DE GRUYTER Adv. Nonlinear Stud. 2020; 20(3): 579–597

Research Article

Guozhen Lu and Yansheng Shen*

Existence of Solutions to Fractional p-Laplacian Systems with Homogeneous Nonlinearities of Critical Sobolev Growth

https://doi.org/10.1515/ans-2020-2098
Received May 8, 2020; revised May 30, 2020; accepted May 31, 2020

Abstract: In this paper, we investigate the existence of nontrivial solutions to the following fractional p-Laplacian system with homogeneous nonlinearities of critical Sobolev growth:

$$\begin{cases} (-\Delta_p)^s u = Q_u(u,v) + H_u(u,v) & \text{in } \Omega, \\ (-\Delta_p)^s v = Q_v(u,v) + H_v(u,v) & \text{in } \Omega, \\ \quad u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ \qquad u, v \geq 0, \quad u, v \neq 0 & \text{in } \Omega, \end{cases}$$

where $(-\Delta_p)^s$ denotes the fractional p-Laplacian operator, $p > 1$, $s \in (0,1)$, $ps < N$, $p_s^* = \frac{Np}{N-ps}$ is the critical Sobolev exponent, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and Q and H are homogeneous functions of degrees p and q with $p < q \leq p_s^*$ and Q_u and Q_v are the partial derivatives with respect to u and v, respectively. To establish our existence result, we need to prove a concentration-compactness principle associated with the fractional p-Laplacian system for the fractional order Sobolev spaces in bounded domains which is significantly more difficult to prove than in the case of single fractional p-Laplacian equation and is of its independent interest (see Lemma 5.1). Our existence results can be regarded as an extension and improvement of those corresponding ones both for the nonlinear system of classical p-Laplacian operators (i.e., $s = 1$) and for the single fractional p-Laplacian operator in the literature. Even a special case of our main results on systems of fractional Laplacian $(-\Delta)^s$ (i.e., $p = 2$ and $0 < s < 1$) has not been studied in the literature before.

Keywords: Fractional p-Laplacian, Elliptic System, Concentration Compactness Principle

MSC 2010: 35B33, 35J50, 39B72, 45G15

Communicated by: Julián López-Gómez

1 Introduction and main results

In recent years, there has been an increasing amount of attention to problems involving nonlocal diffusion operators. These problems usually include the fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$ or fractional p-Laplacian operators $(-\Delta_p)^s$ $(p > 1)$ with subcritical or critical nonlinearities and they arise in a quite natural way in many different applications, for instance in physical models, finances, fluid dynamics and image processing, etc.

***Corresponding author: Yansheng Shen** School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P. R. China, e-mail: ysshen@mail.bnu.edu.cn
Guozhen Lu Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA, e-mail: guozhen.lu@uconn.edu

The fractional Laplacian in $\mathbb{R}^N$ is a nonlocal pseudo-differential operator defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(z)}{|x - z|^{N+\alpha}}\, dz, \tag{1.1}$$

where α is any real number between 0 and 2. This operator is well defined in $\mathcal{S}(\mathbb{R}^N)$, the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^N$. One can extend this operator to a wider space of functions. Let

$$L_\alpha(\mathbb{R}^N) = \Big\{ u : \mathbb{R}^N \to \mathbb{R}, \int\limits_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|)^{N+\alpha}}\, dx < \infty \Big\}.$$

Then it is easy to verify that for $u \in L_\alpha(\mathbb{R}^N) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, the integral on the right-hand side of (1.1) is well defined. The non-locality of the fractional Laplacian makes it difficult to study. To circumvent this difficulty, Caffarelli and Silvestre introduced in their celebrated work [11] the extension method that reduced this nonlocal problem into a local one in higher dimensions. This fundamental extension method has been applied successfully to study equations involving the fractional Laplacian and many results have been obtained since then.

On the other hand, for the nonlocal operators such as fractional Laplacian and its nonlinear generalization, the fractional p-Laplacian, they can be regarded as an extension of the traditional local operators such as the Laplacian $-\Delta$ and p-Laplacian $-\Delta_p$. The corresponding elliptic problems can be seen as an extension of many classical problems such as the well-known Brezis–Nirenberg problem. In 1983, Brezis and Nirenberg, in their celebrated paper [10], showed that the critical growth semi-linear problem

$$\begin{cases} -\Delta u = \lambda u + u|u|^{2^*-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

admits a classical solution provided that $\lambda \in (0, \lambda_1(2))$ and $N \geq 4$, where $\lambda_1(2)$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Furthermore, in dimension $N = 3$, the same existence result holds provided that $\mu < \lambda < \lambda_1(2)$, for a suitable $\mu > 0$ (if Ω is a ball, then $\mu = \frac{1}{4}\lambda_1(2)$ is sharp). By Pohožaev identity, if $\lambda \notin (0, \lambda_1(2))$ and Ω is a star-sharped domain, then problem (1.2) admits no solution. Later on, Capozzi, Fortunato and Palmieri [12] established the existence of nontrivial solutions to the following problem for $\lambda > 0$ when $N \geq 4$:

$$\begin{cases} -\Delta u = \lambda u + u|u|^{2^*-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Subsequently, Ambrosetti and Struwe [3] used the dual approach to the problem (1.3), which allows a direct use of the celebrated Mountain-Pass Theorem and critical point theory of Ambrosetti and Rabinowitz [2, 39], and they presented a new and simpler proof for the existence of nontrivial solutions to (1.3). Furthermore, as an extension of problem (1.2), Azorero and Alonzo in [23] studied the existence of nontrivial solution for the problem

$$\begin{cases} -\Delta_p u = \lambda u|u|^{p-2} + u|u|^{p^*-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases}$$

where $0 < \lambda < \lambda_1(p)$ and $N \geq p^2$, where $\lambda_1(p)$ is the first eigenvalue of the p-Laplacian with the Dirichlet boundary condition. Then in 2015, Servadei and Valdinoci [40], among other things, generalized the well-known Brezis–Nirenberg problem to the critical fractional Laplacian:

$$\begin{cases} (-\Delta)^s u = \lambda u + u|u|^{2_s^*-2} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $s \in (0, 1)$ and $2_s^* = \frac{2N}{N-2s}$. The authors showed that if $\lambda_{1,s}$ is the first eigenvalue of the nonlocal operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum, then for any $\lambda \in (0, \lambda_{1,s})$ there exists a nontrivial solution to (1.4) provided $N \geq 4s$. In [18], W. X. Chen and Zhu consider an indefinite fractional Laplacian problem. A corresponding Liouville-type theorem is established. Furthermore, they obtain a priori estimate for solutions in a bounded domain by blowing-up and rescaling. Substantially improved results for the indefinite fractional Laplacian problem have been further obtained by W. X. Chen, Li and Zhu [16].

Recently, Mosconi, Perera, Squassina and Yang [36] considered the Brezis–Nirenberg problem for the fractional p-Laplacian. More precisely, they investigated the problem

$$\begin{cases} (-\Delta_p)^s u = \lambda u|u|^{p-2} + u|u|^{p_s^*-2} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

where $\lambda > 0$, $s \in (0,1)$, $p > 1$ and $p_s^* = \frac{pN}{N-ps}$. Based on the results in [6] obtained by Brasco, Mosconi and Squassina and an abstract linking theorem based on the cohomological index proved in [43] by Yang and Perera, the authors obtained the existence of nontrivial solutions to problem (1.5).

On the other hand, as another form of the extension to the classical Brezis–Nirenberg problem, Alves, de Morais Filho and Souto in [1] considered a certain system of Laplacian equations with nonlinearities of critical or subcritical Sobolev growth and established the existence and nonexistence results of nontrivial solutions to such a system. Then de Morais Filho and Souto in [19] generalized the aforesaid results to a more general class of nonlinear systems of p-Laplacian operators:

$$\begin{cases} -\Delta_p u = Q_u(u,v) + H_u(u,v) & \text{in } \Omega, \\ -\Delta_p v = Q_v(u,v) + H_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u, v \geq 0, \quad u, v \neq 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $p > 1$ and Ω is a bounded domain and Q and H satisfy certain homogeneity conditions. By using the concentration compactness principle and certain asymptotic estimates for minimizers of the Sobolev inequality, the authors obtained the existence of nontrivial weak solutions to problem (1.6).

Motivated by the aforementioned works, it is natural to ask whether system (1.6) has a nontrivial solution when the p-Laplacian is replaced by the fractional p-Laplacian. As far as we know, there is no related work in this direction so far. In this paper, we consider the following problem:

$$\begin{cases} (-\Delta_p)^s u = Q_u(u,v) + H_u(u,v) & \text{in } \Omega, \\ (-\Delta_p)^s v = Q_v(u,v) + H_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u, v \geq 0, \quad u, v \neq 0 & \text{in } \Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0,1)$, $1 < p < \infty$ and $ps < N$, the fractional p-Laplacian $(-\Delta_p)^s$ is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta_p)^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{B_\varepsilon(x)^c} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ when $p = 2$. It is worth noting that in recent years, there is a rapidly growing literature on the investigation for the fractional p-Laplacian operator. For instance, the fractional p-eigenvalue problems have been studied in Brasco, Parini and Squassina [8], Brasco and Parini [7], Franzina and Palatucci [22] and Lindgren and Lindqvist [32], etc. Regularity of solutions was obtained in Brasco and Lindren [5], Chen, Li and Qi [15], Di Castro, Kuusi and Palatucci [20, 21], Iannizzotto, Mosconi and Squassina [25], Kuusi, Mingione and Sire [26]. Monotonicity and symmetry of solutions was studied by Chen, Li and Wu [14, 42]. Existence via Morse theory was investigated in Iannizzotto, Liu, Perera and Squassina [24]. This operator has been studied extensively in recent years. We refer the reader to, for example, [13, 17, 37] and many references therein for details. In particular, Brasco, Mosconi and Squassina [6] obtained the optimal decay of extremal functions for the fractional Sobolev inequality, which plays an important role in studying the problem (1.7). Moreover, Marano and Mosconi extended the results to the fractional Hardy–Sobolev inequality. The authors proved the existence of optimizers for the fractional Hardy–Sobolev inequality and they studied the asymptotic properties of the optimizers. The reader is referred to [35] for details. For more recent progress on investigations of the Hardy–Sobolev-type inequalities, we refer the interested reader to [34] and the references therein.

