DE GRUYTER Adv. Nonlinear Stud. 2020; 20(3): 511–538

Research Article

Lin Li, Patrizia Pucci* and Xianhua Tang

Ground State Solutions for the Nonlinear Schrödinger–Bopp–Podolsky System with Critical Sobolev Exponent

https://doi.org/10.1515/ans-2020-2097
Received February 24, 2020; revised May 6, 2020; accepted May 8, 2020

Abstract: In this paper, we study the existence of ground state solutions for the nonlinear Schrödinger–Bopp–Podolsky system with critical Sobolev exponent

$$\begin{cases} -\Delta u + V(x)u + q^2\phi u = \mu|u|^{p-1}u + |u|^4 u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$ is a parameter and $2 < p < 5$. Under certain assumptions on V, we prove the existence of a nontrivial ground state solution, using the method of the Pohozaev–Nehari manifold, the arguments of Brézis–Nirenberg, the monotonicity trick and a global compactness lemma.

Keywords: Schrödinger–Bopp–Podolsky System, Pohozaev's Identity, Concentration-Compactness Principle, Ground State Solution

MSC 2010: 35J50, 35J48, 35Q60

Communicated by: David Ruiz

1 Introduction

In the paper [14], d'Avenia and Siciliano have attracted their attention on a new kind of elliptic system, called Schrödinger–Bopp–Podolsky system, which, to the best of our knowledge, was never been considered before in the mathematical literature, although the problem was known among the physicists. The Schrödinger–Bopp–Podolsky system has the following form:

$$\begin{cases} -\Delta u + V(x)u + q^2\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, $\omega, a > 0$, $q \neq 0$. Such a system appears when we couple a Schrödinger field $\psi = \psi(t, x)$ with its electromagnetic field in the Bopp–Podolsky electromagnetic theory, and, in particular, in the electrostatic case for standing waves $\psi(t, x) = e^{i\omega t}u(x)$. We refer to the paper of d'Avenia and Siciliano [14] for more details.

***Corresponding author: Patrizia Pucci** Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy, e-mail: patrizia.pucci@unipg.it. https://orcid.org/0000-0001-7242-8485
Lin Li School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan Province, P. R. China; and School of Mathematics and Statistics & Chongqing Key Laboratory of Social Economy and Applied Statistics, Chongqing Technology and Business University, Chongqing 400067, P. R. China, e-mail: linli@ctbu.edu.cn, lilin420@gmail.com. https://orcid.org/0000-0002-9565-5880
Xianhua Tang School of Mathematics and Statistics, Central South University, Changsha, 410083, Hunan Province, P. R. China, e-mail: tangxh@mail.csu.edu.cn

The existence of standing waves for scalar fields in dimension 3 has extensively been studied by many authors. The most important scalar fields equation is the Schrödinger equation. In this paper we want to investigate about the existence of nonlinear Schrödinger fields interacting with the electromagnetic field $(\mathbf{E}, \mathbf{H})$. Since $\mathbf{E}, \mathbf{H}$ are not assigned, we have to study a system of equations whose unknowns are the Schrödinger field $\psi(x, t)$ and the gauge potentials $\mathbf{A} = \mathbf{A}(x, t)$, $\phi = \phi(x, t)$ related to the electromagnetic field. In order to construct such a system we shall describe, as usual, the interaction between ψ and $\mathbf{E}, \mathbf{H}$ by using the so called gauge covariant derivatives. The Bopp–Podolsky theory is a second order gauge theory for the electromagnetic field. As the Mie theory [21] and its generalizations given by Born and Infeld [4–7], it was introduced to solve the so called infinity problem that appears in the classical Maxwell theory. The Bopp–Podolsky theory is developed by Bopp [3] and Podolsky [22], independently. We also shall investigate the case in which $\mathbf{A}$ and ϕ do not depend on the time t and $\psi(x, t)$ is a standing wave. In this situation we can assume $\mathbf{A} = 0$ and we are reduced to study system (1.1) (see Section 2).

By the well-known Gauss law (or Poisson's equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $\mathcal{G}(x - x_0)$, where

$$\mathcal{G}(x) = \frac{1}{|x|},$$

and the electrostatic energy is

$$\mathcal{E}_{\mathrm{M}}(\mathcal{G}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{G}|^2 \, dx = \infty.$$

Thus, equation (1.2) is replaced by

$$-\operatorname{div}\Big(\frac{\nabla\phi}{\sqrt{1 - |\nabla\phi|^2}}\Big) = \rho \quad \text{in } \mathbb{R}^3$$

in the Born–Infeld theory and by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Podolsky one. In both cases, if $\rho = 4\pi\delta_{x_0}$, we are able to write explicitly the solutions of the respective equations and to see that their energy is finite. In particular, the fundamental solution of the equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0}$$

is $\mathcal{K}(x - x_0)$, where

$$\mathcal{K}(x) := \frac{1 - e^{-\frac{|x|}{a}}}{|x|}, \tag{1.3}$$

which presents no singularities at x_0, since

$$\lim_{x \to x_0} \mathcal{K}(x - x_0) = \frac{1}{a}.$$

Furthermore, its energy is

$$\mathcal{E}_{\mathrm{BP}}(\mathcal{K}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{K}|^2 \, dx + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{K}|^2 \, dx < \infty.$$

We refer to [14] for more details.

Moreover, the Bopp–Podolsky theory may be interpreted as an effective theory for short distances (see [15]) and for large distances it is experimentally indistinguishable from the Maxwell one. Thus, the Bopp–Podolsky parameter $a > 0$, which has dimension of the inverse of mass, can be interpreted as a cut-off distance or can be linked to an effective radius for the electron. For more physical features we refer the interested reader to the recent papers [2, 8, 9, 12, 13] and to the references therein.

In the novel paper [14], d'Avenia and Siciliano deal with the following special form of (1.1):

$$\begin{cases} -\Delta u + V_\infty u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

and study the existence, nonexistence and the behavior of the solution as $a \to 0$. Again the solutions converge to the solution of the "limit" problem with $a = 0$. However, d'Avenia and Scilliano in [14] do not cover the critical case. As far as we know this paper is the first attempt to solve this delicate challenging problem in which lack of compactness appears together with the lack of translation invariance. To overcome these difficulties, we have to use a global compactness lemma as well as introduce new inequalities and techniques. In particular, the main results of the present paper extend [14] to the critical case.

This paper is concerned with the existence of ground state solutions for the following Schrödinger–Bopp–Podolsky system with critical Sobolev exponent:

$$\begin{cases} -\Delta u + V(x)u + q^2\phi u = \mu|u|^{p-1}u + |u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\mu > 0$ is a parameter, $2 < p < 5$ and the potential V satisfies the following conditions:

(V_1) $V \in C^1(\mathbb{R}^3)$, $(\cdot, \nabla V) \in L^\infty(\mathbb{R}^3)$ and

$$2V(x) + (x, \nabla V(x)) \geq 0, \quad x \in \mathbb{R}^3,$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbb{R}^3$.

(V_2) For all $x \in \mathbb{R}^3$ it results $V(x) \leq \liminf_{|\xi|\to\infty} V(\xi) = V_\infty \in \mathbb{R}^+$ and the inequality is strict in a subset of positive Lebesgue measure.

(V_3) There exists a positive number α_0 such that

$$\alpha_0 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2\,dx}{\int_{\mathbb{R}^3} |u|^2\,dx} > 0.$$

From now on we assume, without further mentioning, that (V_1)–(V_3) hold. Then the main results of the paper are stated as follows.

