Research Article

Ahmed Mohammed* and Giovanni Porru

A sharp global estimate and an overdetermined problem for Monge-Ampère type equations

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Abstract: We consider Monge-Ampère type equations involving the gradient that are elliptic in the framework of convex functions. Through suitable symmetrization we find sharp estimates to solutions of such equations. An overdetermined problem related to our Monge-Ampère type operators is also considered and we show that such a problem may admit a solution only when the underlying domain is a ball.

Keywords: Sharp global estimate, Monge-Ampère type equations, Overdetermined problem, symmetrization

MSC: 35B45, 35B51, 35J96, 35N25, 49J40

1 Introduction

About forty years ago, G. Talenti [14] pioneered a method for establishing the following result. Let Ω be a convex domain in $\mathbb{R}^2$, and let $u = u(x, y)$ be a (negative) convex solution to the problem

$$u_{xx}u_{yy} - u_{xy}^2 = 1 \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega.$$

Moreover, let $\Omega^\star_{(1)}$ be the disc centered at the origin, having the same perimeter as Ω, and let $v = v(x, y)$ be the negative solution to the problem

$$v_{xx}v_{yy} - v_{xy}^2 = 1 \text{ in } \Omega^\star_{(1)}, \ \ v = 0 \text{ on } \partial\Omega^\star_{(1)}.$$

Then

$$u^\star(x) \geq v(x) \ \text{ in } \ \Omega^\star_{(1)},$$

where $u^\star$ is a suitable symmetrization of u which is defined in $\Omega^\star_{(1)}$. Clearly, the solution u depends on the shape of Ω, and cannot be written explicitly in general, whereas v is radially symmetric and can be computed explicitly as a solution to an ordinary differential equation. The previous result has been extended to arbitrary dimension n by K. Tso [18, 19] and by N. Trudinger [15, 16], by using a suitable (depending on the dimension n) symmetrization of u. See also [1, 10, 17]. One purpose of the present paper is to extend a similar result to equations which also depend on the gradient Du.

The second problem we consider in this paper is inspired by the following well known result. Let Ω be a bounded smooth domain in $\mathbb{R}^n$, and let u be the solution to the problem

$$\Delta u = 1 \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega.$$

If there is a constant $c > 0$ such that $|Du| = c$ on $\partial\Omega$, then it is well-known that Ω must be a ball (see [13]). As far as we know, a problem of this kind, usually referred to as an overdetermined problem, was first considered by J. Serrin, [13]. The above result is a special case of Serrin's problem. The paper [13] has inspired

***Corresponding Author: Ahmed Mohammed:** Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA, E-mail: amohammed@bsu.edu
Giovanni Porru: Dipartimento di Matematica e Informatica, University of Cagliari, Cagliari, Italy, E-mail: porru@unica.it

numerous investigations involving linear, nonlinear and fully nonlinear operators. We refer to [3, 4, 7, 20] and the references therein.

We now introduce the Monge-Ampère type equations considered in this paper. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with a smooth boundary $\partial\Omega$. For $x \in \Omega$, we write $x = (x^1, \cdots, x^n)$. We use subscripts to denote partial differentiation. For example, we write $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, etc. The familiar Monge-Ampère operator can be written as

$$\det(D^2 u) = \frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 u) u_i\right)_j,$$

where $D^2 u$ denotes the Hessian matrix of the function u, $\det(D^2 u)$ is the determinant of $D^2 u$, and $T_{(n-1)} = T_{(n-1)}(D^2 u)$ is the $(n-1)$-order Newton tensor of $D^2 u$ ([11, 12]). Here and in what follows, the summation convention from 1 to n over repeated indices is in effect. Note that $T_{(n-1)}$ corresponds to the cofactor matrix of $D^2 u$.

Let $g : [0, \infty) \to (0, \infty)$ be a smooth real function satisfying

$$G(s^2) := g(s^2) + 2s^2 g'(s^2) > 0 \quad \forall s \geq 0.$$

Since

$$G(s^2) = \frac{d}{ds}\left(g(s^2)s\right),$$

the function $g(s^2)s$ is positive and strictly increasing for $s > 0$. Note that we are assuming that $g(0)$ and $G(0)$ are positive numbers.

We define the g-Monge-Ampère operator as

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 u) g^n(|Du|^2) u_i\right)_j.$$

Recalling that $T_{(n-1)}(D^2 u)$ is divergence free (see [11, 12]), we have

$$\left(T^{ij}_{(n-1)}(D^2 u)\right)_j = 0, \quad i = 1, \cdots, n. \tag{1}$$

Moreover, since $T_{(n-1)}(D^2 u)$ is the cofactor matrix of $D^2 u$ we have

$$T_{(n-1)}(D^2 u) D^2 u = I \det(D^2 u), \tag{2}$$

where I denotes the $n \times n$ identity matrix. By using (1) and (2), we find

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 u) g^n(|Du|^2) u_i\right)_j = g^{n-1}(|Du|^2) G(|Du|^2) \det(D^2 u). \tag{3}$$

Since the operator $\det(D^2 u)$ is elliptic (in the framework of convex functions), our g-Monge-Ampère operator is elliptic.