In this paper, we consider problem (1.7) in the Banach space $W = W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$. The methods we use here are through the variational arguments and combining with the concentration compactness principle. More precisely, we introduce the nontrivial solution for problem (1.7) by finding a minimum point of the energy functional $I : W \to \mathbb{R}$

$$I(u,v) = \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_{\Omega} Q(u,v)\,dx, \tag{1.8}$$

subjected to the constraint-manifold

$$\mathcal{H} = \Big\{(u,v) \in W : \int_{\Omega} H(u^+,v^+)\,dx = 1\Big\}.$$

Moreover, we say $(u,v) \in W$ is a weak solution of problem (1.7) if

$$\begin{aligned}\int_{\mathbb{R}^{2N}} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\ &+ \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}}\,dx\,dy\\ &= \int_{\Omega} [\varphi Q_u(u,v) + \psi Q_v(u,v)]\,dx + \eta \int_{\Omega} [\varphi H_u(u,v) + \psi H_v(u,v)]\,dx,\end{aligned}$$

where $\eta \in \mathbb{R}$ is a constant satisfying $[I'(u,v) - \eta J'(u,v)](\varphi,\psi) = 0$ and $J(u,v) = \int_\Omega H(u,v)\,dx - 1$. Namely, $\eta \in \mathbb{R}$ is a Lagrange multiplier satisfying the Euler–Lagrange equation corresponding to the functional I defined in (1.8) among functions in the constraint manifold $\mathcal{H}$. For further details of this variational framework, we refer the reader to Section 2.

Before we state our main results, we introduce several notations. In the sequel, we shall denote by $\lambda_1(p)$ the positive eigenvalue of the fractional p-Laplacian eigenvalue problem, subjected to Dirichlet boundary condition, and use its variational characterization

$$\lambda_1 = \lambda_1(p) = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{\int_\Omega |u|^p\,dx}.$$

Let us also denote

$$\frac{\mu_1}{p} = \min\{Q(u,v) : u, v \geq 0,\ u^p + v^P = 1\} \tag{1.9}$$

and

$$\frac{\mu_2}{p} = \max\{Q(u,v) : u, v \geq 0,\ u^p + v^P = 1\}. \tag{1.10}$$

We are ready to state our first result of existence of nontrivial solutions to problem (1.7) when the system of the fractional p-Laplacian has homogeneous nonlinearities of subcritical Sobolev growth. This extends the result in the local case when $s = 1$ in [19].

Theorem 1.1 (Subcritical Case). *Let Q be a $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ function such that*

$$Q(\lambda u, \lambda v) = \lambda^p Q(u,v) \quad \text{for all } \lambda > 0,\ u, v \geq 0 \quad (p\text{-homogenity}) \tag{1.11}$$

and

$$Q_u(0,1) > 0, \quad Q_v(1,0) > 0. \tag{1.12}$$

Let H be another $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ function such that

$$\begin{aligned} H(\lambda u, \lambda v) &= \lambda^q H(u, v) \quad && \text{for all } \lambda > 0, \\ u, v &\geq 0, && p < q < p_s^*, \\ H(u, v) &\geq 0 && \text{for all } u, v \geq 0, \end{aligned} \tag{1.13}$$

and

$$H_u(0, 1) = 0, \quad H_v(1, 0) = 0. \tag{1.14}$$

Then, for $\mu_2 < \lambda_1$, system (1.7) has a nontrivial solution.

Next is the main result of this paper. Namely, we will establish the following existence result of nontrivial solutions to the problem (1.7) when the system of the fractional p-Laplacian has homogeneous nonlinearities of critical Sobolev growth.

Theorem 1.2 (Critical Case). *Let Q be a $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ function defined as in Theorem 1.1 and let H be a $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ function satisfying (1.13) and (1.14). In addition, H is p_s^*-homogeneous, i.e.,*

$$H(\lambda u, \lambda v) = \lambda^{p_s^*} H(u, v) \quad \text{for all } \lambda > 0,\ u, v \geq 0. \tag{1.15}$$

The 1-homogeneous function G defined by

$$G(s^{p_s^*}, t^{p_s^*}) = H(s, t) \quad \text{for all } s, t \geq 0,$$

is concave. Suppose also that $N > p^2 s$ and $0 < \mu_1 \leq \mu_2 < \lambda_1$. Then system (1.7) has a nontrivial solution.

For the critical case, as p_s^* is the limiting fractional Sobolev exponent for the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$, the lack of compactness arise new difficulties. In order to overcome these difficulties, we work with certain asymptotic estimates for minimizers obtained in [6] to establish a relationship for the fractional Sobolev-type constants which ensure the compactness occurs. On the other hand, we establish a concentration compactness principle associated with the system of the fractional p-Laplacian in bounded domain (see Lemma 5.1) which is of its independent interest. This principle is in the spirit to the work for a single fractional p-Laplacian equation [4] recently obtained by Bonder, Saintier and Silva, but nevertheless is more difficult to prove in our case of the system of the fractional p-Laplacians.

The paper is organized as follows: we give some preliminary results in Section 2. In Section 3, we consider the subcritical case and give the proof of Theorem 1.1. In Section 4, we will prove some technical lemmas and in the Section 5 we will prove a version of the concentration compactness lemmas associated with the system of the fractional p-Laplacian in bounded domain (Lemma 5.1). Finally, we complete the proof of Theorem 1.2.

2 Preliminary Results

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. In [17, 36, 37], Mosconi, Perera, Squassina and Chen et al. discussed the Dirichlet boundary value problem for the fractional p-Laplacian by means of variational methods. Since the problems they considered are in a bounded domain, they introduced the function space $(W_0^{s,p}(\Omega), [\,\cdot\,]_{s,p})$. For the entire space, it is well known that the usual fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

endowed with the norm

$$\|u\|_{s,p} = (|u|^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$ and

$$[u]_{s,p} = \left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}$$

is the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$. In this paper, we continue to work in the closed subspace $W_0^{s,p}(\Omega)$ which is defined as

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

it is a uniformly convex Banach space for any $p > 1$ and equivalently renormed by setting

$$\|\cdot\|_{W_0^{s,p}(\Omega)} = [\,\cdot\,]_{s,p} = \left(\int_X \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}, \tag{2.1}$$

where $X = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$. The second equality holds in (2.1) since $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and the integral in (2.1) can be reduced to over X. The embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. It is worth noting that when $s = 1, p = N$, the corresponding Sobolev space $W_0^{1,N}(\Omega)$ is a borderline case for Sobolev embedding and the Sobolev inequality transforms into the well-known Moser-Trudinger inequality (see [38, 41]). In this regard, many authors have contributed to the study of nonlinear PDEs with nonlinearity of exponential growth and there is a vast literature in the subject. We only refer the reader to [27–30] and many references therein.

In the following we shall find weak solutions of (1.7) in the space

$$W = W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega),$$

endowed with the norm

$$\|(u, v)\|^p = [u]_{s,p}^p + [v]_{s,p}^p = \int_X \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + \int_X \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy.$$

Under the assumption of (1.11)–(1.12), we can extend the function Q to the whole plane as a $\mathcal{C}^1$-function as

$$Q(s, t) = \begin{cases} Q(s, t), & s, t \geq 0, \\ Q(0, t) + Q_s(0, t)s, & s \leq 0 \leq t, \\ Q(s, 0) + Q_t(s, 0)t, & t \leq 0 \leq s, \\ 0, & s, t \leq 0. \end{cases}$$

It is particularly noteworthy that under the above extension, we have

$$\begin{aligned} Q_u(u, v) &\geq 0 \quad \text{for all } u \leq 0, \; v \in \mathbb{R}, \\ Q_v(u, v) &\geq 0 \quad \text{for all } v \leq 0, \; u \in \mathbb{R}. \end{aligned} \tag{2.2}$$

(We note that the continuous differentiability of Q comes from Remark 2.1 (iii) and (iv) below.)