Theorem 1.1. *If $p \in (3, 5)$, system* (1.4) *has a ground state solution for any $\mu > 0$, while if $p \in (2, 3]$, system* (1.4) *possesses a ground state solution for $\mu > 0$ large enough.*

Let us give the main ideas under the proof of Theorem 1.1. The existence of ground state solutions for the Schrödinger–Bopp–Podolsky system (1.4), namely of the couples (u, ϕ) which solve (1.4), is obtained by minimizing the action functional associated to (1.4) among all possible solutions. Motivated by [1, 10, 20, 26], we choose the usual Sobolev space $H^1(\mathbb{R}^3)$ to prove the existence of ground state solutions for the "limit" problem

$$\begin{cases} -\Delta u + V_\infty u + q^2\phi u = \mu|u|^{p-1}u + |u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

Then we look for a minimizer of the reduced functional restricted to a suitable manifold $\mathcal{M}_\mu^\infty$, which was introduced by Ruiz in [24] when $a = 0$. Such a manifold consists of the linear combination of the Pohozaev manifold and the Nehari manifold and is called the Pohozaev–Nehari manifold. It has two perfect characteristics: it is a natural constraint for the reduced functional and it contains every solution of problem (1.5). We shall use the concentration-compactness lemma to establish the following result.

Theorem 1.2. *When $p \in (3, 5)$, problem* (1.5) *has a ground state solution for any $\mu > 0$, and when $p \in (2, 3]$, problem* (1.5) *has a ground state solution for $\mu > 0$ large enough.*

Next, in order to use the monotonicity trick of [17], we introduce a family of functionals $\mathcal{I}_\lambda$, which satisfies all the assumptions of Theorem 1.2 and which possesses a bounded $(PS)_{c_\lambda}$ sequence. A global compactness lemma, applied to the functional $\mathcal{I}_{V,\lambda}$ and its limit functional $\mathcal{I}_\lambda^\infty$, allows us to prove that the Palais-Smale condition $(PS)_{c_\lambda}$ holds. Finally, choosing a sequence $(\lambda_n)_n$ approaching 1 at infinity, we show that $(u_{\lambda_n})_n$ is a bounded $(PS)_{c_1}$ sequence for $\mathcal{J}_V$. An application of the global compactness lemma completes the proof of Theorem 1.1.

The paper is organized as follows. In Section 2, we present some preliminaries results. In Section 3, we prove Theorem 1.2. Finally, Section 4 is devoted to the proof of the main Theorem 1.1.

Last, we will mention the very recently paper by Chen and Tang [11]. They also study this type of system, but our method is different with theirs.

2 Variational Setting

We start with some preliminary basic results. Let us consider the nonlinear Schrödinger Lagrangian density

$$\mathcal{L}_{\mathrm{Sc}} = i\hbar\bar{\psi}\partial_t\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + 2F(\psi),$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$, $\hbar, m > 0$. Let $(\phi, \mathbf{A})$ be the gauge potential of the electromagnetic field $(\mathbf{E}, \mathbf{H})$, namely $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{H} = \nabla\times\mathbf{A}.$$

The coupling of the field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$ via the minimal coupling rule, namely the study of the interaction between ψ and its own electromagnetic field, can be obtained replacing in $\mathcal{L}_{\mathrm{Sc}}$ the derivatives ∂_t and ∇ with the covariant ones

$$D_t = \partial_t + \frac{iq}{\hbar}\phi, \quad \mathbf{D} = \nabla - \frac{iq}{\hbar c}\mathbf{A},$$

respectively. Here q is a coupling constant. This leads to consider

$$\begin{aligned}\mathcal{L}_{\mathrm{CSc}} &= i\hbar\overline{\psi}D_t\psi - \frac{\hbar^2}{2m}|\mathbf{D}\psi|^2 + 2F(\psi)\\ &= i\hbar\overline{\psi}\Big(\partial_t + \frac{iq}{\hbar}\phi\Big)\psi - \frac{\hbar^2}{2m}\Big|\Big(\nabla - \frac{iq}{\hbar c}\mathbf{A}\Big)\psi\Big|^2 + 2F(\psi).\end{aligned}$$

Now, to get the total Lagrangian density, we have to add to $\mathcal{L}_{\mathrm{CSc}}$ the Lagrangian density of the electromagnetic field. The Bopp–Podolsky Lagrangian density (see [22, formula (3.9)]) is

$$\begin{aligned}\mathcal{L}_{\mathrm{BP}} &= \frac{1}{8\pi}\Big\{|\mathbf{E}|^2 - |\mathbf{H}|^2 + a^2\Big[(\operatorname{div}\mathbf{E})^2 - \Big|\nabla\times\mathbf{H} - \frac{1}{c}\partial_t\mathbf{E}\Big|^2\Big]\Big\}\\ &= \frac{1}{8\pi}\Big\{|\nabla\phi + \frac{1}{c}\partial_t\mathbf{A}|^2 - |\nabla\times\mathbf{A}|^2 + a^2\Big[\Big(\Delta\phi + \frac{1}{c}\operatorname{div}\partial_t\mathbf{A}\Big)^2 - \Big|\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\partial_t(\nabla\phi + \frac{1}{c}\partial_t\mathbf{A})\Big|^2\Big]\Big\}.\end{aligned}$$

Thus the total action is

$$\mathcal{S}(\psi, \phi, \mathbf{A}) = \int_{\mathbb{R}^3} \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t,$$

where $\mathcal{L} := \mathcal{L}_{\mathrm{CSc}} + \mathcal{L}_{\mathrm{BP}}$ is the total Lagrangian density. We refer the interested readers to [14] for a detailed deduction of (1.4).

Thanks to assumptions (V_2) and (V_3), the Sobolev space $H^1(\mathbb{R}^3)$ can be equipped with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u\nabla v + V(x)uv)\,\mathrm{d}x$$

and the corresponding norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x\Big)^{\frac{1}{2}}.$$

Actually, (V_2) and (V_3) yield that the above norm is equivalent to the usual norm $\|\cdot\|_{H^1}$. Indeed, from (V_3), similar to [17, proof of Lemma 3.4], there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,dx \geq \frac{\alpha_0}{2} \int_{\mathbb{R}^3} |u|^2\,dx + C \int_{\mathbb{R}^3} |\nabla u|^2\,dx,$$

while (V_2) implies that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,dx \leq \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} V_\infty u^2\,dx.$$

The above two estimates imply that $\|\cdot\|$ is an equivalent norm on $H^1(\mathbb{R}^3)$.

It is well known that $H^1(\mathbb{R}^3)$ is continuously embedded into $L^s(\mathbb{R}^3)$ when $2 \leq s \leq 6$, and there exists the best constant $\mathcal{S} > 0$ such that

$$\mathcal{S} = \inf_{u \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^3} |u|^6\,dx)^{\frac{1}{3}}}, \tag{2.1}$$

where

$$\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}.$$

Let $\mathcal{D}$ be the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} (\nabla\varphi\nabla\psi + a^2\Delta\varphi\Delta\psi)\,dx.$$

Then $\mathcal{D}$ is a Hilbert space continuously embedded into $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. In the sequel, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, $p \geq 1$, the letter c_i $(i = 1, 2, \dots)$ or C_i $(i = 1, 2, \dots)$ denote just positive constants. It is interesting to point out the following properties.

Lemma 2.1 ([14, Lemma 3.1]). *The space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$.*

The next lemma gives a useful characterization of the space $\mathcal{D}$.

Lemma 2.2 ([14, Lemma 3.2]). *The space $C_c^\infty(\mathbb{R}^3)$ is dense in*

$$\mathcal{A} := \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$

normed by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

For every fixed $u \in H^1(\mathbb{R}^3)$, the Riesz theorem implies that there exists a unique solution $\phi_u \in \mathcal{D}$ of the second equation in (1.4). To write explicitly such a solution (see also [22, formula (2.6)]), we take $\mathcal{K}$ as defined by (1.3). Then the next fundamental properties hold.