A motivation for the definition of the g-Monge-Ampère operator is the following. Using the Kronecker delta $\delta^{i\ell}$, define the $n \times n$ matrix $\mathcal{A} = [\mathcal{A}^{ij}]$ with

$$\mathcal{A}^{ij} = (g(|Du|^2) u_i)_j = \left[g(|Du|^2)\delta^{i\ell} + 2g'(|Du|^2) u_i u_\ell\right] u_{\ell j}.$$

The trace of the matrix $\mathcal{A}$ is the familiar operator $(g(|Du|^2)u_i)_i$. We claim that the determinant of the matrix $\mathcal{A}$ coincides with our operator (3). Indeed, the eigenvalues Λ^i, $i = 1, \cdots, n$, of the $n \times n$ matrix

$$\mathcal{B} = \left[g(|Du|^2)\delta^{i\ell} + 2g'(|Du|^2) u_i u_\ell\right]$$

are the following

$$\Lambda^1 = \cdots = \Lambda^{n-1} = g(|Du|^2), \; \Lambda^n = G(|Du|^2).$$

Since $\det\mathcal{A} = \det\mathcal{B} \cdot \det(D^2 u)$, we find

$$\det\mathcal{A} = g^{n-1}(|Du|^2) G(|Du|^2) \det(D^2 u).$$

The claim follows from the latter equation and (3).

In this paper we show that many results which hold for Monge-Ampère equations can be extended to the corresponding g-Monge-Ampère equations. To state our main results we begin by introducing two assumptions. We need the following notations.

$$\mathcal{K}(s^2) = g^{n-1}(s^2)G(s^2), \quad \text{and} \quad \mathcal{G}(s) := g^n(s^2)s^{n+1}, \qquad s > 0. \tag{4}$$

We make the following assumptions.

There is a constants $\gamma > 0$ such that

$$t^{\gamma-n}\mathcal{K}(s^2) \ge \mathcal{K}(t^2s^2) \qquad \forall\,(t,s) \in (0,1)\times(0,\infty). \tag{5}$$

Let $f(t)$ be a non-decreasing function. With γ as in (5), suppose there is $0 \le q < \gamma$ such that

$$\frac{f(t)}{t^q} \quad \text{is non-increasing on } (0,\infty). \tag{6}$$

In stating our results we will consider functions from the class

$$\Phi(\Omega) := \{u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) : \ u \text{ is convex in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

We now state our first result. We refer to Section 2 for the definitions of $(n-1)$-symmetrand of a function on Ω, the one-mean radius $\eta_{(1)}(\Omega)$, and the Gauss curvature $\mathcal{H}_{(n-1)}$ of $\partial\Omega$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex and smooth domain, and suppose that $\mathcal{G}$ defined as in (4) is convex in $(0,\infty)$. Suppose f is positive, non-decreasing. We further assume that conditions (5) and (6) hold. Let $v \in \Phi(\Omega)$ be a super-solution of*

$$\mathcal{K}(|Du|^2)\det(D^2u) = f(-u) \ \ in \ \ \Omega, \tag{7}$$

and let $v^\star$ be its $(n-1)$-symmetrand. Let $z \in \Phi(B_R)$ be a sub-solution of (7) in the ball B_R, where $R := \eta_{(1)}(\Omega)$. If $v^\star(r) = v^\star(x)$ and $z(r) = z(x)$ for $r = |x|$, then we have

$$v^\star(r) \ge z(r), \quad 0 < r < R.$$

For our second main result we consider an overdetermined system that may admit a convex solution only when the underlying domain is a ball. More specifically we have the following. Recall that functions in $\Phi(\Omega)$ vanish on the boundary $\partial\Omega$.

Theorem 1.2. *Let $c > 0$ be a given constant. If there is a solution $u \in \Phi(\Omega)$ of the system*

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = c \qquad in \ \ \Omega, \tag{8}$$

$$\mathcal{H}_{(n-1)}\left(g(|Du|^2)|Du|\right)^{n-1} = c^{\frac{n-1}{n}} \qquad on \ \ \partial\Omega, \tag{9}$$

then Ω is a ball.

For results of existence and regularity of Monge-Ampère equations we refer to [2, 5] and the references therein.

We organize the rest of the paper as follows. Section 2 provides a brief preliminary in which basic notions are recalled. In Section 3 we give an estimate of $(n-1)$-symmetrands of supersolutions $v \in \Phi(\Omega)$ to (7). This will be followed by a comparison principle for the equation (7), which may be of independent interest. In the same section we will also present the proof of Theorem 1.1. As an application of Theorem 1.1, we give an estimate of an Hessian integral. In Section 4, we will give the proof of Theorem 1.2.

2 Preliminaries

Let $\kappa^1, \cdots, \kappa^{n-1}$ be the principal curvatures of $\partial\Omega$. For $j = 1, \cdots, n-1$ we define the j-th mean curvature of $\partial\Omega$ by

$$\mathcal{H}_{(j)}(\partial\Omega) = S_{(j)}(\kappa^1, \cdots, \kappa^{n-1}),$$

where $S_{(j)}$ denotes the elementary symmetric function of order j of $\kappa^1, \cdots, \kappa^{n-1}$. Let $u \in \Phi(\Omega)$. From Sard's theorem it follows that for almost all $t \in (m_0, 0)$, $m_0 = \min_\Omega u$, the sub-level sets

$$\Omega_t = \{x \in \Omega : u(x) < t\}$$

will have smooth boundary Σ_t given by the level surface

$$\Sigma_t = \{x \in \Omega : u(x) = t\}.$$

Clearly, $\Omega_0 = \Omega$. Furthermore, we denote with ω_n the volume of the unit ball in $\mathbb{R}^n$.