On the other hand, under assumption (1.14), by taking a similar argument as above, we can give a $\mathcal{C}^1$ extension of H to the whole plane as

$$H(u, v) = H(u^+, v^+) \quad \text{for all } u, v \in \mathbb{R},$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Hence, in this paper, we shall find the solutions of system (1.7) as a minimum of the functional $I : W \to \mathbb{R}$

$$I(u, v) = \frac{1}{p} \left(\int_X \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + \int_X \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right) - \int_\Omega Q(u, v) \, dx,$$

subjected to the constraint-manifold

$$\mathcal{H} = \left\{ (u, v) \in W : \int_\Omega H(u^+, v^+) \, dx = 1 \right\}.$$

Remark 2.1. Since the $\mathcal{C}^1$-functions Q and H are homogeneous functions, for convenience of the reader we list several properties of homogeneous functions: Let F be a q ($q \geq 1$) homogeneous $\mathcal{C}^1$ function. Then:

(i) There exists $M_F > 0$ such that

$$|F(s, t)| \leq M_F(|s|^q + |t|^q) \quad \text{for all } s, t \in \mathbb{R}, \qquad M_F = \max\{F(s, t) : s, t \in \mathbb{R}, \; |s|^q + |t|^q = 1\}.$$

(ii) The maximum M_F is attained for some $(s_0, t_0) \in \mathbb{R}^2$ (both not vanishing), since

$$\{(s,t) : s, t \in \mathbb{R}^2, |s|^q + |t|^q = 1\},$$

is a compact set and F continuous on it.

(iii) For all $s, t \in \mathbb{R}$,

$$sF_s(s,t) + tF_t(s,t) = qF(s,t). \tag{2.3}$$

(iv) ∇F is a $(q-1)$ homogeneous function and by (2.3), when $q = 1$, we have: F is concave if and only if $F_{st}(s,t) \geq 0$.

Remark 2.2. For the convenience of the reader, we list some explicit examples below for the homogeneous functions Q and H, which can provide us better understanding of the problem we consider in this paper. Let us denote

$$P_l(u,v) = au^l + \sum_{\alpha_i+\beta_i=l} b_i u^{\alpha_i} v^{\beta_i} + cv^l,$$

where $i \in \mathcal{I}$ ($\sharp\mathcal{I} < \infty$), $l > 0$, $\alpha_i \geq 1$, $\beta_i \geq 1$, $a, b_i, c \in \mathbb{R}$. With suitable b_i and l, sums $\sum^\infty$ may be allowed. Then the following elementary functions and some others possible combinations of them, with convenient a, b_i, c, satisfy hypotheses (1.11) and (1.12):

$$\begin{aligned} Q_1(u,v) &= P_p(u,v), \\ Q_2(u,v) &= \frac{P_{l_2}(u,v)}{P_{l_3}(u,v)} \quad \text{with } l_2 - l_3 = p. \end{aligned}$$

Moreover, $H(u,v) = P_{p_s^*}(u,v)$ satisfies (1.15) and additional assumption given in Theorem 1.2. For more examples and properties of the homogeneous functions Q and H, we can see [19] for more details.

3 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We first introduce a lemma which will be used in our proof of Theorem 1.1.

Lemma 3.1 (Chen, Mosconi and Squassina [17]). *For any $u \in W_0^{s,p}(\Omega)$ it holds*

$$\langle(-\Delta_p)^s u^\pm, u^\pm\rangle \leq \langle(-\Delta_p)^s u, u^\pm\rangle \leq \langle(-\Delta_p)^s u, u\rangle,$$

with strict inequality as long as u is sign-changing.

Proof of Theorem 1.1. Let $\{(u_n, v_n)\} \in \mathcal{H}$ be a minimizing sequence for $R_Q := \min_{(u,v)\in\mathcal{H}} I(u,v)$, i.e.,

$$R_Q = I(u_n, v_n) + o_n(1), \tag{3.1}$$

where $o_n(1)$, from now on, is such that $o_n(1) \to 0$ as $n \to \infty$.

From (1.10) we get that

$$I(u_n, v_n) \geq \frac{1}{p}\min\Big\{1, \Big(1 - \frac{\mu_2}{\lambda_1}\Big)\Big\}\Big(\int_X \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_X \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big). \tag{3.2}$$

Hence $\|(u_n, v_n)\|$ is bounded and (3.2) implies that $I \geq 0$ and $I > 0$ on $\mathcal{H}$.

As we are in the subcritical case, there are subsequence $\{u_n\}$, $\{v_n\}$ in $W_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $W_0^{s,p}(\Omega)$ and these convergences also hold pointwisely on Ω and $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ strongly in $L^r(\Omega)$ for $r \in [1, p_s^*)$. Moreover, there is $h \in L^r(\Omega)$ such that $|u_n(x)|, |v_n(x)| \leq h(x)$, a.e. in Ω.

Therefore, passing to the limit in (3.1) we get $R_Q = I(u,v)$ and $(u,v) \in \mathcal{H}$.

By the Lagrange multiplier theorem, if $J(u,v) = \int_\Omega H(u,v)\,dx - 1$, there exists $\eta \in \mathbb{R}$ such that

$$[I'(u,v) - \eta J'(u,v)] \cdot (\varphi, \psi) = 0 \quad \text{for all } (\varphi, \psi) \in \mathcal{H}. \tag{3.3}$$

Here I', J' are the Fréchet derivatives, i.e., from (3.3), we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ = \int_\Omega (\varphi Q_u(u,v)+\psi Q_v(u,v))\,dx + \eta\int_\Omega(\varphi H_u(u,v)+\psi H_v(u,v))\,dx\end{aligned}$$

for all $(\varphi,\psi)\in W$. Particularly, setting $\varphi=u$, $\psi=v$ in the above equation, and using the homogeneity properties (2.3), we get $pI(u,v)=q\eta$, which implies that $\eta=\frac{p}{q}R_Q$ is a positive number. In addition, it is easy to verify that $(\rho u,\rho v)$, with $\rho=(\frac{p}{q}R_Q)^{\frac{1}{p-q}}>0$, is the desired solution. This solution, now denoted by (u,v), is nonnegative. In fact, replacing (φ,ψ) by (u^-,v^-), combining with Lemma 3.1, we have

$$\begin{aligned}0\le \|(u^-,v^-)\| &= \langle(-\Delta_p)^s u^-,u^-\rangle + \langle(-\Delta_p)^s v^-,v^-\rangle\\ &\le \langle(-\Delta_p)^s u,u^-\rangle + \langle(-\Delta_p)^s v,v^-\rangle\\ &= \int_\Omega u^-[Q_u(u,v)+H_u(u,v)]\,dx + \int_\Omega v^-[Q_v(u,v)+H_v(u,v)]\,dx\\ &= \int_\Omega u^-[Q_u(u^-,v)+H_u(u,v)]\,dx + \int_\Omega v^-[Q_v(u,v^-)+H_v(u,v)]\,dx.\end{aligned}$$

Notice that $u^-, v^-\le 0$, $H(u,v)=H(u^+,v^+)\ge 0$ for all $u,v\in\mathbb{R}$, then combining (2.2), we get that

$$\int_\Omega u^-[Q_u(u^-,v)+H_u(u,v)]\,dx + \int_\Omega v^-[Q_v(u,v^-)+H_v(u,v)]\,dx \le 0,$$

which implies that

$$u^-=v^-=0.$$

By (1.7), (1.12) and (1.14), we see that $u=0$ if and only if $v=0$. Hence, $u,v\ge 0$, $u,v\ne 0$. Thus Theorem 1.1 is proved. □

4 Some Technical Lemmas

In this section, we establish several auxiliary estimates that will be used in the proof of Theorem 1.2. First, we recall the fractional Sobolev inequality. Let the symbol $W_0^{s,p}(\mathbb{R}^N)$ denotes the homogeneous fractional Sobolev space

$$W_0^{s,p}(\mathbb{R}^N) := \{u\in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p}<\infty\},$$

endowed with the norm $\|\cdot\|_{W_0^{s,p}(\mathbb{R}^N)}=[\cdot]_{s,p}$, and let

$$S=S(\mathbb{R}^N)=\inf_{u\in W_0^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{(\int_{\mathbb{R}^N}|u|^{p_s^*}\,dx)^{\frac{p}{p_s^*}}}, \tag{4.1}$$

it is well known that $S(\mathbb{R}^N)$ is achieved in $\mathbb{R}^N$. In particular, for $p=2$, the explicit formula for the minimizers for S has been given by Lieb in [31]. More precisely, the minimizers are of the form $cU(\frac{|x-x_0|}{\varepsilon})$, where

$$U(x)=\frac{1}{(1+|x|^2)^{\frac{N-2s}{2}}},\quad x\in\mathbb{R}^N,$$

$c\ne 0$, $x_0\in\mathbb{R}^N$ and $\varepsilon>0$. But for $p\ne 2$, it is not known the explicit formula for the minimizers of $S(\mathbb{R}^N)$ till now. On the other hand, in [6], Brasco, Mosconi and Squassina investigated the existence and properties for the minimization problem (4.1). Recently, Marano, Mosconi extended the results in [6] and they obtained the existence and asymptotic properties for optimizers of the fractional Hardy–Sobolev inequality, see [35] and the references therein.