Lemma 2.3 ([14, Lemma 3.3]). *For all $y \in \mathbb{R}^3$ the function $\mathcal{K}(\cdot - y)$ solves*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y$$

in the sense of distributions. Moreover,
(i) if $g \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and the map $\mathbb{R}^3 \ni y \mapsto g(y)/|x - y|$ is summable in $\mathbb{R}^3$ for a.e. $x \in \mathbb{R}^3$, then $\mathcal{K} * g \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$,
(ii) if $g \in L^s(\mathbb{R}^3)$, with $1 \leq s < \frac{3}{2}$, then $\mathcal{K} * g \in L^r(\mathbb{R}^3)$ for all $r \in (\frac{3s}{3-2s}, \infty]$.
In both cases $\mathcal{K} * g$ solves

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi g$$

in the sense of distributions, and has distributional derivatives

$$\nabla(\mathcal{K} * g) = (\nabla\mathcal{K}) * g \quad \text{and} \quad \Delta(\mathcal{K} * g) = (\Delta\mathcal{K}) * g \quad \text{a.e. in } \mathbb{R}^3.$$

Fix $u \in H^1(\mathbb{R}^3)$, the unique solution in $\mathcal{D}$ of the second equation in (1.4) is

$$\phi_u := \mathcal{K} * u^2.$$

Furthermore, we define

$$\psi_u := e^{-\frac{|x|}{a}} * u^2.$$

The coming properties will be useful.

Lemma 2.4 ([14, Lemma 3.4]). *For every $u \in H^1(\mathbb{R}^3)$ we have:*
(1) $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$ *for every* $y \in \mathbb{R}^3$,
(2) $\phi_u \geq 0$ *in* $\mathbb{R}^3$,
(3) $\phi_u \in L^s(\mathbb{R}^3) \cap C(\mathbb{R}^3)$ *for every* $s \in (3, \infty]$,
(4) $\nabla\phi_u = \nabla\mathcal{K} * u^2 \in L^s(\mathbb{R}^3) \cap C(\mathbb{R}^3)$ *for every* $s \in (\frac{3}{2}, \infty]$,
(5) $\phi_u \in \mathcal{D}$,
(6) $\|\phi_u\|_6 \leq C\|u\|^2$,
(7) ϕ_u *is the unique minimizer in* $\mathcal{D}$ *of the functional*

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{a^2}{2}\|\Delta\phi\|_2^2 - \int_{\mathbb{R}^3} \phi u^2 \, dx, \quad \phi \in \mathcal{D},$$

(8) $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy \leq \mathcal{S}^2\|u\|^4_{\frac{12}{5}}$.
Moreover, if $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{v_n} \rightharpoonup \phi_v$ *in* $\mathcal{D}$.

In view of [14], under (V_1)–(V_3), the energy functional of (1.4), defined in $H^1(\mathbb{R}^3) \times \mathcal{D}$ by

$$\begin{aligned}\mathcal{S}(u, \phi) = \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u|_2^2 + V(x)u^2] \, dx + \frac{q^2}{2}\int_{\mathbb{R}^3} \phi u^2 \, dx - \frac{q^2}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi|^2 \, dx \\ - \frac{a^2q^2}{16\pi}\int_{\mathbb{R}^3} |\Delta\phi|^2 \, dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \, dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \, dx,\end{aligned} \tag{2.2}$$

is continuously differentiable and its critical points correspond to the weak solutions of (1.4). Indeed, if $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $\mathcal{S}$, then

$$0 = \partial_u\mathcal{S}(u, \phi)[v] = \int_{\mathbb{R}^3} [\nabla u\nabla v + V(x)uv] \, dx + q^2\int_{\mathbb{R}^3} \phi uv \, dx - \mu\int_{\mathbb{R}^3} |u|^{p-1}uv \, dx - \int_{\mathbb{R}^3} |u|^4uv \, dx$$

for all $v \in H^1(\mathbb{R}^3)$ and

$$0 = \partial_\phi\mathcal{S}(u, \phi)[\varphi] = \frac{q^2}{2}\int_{\mathbb{R}^3} u^2\varphi \, dx - \frac{q^2}{8\pi}\int_{\mathbb{R}^3} \nabla\phi\nabla\varphi \, dx - \frac{a^2q^2}{8\pi}\int_{\mathbb{R}^3} \Delta\phi\Delta\varphi \, dx \tag{2.3}$$

for all $\varphi \in \mathcal{D}$.

In order to avoid the difficulty originated by the strongly indefiniteness of the functional $\mathcal{S}$, we apply a reduction procedure used in [14]. Since $\partial_\phi\mathcal{S}$ is a C^1 functional, if G_Φ is the graph of the map Φ defined by $H^1(\mathbb{R}^3) \ni u \mapsto \phi_u \in \mathcal{D}$, an application of the Implicit Function Theorem gives

$$G_\Phi = \{(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D} : \partial_\phi\mathcal{S}(u, \phi) = 0\} \quad \text{and} \quad \Phi \in C^1(H^1(\mathbb{R}^3), \mathcal{D}).$$

Jointly with (2.2) and (2.3), the functional $\mathcal{I}(u) := \mathcal{S}(u, \phi_u)$ has the reduced form

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] \, dx + \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \, dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \, dx,$$

which is of class $C^1(H^1(\mathbb{R}^3))$ and for all $u, v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}\mathcal{I}'(u)[v] &= \partial_u\mathcal{S}(u, \Phi(u))[v] + \partial_\phi\mathcal{S}(u, \Phi(u)) \circ \Phi'(u)[v] \\ &= \partial_u\mathcal{S}(u, \Phi(u))[v] \\ &= \int_{\mathbb{R}^3} [\nabla u\nabla v + V(x)uv] \, dx + q^2\int_{\mathbb{R}^3} \phi_u uv \, dx - \mu\int_{\mathbb{R}^3} |u|^{p-1}uv \, dx - \int_{\mathbb{R}^3} |u|^4uv \, dx.\end{aligned}$$

Moreover, the following statements are equivalent:
(i) The pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $\mathcal{S}$. i.e. (u, ϕ) is a solution of (1.4).
(ii) u is a critical point of $\mathcal{I}$ and $\phi = \phi_u$.
Hence, if $u \in H^1(\mathbb{R}^3)$ is a critical point of $\mathcal{I}$, then the pair (u, ϕ_u) is a solution of (1.4). For the sake of simplicity, in many cases we just say $u \in H^1(\mathbb{R}^3)$, instead of $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}$, is a solution of (1.4).

Let us define the function $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^3} \phi_u(x)u^2(x)\,dx.$$

It is clear that for all fixed $u \in H^1(\mathbb{R}^3)$ then $\Psi(u(\cdot + y)) = \Psi(u)$ for any $y \in \mathbb{R}^3$ and that Ψ is weakly lower semi-continuous in $H^1(\mathbb{R}^3)$. The next lemma shows that the functional Ψ and its derivative Ψ' have the B-L splitting property, which is similar to the well-known Brézis–Lieb lemma.

Lemma 2.5. *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*
(i) $\Psi(u_n - u) = \Psi(u_n) - \Psi(u) + o(1)$,
(ii) $\Psi'(u_n - u) = \Psi'(u_n) - \Psi'(u) + o(1)$ *in* $(H^1(\mathbb{R}^3))'$.

Proof. (i) This result is proved in [14, Lemma B.2].

(ii) This property is obtained in [25, Lemma 2.2] step by step, thanks to

$$\phi_u := \frac{1 - e^{-\frac{|x|}{a}}}{|x|} * u^2 \quad \text{and} \quad \frac{1 - e^{-\frac{|x|}{a}}}{|x|} \le \frac{1}{|x|}.$$ □

In the sequel, the Pohozaev identity obtained in [14] will be frequently used.

Proposition 2.6. *Assume that* (V_1)–(V_2) *hold. Let $f \in C^1(\mathbb{R})$ satisfy for some $C > 0$ and p, with $1 \le p \le 5$,*

$$|f(t)| \le C(|t| + |t|^p), \quad t \in \mathbb{R}.$$

Let $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)$ be a solution of the problem

$$\begin{cases} -\Delta u + V(x)u + q^2\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Then the Pohozaev identity holds true

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} [3V(x) + \nabla V(x)\cdot x]u^2\,dx + \frac{5}{4}\int_{\mathbb{R}^3} q^2\phi_u u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx = 3\int_{\mathbb{R}^3} F(u)\,dx,$$

where $F(t) = \int_0^t f(\tau)\,d\tau$.