For an open set $\mathcal{O} \subset \mathbb{R}^n$, the quermassintegral $V_{(1)} = V_{(1)}(\mathcal{O})$ is defined by

$$V_{(1)}(\mathcal{O}) = \frac{1}{n(n-1)} \int_{\partial\mathcal{O}} \mathcal{H}_{(n-2)}(\partial\mathcal{O})\, d\sigma, \tag{10}$$

where $d\sigma$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Following [16] we define the 1-mean radius of $\mathcal{O}$, denoted by $\eta_{(1)}(\mathcal{O})$, as

$$\eta_{(1)}(\mathcal{O}) = \frac{V_{(1)}(\mathcal{O})}{\omega_n}.$$

In case $\mathcal{O}$ is a ball, $\eta_{(1)}(\mathcal{O})$ is the radius of $\mathcal{O}$. For a general $\mathcal{O}$, we denote by $\mathcal{O}^\star_{(n-1)}$ the ball with radius $\eta_{(1)}(\mathcal{O})$.

As usual, we denote by $|E|$ the Lebesgue measure of a subset $E \subset \mathbb{R}^n$. The following isoperimetric inequality is well known.

$$\left(\frac{|\mathcal{O}|}{\omega_n}\right)^{\frac{1}{n}} \le \frac{V_{(1)}(\mathcal{O})}{\omega_n}.$$

It follows that

$$|\mathcal{O}| \le |\mathcal{O}^\star_{(n-1)}|.$$

We define the rearrangement of u with respect to the quermassintegral $V_{(1)}$ as

$$u^\bigstar(s) = \sup\{t \le 0 : V_{(1)}(\Omega_t) \le s, \ \ 0 \le s \le V_{(1)}(\Omega)\}. \tag{11}$$

The function $u^\bigstar(s)$ is negative, increasing and satisfies

$$u^\bigstar(0) = \min_\Omega u(x), \quad u^\bigstar(V_{(1)}(\Omega)) = 0.$$

We also define

$$u^\star(x) = u^\bigstar(\omega_n|x|), \quad 0 \le |x| \le \eta_{(1)}(\Omega). \tag{12}$$

The function $u^\star(x)$ is called the $(n-1)$-symmetrand of u, and can also be defined by (see [16])

$$u^\star(x) = \sup\{t \le 0 : \eta_{(1)}(\Omega_t) \le |x|, \ \ 0 \le |x| \le \eta_{(1)}(\Omega)\}.$$

Since $u^\star(x)$ is radially symmetric we often write $u^\star(x) = u^\star(r)$ for $|x| = r$. We have $u^\star(0) = \min_\Omega u(x)$ and $u^\star(\eta_{(1)}(\Omega)) = 0$.

3 Estimates via symmetrization

Lemma 3.1. *Let $u \in \Phi(\Omega)$. For $t \in (m_0, 0)$, $m_0 = \min_\Omega u(x)$, let*

$$\Sigma_t = \{x \in \Omega : u(x) = t\}.$$

For almost every $t \in (m_0, 0)$ we have

$$T^{ij}_{(n-1)}(D^2 u)u_i u_j = \mathcal{H}_{(n-1)}|Du|^{n+1} \quad on \ \Sigma_t. \tag{13}$$

Furthermore we have

$$\frac{d}{dr}\int_{\Sigma_t} \mathcal{H}_{(n-2)}\,d\sigma = (n-1)\int_{\Sigma_t} \mathcal{H}_{(n-1)}|Du|^{-1}\,d\sigma. \tag{14}$$

Proof. For a proof of (13) we refer to [11, 12]. For a proof of (14) see equation (2.20) of [16].

Theorem 3.2. *Let $v \in \Phi(\Omega)$, let $f > 0$ be non-decreasing and let*

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 v)g^n(|Dv|^2)v_i\right)_j \le f(-v) \quad in \ \Omega. \tag{15}$$

If the function $\mathcal{G}$ defined as in (4) is convex then

$$\left(g((\dot v^\star)^2)\dot v^\star\right)^n \le n\int_0^r t^{n-1} f(-v^\star(t))\,dt, \tag{16}$$

where $v^\star = v^\star(r)$ is the $(n-1)$-symmetrand of v, and $\dot v^\star = \frac{dv^\star}{dr}$.