Now, we define

$$\tilde{S} = S(\Omega) := \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{(\int_\Omega |u|^{p_s^*}\,dx)^{\frac{p}{p_s^*}}} = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\int_X \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{(\int_\Omega |u|^{p_s^*}\,dx)^{\frac{p}{p_s^*}}},$$

$$\tilde{S}_H = \tilde{S}_H(\Omega) := \inf_{(u,v)\in W\setminus\{0\}} \frac{\int_X \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_X \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{(\int_\Omega H(u^+,v^+)\,dx)^{\frac{p}{p_s^*}}}. \tag{4.2}$$

We next prove a relationship between $\tilde{S}$ and $\tilde{S}_H$, which will be used later. To prove it, we introduce some definitions and recall some known results. Let (s_0, t_0) $(s_0, t_0 \geq 0)$ be the one defined in Remark 2.1 (ii) for the p-homogeneous function $F(s,t) = H(s,t)^{\frac{p}{p_s^*}}$, and $m^{-1} = M_F$. Then we have

$$mH(s,t)^{\frac{p}{p_s^*}} \leq |s|^p + |t|^p \quad \text{for all } s, t \in \mathbb{R}$$

and

$$mH(s_0,t_0)^{\frac{p}{p_s^*}} = s_0^p + t_0^p. \tag{4.3}$$

We now state a lemma which will be needed in the proof of Lemma 4.2 below.

Lemma 4.1 (de Morais Filho and Souto [19]). *Let H be a p_s^*-homogeneous continuous function satisfying: The 1-homogeneous function G, defined by*

$$G(s^{p_s^*}, t^{p_s^*}) = H(s,t) \quad \textit{for all } s, t \geq 0,$$

is concave. Then the Hölder-type inequality holds:

$$\int_\Omega H(u,v)\,dx \leq H(\|u\|_{L^{p_s^*}(\Omega)}, \|v\|_{L^{p_s^*}(\Omega)}) \quad \textit{for all } u, v \in L^{p_s^*}(\Omega)\ u, v \geq 0.$$

Lemma 4.2. *Let Ω be a domain (not necessarily bounded). If* (4.3) *holds, then*

$$\tilde{S}_H = m\tilde{S}.$$

Moreover, if $\tilde{S}$ is attained at ω_0, then $\tilde{S}_H$ is attained at $(s_0\omega_0, t_0\omega_0)$ for all $s_0, t_0 > 0$ satisfying (4.3).

Proof. Consider $\{\omega_n\} \subseteq W_0^{s,p}(\Omega)$ a minimizing sequence for $\tilde{S}$ and the sequence $\{(s_0,t_0)\omega_n\}$. Substituting this sequence in quotient (4.2), using (4.3) and taking the limit, we obtain

$$\tilde{S}_H \leq m\tilde{S}.$$

For the reverse inequality, we choose $\{(u_n, v_n)\}$ be a minimizing sequence for $\tilde{S}_H$. By the definition of $\tilde{S}$ and Lemma 4.1, we have

$$\frac{\int_X \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_X \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{(\int_\Omega H(u_n,v_n)\,dx)^{\frac{p}{p_s^*}}} \geq \tilde{S}\frac{\|u_n\|^p_{L^{p_s^*}(\Omega)} + \|v_n\|^p_{L^{p_s^*}(\Omega)}}{H(\|u_n\|_{L^{p_s^*}(\Omega)}, \|v_n\|_{L^{p_s^*}(\Omega)})^{\frac{p}{p_s^*}}} \geq m\tilde{S}.$$

Taking the limit, we obtain the remained inequality. The rest of the proof is straightforward computation and we omit it. □

We also need the following definitions:

$$R_\lambda(u) = \frac{\int_X \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \lambda\int_\Omega |u|^p\,dx}{(\int_\Omega |u|^{p_s^*}\,dx)^{\frac{p}{p_s^*}}}, \quad \lambda > 0,$$

$$\tilde{R}_Q(u,v) = \frac{\frac{1}{p}(\int_X \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_X \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy) - \int_\Omega Q(u,v)\,dx}{(\int_\Omega H(u^+,v^+)\,dx)^{\frac{p}{p_s^*}}},$$

$$R_\lambda = \inf\{R_\lambda(u) : u \in W_0^{s,p}(\Omega) \text{ and } \|u\|_{L^{p_s^*}(\Omega)} = 1\},$$

$$\tilde{R}_Q = \inf\Big\{\tilde{R}_Q(u,v) : (u,v) \in W \text{ and } \int_\Omega H(u^+,v^+)\,dx = 1\Big\}.$$

In the following, we establish a relationship between $\tilde{R}_Q$ and $\tilde{S}_H$ which is crucial to prove Theorem 1.2. In order to establish the result, we need some estimates obtained in [6, 17]. From [6], we know that there exist a minimizer for $S(\mathbb{R}^N)$, and for every minimizer U, there exist $x_0 \in \mathbb{R}^N$ and a constant sign monotone function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = u(|x - x_0|)$. Next we fix a radially nonnegative decreasing minimizer $U = U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta_p)^s U = U^{p_s^*-1}.$$

Lemma 4.3 (Brasco, Mosconi and Squassina [6]). *There exist $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{\frac{N-ps}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{N-ps}{p-1}}}, \quad \frac{U(\theta r)}{U(r)} \leq \frac{1}{2} \quad \text{for all } r \geq 1.$$

For every $\delta \geq \varepsilon > 0$. Let us set

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)}$$

and

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta}^p (t - U_\varepsilon(\theta\delta)) & \text{if } U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta), \\ t + U_\varepsilon(\delta)(m_{\varepsilon,\delta}^{p-1} - 1) & \text{if } t \geq U_\varepsilon(\delta), \end{cases}$$

as well as

$$G_{\varepsilon,\delta}(t) = \int_0^t g'_{\varepsilon,\delta}(\tau)^{\frac{1}{p}}\,d\tau = \begin{cases} 0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta}(t - U_\varepsilon(\theta\delta)) & \text{if } U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta), \\ t & \text{if } t \geq U_\varepsilon(\delta). \end{cases}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider now the radially symmetric nonincreasing function

$$u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r)),$$

which satisfies

$$u_{\varepsilon,\delta}(r) = \begin{cases} U_\varepsilon(r) & \text{if } r \leq \delta, \\ 0 & \text{if } r \geq \theta\delta. \end{cases}$$

Then $u_{\varepsilon,\delta} \in W_0^{s,p}(\Omega)$, for any $\delta < \theta^{-1}\delta_\Omega$ ($\delta_\Omega := \operatorname{dist}(0, \partial\Omega)$).

Lemma 4.4 (Brasco, Mosconi and Squassina [6], Chen, Mosconi and Squassina [17]). *There exists $C > 0$ such that, for any $0 < 2\varepsilon \leq \delta < \theta^{-1}\delta_\Omega$, there holds*

$$\int_{\mathbb{R}^{2N}} \frac{|u_{\varepsilon,\delta}(x) - u_{\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \leq S^{\frac{N}{ps}} + C\Big(\frac{\varepsilon}{\delta}\Big)^{\frac{N-ps}{p-1}}$$

and

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(x)^{p_s^*}\,dx \geq S^{\frac{N}{ps}} - C\Big(\frac{\varepsilon}{\delta}\Big)^{\frac{N}{p-1}}.$$

Moreover, for any $\beta > 0$, there exists C_β such that

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(x)^\beta\,dx \geq C_\beta \begin{cases} \varepsilon^{N - \frac{N-ps}{p}\beta} |\log \frac{\varepsilon}{\delta}| & \text{if } \beta = \frac{p_s^*}{p'}, \\ \varepsilon^{\frac{N-ps}{p(p-1)}\beta} \delta^{N - \frac{N-ps}{p-1}\beta} & \text{if } \beta < \frac{p_s^*}{p'}, \\ \varepsilon^{N - \frac{N-ps}{p}\beta} & \text{if } \beta > \frac{p_s^*}{p'}. \end{cases}$$

Lemma 4.5. *If $\mu_1 > 0$ and $N > p^2 s$, then*

$$\tilde{R}_Q < \frac{\tilde{S}_H}{p}. \tag{4.4}$$

Proof. Without loss of generality we may assume that $0 \in \Omega$. Let us consider the truncated function $u_{\varepsilon,\delta}$, and we let $u_\varepsilon = s_0 u_{\varepsilon,\delta}$, $v_\varepsilon = t_0 u_{\varepsilon,\delta}$. Using (1.9) and (4.2), we have

$$\frac{\frac{1}{p}\int_X \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{p}\int_X \frac{|v_\varepsilon(x)-v_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_\Omega Q(u_\varepsilon, v_\varepsilon)\,dx}{\left(\int_\Omega H(u_\varepsilon, v_\varepsilon)\,dx\right)^{\frac{p}{p_s^*}}} \le \frac{\frac{1}{p}\int_X \frac{|u_{\varepsilon,\delta}(x)-u_{\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{\mu_1}{p}\int_\Omega u_{\varepsilon,\delta}(x)^p\,dx}{m^{-1}\left(\int_\Omega u_{\varepsilon,\delta}(x)^{p_s^*}\,dx\right)^{\frac{p}{p_s^*}}} = \frac{m}{p} R_{\mu_1}(u_{\varepsilon,\delta}),$$

that is,

$$\tilde{R}_Q(u_\varepsilon, v_\varepsilon) \le \frac{m}{p} R_{\mu_1}(u_{\varepsilon,\delta}). \tag{4.5}$$

