha w) = I(\alpha_w w) < I^\infty(\alpha_w w) \leq \max_{\alpha \geq 0} I^\infty(\alpha w) = I^\infty(w) = c^\infty,$$

where α_w is the constant such that $\max_{\alpha \geq 0} I(\alpha w) = I(\alpha_w w)$.

Now, it follows from Lemmas 3.5 and 3.6 that there exists a sequence $\{u_n\} \subset \mathcal{N}_{R_n}$ such that $I(u_n) = c_n \rightarrow c$ as $n \rightarrow \infty$. Since $J(u_n) = 0$, we can deduce that $\{u_n\}$ is uniformly bounded in $W^s(\mathbb{R}^N)$. Thus, for any $\varepsilon > 0$, by Lemma 3.7 there exist $r > 0$ and a sequence $\{\chi_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_r(\chi_n)} |u_n|^p dx \geq \lambda - \varepsilon,$$

where

$$\lambda := \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^p dx > 0.$$

Now, we claim that $\{\chi_n\}$ is bounded in \mathbb{R}^N . If not, then we have

$$J^\infty(u_n) = J(u_n) + o_n(1),$$

which implies the existence of $\alpha_n > 0$ such that $\alpha_n \rightarrow 1$ and $J^\infty(\alpha_n u_n) = 0$. Thus, we have

$$c^\infty \leq \liminf_{n \rightarrow \infty} I^\infty(\alpha_n u_n) = \liminf_{n \rightarrow \infty} I^\infty(u_n) = \liminf_{n \rightarrow \infty} I(u_n) = c,$$

which leads to a contradiction. Therefore, $\{\chi_n\}$ is bounded and thus $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$. Proceeding as the arguments in the proof of Lemma 3.2, we can prove that c is achieved. \square

Lemma 3.9. Assume (V_4) holds, and let $u(x)$ be a positive ground state solution of equation (1.1), then

$$u(x) \leq \frac{C}{1 + |x|^{N+2s}}.$$

Proof. By Lemma 2.4 and Proposition 2.5, we know that $u \in C^{1,2s+\sigma_0-1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ if $2s + \sigma_0 > 1$, while $u \in C^{0,2s+\sigma_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ if $2s + \sigma_0 \leq 1$. Then, by a similar argument to the proof of Lemma 2.6 in [26], one can derive

$$\lim_{|x| \rightarrow \infty} u(x) = 0.$$

Thus, there exists $R > 0$ such that

$$(-\Delta)^s u + V_0 u \leq (-\Delta)^s u + V(x)u = u \log u^2 \leq 0, \quad x \in \mathbb{R}^N \setminus B_R.$$

It follows from Lemma C.1 in [27] that the fundamental solution G_{s,V_0} satisfies

$$((-\Delta)^s + V_0)G_{s,V_0} = \delta_0, \quad x \in \mathbb{R}^N.$$

Furthermore, G_{s,V_0} is radial, positive, strictly decreasing in $|x|$ and satisfies

$$\lim_{|x| \rightarrow \infty} |x|^{N+2s} G_{s,V_0} = V_0^{-2} C_{N,s} \quad (3.29)$$

with some positive constant $C_{N,s}$. Hence, there exists a constant $c_0 > 0$ such that $G_{s,V_0} \geq c_0 > 0$ in B_R . Let $C_0 = c_0^{-1} \|u\|_{L^\infty}$, then we have

$$w(x) := C_0 G_{s,V_0} - u \geq 0, \quad x \in B_R$$

and

$$(-\Delta)^s w + V_0 w = ((-\Delta)^s + V_0)(C_0 G_{s,V_0} - u) \geq 0, \quad x \in \mathbb{R}^N \setminus B_R.$$

Note that the function w is continuous away from the origin and $w(x) > 0$ on B_R holds. We claim that $w(x) > 0$ on B_R^c as well. Suppose on the contrary that w is strictly negative somewhere in B_R^c . Since $w(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ and $w(x) > 0$ on B_R , this implies that w attains a strict global minimum at some point $x_0 \in B_R^c$ with $w(x_0) < 0$. By using the singular integral expression for $(-\Delta)^s$, it is easy to see that $((-\Delta)^s w)(x_0) < 0$. On the other hand, we have $(-\Delta)^s w + V_0 w \geq 0$ on B_R^c , which implies that $((-\Delta)^s w)(x_0) > 0$. This is a contradiction, and we conclude that $w(x) \geq 0$ on \mathbb{R}^N . Thus, we conclude

$$u \leq C_0 G_{s,V_0}. \quad (3.30)$$

Therefore, the result is directly from (3.29) and (3.30). \square

From Lemma 3.8, there exists $u \in \mathcal{N}$, $w \in \mathcal{N}^\infty$ such that $I(u) = c$ and $I^\infty(w) = c^\infty$. Furthermore, with the same argument as that in proving Theorem 1.1, we may assume $u > 0$ and $w > 0$. Similar to Lemma 3.9, one obtains

$$w(x) \leq \frac{C}{1 + |x|^{N+2s}}, \quad \text{for some constant } C > 0.$$

Denote $w_R(x) := w(x_1 + 2R, x')$ with $(x_1, x') \in \mathbb{R}^N$. We have the following estimate.