The vanishing lemma for Sobolev space is stated as follows.

Lemma 2.7 (Vanishing Lemma, [19]). *Assume that $(u_n)_n$ is bounded sequence in $H^1(\mathbb{R}^3)$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n(x)|^2\,dx = 0$$

for some $R > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for every r, with $2 < r < 6$.

The arguments of Ramos, Wang and Willem [23] give sufficient conditions to ensure the convergence to 0 in $L^6(\mathbb{R}^3)$ of a sequence in $H^1(\mathbb{R}^3)$.

Lemma 2.8. *Let $R > 0$ and $(u_n)_n$ be a bounded sequence in $H^1(\mathbb{R}^3)$. If*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^6\,dx = 0,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ as $n \to \infty$ for any $r \in (2, 6]$.

The successive concentration-compactness principle is due to P.-L. Lions [19].

Lemma 2.9 ([19, Lemma 1.1]). *Let $(\rho_n)_n$ be a sequence of nonnegative functions in $L^1(\mathbb{R}^N)$ such that for some $l > 0$ fixed $\int_{\mathbb{R}^N} \rho_n\,dx = l$ for all n. Then there exists a subsequence, still denoted by $(\rho_n)_n$, satisfying one of the* *following three possibilities:*

(i) (Compactness) *There exists* $(y_n)_n \subset \mathbb{R}^N$ *with the property that for any* $\varepsilon > 0$ *there is* $R > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} \rho_n(x)\,dx \geq l - \varepsilon.$$

(ii) (Vanishing) *For any fixed* $R > 0$ *there holds*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)\,dx = 0.$$

(iii) (Dichotomy) *There exists an* $\alpha \in (0, l)$ *and* $(y_n)_n \subset \mathbb{R}^N$ *with the property that for any* $\varepsilon > 0$ *there is* $R > 0$ *such that for all* $r \geq R$ *and* $r' \geq R$ *it holds*

$$\limsup_{n\to\infty} \left(\left| \alpha - \int_{B_r(y_n)} \rho_n\,dx \right| + \left| (l-\alpha) - \int_{\mathbb{R}^N \setminus B_{r'}(y_n)} \rho_n\,dx \right| \right) < \varepsilon.$$

When V is not a constant, it is more difficult to establish the boundedness of any (PS) sequence. To overcome this difficulty, we use a subtle approach developed by Jeanjean in [16].

Theorem 2.10 ([16, Theorem 1.1]). *Let* X *be a Banach space and let* $\Lambda \subset \mathbb{R}^+$ *be an interval. Consider a family* $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ *of* $C^1(X)$ *functionals, with the form*

$$\varphi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in \Lambda,$$

where $B(u) \geq 0$ *for all* $u \in X$*, and such that either* $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$*. If there exist* $v_1, v_2 \in X$ *such that*

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi_\lambda(\gamma(t)) > \max\{\varphi_\lambda(v_1), \varphi_\lambda(v_2)\} \quad \textit{for all } \lambda \in \Lambda,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$*, then for a.e.* $\lambda \in \Lambda$*, there exists a sequence* $(v_n)_n$ *in* X *such that*

(i) $(v_n)_n$ *is bounded,*

(ii) $\varphi_\lambda(v_n) \to c_\lambda$*,*

(iii) $\varphi'_\lambda(v_n) \to 0$ *in the dual space* X' *of* X*.*

At last, we give a fundamental inequality we shall use later.

Lemma 2.11. *Let* $b > 0$*. Then*

$$h(t) := t^3(e^{-\frac{b}{t}} - e^{-b}) + \frac{1-t^3}{3} b e^{-b} \geq 0 \quad \textit{for all } t > 0 \tag{2.4}$$

and

$$1 - e^{-b} - \frac{1}{3} b e^{-b} > 0. \tag{2.5}$$

Proof. A simple calculation shows that $h(0+) = \frac{be^{-b}}{3} > 0$ and for all $t > 0$

$$h'(t) = t^2\Big[3(e^{-\frac{b}{t}} - e^{-b}) + \frac{b}{t} e^{-\frac{b}{t}} - be^{-b}\Big].$$

Consequently, $h'(0+) = 0 = h'(1)$ and $t = 1$ is strict minimum point for h in $\mathbb{R}_0^+$ so that $h(t) > h(1) = 0$ for all $t > 0$, with $t \neq 1$. This proves (2.4). Finally, (2.5) holds actually for all $b \geq 0$ by direct computation. □

3 The Constant Potential Case

In this section we assume that *V is the positive constant V_∞ which appears in* (V_2) and we consider the "limit problem"

$$\begin{cases} -\Delta u + V_\infty u + q^2\phi u = \mu|u|^{p-1}u + |u|^4 u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

associated with system (1.4). The norm on the $H^1(\mathbb{R}^3)$ is taken as

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\,dx \Big)^{\frac{1}{2}}.$$

The underlying energy functional $\mathcal{J}_\mu^\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$, related to (3.1), is defined by

$$\mathcal{J}_\mu^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\,dx + \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |u(x)|^{p+1}\,dx - \frac{1}{6}\int_{\mathbb{R}^3} |u(x)|^6\,dx.$$

Clearly, $\mathcal{J}_\mu^\infty \in C^1(H^1(\mathbb{R}^3))$ and

$$(\mathcal{J}_\mu^\infty)'(u)[\varphi] = \int_{\mathbb{R}^3} (\nabla u \nabla\varphi + V_\infty u\varphi)\,dx + q^2\int_{\mathbb{R}^3} \phi_u u\varphi\,dx - \mu\int_{\mathbb{R}^3} |u|^{p-1}u\varphi\,dx - \int_{\mathbb{R}^3} |u|^4 u\varphi\,dx$$

for every $\varphi \in H^1(\mathbb{R}^3)$. Hence, the critical points of $\mathcal{J}_\mu^\infty$ in $H^1(\mathbb{R}^3)$ are weak solutions of problem (3.1).

Define $\mathcal{G}_\mu^\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{G}_\mu^\infty(u) &= 2(\mathcal{J}_\mu^\infty)'(u)[u] - \mathcal{P}_\mu^\infty(u)\\ &= \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |u|^2\,dx + \frac{3}{4}\int_{\mathbb{R}^3} q^2\phi_u u^2\,dx\\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3} |u|^6\,dx,\end{aligned}$$

where $\mathcal{P}_\mu^\infty$ is given by

$$\begin{aligned}\mathcal{P}_\mu^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} V_\infty u^2\,dx + \frac{5}{4}\int_{\mathbb{R}^3} q^2\phi_u u^2\,dx\\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx - \frac{3\mu}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{1}{2}\int_{\mathbb{R}^3} |u|^6\,dx.\end{aligned}$$

See Proposition 2.6.

We study the functional $\mathcal{J}_\mu^\infty$ restricted on the manifold $\mathcal{M}_\mu^\infty$ defined as

$$\mathcal{M}_\mu^\infty = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{G}_\mu^\infty(u) = 0\}.$$

Obviously, if $u \in H^1(\mathbb{R}^3)$ is a nontrivial critical point of $\mathcal{J}_\mu^\infty$, then $u \in \mathcal{M}_\mu^\infty$. Hence, if $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a solution of (3.1), then $u \in \mathcal{M}_\mu^\infty$. The next result describes the properties of the manifold $\mathcal{M}_\mu^\infty$.