On the other hand, for $\mu_1 > 0$ and $N > p^2 s$, combining Lemma 4.4, we have

$$\begin{aligned}
R_{\mu_1}(u_{\varepsilon,\delta}) &= \frac{\int_X \frac{|u_{\varepsilon,\delta}(x)-u_{\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu_1\int_\Omega u_{\varepsilon,\delta}(x)^p\,dx}{\left(\int_\Omega u_{\varepsilon,\delta}(x)^{p_s^*}\,dx\right)^{\frac{p}{p_s^*}}} \\
&= \frac{\int_{\mathbb{R}^{2N}} \frac{|u_{\varepsilon,\delta}(x)-u_{\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu_1\int_\Omega u_{\varepsilon,\delta}(x)^p\,dx}{\left(\int_\Omega u_{\varepsilon,\delta}(x)^{p_s^*}\,dx\right)^{\frac{p}{p_s^*}}} \\
&\le \frac{S^{\frac{N}{ps}} + C(\frac{\varepsilon}{\delta})^{\frac{N-ps}{p-1}} - C_p\mu_1\varepsilon^{N-\frac{N-ps}{p}p}}{[S^{\frac{N}{ps}} - C(\frac{\varepsilon}{\delta})^{\frac{N}{p-1}}]^{\frac{N-ps}{N}}} \\
&= \frac{[S^{\frac{N}{ps}} + C(\frac{\varepsilon}{\delta})^{\frac{N-ps}{p-1}} - C_p\mu_1\varepsilon^{ps}][S^{\frac{N}{ps}} - C(\frac{\varepsilon}{\delta})^{\frac{N}{p-1}}]^{\frac{ps}{N}}}{S^{\frac{N}{ps}} - C(\frac{\varepsilon}{\delta})^{\frac{N}{p-1}}} \\
&\le \frac{[S^{\frac{N}{ps}} + C(\frac{\varepsilon}{\delta})^{\frac{N-ps}{p-1}} - C_p\mu_1\varepsilon^{ps}]S}{S^{\frac{N}{ps}} - C(\frac{\varepsilon}{\delta})^{\frac{N}{p-1}}}.
\end{aligned}$$

Since $N > p^2 s$, we have $\min\{\frac{N-ps}{p-1}, \frac{N}{p-1}\} > ps$, which implies that for small ε, we have

$$S^{\frac{N}{ps}} + C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-ps}{p-1}} - C_p\mu_1\varepsilon^{ps} < S^{\frac{N}{ps}} - C\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}.$$

Combining Lemma 4.2 and (4.5), we have that $\tilde{R}_Q(u_\varepsilon, v_\varepsilon) < \frac{\tilde{S}_H}{p}$ for a small ε, and hence (4.4) holds. □

5 A Concentration Compactness Principle Associated with the Fractional p-Laplacian System

In the next section we shall prove Theorem 1.2 using the fact that if $\tilde{R}_Q < \frac{\tilde{S}_H}{p}$, a minimizing sequence for $\tilde{R}_Q$ indeed converges, i.e., compactness occurs for $\tilde{R}_Q < \frac{\tilde{S}_H}{p}$. This will be possible due to a concentration compactness principle associated with the fractional p-Laplacian system that we are to prove. The method is based on the seminal paper of P. L. Lions [33], and here we provide a more general proof which is inspired by the results recently obtained by Bonder, Saintier and Silva in [4].

First, we define the fractional (s,p)-gradient of a function $u \in W_0^{s,p}(\mathbb{R}^N)$ as

$$|D^s u(x)|^p = \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^{N+ps}}\,dh.$$

In the following, we shall use the notation $|D^s u|$ denotes the fractional gradient of a function u, and the (s,p)-gradient is well defined a.e. in $\mathbb{R}^N$ and $|D^s u| \in L^p(\mathbb{R}^N)$.

Lemma 5.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary and let $\{(u_n, v_n)\} \subseteq W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$. Let us assume*

$$|D^s u_n|^p \rightharpoonup \mu, \quad |D^s v_n|^p \rightharpoonup \nu, \quad H(u_n, v_n) \rightharpoonup \sigma,$$

in the sense of measure, where μ, ν and σ are bounded nonnegative measure on $\mathbb{R}^N$. Then there exist at most a countable set J, a family of distinct points $\{x_j\}_{j\in J} \subseteq \bar{\Omega}$ and $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ and $\{\sigma_j\}_{j\in J} \subseteq (0, \infty)$ such that

$$\sigma = H(u, v) + \sum_{j\in J} \sigma_j \delta_{x_j}, \tag{5.1}$$

$$\mu \geq |D^s u|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \tag{5.2}$$

$$\nu \geq |D^s v|^p + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{5.3}$$

(Here δ_x indicates the Dirac mass at x.) Moreover, the following relations holds:

$$\tilde{S}_H \cdot (\sigma_j)^{\frac{p}{p_s^*}} \leq \mu_j + \nu_j.$$

Before proving Lemma 5.1, we need the following lemmas.

Lemma 5.2. *Given $\varepsilon > 0$, there exists C_ε such that*

$$|H(s + a, t + b) - H(s, t)| \leq \varepsilon(|s|^{p_s^*} + |t|^{p_s^*}) + C_\varepsilon(|a|^{p_s^*} + |b|^{p_s^*}).$$

Proof. Using the mean value theorem,

$$|H(s + a, t + b) - H(s, t)| = |\nabla H(s + \theta a, t + \theta b) \cdot (a, b)|, \tag{5.4}$$

for some $\theta \in (0, 1)$. Using the homogeneity properties (Remark 2.1), and the inequality

$$|A + \theta B|^{p_s^*-1} \leq C_{p_s^*-1}(|A|^{p_s^*-1} + |B|^{p_s^*-1}) \quad \text{for all } A, B \in \mathbb{R},$$

in (5.4), we obtain

$$\begin{aligned} |H(s + a, t + b) - H(s, t)| \leq C\big(&|s|^{p_s^*-1}|a| + |s|^{p_s^*-1}|b| + |t|^{p_s^*-1}|a| + |t|^{p_s^*-1}|b| \\ &+ |a|^{p_s^*} + |b|^{p_s^*} + |a|^{p_s^*-1}|b| + |b|^{p_s^*-1}|a|\big). \end{aligned}$$

Applying Young's inequality to the last inequality, we get our desired result. □

Lemma 5.3. *Let $\Omega \subseteq \mathbb{R}^N$ and suppose that σ is a measure on Ω, $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ a.e. on Ω and $\|u_n\|_{L^{p_s^*}(\Omega,d\sigma)}$, $\|v_n\|_{L^{p_s^*}(\Omega,d\sigma)}$ are bounded sequence. Then we have*

$$\int_\Omega H(u_n, v_n)\, d\sigma - \int_\Omega H(u_n - u, v_n - v)\, d\sigma = \int_\Omega H(u, v)\, d\sigma + o_n(1). \tag{5.5}$$

Proof. The idea of this proof was borrowed from [9, 19]. Let us fix some $\varepsilon > 0$ and let C_ε be as in Lemma 5.2. Let us define the function

$$g_n = |H(u_n, v_n) - H(u_n - u, v_n - v) - H(u, v)|.$$

Notice that

$$g_n \leq H(u, v) + \varepsilon(|u_n - u|^{p_s^*} + |v_n - v|^{p_s^*}) + C_\varepsilon(|u|^{p_s^*} + |v|^{p_s^*}),$$

where we have used Lemma 5.2 with $s = u_n - u$, $t = v_n - v$, $u = a$ and $v = b$. Define

$$w_{n,\varepsilon} = g_n - \varepsilon(|u_n - u|^{p_s^*} + |v_n - v|^{p_s^*}),$$

then Remark 2.1 (i) yields

$$|w_{n,\varepsilon}| = |g_n - \varepsilon(|u_n - u|^{p_s^*} + |v_n - v|^{p_s^*})| \leq (M_H + C_\varepsilon)(|u|^{p_s^*} + |v|^{p_s^*}) \in L^1(\Omega, d\sigma).$$

Since $u_n \to u$, $v_n \to v$ a.e. $x \in \Omega$, we have $w_{n,\varepsilon}(x) \to 0$ a.e. in Ω as $n \to \infty$. The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_\Omega w_{n,\varepsilon}\, d\sigma = 0.$$

Therefore, since $g_n = w_{n,\varepsilon} + \varepsilon(|u_n - u|^{p_s^*} + |v_n - v|^{p_s^*})$ and $\|u_n\|_{L^{p_s^*}(\Omega,d\sigma)}$, $\|v_n\|_{L^{p_s^*}(\Omega,d\sigma)}$ are bounded sequences, we have

$$\limsup_{n\to\infty} \int_\Omega g_n \, d\sigma \le \varepsilon \limsup_{n\to\infty} \int_\Omega (|u_n - u|^{p_s^*} + |v_n - v|^{p_s^*}) \, d\sigma \le C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (5.5) holds. □

In the following, we give two properties of the nonlocal (s, p)-gradient. The first one is a scaling property and the second one is a decay estimate for the nonlocal gradient of a function with compact support. Via these two techniques, we can give the proof of Lemma 5.1. Now, for $u \in W_0^{s,p}(\mathbb{R}^N)$, we define $u_{r,x_0}(x) = u(\frac{x-x_0}{r})$, where $r > 0$ and $x_0 \in \mathbb{R}^N$. By a straightforward computation, we have

$$|D^s u_{r,x_0}(x)|^p = \frac{1}{r^{ps}} \left| D^s u\left(\frac{x - x_0}{r}\right)\right|^p. \tag{5.6}$$

Lemma 5.4 (Bonder, Saintier and Silva [4]). *Let $v \in W^{1,\infty}(\mathbb{R}^N)$ be such that* $\operatorname{supp}(v) \subset B_1(0)$. *Then there exists a constant $C > 0$ depending on N, s, p and $\|v\|_{1,\infty}$ such that*

$$|D^s v(x)|^p \le C \min\{1, |x|^{-(N+ps)}\}.$$

Combining (5.6) and Lemma 5.4, we obtain the following:

Lemma 5.5. *Let $\phi \in W^{1,\infty}(\mathbb{R}^N)$ be such that* $\operatorname{supp}(\phi) \subset B_1(0)$, *and given $r > 0$ and $x_0 \in \mathbb{R}^N$, we define*

$$\phi_{r,x_0} = \phi\left(\frac{x - x_0}{r}\right).$$

Then

$$|D^s \phi_{r,x_0}(x)|^p \le C \min\{r^{-ps}, r^N |x - x_0|^{-(N+ps)}\},$$

where $C > 0$ depends on N, s, p and $\|\phi\|_{1,\infty}$.