Proof. We note that for $g(s^2) = 1$ this result is proved in [10]. Integrating (15) over Ω_t we find

$$\frac{1}{n}\int_{\Sigma_t} T^{ij}_{(n-1)}(D^2 v)g^n(|Dv|^2)|Dv|^{-1}v_i v_j\,d\sigma \le \int_{\Omega_t} f(-v)\,dx.$$

On using (13), this inequality yields

$$\int_{\Sigma_t} \mathcal{H}_{(n-1)}g^n(|Dv|^2)|Dv|^n\,d\sigma \le n\int_{\Omega_t} f(-v)\,dx.$$

In terms of $\mathcal{G}$, the latter inequality reads as

$$\int_{\Sigma_t} \mathcal{H}_{(n-1)}\mathcal{G}(|Dv|)|Dv|^{-1}\,d\sigma \le n\int_{\Omega_t} f(-v)\,dx. \tag{17}$$

Now we use Jensen's inequality in the following form

$$\mathcal{G}\left(\frac{\int_{\Sigma_t}|Dv|\,dm}{\int_{\Sigma_t}dm}\right) \le \frac{\int_{\Sigma_t}\mathcal{G}(|Dv|)\,dm}{\int_{\Sigma_t}dm}.$$

With $dm = \mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma$ we find

$$\mathcal{G}\left(\frac{\int_{\Sigma_t}\mathcal{H}_{(n-1)}\,d\sigma}{\int_{\Sigma_t}\mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma}\right) \le \frac{\int_{\Sigma_t}\mathcal{G}(|Dv|)\,\mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma}{\int_{\Sigma_t}\mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma}.$$

Recalling that

$$\int_{\Sigma_t}\mathcal{H}_{(n-1)}\,d\sigma = n\omega_n,$$

by the previous inequality and (17) we find

$$\mathcal{G}\left(\frac{n\omega_n}{\int_{\Sigma_t}\mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma}\right) \le \frac{n\int_{\Omega_t} f(-v)\,dx}{\int_{\Sigma_t}\mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma}. \tag{18}$$

On using (14) and putting $V_{(1)}(\Omega_t) = V_{(1)}(t)$ we have

$$\int_{\Sigma_t} \mathcal{H}_{(n-1)}|Dv|^{-1}\,d\sigma = nV'_{(1)}(t).$$

Insertion of this equation into (18) yields

$$\mathcal{G}\left(\frac{\omega_n}{V'_{(1)}(t)}\right) \le \frac{\int_{\Omega_t} f(-v)\,dx}{V'_{(1)}(t)}. \tag{19}$$

On the other hand we have (see [6], Theorem 3.36 and Proposition 6.23; or [10], pag 190)

$$\int_{\Omega_t} f(-v)\,dx \le n\omega_n^{1-n}\int_{m_0}^{t} f(-\tau)V_{(1)}^{n-1}(\tau)V'_{(1)}(\tau)\,d\tau.$$

Putting $V_{(1)}(\tau) = \rho$, and recalling that $v^\star(\rho)$ is the inverse of $V_{(1)}(\tau)$ (see (11)) we find

$$\int_{\Omega_t} f(-v)\,dx \le n\omega_n^{1-n}\int_0^{V_{(1)}(t)} f(-v^\star(\rho))\rho^{n-1}\,d\rho.$$

Inserting this into (19) we get

$$\mathcal{G}\left(\frac{\omega_n}{V'_{(1)}(t)}\right) \le \frac{n\omega_n^{1-n}\int_0^{V_{(1)}(t)} f(-v^\star(\rho))\rho^{n-1}\,d\rho}{V'_{(1)}(t)}.$$

Putting $V_{(1)}(t) = s$ we have

$$V'_{(1)}(t) = \left(\frac{dv^\star(s)}{ds}\right)^{-1}.$$

Hence,

$$\left(\frac{dv^\star(s)}{ds}\right)^{-1}\mathcal{G}\left(\omega_n\frac{dv^\star(s)}{ds}\right) \le n\omega_n^{1-n}\int_0^{s}\rho^{n-1}f(-v^\star(\rho))\,d\rho.$$

Putting $s := \omega_n r$, recalling (12) and writing $\frac{dv^*(r)}{dr} := \dot{v}^*$, the latter inequality yields

$$(\dot{v}^*)^{-1}\mathcal{G}(\dot{v}^*) \le n\omega_n^{-n}\int_0^{\omega_n r}\rho^{n-1}f(-v^\star(\rho))\,d\rho. \tag{20}$$

Finally, putting $\rho := \omega_n t$ and recalling the definition of $\mathcal{G}$, (20) yields

$$\left(g((\dot{v}^*)^2)\dot{v}^*\right)^n \le n\int_0^{r} t^{n-1}f(-v^*(t))\,dt. \tag{21}$$

The theorem is proved.

If Ω is a ball and v is a (radial) solution of (15) with equality sign, then, all the inequalities used in the proof of the latter theorem are equalities. In this situation we have $v^*(r) = v(r)$ and

$$\left(g(\dot{v}^2)\dot{v}\right)^n = n\int_0^{r} t^{n-1}f(-v(t))\,dt. \tag{22}$$

Now we prove a comparison principle.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $f = f(t)$ be positive, non-decreasing and such that conditions (5) and (6) hold. Let $u, v \in \Phi(\Omega)$ satisfy*

$$\mathcal{K}(|Du|^2)\det(D^2u) \geq f(-u) \quad in \;\; \Omega, \tag{23}$$

$$\mathcal{K}(|Dv|^2)\det(D^2v) \leq f(-v) \quad in \;\; \Omega. \tag{24}$$

Then $u \leq v$ in Ω.