Lemma 3.1. *Let $p > 2$. Then the following properties hold for the manifold $\mathcal{M}_\mu^\infty$:*
(1) *For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ there exists a unique number $\theta_0 > 0$ such that $u_{\theta_0} := \theta^2 u(\theta_0\,\cdot)$ is in $\mathcal{M}_\mu^\infty$ and*

$$\mathcal{J}_\mu^\infty(u_{\theta_0}) = \max_{\theta \ge 0} \mathcal{J}_\mu^\infty(u_\theta).$$

(2) $0 \notin \partial\mathcal{M}_\mu^\infty$.
(3) $\mathcal{J}_\mu^\infty(u) > 0$ *for all* $u \in \mathcal{M}_\mu^\infty$.
(4) $(\mathcal{G}_\mu^\infty)'(u) \neq 0$ *for any* $u \in \mathcal{M}_\mu^\infty$*, that is,* $\mathcal{M}_\mu^\infty$ *is a* C^1*-manifold.*
(5) $\mathcal{M}_\mu^\infty$ *is a natural constraint of* $\mathcal{J}_\mu^\infty$*, namely every critical point of* $\mathcal{J}_\mu^\infty|_{\mathcal{M}_\mu^\infty}$ *is a critical point of* $\mathcal{J}_\mu^\infty$.
(6) *There exists a positive constant* $C > 0$ *such that* $\|u\|_{p+1} \ge C$ *for any* $u \in \mathcal{M}_\mu^\infty$.

Proof. (1) Fix $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and note that for $\theta > 0$,

$$u_\theta \in \mathcal{M}_\mu^\infty \iff \theta g'(\theta) = 0 \iff g'(\theta) = 0,$$

where g is given in $\mathbb{R}_0^+$ by

$$\begin{aligned}g(\theta) &= \frac{\theta^3}{2}\int |\nabla u|^2\,dx + \frac{\theta}{2}\int V_\infty |u|^2\,dx + \frac{\theta^3 q^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{\theta a}}}{|x-y|} u^2(x)u^2(y)\,dx\,dy\\ &\quad - \mu\frac{\theta^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{\theta^9}{6}\int_{\mathbb{R}^3} |u|^6\,dx.\end{aligned}$$

Clearly, $g(\theta)$ is positive for small $\theta > 0$ and tends to $-\infty$ as $\theta \to \infty$. Since g' is continuous in $\mathbb{R}_0^+$, there exists at least one $\theta_0 = \theta_0(u) > 0$ such that $g'(\theta_0) = 0$, which means that $u_{\theta_0} \in \mathcal{M}_\mu^\infty$.

To show the uniqueness of θ_0, note that $g'(\theta) = 0$ and $\theta > 0$ imply that

$$\begin{aligned}\frac{3}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx &= \mu\frac{(2p-1)\theta^{2(p-2)}}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx + \frac{3\theta^6}{2}\int_{\mathbb{R}^3}|u|^6\,dx + \frac{q^2}{4a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{\theta a}}u^2(x)u^2(y)\,dx\,dy\\ &\quad - \frac{3q^2\theta^{-1}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-e^{-\frac{|x-y|}{\theta a}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy - \frac{\theta^{-2}}{2}\int_{\mathbb{R}^3}V_\infty|u|^2\,dx\\ &=: h(\theta).\end{aligned}\tag{3.2}$$

Now the derivative h' of h is strictly positive in $\mathbb{R}^+$, with $h(0+) = -\infty$ and $h(\infty) = \infty$. As a consequence, there exists a unique $\theta_0 > 0$ such that (3.2) holds true. The uniqueness of θ_0 is verified and (1) is proved.

(2) The Sobolev embedding theorem and (2.5) give

$$\begin{aligned}&\frac{3}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty u^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx - \frac{q^2}{4a}\int_{\mathbb{R}^3}\psi_u u^2\,dx - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3}|u|^6\,dx\\ &= \frac{3}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty u^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-e^{-\frac{|x-y|}{a}} - \frac{|x-y|}{3a}e^{-\frac{|x-y|}{a}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy\\ &\quad - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3}|u|^6\,dx\\ &\geq \frac{1}{2}\|u\|^2 - C_1\|u\|^{p+1} - C_2\|u\|^6,\end{aligned}$$

which is strictly positive for $\|u\|$ small. Hence $0 \notin \partial\mathcal{M}_\mu^\infty$.

(3) Note that if $u \in \mathcal{M}_\mu^\infty$ and $2 < p < 5$, then

$$\begin{aligned}(2p-1)\mathcal{J}_\mu^\infty(u) &= (2p-1)\mathcal{J}_\mu^\infty(u) - \mathcal{G}_\mu^\infty(u)\\ &= (p-2)\int_{\mathbb{R}^3}|\nabla u|^2\,dx + (p-1)\int_{\mathbb{R}^3}V_\infty u^2\,dx + \frac{p-2}{2}\int_{\mathbb{R}^3}q^2\phi_u u^2\,dx\\ &\quad + \frac{q^2}{4a}\int_{\mathbb{R}^3}\psi_u u^2\,dx + \frac{(5-p)}{3}\int_{\mathbb{R}^3}|u|^6\,dx > 0,\end{aligned}\tag{3.3}$$

as required.

(4) Suppose by contradiction that $(\mathcal{G}_\mu^\infty)'(u) = 0$ for some $u \in \mathcal{M}_\mu^\infty$. Then the equation $(\mathcal{G}_\mu^\infty)'(u) = 0$ can be written in a weak sense as

$$-3\Delta u + V_\infty u + 3q^2\phi_u u - \frac{q^2}{a}\psi_u u = \mu(2p-1)|u|^{p-1}u + 9|u|^4u.\tag{3.4}$$

Define

$$\begin{aligned} a_1 &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx, & b_1 &= \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|u|^2\,dx,\\ c_1 &= \frac{1}{4}\int_{\mathbb{R}^3}q^2\phi_u u^2\,dx, & d_1 &= \frac{1}{4}\int_{\mathbb{R}^3}\frac{q^2}{a}\psi_u u^2\,dx,\\ e_1 &= \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx, & f_1 &= \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\,dx.\end{aligned}$$

Then we have

$$\begin{cases} 3a_1 + b_1 + 3c_1 - d_1 - (2p-1)e_1 - 9f_1 = 0,\\ 6a_1 + 2b_1 + 12c_1 - 4d_1 - (p+1)(2p-1)e_1 - 54f_1 = 0,\\ 3a_1 + 3b_1 + 15c_1 - 2d_1 - 3(2p-1)e_1 - 27f_1 = 0,\\ a_1 + b_1 + c_1 - e_1 - f_1 = k,\end{cases}\tag{3.5}$$

where the first equation is $u \in \mathcal{M}_\mu^\infty$, the second one is $(\mathcal{G}_\mu^\infty)'(u)[u] = 0$, the third one comes from the Pohozaev identity of (3.4) and the last one is the functional $\mathcal{J}_\mu^\infty$ on $\mathcal{M}_\mu^\infty$. From (3.5), fixed f_1 and d_1 as data, due to the Cramer rule, we get

$$e_1 = -\frac{48f_1 + 2d_1 + 3k}{4(p-1)(p-2)} < 0$$

for any f_1, d_1, k. This is not possible, since $p \in (2,5)$, $f_1 > 0$, $d_1 > 0$ and $k > 0$ when $u \in \mathcal{M}_\mu^\infty$. Thus, we have $(\mathcal{G}_\mu^\infty)'(u) \neq 0$ for every $u \in \mathcal{M}_\mu^\infty$ and by the implicit function theorem, $\mathcal{M}_\mu^\infty$ is a C^1-manifold.