Lemma 5.6 (Bonder, Saintier and Silva [4]). *Let $0 < s < 1 < p$ be such that $ps < N$ and let $p \le q < p_s^*$. Let $\omega \in L^\infty(\mathbb{R}^N)$ be such that there exists $\alpha > 0$ and $C > 0$ such that*

$$0 \le \omega(x) \le C|x|^{-\alpha}.$$

Then if $\alpha > sq - N\frac{q-p}{p}$, $W_0^{s,p}(\mathbb{R}^N) \subset\subset L^q(\omega\, dx; \mathbb{R}^N)$. That is, for any bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\mathbb{R}^N)$, there exist a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ and a function $u \in W_0^{s,p}(\mathbb{R}^N)$ such that $u_{n_j} \rightharpoonup u$ weakly in $W_0^{s,p}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |u_{n_j} - u(x)|^q \, \omega(x) \, dx \to 0 \quad \text{as } j \to \infty.$$

For the detailed proofs of Lemma 5.4, Lemma 5.5 and Lemma 5.6, we refer to [4] and the references therein. Furthermore, it is worth noting that in the case $p = q$, we need $\alpha > ps$ in Lemma 5.6. So if $\phi \in W^{1,\infty}(\mathbb{R}^N)$ has compact support, then $\omega = |D^s\phi|^p$ verifies the hypotheses of Lemma 5.6 with $p = q$.

Proof of Lemma 5.1. The idea of proving (5.1) is passing to the limit in the following identity resulting from Lemma 5.3:

$$\int_\Omega |\phi|^{p_s^*} H(u_n, v_n) \, dx = \int_\Omega |\phi|^{p_s^*} H(u, v) \, dx + \int_\Omega |\phi|^{p_s^*} H(u_n - u, v_n - v) \, dx + o_n(1). \tag{5.7}$$

Thus we consider the sequence $\{(\tilde{u}_n, \tilde{v}_n)\} \subseteq W$, where $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. In this particular case, from the assumption of Lemma 5.1, we know that $\tilde{u}_n \to 0$, $\tilde{v}_n \to 0$ a.e. on $\mathbb{R}^N$ and in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ $(1 \le q < p_s^*)$. Moreover, we assume that $|D^s\tilde{u}_n|^p \rightharpoonup \tilde{u}$, $|D^s\tilde{v}_n|^p \rightharpoonup \tilde{v}$ and $H(\tilde{u}_n, \tilde{v}_n) \rightharpoonup \tilde{\sigma}$ in the sense of measure. Firstly, we show that the measures $\tilde{\mu}$, $\tilde{v}$ and $\tilde{\sigma}$ verify a reverse Hölder inequality. In fact, given $\phi \in C_0^\infty(\mathbb{R}^N)$, we will prove that

$$\tilde{S}_H \left(\int_{\mathbb{R}^N} |\phi|^{p_s^*} \, d\tilde{\sigma} \right)^{\frac{p}{p_s^*}} \le \int_{\mathbb{R}^N} |\phi|^p (d\tilde{\mu} + d\tilde{v}). \tag{5.8}$$

From (4.2), we have

$$
\begin{aligned}
\Big[\int_{\mathbb{R}^N} |\phi|^{p_s^*} H(u_n - u, v_n - v)\,dx\Big]^{\frac{p}{p_s^*}} &= \Big[\int_{\Omega} |\phi|^{p_s^*} H(u_n - u, v_n - v)\,dx\Big]^{\frac{p}{p_s^*}} \\
&= \Big[\int_{\Omega} |\phi|^{p_s^*} H(\tilde{u}_n, \tilde{v}_n)\,dx\Big]^{\frac{p}{p_s^*}} \\
&\le \tilde{S}_H^{-1}\Big[\int_X \frac{|\phi(x)\tilde{u}_n(x) - \phi(y)\tilde{u}_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\
&\qquad + \int_X \frac{|\phi(x)\tilde{v}_n(x) - \phi(y)\tilde{v}_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big] \\
&= \tilde{S}_H^{-1}\Big[\int_{\mathbb{R}^{2N}} \frac{|\phi(x)\tilde{u}_n(x) - \phi(y)\tilde{u}_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\
&\qquad + \int_{\mathbb{R}^{2N}} \frac{|\phi(x)\tilde{v}_n(x) - \phi(y)\tilde{v}_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big] \\
&= \tilde{S}_H^{-1}\Big[\int_{\mathbb{R}^N} |D^s(\phi\tilde{u}_n)|^p\,dx + \int_{\mathbb{R}^N} |D^s(\phi\tilde{v}_n)|^p\,dx\Big].
\end{aligned}
\tag{5.9}
$$

For the right-hand side of (5.9), we consider the term $\int_{\mathbb{R}^N} |D^s(\phi\tilde{u}_n)|^p\,dx$. By using Minkowski's inequality, we have

$$
\begin{aligned}
\|D^s(\phi\tilde{u}_n)\|_p &= \Big(\int_{\mathbb{R}^N} |D^s(\phi\tilde{u}_n)|^p\,dx\Big)^{\frac{1}{p}} \\
&\le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(y)(\tilde{u}_n(x) - \tilde{u}_n(y))|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)(\phi(x) - \phi(y))|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(x)|^p \frac{|\tilde{u}_n(x+h) - \tilde{u}_n(x)|^p}{|h|^{N+ps}}\,dh\,dx\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\tilde{u}_n(x+h)|^p \frac{|\phi(x+h) - \phi(x)|^p}{|h|^{N+ps}}\,dh\,dx\Big)^{\frac{1}{p}}.
\end{aligned}
$$

By a simple change of variable, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\tilde{u}_n(x+h)|^p \frac{|\phi(x+h) - \phi(x)|^p}{|h|^{N+ps}}\,dh\,dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\tilde{u}_n(y)|^p \frac{|\phi(y) - \phi(y+\hat{h})|^p}{|\hat{h}|^{N+ps}}\,d\hat{h}\,dy \\
&= \int_{\mathbb{R}^N} |\tilde{u}_n(y)|^p |D^s\phi(y)|^p\,dy.
\end{aligned}
$$

Hence, we get

$$
\|D^s(\tilde{u}_n\phi)\|_p \le \Big(\int_{\mathbb{R}^N} |\phi(x)|^p |D^s\tilde{u}_n(x)|^p\,dx\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N} |\tilde{u}_n(x)|^p |D^s\phi(x)|^p\,dx\Big)^{\frac{1}{p}}.
$$

Now, from Lemma 5.4, we know that the weight $\omega(x) = |D^s\phi(x)|^p$ satisfies the hypotheses of Lemma 5.6, and hence $\tilde{u}_n \to 0$ strongly in $L^p(\omega)$. Therefore,

$$
\limsup_{n\to\infty} \|D^s(\phi\tilde{u}_n)\|_p \le \Big(\int_{\mathbb{R}^N} |\phi|^p\,d\tilde{\mu}\Big)^{\frac{1}{p}}.
$$

Similarly, we have

$$
\limsup_{n\to\infty} \|D^s(\phi\tilde{v}_n)\|_p \le \Big(\int_{\mathbb{R}^N} |\phi|^p\,d\tilde{\nu}\Big)^{\frac{1}{p}}.
$$

Using the above facts and passing to the limit in (5.9), we get that

$$\tilde{S}_H\Big(\int_{\mathbb{R}^N} |\phi|^{p_s^*}\,d\tilde{\sigma}\Big)^{\frac{p}{p_s^*}} \le \int_{\mathbb{R}^N} |\phi|^p (d\tilde{\mu} + d\tilde{\nu}).$$