Proof. Since u is convex, and hence sub-harmonic on Ω, it follows from Hopf's lemma that, for some constant $c^1 > 0$, $-u(x) \geq c^1 d(x)$ in Ω, where $d(x)$ denotes the distance of $x \in \Omega$ to the boundary $\partial\Omega$, and c^1 is a suitable positive constant. Moreover, since $v \in C^{0,1}(\overline{\Omega})$, there is a constant $c^2 > 0$ such that $-v(x) \leq c^2 d(x)$ on Ω. Consequently,

$$u(x) \leq \frac{c^1}{c^2} v(x) \quad \forall x \in \overline{\Omega}.$$

We can take $c^1 > 0$ sufficiently small such that $c^1 < c^2$. To produce a contradiction, suppose that $u \leq v$ in Ω does not hold. Let

$$S := \{\lambda \in [0,1] : u(x) \leq \lambda v(x) \;\; \forall x \in \Omega\}.$$

Let $\Lambda := \sup S$. Note that $0 < \Lambda < 1$, and $u(x) \leq \Lambda v(x)$ for all $x \in \Omega$. By condition (6) we have

$$f(-\Lambda v(x)) \geq \Lambda^q f(-v(x)) \quad \forall x \in \overline{\Omega}, \tag{25}$$

Since $0 \leq q < \gamma$, we can choose $\epsilon > 0$ sufficiently small that $\Lambda + \epsilon < 1$ and $\Lambda^q > (\Lambda + \epsilon)^\gamma$. Let $w := (\Lambda + \epsilon)v$. Then we find

$$\begin{aligned}
\mathcal{K}(|Du|^2)\det(D^2u) &\geq f(-u) \quad \text{(by (23))} \\
&\geq f(-\Lambda v) \quad \text{(since } f \text{ is non decreasing)} \\
&\geq \Lambda^q f(-v) \quad \text{(by (25))} \\
&> (\Lambda + \epsilon)^\gamma f(-v) \quad \text{(since } \Lambda^q > (\Lambda + \epsilon)^\gamma) \\
&\geq (\Lambda + \epsilon)^{\gamma - n} \mathcal{K}(|Dv|^2)(\Lambda + \epsilon)^n \det(D^2 v) \quad \text{(by (24))} \\
&\geq \mathcal{K}(|Dw|^2)\det(D^2w) \quad \text{(by (5))}.
\end{aligned}$$

In conclusion, we find that

$$\mathcal{K}(|Du|^2)\det(D^2u) > \mathcal{K}(|Dw|^2)\det(D^2w) \;\; \forall x \in \Omega. \tag{26}$$

We claim that

$$u(x) \leq w(x) \;\; \text{in} \;\; \Omega.$$

If not, let $\bar{x} \in \Omega$ such that $0 < (u - w)(\bar{x})$ is the maximum of $u - w$ in Ω. Then $Du(\bar{x}) = Dw(\bar{x})$ and, recalling that u and w are convex in Ω, we have

$$\det(D^2u(\bar{x})) \leq \det(D^2w(\bar{x})).$$

On noting that $\mathcal{K}(|Du(\bar{x})|^2) = \mathcal{K}(|Dw(\bar{x})|^2)$ we find

$$\mathcal{K}(|Du(\bar{x})|^2)\det(D^2u(\bar{x})) \leq \mathcal{K}(|Dw(\bar{x})|^2)\det(D^2w(\bar{x})),$$

which contradicts the strict inequality in (26). Hence,

$$u(x) \le w(x) = (\Lambda + \epsilon)v(x),$$

which implies the contradictory conclusion $\Lambda + \epsilon \in S$. The lemma is proved.

Proof of Theorem 1.1. Let $w < 0$ be a (radial) solution of

$$\mathcal{K}(|Dw|^2)\det(D^2 w) = f(-v^\star) \ \text{ in } B_R, \quad w = 0 \ \text{ on } \partial B_R. \tag{27}$$

Recalling the definition of $\mathcal{K}$ given in (4), and since

$$\det(D^2 w) = \ddot{w}\left(\frac{\dot{w}}{r}\right)^{n-1},$$

from (27) we find

$$g^{n-1}(\dot{w}^2)G(\dot{w}^2)\dot{w}^{n-1}\ddot{w} = r^{n-1}f(-v^\star),$$

$$\left(g(\dot{w}^2)\dot{w}\right)^{n-1}\frac{d}{dr}\left(g(\dot{w}^2)\dot{w}\right) = r^{n-1}f(-v^\star),$$

$$\frac{1}{n}\frac{d}{dr}\left(g(\dot{w}^2)\dot{w}\right)^n = r^{n-1}f(-v^\star).$$

Integration over $(0, r)$ yields

$$\left(g(\dot{w}^2)\dot{w}\right)^n = n\int_0^r t^{n-1}f(-v^\star(t))\,dt.$$

Comparing this equation with inequality (16) we find

$$\left(g((\dot{v}^\star)^2)\dot{v}^\star\right)^n \le \left(g(\dot{w}^2)\dot{w}\right)^n,$$

which implies

$$\dot{v}^\star(r) \le \dot{w}(r), \quad 0 < r < R.$$

Integrating over (r, R) and making use of the conditions $v^\star(R) = w(R) = 0$, we find

$$-v^\star(r) \le -w(r).$$

With $v^\star(x) = v^\star(r)$ and $w(x) = w(r)$ for $r = |x|$ we have

$$-v^\star(x) \le -w(x). \tag{28}$$

By (27) and (28) we find

$$\mathcal{K}(|Dw|^2)\det(D^2 w) = f(-v^\star) \le f(-w) \ \text{ in } B_R.$$

In summary, we see that w and z satisfy

$$\mathcal{K}(|Dw|^2)\det(D^2 w) \le f(-w) \ \text{ in } B_R, \quad w = 0 \ \text{ on } \partial B_R,$$

$$\mathcal{K}(|Dz|^2)\det(D^2 z) \ge f(-z) \ \text{ in } B_R, \quad z = 0 \ \text{ on } \partial B_R.$$