(5) Let u be a critical point of the functional $\mathcal{J}_\mu^\infty$, restricted to the manifold $\mathcal{M}_\mu^\infty$. By the Lagrange multiplier theorem there exists a $\nu \in \mathbb{R}$ such that

$$(\mathcal{J}_\mu^\infty)'(u) + \nu(\mathcal{G}_\mu^\infty)'(u) = 0.$$

We claim that $\nu = 0$. Evaluating the linear functional above at $u \in \mathcal{M}_\mu^\infty$, we obtain

$$(\mathcal{J}_\mu^\infty)'(u)[u] + \nu(\mathcal{G}_\mu^\infty)'(u)[u] = 0,$$

which is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty |u|^2 + \phi_u u^2 - \mu |u|^{p+1} - |u|^6)\,dx + \nu\Big(3\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} V_\infty |u|^2\,dx \\ + 3q^2 \int_{\mathbb{R}^3} \phi_u u^2\,dx - \int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx - (2p-1)\int_{\mathbb{R}^3} \mu|u|^{p+1}\,dx - 9\int_{\mathbb{R}^3} |u|^6\,dx\Big) = 0.\end{aligned}$$

The above equality is associated with the equation

$$-\Delta u + V_\infty u + q^2\phi_u u - |u|^{p-1}u - |u|^4 u + \nu\Big(3(-\Delta u) + V_\infty u + 3q^2\phi_u u - \frac{q^2}{a}\psi_u u - (2p-1)|u|^{p-1}u - 9|u|^4 u\Big) = 0,$$

which can be rewritten as

$$(1+3\nu)(-\Delta u) + (1+\nu)V_\infty u + (1+3\nu)q^2\phi_u u = \nu\Big(\frac{q^2}{a}\Big)\psi_u u + (1+\nu(2p-1))|u|^{p-1}u + (1+9\nu)|u|^4 u.$$

The solutions of this equation satisfy the following Pohozaev identity

$$\begin{aligned}\frac{1+3\nu}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3(1+\nu)}{2}\int_{\mathbb{R}^3} V_\infty |u|^2\,dx + \frac{5(1+3\nu)}{4}\int_{\mathbb{R}^3} q^2\phi_u u^2\,dx + \frac{1-2\nu}{4}\int_{\mathbb{R}^3}\frac{q^2}{a}\psi_u u^2\,dx \\ = \frac{3[1+\nu(2p-1]}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx + \frac{1+9\nu}{2}\int_{\mathbb{R}^3} |u|^6\,dx.\end{aligned}$$

Using the notations of (3), recalling that $u \in \mathcal{M}_\mu^\infty$, by multiplying the above equation by u and integrating, and by the Pohozaev identity for the above equation, we get the following linear systems of $a_1, b_1, c_1, d_1, e_1, f_1$. Namely,

$$\begin{cases} a_1 + b_1 + c_1 - e_1 - f_1 = k > 0, \\ 3a_1 + b_1 + 3c_1 - d_1 - (2p-1)e_1 - 9f_1 = 0, \\ 2(1+3\nu)a_1 + 2(1+\nu)b_1 + 4(1+3\nu)c_1 - 4\nu d_1 - (p+1)(1+\nu(2p-1))e_1 - 6(1+9\nu)f_1 = 0, \\ (1+3\nu)a_1 + 3(1+\nu)b_1 + 5(1+3\nu)c_1 + (1-2\nu)d_1 - 3(1+\nu(2p-1))e_1 - 3(1+9\nu)f_1 = 0. \end{cases} \tag{3.6}$$

Indeed, fixed d_1 and f_1 as data, we see that the coefficient matrix A of (3.6) is

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 3 & 1 & 3 & -(2p-1) \\ 2(1+3\nu) & 2(1+\nu) & 4(1+3\nu) & -(p+1)(1+\nu(2p-1)) \\ (1+3\nu) & 3(1+\nu) & 5(1+3\nu) & -3(1+\nu(2p-1)) \end{pmatrix}.$$

By computation, the determinant of the coefficient matrix A of (3.6) is

$$\det(A) = -16\nu(1+3\nu)(p-1)(p-2).$$

Then

$$\det(A) = 0 \iff p = 1,\ p = 2,\ \nu = 0,\ \nu = -\frac{1}{3}.$$

We claim that ν must be equal to zero by excluding the other two possibilities.

(i) If $\nu \neq 0$, $\nu \neq -\frac{1}{3}$, the linear system (3.6) has a unique solution. Using the Cramer rule, we find that

$$e_1 = -\frac{3k + 2d_1 + 48f_1}{4(p-1)(p-2)} < 0,$$

which contradicts the fact that $e_1 > 0$.

(ii) Assume that $\nu = -\frac{1}{3}$. In such case, the third equation of (3.6) changes into the following one:

$$2b_1 + 2d_1 + 3(p+1)(p-2)e_1 + 18f_1 = 0,$$

which is also impossible, since all b_1, d_1, e_1 and f_1 are positive.

In conclusion, $\nu = 0$, and as a result, $(\mathcal{I}_\mu^\infty)'(u) = 0$, i.e., u is a critical point of the functional $\mathcal{I}_\mu^\infty$.

(6) Fix $u \in \mathcal{M}_\mu^\infty$, so that $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $\mathcal{G}_\mu^\infty(u) = 0$. The Sobolev embedding inequality and (2.5) yield

$$\begin{aligned} 0 &= \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |u|^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx \\ &\quad - \frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3} |u|^6\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx \\ &\geq \frac{1}{2}C_p\|u\|_{p+1}^2 - \frac{2p-1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned}$$

Then

$$\|u\|_{p+1} \geq \left[\frac{C_p(p+1)}{2(2p-1)}\right]^{\frac{1}{p-1}} := C,$$

as required. □

Property (5) of Lemma 3.1 insures that it is enough to find critical points of $\mathcal{I}_\mu^\infty$ restricted to $\mathcal{M}_\mu^\infty$. Set

$$c_1^\infty = \inf_{g\in\Gamma}\max_{\theta\in[0,1]} \mathcal{I}_\mu^\infty(g(\theta)), \quad c_2^\infty = \inf_{u\neq 0}\max_{\theta\geq 0} \mathcal{I}_\mu^\infty(u_\theta), \quad c_3^\infty = \inf_{u\in\mathcal{M}_\mu^\infty} \mathcal{I}_\mu^\infty(u),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0,\ \mathcal{I}_\mu^\infty(\gamma(1)) \leq 0,\ \gamma(1) \neq 0\}$.

Lemma 3.2. *The following relations hold:*

$$c^\infty := c_1^\infty = c_2^\infty = c_3^\infty > 0.$$

Proof. When $p \in (2,5)$, we have

$$\begin{aligned} \mathcal{I}_\mu^\infty(u_\theta) &= \frac{\theta^3}{2}\int |\nabla u|^2\,dx + \frac{\theta}{2}\int V_\infty |u|^2\,dx + \frac{\theta^3 q^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-\frac{|x-y|}{\theta a}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy \\ &\quad - \mu\frac{\theta^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{\theta^9}{6}\int_{\mathbb{R}^3} |u|^6\,dx \to -\infty, \end{aligned} \tag{3.7}$$

as $\theta \to \infty$. This implies that $c_2^\infty = c_3^\infty$.

From (3.7) we see that $\mathcal{I}_\mu^\infty(u_\theta) < 0$ for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and θ large enough. Thus, $c_1^\infty \leq c_2^\infty$.

On the other hand, for any $\gamma \in \Gamma$, we claim that $\gamma([0,1]) \cap \mathcal{M}_\mu^\infty \neq \emptyset$. Indeed, for every u in

$$\mathcal{G} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{G}_\mu^\infty(u) \geq 0\} \cup \{0\}$$

the Sobolev embedding theorem and (2.5) give

$$\begin{aligned}&\frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |u|^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_u u^2\,dx - \frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3} |u|^6\,dx\\ &\geq \frac{1}{2}\|u\|^2 - C_1\|u\|^{p+1} - C_2\|u\|^6.\end{aligned}$$

This implies that there exists a small neighborhood of 0 such that it is contained in $\mathcal{G}$. Furthermore, for every $u \in \mathcal{G}$, we have

$$3\mathcal{J}_\mu^\infty(u) = \mathcal{G}_\mu^\infty(u) + \int_{\mathbb{R}^3} V_\infty u^2\,dx + \frac{2(p-2)}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx + \int_{\mathbb{R}^3} |u|^6\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx \geq 0$$

and $\mathcal{J}_\mu^\infty(u) > 0$ if $u \neq 0$. Hence, for any $\gamma \in \Gamma$, satisfying $\gamma(0) = 0$, $\mathcal{J}_\mu^\infty(\gamma(1)) \leq 0$ and $\gamma(1) \neq 0$, the curve γ must across the manifold $\mathcal{M}_\mu^\infty$. Therefore, $c_1^\infty \geq c_3^\infty$. □

Now, we present an upper bound estimate for c^∞, which is very important for prove the (PS) condition.