This concludes the proof of the reverse Hölder inequality (5.8). Since $u_n = 0$ and $v_n = 0$ in Ω^c, it is clearly that $\operatorname{supp}(\tilde{\sigma}) \subseteq \bar{\Omega}$. Then, from [33, Lemma 1.2], there exists a countable set J, points $\{x_j\}_{j\in J} \subseteq \bar{\Omega}$ and $\{\sigma_j\}_{j\in J} \subseteq (0,\infty)$ such that

$$\tilde{\sigma} = \sum_{j\in J} \sigma_j \delta_{x_j}. \tag{5.10}$$

From (5.7) we have $\sigma = H(u,v) + \tilde{\sigma}$ and combining (5.10) we get that

$$\sigma = H(u,v) + \sum_{j\in J} \sigma_j \delta_{x_j}.$$

Now, to prove the relation between the weights σ_j and μ_j, ν_j, we take $\phi \in C_0^\infty(\mathbb{R}^N)$, with $0 \le \phi \le 1$, $\phi(0) = 1$ and $\operatorname{supp}\phi = B(0,1)$. For given $\varepsilon > 0$ and $x_j \in \bar{\Omega}$, we consider the rescaled functions

$$\phi_{\varepsilon,x_i}(x) = \phi\Big(\frac{x - x_j}{\varepsilon}\Big).$$

Arguing as in the proof of inequality (5.9) again, we have

$$\tilde{S}_H\Big[\int_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}|^{p_s^*} H(u_n, v_n)\,dx\Big]^{\frac{p}{p_s^*}} \le \int_{\mathbb{R}^N} |D^s(\phi_{\varepsilon,x_j} u_n)|^p\,dx + \int_{\mathbb{R}^N} |D^s(\phi_{\varepsilon,x_j} v_n)|^p\,dx. \tag{5.11}$$

As before, we have

$$\|D^s(\phi_{\varepsilon,x_j} u_n)\|_p \le \Big(\int_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}(x)|^p |D^s u_n(x)|^p\,dx\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N} |u_n(x)|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

Recall that from Lemma 5.5 we obtain that

$$|D^s \phi_{\varepsilon,x_j}(x)|^p \le C\min\{\varepsilon^{-sp}, \varepsilon^N |x - x_j|^{-(N+ps)}\}. \tag{5.12}$$

Now, (5.12) implies that $|D^s \phi_{\varepsilon,x_j}|^p$ satisfies the hypotheses of Lemma 5.6. Moreover, since $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n(x)|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx = \int_{\mathbb{R}^N} |u(x)|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx. \tag{5.13}$$

Now we check that

$$\int_{\mathbb{R}^N} |u(x)|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.14}$$

Utilizing (5.12) again, we get

$$\int_{\mathbb{R}^N} |u(x)|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx \le C\Big(\varepsilon^{-ps} \int_{|x-x_j|<\varepsilon} |u|^p\,dx + \varepsilon^N \int_{|x-x_j|\ge\varepsilon} \frac{|u|^p}{|x - x_j|^{N+ps}}\,dx\Big) = C(I + II).$$

The first one is the easiest one,

$$I \lesssim \varepsilon^{-ps}\Big(\int_{|x-x_j|<\varepsilon} |u|^{p_s^*}\,dx\Big)^{\frac{p}{p_s^*}} \cdot |B_\varepsilon|^{\frac{ps}{N}} \lesssim \Big(\int_{|x-x_j|<\varepsilon} |u|^{p_s^*}\,dx\Big)^{\frac{p}{p_s^*}}.$$

Since $u \in L^{p_s^*}(\mathbb{R}^N)$, thus I goes to zero as $\varepsilon \to 0$.

For the second term we proceed as follows:

$$\begin{aligned}
II &= \sum_{k=0}^{\infty} \varepsilon^N \int\limits_{2^k\varepsilon \le |x-x_j| \le 2^{k+1}\varepsilon} \frac{|u|^p}{|x-x_j|^{N+ps}}\,dx \\
&\le \sum_{k=0}^{\infty} \frac{1}{2^{k(N+ps)}\varepsilon^{ps}} \int\limits_{|x-x_j|\le 2^{k+1}\varepsilon} |u|^p\,dx \\
&\le \sum_{k=0}^{\infty} \frac{1}{2^{k(N+ps)}\varepsilon^{ps}} \Big(\int\limits_{|x-x_j|\le 2^{k+1}\varepsilon} |u|^{p_s^*}\,dx \Big)^{\frac{p}{p_s^*}} |B_{2^{k+1}\varepsilon}|^{\frac{ps}{N}} \\
&= c \sum_{k=0}^{\infty} \frac{1}{2^{kN}} \Big(\int\limits_{|x-x_j|\le 2^{k+1}\varepsilon} |u|^{p_s^*}\,dx \Big)^{\frac{p}{p_s^*}},
\end{aligned}$$

where c depends only on N, s, p. Now, given $\delta > 0$, take $k_0 \in \mathbb{N}$ such that $c \sum_{k=k_0+1}^{\infty} 2^{-Nk} < \delta$. So

$$\begin{aligned}
II &\le \|u\|_{p_s^*}^p \delta + c \sum_{k=0}^{k_0} \frac{1}{2^{Nk}} \Big(\int\limits_{|x-x_j|<2^{k_0+1}\varepsilon} |u|^{p_s^*}\,dx \Big)^{\frac{p}{p_s^*}} \\
&= \|u\|_{p_s^*}^p \delta + c(s,p,N,k_0) \Big(\int\limits_{|x-x_j|<2^{k_0+1}\varepsilon} |u|^{p_s^*}\,dx \Big)^{\frac{p}{p_s^*}}.
\end{aligned}$$

Therefore, we obtain that $\limsup_{\varepsilon\to 0} II \le \delta \|u\|_{p_s^*}^p$, for any $\delta > 0$. This concluded the proof of (5.14).

Since $|D^s u_n|^p \rightharpoonup \mu$, combined with (5.12)–(5.14), yields

$$\lim_{n\to\infty} \|D^s(\phi_{\varepsilon,x_j} u_n)\|_p \le \Big(\int\limits_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}(x)|^p\,d\mu \Big)^{\frac{1}{p}} + \Big(\int\limits_{\mathbb{R}^N} |u|^p |D^s \phi_{\varepsilon,x_j}(x)|^p\,dx \Big)^{\frac{1}{p}} \to \mu_j^{\frac{1}{p}} \quad \text{as } \varepsilon \to 0.$$

Thus,

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |D^s(\phi_{\varepsilon,x_j} u_n)|^p\,dx \le \mu_j \quad \text{as } \varepsilon \to 0.$$

Similarly, we have

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |D^s(\phi_{\varepsilon,x_j} v_n)|^p\,dx \le \nu_j \quad \text{as } \varepsilon \to 0.$$

For the left-hand side of (5.11), (5.1) implies that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}|^{p_s^*} H(u_n, v_n)\,dx = \int\limits_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}|^{p_s^*} H(u,v)\,dx + \sigma_j \quad \text{as } \varepsilon \to 0.$$

Furthermore, it is easy to verify that

$$\int\limits_{\mathbb{R}^N} |\phi_{\varepsilon,x_j}|^{p_s^*} H(u,v)\,dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Using the above fact and passing to the limit in (5.11), we obtain that

$$\tilde{S}_H \sigma_j^{\frac{p}{p_s^*}} \le \mu_j + \nu_j \quad \text{for all } j \in J.$$

Finally, we have

$$\mu \ge \sum_{j\in J} \mu_j \delta_{x_j}, \quad \nu \ge \sum_{j\in J} \nu_j \delta_{x_j}.$$

By the Fatou Lemma, we obtain that $\mu \ge |D^s u|^p$ and $\nu \ge |D^s v|^p$. Since $|D^s u|^p$ and $|D^s v|^p$ are orthogonal to $\sum_{j\in J} \mu_j \delta_{x_j}$ and $\sum_{j\in J} \nu_j \delta_{x_j}$, respectively, it follows that (5.2) and (5.3) hold. □

6 Proof of Theorem 1.2

Following the subcritical case (Theorem 1.1), it only remains to show that the infimum

$$\tilde{R}_Q := \min_{(u,v)\in\mathcal{H}} I(u,v)$$

is attained. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a minimizing sequence for $\tilde{R}_Q$, i.e.,

$$\begin{aligned}&\frac{1}{p}\bigg[\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg] - \int_\Omega Q(u_n,v_n)\,dx\\ &\quad= \frac{1}{p}\bigg(\int_{\mathbb{R}^N} |D^s u_n|^p\,dx + \int_{\mathbb{R}^N} |D^s v_n|^p\,dx\bigg) - \int_\Omega Q(u_n,v_n)\,dx = \tilde{R}_Q + o_n(1).\end{aligned}$$

As in the proof of Theorem 1.1, $\|(u_n, v_n)\|$ is bounded. Let us suppose that $u_n \rightharpoonup u,\ v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and assume the assumptions of Lemma 4.2. Moreover, since H nonnegative, by the above convergences we have $0 \le \int_\Omega H(u,v)\,dx \le 1$. On the other hand, by Lemma 5.1, we have

$$\begin{aligned}I(u,v) &= \frac{1}{p}\bigg[\int_{\mathbb{R}^N} |D^s u|^p\,dx + \int_{\mathbb{R}^N} |D^s v|^p\,dx\bigg] - \int_\Omega Q(u,v)\,dx\\ &= \frac{1}{p}\bigg[\int_{\mathbb{R}^N} |D^s u_n|^p\,dx + \int_{\mathbb{R}^N} |D^s v_n|^p\,dx\bigg] - \frac{1}{p}\bigg[\int_{\mathbb{R}^N} |D^s u_n|^p\,dx - \int_{\mathbb{R}^N} |D^s u|^p\,dx\\ &\qquad + \int_{\mathbb{R}^N} |D^s v_n|^p\,dx - \int_{\mathbb{R}^N} |D^s v|^p\,dx\bigg] - \int_\Omega Q(u_n,v_n)\,dx\\ &\le \tilde{R}_Q - \frac{1}{p}\bigg[\int_{\mathbb{R}^N} d\mu - \int_{\mathbb{R}^N} |D^s u|^p\,dx + \int_{\mathbb{R}^N} d\nu - \int_{\mathbb{R}^N} |D^s v|^p\,dx\bigg] + o_n(1)\\ &\le \tilde{R}_Q - \frac{1}{p}\bigg(\sum_{j\in J}\mu_j + \sum_{j\in J}\nu_j\bigg) + o_n(1)\\ &\le \tilde{R}_Q - \frac{\tilde{S}_H}{p}\sum_{j\in J}\sigma_j^{\frac{p}{p_s^*}}.\end{aligned} \tag{6.1}$$

In addition, by (5.1), we have

$$1 = \int_\Omega d\sigma = \int_\Omega H(u,v)\,dx + \sum_{j\in J}\sigma_j.$$

Case I. We have $\sigma_j = 0$ for all $j \in J$. In this case, we get that $I(u,v) \le \tilde{R}_Q$ and $\int_\Omega H(u,v)\,dx = 1$, i.e., $(u,v) \in \mathcal{H}$ is the wanted minimum point.