By Lemma 3.3 we have,

$$w(x) \ge z(x) \ \text{ in } B_R.$$

Thus, from (28) and the latter inequality we conclude

$$v^\star(x) \ge z(x) \ \text{ in } B_R.$$

The theorem follows.

Corollary 3.4. *Let $h : [0, \infty) \to [0, \infty)$ be a non-decreasing smooth function such that $h(t) > 0$ for $t > 0$. Under the assumptions and notation of Theorem 1.1, we have*

$$\int_{\Omega} h(-v)\,dx \leq \int_{\Omega^\star_{(n-1)}} h(-z)\,dx.$$

Proof. Let

$$\mu(t) = |\{x \in \Omega : v(x) < t\}|, \quad \mu^\star(t) = |\{x \in \Omega^\star_{(n-1)} : v^\star(x) < t\}|.$$

We know that (see (2.24) of [16])

$$\mu(t) \leq \mu^\star(t) \quad \forall t \in (m_0, 0), \quad m_0 = \inf_{\Omega} v(x).$$

Recalling that $m_0 = v(0) = v^\star(0)$ we find

$$\begin{aligned}\int_{\Omega} h(-v)\,dx &= \int_{m_0}^{0} h(-t)\mu'(t)\,dt = h(0)\mu(0) + \int_{m_0}^{0} h'(-t)\mu(t)\,dt \\ &\leq h(0)\mu^\star(0) + \int_{m_0}^{0} h'(-t)\mu^\star(t)\,dt = \int_{m_0}^{0} h(-t)(\mu^\star)'(t)\,dt \\ &= \int_{\Omega^\star_{(n-1)}} h(-v^\star)\,dx.\end{aligned}$$

Since by Theorem 1.1 we have $-v^\star(x) \leq -z(x)$, and h is non-decreasing, the corollary follows.

Let β be a non-negative real number. For $u \in \Phi(\Omega)$, we define the Hessian integral

$$I(\Omega, \beta, v) = \int_{\Omega} (-v)^{\beta} \mathcal{K}(|Dv|^2)\det(D^2 v)\,dx.$$

Special cases.

If $\beta = 1$, recalling (4) and (3), we find

$$I(\Omega, 1, v) = \int_{\Omega} (-v)\mathcal{K}(|Dv|^2)\det(D^2 v)\,dx = \frac{1}{n}\int_{\Omega} T^{ij}_{(n-1)} g^n(|Dv|^2) v_i v_j\,dx.$$

If $\beta = 0$, recalling (4), (3) and (13), we find

$$I(\Omega, 0, v) = \int_{\Omega} \mathcal{K}(|Dv|^2)\det(D^2 v)\,dx = \frac{1}{n}\int_{\partial\Omega} \mathcal{H}_{(n-1)} g^n(|Dv|^2)|Dv|^n\,d\sigma.$$

Proposition 3.5. *Assume the conditions of Theorem 1.1, and let $v \in \Phi(\Omega)$ be a super-solution of the equation*

$$\mathcal{K}(|Du|^2)\det(D^2 u) = f(-u) \ \ in \ \ \Omega.$$

If $z \in \Phi(\Omega^\star_{(n-1)})$ is a sub-solution of the above equation in the ball $\Omega^\star_{(n-1)}$, we have

$$I(\Omega, \beta, v) \leq I(\Omega^\star_{(n-1)}, \beta, z).$$

Proof. By using Corollary 3.4 we find

$$\begin{aligned}
I(\Omega,\beta,v) &= \int_{\Omega} (-v)^{\beta}\mathcal{K}(|Dv|^2)\det(D^2 v)\,dx \\
&\le \int_{\Omega} (-v)^{\beta} f(-v)\,dx \quad \text{(since } v \text{ is a super-solution)} \\
&\le \int_{\Omega^{\star}_{(n-1)}} (-z)^{\beta} f(-z)\,dx \quad \text{(by Corollary 3.4)} \\
&\le \int_{\Omega^{\star}_{(n-1)}} (-z)^{\beta}\mathcal{K}(|Dz|^2)\det(D^2 z)\,dx \quad \text{(since } z \text{ is a sub-solution)} \\
&= I(\Omega^{\star}_{(n-1)},\beta,z).
\end{aligned}$$

The proposition is proved.