Lemma 3.3. *The following inequalities hold:*

$$0 < c^\infty < \frac{1}{3}\mathcal{S}^{\frac{3}{2}},$$

where $\mathcal{S}$ is defined in (2.1), *if one of the next conditions is satisfied:*
- $3 < p < 5$ *and* $\mu > 0$,
- $2 < p \leq 3$ *and* $\mu > 0$ *large enough.*

Proof. Let us define for all $x \in \mathbb{R}^3$,

$$u_\varepsilon(x) = \psi(x)U_\varepsilon(x), \quad \text{where } U_\varepsilon(x) = \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \tag{3.8}$$

and $\psi \in C_c^\infty(\mathbb{R}^3)$ is such that $0 \leq \psi \leq 1$ in $\mathbb{R}^3$, $\psi(x) \equiv 1$ in B_δ and $\psi \equiv 0$ in $\mathbb{R}^3 \setminus B_{2\delta}$, for some $\delta > 0$. We know from (2.1) and (3.8) that

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon(x)|^2\,dx \leq \mathcal{S}^{\frac{3}{2}} + O(\varepsilon), \qquad \int_{\mathbb{R}^3} |u_\varepsilon(x)|^6\,dx \geq \mathcal{S}^{\frac{3}{2}} + O(\varepsilon^3). \tag{3.9}$$

By computation, we can deduce that

$$\int_{\mathbb{R}^3} |u_\varepsilon(x)|^p\,dx = \begin{cases} O(\varepsilon^{\frac{6-p}{2}}), & p > 3,\\ O(\varepsilon^{\frac{p}{2}}|\ln\varepsilon|), & p = 3,\\ O(\varepsilon^{\frac{p}{2}}), & p < 3.\end{cases} \tag{3.10}$$

From the definition of c^∞, it is clear that there holds

$$0 < c^\infty \leq \max_{\theta \geq 0} \mathcal{J}_\mu^\infty((u_\varepsilon)_\theta). \tag{3.11}$$

Set

$$g(\theta) = \frac{\theta^3}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx - \frac{\theta^9}{6}\int_{\mathbb{R}^3} |u_\varepsilon|^6\,dx, \quad \theta \geq 0.$$

A direct calculation shows that g attains its maximum at

$$\theta_0 = \left(\frac{\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx}{\int_{\mathbb{R}^3} |u_\varepsilon|^6\,dx}\right)^{\frac{1}{6}}.$$

Moreover, by (3.9), using the inequality $(a+b)^p \le a^p + p(a+b)^{p-1}b$, which holds for any $p \ge 1$ and $a, b \ge 0$, we deduce that

$$\begin{aligned}\max_{\theta\ge 0} g(\theta) &= g(\theta_0)\\ &= \frac{1}{2}\left(\frac{\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\,dx}{\int_{\mathbb{R}^3}|u_\varepsilon|^6\,dx}\right)^{\frac{3}{6}}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\,dx - \frac{1}{6}\left(\frac{\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\,dx}{\int_{\mathbb{R}^3}|u_\varepsilon|^6\,dx}\right)^{\frac{3}{2}}\int_{\mathbb{R}^3}|u_\varepsilon|^6\,dx\\ &= \frac{1}{3}\cdot\frac{\|\nabla u_\varepsilon\|_2^3}{\|u_\varepsilon\|_6^3}\\ &\le \frac{1}{3}\cdot\frac{[\mathcal{S}^{\frac{3}{2}}+O(\varepsilon)]^{\frac{3}{2}}}{[\mathcal{S}^{\frac{3}{2}}+O(\varepsilon^3)]^{1/2}}\\ &\le \frac{1}{3}\mathcal{S}^{\frac{3}{2}}+O(\varepsilon).\end{aligned}\tag{3.12}$$

Since $\mathcal{J}_\mu^\infty((u_\varepsilon)_\theta) \to -\infty$ as $\theta \to \infty$, there exists $\theta_\varepsilon > 0$ such that

$$\mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) = \max_{\theta\ge 0}\mathcal{J}_\mu^\infty((u_\varepsilon)_\theta) > 0.$$

Since 0 is a local minimum of $\mathcal{J}_\mu^\infty$, there exists a constant $C>0$, independent of ε, such that $\mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) \ge C > 0$. This implies that $\theta_\varepsilon \ge \theta_1 > 0$, where θ_1 is a positive constant. Otherwise, there should exist a sequence $(\varepsilon_n)_n$ such that

$$\lim_{n\to\infty}\varepsilon_n = 0 = \lim_{n\to\infty}\theta_{\varepsilon_n}.$$

Then Lemma 2.4 and (3.9)–(3.11) imply as $n\to\infty$

$$\begin{aligned}c_0 < c^\infty &\le \mathcal{J}_\mu^\infty((u_{\varepsilon_n})_{\theta_{\varepsilon_n}}) + o(1)\\ &= \frac{\theta_{\varepsilon_n}^3}{2}\int|\nabla u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}}{2}\int_{\mathbb{R}^3}V_\infty|u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}^3}{4}\int_{\mathbb{R}^3}\phi_{u_{\varepsilon_n}}u_{\varepsilon_n}^2(x)\,dx + o(1)\\ &\le \frac{\theta_{\varepsilon_n}^3}{2}\int|\nabla u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}}{2}\int_{\mathbb{R}^3}V_\infty|u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_{\varepsilon_n}^2(x)u_{\varepsilon_n}^2(y)}{|x-y|}\,dx\,dy\\ &\qquad - \mu\frac{\theta_{\varepsilon_n}^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^{p+1}\,dx - \frac{\theta_{\varepsilon_n}^9}{6}\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^6\,dx + o(1)\\ &\le \frac{\theta_{\varepsilon_n}^3}{2}\int_{\mathbb{R}^3}|\nabla u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}}{2}\int_{\mathbb{R}^3}V_\infty|u_{\varepsilon_n}|^2\,dx + \frac{\theta_{\varepsilon_n}^3}{4}\left(\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^{\frac{12}{5}}\,dx\right)^{\frac{5}{3}} - \mu\frac{\theta_{\varepsilon_n}^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^{p+1}\,dx\\ &\qquad - \frac{\theta_{\varepsilon_n}^9}{6}\int_{\mathbb{R}^3}|u_{\varepsilon_n}|^6\,dx + o(1)\\ &= o(1).\end{aligned}$$

This is clearly impossible and the claim follows.