Lemma 3.10. *Suppose that $V(x)$ satisfies (V_4) – (V_5) , then $m < c + c^\infty$.*

Proof. Direct computation yields

$$\begin{aligned} I(\alpha u + \beta w_R) &= \frac{\alpha^2}{2} \left([u]_s^2 + \int_{\mathbb{R}^N} (V(x) + 1) u^2 dx \right) + \frac{\beta^2}{2} \left([w_R]_s^2 + \int_{\mathbb{R}^N} (V(x) + 1) w_R^2 dx \right) \\ &\quad + \alpha\beta \int_{\mathbb{R}^N} (V(x) + 1) u w_R dx - \frac{1}{2} \int_{\mathbb{R}^N} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx \\ &\quad + \alpha\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(w_R(x) - w_R(y))}{|x - y|^{N+2s}} dx dy. \end{aligned} \quad (3.31)$$

To prove this lemma, it is sufficient to show that

$$\sup_{(\alpha, \beta) \in \mathbb{R}^2} I(\alpha u + \beta w_R) < c + c^\infty. \quad (3.32)$$

In fact, by Miranda theorem, there exist $\bar{\alpha} > 0$ and $\bar{\beta} < 0$ such that $\bar{\alpha} u + \bar{\beta} w_R \in \mathcal{M}$. Thus, $m < c + c^\infty$.

Claim 1. There exists $R_0, r_0 > 0$ such that for all $R \geq R_0$, $\alpha^2 + \beta^2 = r^2 > r_0^2$,

$$\sup_{(\alpha, \beta) \in \mathbb{R}^2} I(\alpha u + \beta w_R) \leq 0.$$

Indeed, let $\tilde{\alpha} = \alpha/r$, $\tilde{\beta} = \beta/r$ and $\varphi = \tilde{\alpha}u + \tilde{\beta}w_R$. We first find $R' > 0$ such that for all $R > R'$, $\|\varphi\|_s$ and $\int_{\mathbb{R}^N} \varphi^2 \log \varphi^2 dx$ are bounded from above and below by two positive constants. Thus, we have

$$I(\alpha u + \beta w_R) = I(r\varphi) = r^2 I(\varphi) - \frac{1}{2} r^2 \log r^2 \int_{\mathbb{R}^N} \varphi^2 dx,$$

which implies the conclusion if $r > 0$ is large enough.

Claim 2. There exists $\varepsilon > 0$ small and $R_1 > 0$ large such that for $R > R_1$, it holds

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(w_R(x) - w_R(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (V(x) + 1) u w_R dx = O(R^{-2s+\varepsilon(N+2s)}). \quad (3.33)$$

In fact, $w_R > 0$ solves the equation

$$(-\Delta)^s w_R + V_{\infty} w_R = w_R \log w_R^2, \quad x \in \mathbb{R}^N.$$

Thus,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(w_R(x) - w_R(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V_{\infty} u w_R dx = \int_{\mathbb{R}^N} u w_R \log w_R^2 dx.$$

For arbitrarily ε small, there exists $C_\varepsilon > 0$ such that $|t \log t^2| \leq C_\varepsilon t^{1-\varepsilon}$ for all $0 < t < 1$. Thus, applying Lemma 3.9, we derive

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u w_R \log w_R^2 dx \right| &\leq \int_{\mathbb{R}^N \setminus B_R} |u w_R \log w_R^2| dx + \int_{B_R} |u w_R \log w_R^2| dx \\ &\leq C \left(\int_{\mathbb{R}^N \setminus B_R} |u| dx + \int_{\mathbb{R}^N \setminus B_R} |w \log w^2| dx \right) \\ &\leq C \left(R^{-2s} + \int_{\mathbb{R}^N \setminus B_R} |x|^{-(1-\varepsilon)(N+2s)} dx \right) \\ &\leq C R^{-2s+\varepsilon(N+2s)}. \end{aligned}$$

Therefore, Claim 2 follows.

Claim 3. For α and β bounded, there exists $R_2 > 0$ such that, for $R > R_2$, it holds

$$\int_{\mathbb{R}^N} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx = \int_{\mathbb{R}^N} (\alpha u)^2 \log(\alpha u)^2 dx + \int_{\mathbb{R}^N} (\beta w_R)^2 \log(\beta w_R)^2 dx + O(R^{-2s}).$$

In fact, applying Lemma 3.9, one can obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} (\alpha u)^2 \log(\alpha u)^2 dx &= O \left(\int_{\mathbb{R}^N \setminus B_R(0)} |u|^{2-\varepsilon} dx \right) = O(R^{-N-4s+\varepsilon(N+2s)}), \\ \int_{\mathbb{R}^N \setminus B_R(2Re_1)} (\beta w_R)^2 \log(\beta w_R)^2 dx &= O \left(\int_{\mathbb{R}^N \setminus B_R(0)} |w|^{2-\varepsilon} dx \right) = O(R^{-N-4s+\varepsilon(N+2s)}), \end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus (B_R(0) \cup B_R(2Re_1))} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx = O(R^{-N-4s+\varepsilon(N+2s)}).$$