We end this section with the following remark. In [5] (Section 5) and in [8] (Chapter 17), the following equation has been investigated.

$$\det(D^2 v) = \frac{f(-v)}{Q(|Dv|^2)}, \quad Q(s^2) > 0 \quad \forall s \ge 0.$$

We note that this equation coincides with our equation (7) with

$$\mathcal{K}(s^2) = g^{n-1}(s^2)G(s^2) = Q(s^2).$$

Since

$$g^{n-1}(s^2)G(s^2) = g^{n-1}(s^2)\frac{d}{ds}\big(g(s^2)s\big) = s^{1-n}\frac{1}{n}\frac{d}{ds}\big(g(s^2)s\big)^n,$$

we find

$$s^{1-n}\frac{1}{n}\frac{d}{ds}\big(g(s^2)s\big)^n = Q(s^2)$$

and

$$g^n(s^2) = \frac{n}{s^n}\int_0^s t^{n-1}Q(t^2)\,dt.$$

Using the latter function $g(s^2)$ we can apply our previous results to the present equation. For instance, (16) reads as

$$\int_0^{v^{\star}(r)} t^{n-1}Q(t^2)\,dt \le \int_0^r t^{n-1} f(-v^{\star}(t))\,dt.$$

4 Overdetermined problem

Proof of Theorem 1.2. We note that in case of $g(s^2) = 1$ and $c = 1$, this theorem is proved in [4]. On using (13) we find

$$\begin{aligned}
&\int_\Omega \left[\frac{c^{\frac{n-1}{n}}}{n}\left(g(|Du|^2)u_j\right)_j - \frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j\right] dx \\
&= \frac{1}{n}\int_\Omega \left(c^{\frac{n-1}{n}} g(|Du|^2)u_j - T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j dx \\
&= \frac{1}{n}\int_{\partial\Omega} \left(c^{\frac{n-1}{n}} g(|Du|^2)u_j - T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)\frac{u_j}{|Du|}\, d\sigma \\
&= \frac{1}{n}\int_{\partial\Omega} \left(c^{\frac{n-1}{n}} g(|Du|^2)|Du| - T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)|Du|^{-1}u_iu_j\right) d\sigma \\
&= \frac{1}{n}\int_{\partial\Omega} \left(c^{\frac{n-1}{n}} g(|Du|^2)|Du| - \mathcal{H}_{(n-1)}g^n(|Du|^2)|Du|^n\right) d\sigma.
\end{aligned}$$

From this and (9) we find

$$\begin{aligned}
&\int_\Omega \left[\frac{c^{\frac{n-1}{n}}}{n}\left(g(|Du|^2)u_j\right)_j - \frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j\right] dx \\
&= \frac{1}{n}\int_{\partial\Omega} g(|Du|^2)|Du|\left(c^{\frac{n-1}{n}} - \mathcal{H}_{(n-1)}\left(g(|Du|^2)|Du|\right)^{n-1}\right) d\sigma = 0.
\end{aligned} \tag{29}$$

Now we prove that for a solution u to (8) we have

$$\frac{c^{\frac{n-1}{n}}}{n}\left(g(|Du|^2)u_j\right)_j - \frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j \geq 0. \tag{30}$$

Indeed, in view of (8), inequality (30) can be written as

$$\frac{1}{n}\left(g(|Du|^2)u_j\right)_j \geq c^{\frac{1-n}{n}}c = c^{\frac{1}{n}}. \tag{31}$$

Recall that, if $\mathcal{A} = [\mathcal{A}^{ij}]$ with $\mathcal{A}^{ij} = \left(g(|Du|^2)u_i\right)_j$, we have

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = \det\mathcal{A}.$$

By the latter equation and (8), (31) can be written as

$$\frac{1}{n}\left(g(|Du|^2)u_j\right)_j \geq (\det\mathcal{A})^{\frac{1}{n}}. \tag{32}$$

To prove (32), we first apply the arithmetic-geometric mean inequality

$$\frac{1}{n}(g(|Du|^2)u_j)_j \geq \left(\left(g(|Du|^2)u_1\right)_1 \cdots \left(g(|Du|^2)u_n\right)_n\right)^{\frac{1}{n}}. \tag{33}$$

We have equality in (33) if and only if

$$\left(g(|Du|^2)u_1\right)_1 = \cdots = \left(g(|Du|^2)u_n\right)_n. \tag{34}$$

Now we apply the following Hadamard inequality to the positive definite matrix $\mathcal{A}$ (see Theorem 7.8.1 of [9])

$$\mathcal{A}^{11}\cdots\mathcal{A}^{nn} \geq \det\mathcal{A}, \tag{35}$$

with equality sign if and only if

$$\mathcal{A}^{ij} = 0, \quad i \neq j. \tag{36}$$

With $\mathcal{A}^{ij} = \big(g(|Du|^2)u_i\big)_j$, (35) reads as

$$\big(g(|Du|^2)u_1\big)_1 \cdots \big(g(|Du|^2)u_n\big)_n \geq \det\mathcal{A}. \tag{37}$$

Conditions (36) become

$$\big(g(|Du|)^2 u_i\big)_j = 0, \quad i \neq j. \tag{38}$$

Inequality (32) follows by (33) and (37). Hence (30) holds. Then, by (29) we must have

$$\frac{c^{\frac{n-1}{n}}}{n}\Big(g(|Du|^2)u_j\Big)_j - \frac{1}{n}\Big(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\Big)_j = 0. \tag{39}$$

But, as equality holds in (30), also equations (34) and (38) must hold.