Case II. Suppose on the contrary, that there is at least one $\sigma_j > 0$ for some $j \in J$. By Lemma 4.5, $\tilde{R}_Q < \frac{\tilde{S}_H}{p}$ and hence (6.1) yields

$$I(u,v) < \tilde{R}_Q - \tilde{R}_Q\sum_{j\in J}\sigma_j^{\frac{p}{p_s^*}} = \tilde{R}_Q\bigg(1 - \sum_{j\in J}\sigma_j^{\frac{p}{p_s^*}}\bigg). \tag{6.2}$$

Since $0 < \sigma_j < 1$, the following holds:

$$1 - \sum_{j\in J}\sigma_j^{\frac{p}{p_s^*}} \le \bigg(1 - \sum_{j\in J}\sigma_j\bigg)^{\frac{p}{p_s^*}} = \bigg(\int_\Omega H(u,v)\,dx\bigg)^{\frac{p}{p_s^*}}.$$

The last inequality in (6.2) gives

$$I(u,v) < \tilde{R}_Q\bigg(\int_\Omega H(u,v)\,dx\bigg)^{\frac{p}{p_s^*}}. \tag{6.3}$$

Since $I(u,v) \geq 0$, we have $A = \int_\Omega H(u,v)\,dx > 0$. Therefore $(u,v) \neq (0,0)$. Taking $\tilde{u} = \frac{u}{A^{1/p_s^*}}$ and $\tilde{v} = \frac{v}{A^{1/p_s^*}}$, inequality (6.3) is reduced to

$$I(\tilde{u},\tilde{v}) = \frac{1}{A^{\frac{p}{p_s^*}}} I(u,v) < \tilde{R}_Q\bigg(\frac{1}{A}\int_\Omega H(u,v)\,dx\bigg)^{\frac{p}{p_s^*}} = \tilde{R}_Q,$$

which contradicts the definition of $\tilde{R}_Q$. Hence, $\sigma_j = 0$ for all $j \in J$ and the proof of Theorem 1.2 is now completed.

Acknowledgment: The authors wish to thank the referee for his many useful comments which have improved the exposition of the paper.

Funding: The first author was partially supported by a collaboration grant from the Simons Foundation, Grant/Award Number: 519099.

References

[1] C. O. Alves, D. C. de Morais Filho and M. A. S. Souto, On systems of elliptic equations involving subcritical or critical Sobolev exponents, *Nonlinear Anal.* **42** (2000), no. 5, 771–787.
[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
[3] A. Ambrosetti and M. Struwe, A note on the problem $-\Delta u = \lambda u + u|u|^{2^*-2}$, *Manuscripta Math.* **54** (1986), no. 4, 373–379.
[4] J. F. Bonder, N. Saintier and A. Silva, The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional Brezis–Nirenberg problem, *NoDEA Nonlinear Differential Equations Appl.* **25** (2018), no. 6, Paper No. 52.
[5] L. Brasco and E. Lindgren, Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case, *Adv. Math.* **304** (2017), 300–354.
[6] L. Brasco, S. Mosconi and M. Squassina, Optimal decay of extremals for the fractional Sobolev inequality, *Calc. Var. Partial Differential Equations* **55** (2016), no. 2, Article No. 23.
[7] L. Brasco and E. Parini, The second eigenvalue of the fractional p-Laplacian, *Adv. Calc. Var.* **9** (2016), no. 4, 323–355.
[8] L. Brasco, E. Parini and M. Squassina, Stability of variational eigenvalues for the fractional p-Laplacian, *Discrete Contin. Dyn. Syst.* **36** (2016), no. 4, 1813–1845.
[9] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983), no. 3, 486–490.
[10] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), no. 4, 437–477.
[11] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), no. 7–9, 1245–1260.
[12] A. Capozzi, D. Fortunato and G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2** (1985), no. 6, 463–470.
[13] W. Chen and Y. Gui, Multiple solutions for a fractional p-Kirchhoff problem with Hardy nonlinearity, *Nonlinear Anal.* **188** (2019), 316–338.
[14] W. Chen and C. Li, Maximum principles for the fractional p-Laplacian and symmetry of solutions, *Adv. Math.* **335** (2018), 735–758.
[15] W. Chen, C. Li and S. Qi, A Hopf lemma and regularity for fractional p-Laplacian, preprint (2018), https://arxiv.org/abs/1805.02776v2.
[16] W. Chen, C. Li and J. Zhu, Fractional equations with indefinite nonlinearities, *Discrete Contin. Dyn. Syst.* **39** (2019), no. 3, 1257–1268.
[17] W. Chen, S. Mosconi and M. Squassina, Nonlocal problems with critical Hardy nonlinearity, *J. Funct. Anal.* **275** (2018), no. 11, 3065–3114.
[18] W. Chen and J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, *J. Differential Equations* **260** (2016), no. 5, 4758–4785.
[19] D. C. de Morais Filho and M. A. S. Souto, Systems of p-Laplacean equations involving homogeneous nonlinearities with critical Sobolev exponent degrees, *Comm. Partial Differential Equations* **24** (1999), no. 7–8, 1537–1553.
[20] A. Di Castro, T. Kuusi and G. Palatucci, Nonlocal Harnack inequalities, *J. Funct. Anal.* **267** (2014), no. 6, 1807–1836.

[21] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), no. 5, 1279–1299.
[22] G. Franzina and G. Palatucci, Fractional p-eigenvalues, *Riv. Math. Univ. Parma (N.S.)* **5** (2014), no. 2, 373–386.
[23] J. P. García Azorero and I. Peral Alonso, Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues, *Comm. Partial Differential Equations* **12** (1987), no. 12, 1389–1430.
[24] A. Iannizzotto, S. Liu, K. Perera and M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, *Adv. Calc. Var.* **9** (2016), no. 2, 101–125.
[25] A. Iannizzotto, S. Mosconi and M. Squassina, Global Hölder regularity for the fractional p-Laplacian, *Rev. Mat. Iberoam.* **32** (2016), no. 4, 1353–1392.
[26] T. Kuusi, G. Mingione and Y. Sire, Nonlocal equations with measure data, *Comm. Math. Phys.* **337** (2015), no. 3, 1317–1368.
[27] N. Lam and G. Lu, Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$, *J. Funct. Anal.* **262** (2012), no. 3, 1132–1165.
[28] N. Lam and G. Lu, Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications, *Adv. Math.* **231** (2012), no. 6, 3259–3287.
[29] N. Lam and G. Lu, N-Laplacian equations in $\mathbb{R}^N$ with subcritical and critical growth without the Ambrosetti–Rabinowitz condition, *Adv. Nonlinear Stud.* **13** (2013), no. 2, 289–308.
[30] N. Lam and G. Lu, Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition, *J. Geom. Anal.* **24** (2014), no. 1, 118–143.
[31] E. H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, *Ann. of Math. (2)* **118** (1983), no. 2, 349–374.
[32] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differential Equations* **49** (2014), no. 1–2, 795–826.
[33] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana* **1** (1985), no. 1, 145–201.
[34] G. Lu and Q. Yang, Paneitz operators on hyperbolic spaces and high order Hardy–Sobolev–Maz'ya inequalities on half spaces, *Amer. J. Math.* **141** (2019), no. 6, 1777–1816.
[35] S. A. Marano and S. J. N. Mosconi, Asymptotics for optimizers of the fractional Hardy–Sobolev inequality, *Commun. Contemp. Math.* **21** (2019), no. 5, Article ID 1850028.
[36] S. Mosconi, K. Perera, M. Squassina and Y. Yang, The Brezis-Nirenberg problem for the fractional p-Laplacian, *Calc. Var. Partial Differential Equations* **55** (2016), no. 4, Article ID 105.
[37] S. Mosconi and M. Squassina, Nonlocal problems at nearly critical growth, *Nonlinear Anal.* **136** (2016), 84–101.
[38] J. Moser, A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.* **20** (1970/71), 1077–1092.
[39] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.
[40] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.* **367** (2015), no. 1, 67–102.
[41] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.* **17** (1967), 473–483.
[42] L. Wu and W. Chen, The sliding methods for the fractional p-Laplacian, *Adv. Math.* **361** (2020), Article ID 106933.
[43] Y. Yang and K. Perera, N-Laplacian problems with critical Trudinger–Moser nonlinearities, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **16** (2016), no. 4, 1123–1138.