Furthermore, direct computation yields

$$\begin{aligned} & \left| \int_{B_R(0)} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx - \int_{B_R(0)} (\alpha u)^2 \log(\alpha u)^2 dx \right| \\ & \leq \left| \int_{B_R(0)} (\alpha u)^2 (\log(\alpha u + \beta w_R)^2 - \log(\alpha u)^2) dx \right| + \left| \int_{B_R(0)} (2\alpha\beta u w_R + \beta^2 w_R^2) \log(\alpha u + \beta w_R)^2 dx \right| \\ & = O(R^{-2s}). \end{aligned}$$

Hence, Claim 3 follows.

Now, we prove (3.32). From Claim 1, we may suppose that both α and β are bounded. Combining Claim 2, Claim 3 with (3.31), we can deduce that for $R > \max\{R_0, R_1, R_2\}$,

$$I(\alpha u + \beta w_R) = I(\alpha u) + I(\beta w_R) + O(R^{-2s+\varepsilon(N+2s)}).$$

It is easy to check that $J(w_R) \rightarrow 0$ as $R \rightarrow \infty$. Thus, there exists β_R satisfying $\beta_R \rightarrow 1$ as $R \rightarrow +\infty$ and $J(\beta_R w_R) = 0$. Moreover, for R is large enough, by (V_3) and (V_4) , we deduce that

$$\begin{aligned} I(\alpha u + \beta w_R) &= I(\alpha u) + I(\beta w_R) + O(R^{-2s+\varepsilon(N+2s)}) \\ &\leq I(u) + I^\infty(\beta_R w_R) + \frac{1}{2} \beta_R^2 \int_{\mathbb{R}^N} (V(x) - V_\infty) w_R^2 dx + O(R^{-2s+\varepsilon(N+2s)}) \\ &\leq c + c^\infty - \frac{1}{2} \int_{B_1(0)} (V_\infty - V(x - 2Re_1)) w^2 dx + O(R^{-2s+\varepsilon(N+2s)}) \\ &\leq c + c^\infty - C \int_{B_1(0)} \frac{w^2}{|x - 2Re_1|^m} dx + O(R^{-2s+\varepsilon(N+2s)}) \\ &\leq c + c^\infty - \frac{C}{R^m} + O(R^{-2s+\varepsilon(N+2s)}) \\ &< c + c^\infty. \end{aligned}$$

Therefore, we complete the proof. \square

Proposition 3.11. Assume (V_4) – (V_5) hold, then m is achieved.

Proof. It follows from Lemmas 3.5 and 3.6 that there exists a sequence $\{u_n\}$ such that $u_n \in \mathcal{M}_{R_n}$ and $I(u_n) = m_{R_n} \rightarrow m$ as $n \rightarrow \infty$. Since $J(u_n) = 0$, we deduce that $\{u_n\}$ is bounded in $W^s(\mathbb{R}^N)$. Thus, by Lemma 3.7, we find a sequence $\{x_n\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $r > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_r(x_n)} |u_n|^p dx \geq \lambda - \varepsilon,$$

where

$$\lambda := \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^p dx > 0.$$

Now, we claim that $\{x_n\}$ is bounded. If not, then

$$\langle I'(u_n), (u_n)^+ \rangle = \langle I^{\infty'}(u_n), (u_n)^+ \rangle + o_n(1)$$

and

$$\langle I'(u_n), (u_n)^- \rangle = \langle I^{\infty'}(u_n), (u_n)^- \rangle + o_n(1).$$

Thus, there exist $\alpha_n, \beta_n > 0$ with $\alpha_n \rightarrow 1, \beta_n \rightarrow 1$ such that

$$\langle I^{\infty'}(\alpha_n(u_n)^+ + \beta_n(u_n)^-), \alpha_n(u_n)^+ \rangle = 0$$

and

$$\langle I^{\infty'}(\alpha_n(u_n)^+ + \beta_n(u_n)^-), \beta_n(u_n)^- \rangle = 0.$$

This implies $\alpha_n(u_n)^+ + \beta_n(u_n)^- \in \mathcal{M}^\infty$. Arguing as we prove Theorem 1.1, we infer $2c^\infty \leq m^\infty$. Therefore,

$$c + c^\infty \leq 2c^\infty \leq \liminf_{n \rightarrow \infty} I^\infty(\alpha_n(u_n)^+ + \beta_n(u_n)^-) \leq \liminf_{n \rightarrow \infty} I^\infty(u_n) = \liminf_{n \rightarrow \infty} I(u_n) = m,$$

which leads to a contradiction. This implies that $\{x_n\}$ is bounded. Therefore, $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$. Proceeding as the proof of Lemma 3.2, we can verify that m is achieved. \square

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