To finish the proof, we note that by (39) and (8) we find

$$\frac{c^{\frac{n-1}{n}}}{n}\Big(g(|Du|^2)u_j\Big)_j = c.$$

Recall that the summation convention is in effect in the latter equation. Therefore, on using (34), for j fixed we find

$$\Big(g(|Du|^2)u_j\Big)_j = c^{\frac{1}{n}}, \quad j = 1, \cdots, n.$$

Integrating and using (38) we get

$$g(|Du|^2)u_j = c^{\frac{1}{n}}(x^j - \bar{x}^j), \quad j = 1, \cdots, n, \tag{40}$$

where $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^n)$ is a point of minimum for $u(x)$. It follows that

$$g^2(|Du|^2)u_j^2 = c^{\frac{2}{n}}(x^j - \bar{x}^j)^2, \quad j = 1, \cdots, n.$$

Adding from $j = 1$ to $j = n$ we find

$$g^2(|Du|^2)|Du|^2 = c^{\frac{2}{n}}r^2, \quad r^2 = \sum_1^n (x^j - \bar{x}^j)^2,$$

and

$$g(|Du|^2)|Du| = c^{\frac{1}{n}}r.$$

Since $g(s^2)s$ is strictly increasing, $|Du|$ must be radial around the point of minimum of $u(x)$. Then, from (40) we get

$$u_j = F(r)(x^j - \bar{x}^j), \quad j = 1, \cdots, n, \tag{41}$$

where F is a suitable function. Note that (41) can be written as

$$u_j = \big(K(r)\big)_j, \quad j = 1, \cdots, n, \tag{42}$$

with

$$K(r) = \int_0^r F(t)t\,dt.$$

By (42) we find

$$u(r) = K(r) + c.$$

Hence, u is radially symmetric. Since u is strictly convex, Ω must be a ball centered at $\bar{x}$. The theorem is proved.

In [21], the Monge-Ampère equation $\det(D^2u) = 1$ in two dimensions has been discussed. In particular, it is proved that the P-function $P(x) = |Du|^2 - 2u$ attains its maximum value on $\partial\Omega$.

Appendix

If $g(s^2) = 1$, we find $G(s^2) = 1$.

If $g(s^2) = (1 + s^2)^{-\frac{\alpha}{2}}$, $\alpha \leq 1$, then $G(s^2) = (1 + s^2)^{-\frac{\alpha+2}{2}}(1 + (1 - \alpha)s^2) > 0$.

Comments on the convexity of the function $\mathcal{G}(s)$ defined in (4).
If $g(s^2) = 1$, we find

$$\mathcal{G}(s) = s^{n+1},$$

which is convex for $s > 0$.

If $g(s^2) = (1 + s^2)^{-\frac{\alpha}{2}}$, then

$$\mathcal{G}(s) = (1 + s^2)^{-\frac{n\alpha}{2}} s^{n+1}, \quad s > 0.$$

We have

$$\mathcal{G}'(s) = (1 + s^2)^{-\frac{n\alpha+2}{2}} \left[((1 - \alpha)n + 1)s^{n+2} + (n + 1)s^n\right],$$

and

$$\begin{aligned}\mathcal{G}''(s) = (1 + s^2)^{-\frac{n\alpha+4}{2}} \big\{&-(n\alpha + 2)\left[((1 - \alpha)n + 1)s^{n+3} + (n + 1)s^{n+1}\right] \\ &+ (n + 2)((1 - \alpha)n + 1)s^{n+1} + n(n + 1)s^{n-1} \\ &+ (n + 2)((1 - \alpha)n + 1)s^{n+3} + n(n + 1)s^{n+1}\big\}.\end{aligned}$$

Simplification yields

$$\begin{aligned}\mathcal{G}''(s) = n(1 + s^2)^{-\frac{n\alpha+4}{2}} \big\{&((1 - \alpha)n + 1)(1 - \alpha)s^{n+3} \\ &+ (2n(1 - \alpha) + 2 - 3\alpha)s^{n+1} + (n + 1)s^{n-1}\big\}.\end{aligned}$$

By computations one finds

$$\mathcal{G}''(s) > 0 \;\text{ for }\; \alpha < \frac{8n + 8}{8n + 9}.$$

Comments on the condition (5).
If $g(s^2) = 1$, condition (5) with $\gamma = 2n$ reads as

$$t^n s^n \geq (ts)^n,$$

which is trivially satisfied.
If $g(s^2) = (1 + s^2)^{-\frac{\alpha}{2}}$ with $n \geq 3$, $0 < \alpha < (n - 2)/n$, condition (5) with

$$\gamma = n(1 - \alpha) - 2,$$

reads as

$$t^{-(n\alpha+2)}(1 + s^2)^{-\frac{n\alpha+2}{2}}(1 + (1 - \alpha)s^2) \geq (1 + t^2 s^2)^{-\frac{n\alpha+2}{2}}(1 + (1 - \alpha)t^2 s^2).$$

Simplifying we find

$$(t^{-2} + s^2)^{\frac{n\alpha+2}{2}}(1 + (1 - \alpha)s^2) \geq (1 + s^2)^{\frac{n\alpha+2}{2}}(1 + (1 - \alpha)t^2 s^2),$$

which is clearly satisfied for $0 < t < 1$.

Conflict of Interest: The authors declare that they have no conflict of interest.

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