On the other hand, from (3.9)–(3.10), for any $\varepsilon > 0$,

$$0 < C \le \mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) \le C_1\theta_\varepsilon^3 + C_2\theta_\varepsilon - C_3\theta_\varepsilon^9,$$

which implies that there exists $\theta_2 > 0$ such that $\theta_\varepsilon \le \theta_2$. Thus, $0 < \theta_1 \le \theta_\varepsilon \le \theta_2$ for any $\varepsilon > 0$. Now, by (3.12), we deduce that

$$\begin{aligned}\mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) &\le \frac{1}{3}\mathcal{S}^{\frac{3}{2}} + O(\varepsilon) + \frac{\theta_\varepsilon}{2}\int_{\mathbb{R}^3}|u_\varepsilon|^2\,dx + \frac{\theta_\varepsilon^3}{4}\int_{\mathbb{R}^3}\phi_{u_\varepsilon}u_\varepsilon^2\,dx - \mu\frac{\theta_\varepsilon^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u_\varepsilon|^{p+1}\,dx - \frac{\theta_\varepsilon^9}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6\,dx\\ &\le \frac{1}{3}\mathcal{S}^{\frac{3}{2}} + O(\varepsilon) + \frac{\theta_2}{2}\int_{\mathbb{R}^3}|u_\varepsilon|^2\,dx + \frac{\theta_2^3\mathcal{S}}{4}\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{5}}\,dx\right)^{\frac{5}{3}} - \mu\frac{\theta_1^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u_\varepsilon|^{p+1}\,dx.\end{aligned}$$

Next,

$$\mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) \le \frac{1}{3}\mathcal{S}^{\frac{3}{2}} + O(\varepsilon) + C\Big(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{5}}\,\mathrm{d}x\Big)^{\frac{5}{3}} - C\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1}\,\mathrm{d}x.$$

Observing that $p > 2$, we have

$$\lim_{\varepsilon\to 0^+} \frac{(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{5}}\,\mathrm{d}x)^{\frac{5}{3}}}{\varepsilon} \le \lim_{\varepsilon\to 0^+} \frac{O(\varepsilon^2)}{\varepsilon} = 0,$$

and noting that $2 - \frac{1}{2}(p+1) < 0$ if $4 < p+1 < 6$, we have

$$\lim_{\varepsilon\to 0} \mu \frac{\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1}\,\mathrm{d}x}{\varepsilon} = \begin{cases} \lim\limits_{\varepsilon\to 0} \mu O(\varepsilon^{2-\frac{1}{2}(p+1)}) = \infty, & 4 < p+1 < 6,\\ \lim\limits_{\varepsilon\to 0} \mu O(\varepsilon^{2-\frac{1}{2}(p+1)}), & 3 < p+1 < 4,\\ \lim\limits_{\varepsilon\to 0} \mu O(\varepsilon^{2-\frac{1}{2}(p+1)}|\ln\varepsilon|), & p+1 = 3.\end{cases}$$

We can choose μ large enough such that the other two limits are equal to ∞, for instance, $\mu = \varepsilon^{-2}$.

From the above inequalities, we conclude that

$$\mathcal{J}_\mu^\infty((u_\varepsilon)_{\theta_\varepsilon}) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}$$

for ε small enough and, combining with (3.11), we complete the proof. □

For a minimizing sequence for c^∞ on the manifold $\mathcal{M}_\mu^\infty$ we obtain the following compactness result.

Lemma 3.4. *Let $(u_n)_n \subset \mathcal{M}_\mu^\infty$ be a minimizing sequence for c^∞, given in Lemma 3.2. Then there exists a sequence $(y_n)_n \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there is an $R > 0$ satisfying*

$$\int_{\mathbb{R}^3\setminus B_R(y_n)} (|\nabla u_n|^2 + V_\infty u_n^2)\,\mathrm{d}x \le \varepsilon$$

for all n sufficiently large.

Proof. Following the idea of [18], let $(u_n)_n \subset \mathcal{M}_\mu^\infty$ be such that

$$0 < \lim_{n\to\infty} \mathcal{J}_\mu^\infty(u_n) = c^\infty < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}. \tag{3.13}$$

Since $(u_n)_n \subset \mathcal{M}_\mu^\infty$, by (3.3) we see that

$$\begin{aligned}\mathcal{J}_\mu^\infty(u_n) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V_\infty |u_n|^2)\,\mathrm{d}x + \frac{1}{4}\int \phi_{u_n} u_n^2\,\mathrm{d}x - \frac{\mu}{p+1}\int |u_n|^{p+1}\,\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6\,\mathrm{d}x\\ &= \frac{p-2}{2p-1}\int_{\mathbb{R}^3} |\nabla u_n|^2\,\mathrm{d}x + \frac{p-1}{2p-1}\int_{\mathbb{R}^3} V_\infty |u_n|^2\,\mathrm{d}x + \frac{q^2(p-2)}{2(2p-1)}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,\mathrm{d}x\\ &\quad + \frac{q^2}{4a(2p-1)}\int_{\mathbb{R}^3} \psi_u u^2\,\mathrm{d}x + \frac{5-p}{3(2p-1)}\int_{\mathbb{R}^3} |u_n|^6\,\mathrm{d}x\\ &\triangleq J(u_n) \ge 0.\end{aligned} \tag{3.14}$$

From (3.13), it follows that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$. Next, we use Lemma 2.9 to conclude the compactness of the sequence $(u_n)_n$. Let

$$\rho_n = \frac{p-2}{2p-1}|\nabla u_n|^2 + \frac{p-1}{2p-1}V_\infty |u_n|^2 + \frac{q^2(p-2)}{2(2p-1)}\phi_{u_n} u_n^2 + \frac{q^2}{4a(2p-1)}\psi_u u^2 + \frac{5-p}{3(2p-1)}|u_n|^6.$$

Then $(\rho_n)_n$ is a sequence of nonnegative L^1 functions on $\mathbb{R}^3$ by (3.14), and as $n\to\infty$

$$\int_{\mathbb{R}^3} \rho_n\,\mathrm{d}x \to c^\infty, \tag{3.15}$$

thanks to (3.13).

(i) Vanishing does not occur. Suppose by contradiction that for all $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \rho_n(x)\,dx = 0.$$

Hence

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^6\,dx = 0.$$

The Vanishing Lemma 2.7 and Lemma 2.8 yield that

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^3) \text{ for any } t \in (2, 6]. \tag{3.16}$$

A consequence of Lemma 2.4 gives as $n \to \infty$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \psi_{u_n} u_n^2\,dx \to 0. \tag{3.17}$$

Since $u_n \in \mathcal{M}_\mu^\infty$, properties (3.14), (3.16) and (3.17) imply that

$$\lim_{n\to\infty} \mathcal{J}_\mu^\infty(u_n) = 0.$$

This contradicts either (3.13) or (3.15).

(ii) Dichotomy does not occur. Suppose by contradiction that there exist an $\alpha \in (0, c^\infty)$ and $(y_n)_n \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$ there is $(R_n)_n \subset \mathbb{R}^+$, with $R_n \to \infty$, satisfying

$$\limsup_{n\to\infty} \left(\left| \alpha - \int_{B_{R_n}(y_n)} \rho_n\,dx \right| + \left| c^\infty - \alpha - \int_{\mathbb{R}^3 \setminus B_{2R_n}(y_n)} \rho_n\,dx \right| \right) < \varepsilon. \tag{3.18}$$

Let $\xi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ be a cut-off function such that $0 \le \xi \le 1$, $\xi(t) = 1$ for $t \le 1$, $\xi(t) = 0$ for $t \ge 2$ and $|\xi'(t)| \le 2$. Set

$$v_n(x) = \xi\Big(\frac{|x - y_n|}{R_n}\Big) u_n(x), \quad w_n(x) = \Big(1 - \xi\Big(\frac{|x - y_n|}{R_n}\Big)\Big) u_n(x).$$

Then by (3.14) and (3.18), we see that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} J(v_n)\,dx \ge \alpha, \quad \liminf_{n\to\infty} \int_{\mathbb{R}^3} J(w_n)\,dx \ge c^\infty - \alpha.$$

Put $\Omega_n = B_{2R_n}(y_n) \setminus B_{R_n}(y_n)$. The above inequalities and (3.14) give

$$\int_{\Omega_n} (|\nabla u_n|^2 + V_\infty u_n^2)\,dx \to 0, \quad \int_{\Omega_n} \phi_{u_n} u_n^2\,dx \to 0, \quad \int_{\Omega_n} |u_n|^6\,dx \to 0,$$

as $n \to \infty$. By direct computation we have as $n \to \infty$

$$\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx = \int_{\mathbb{R}^3} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^3} |\nabla w_n|^2\,dx + o_n(1), \tag{3.19}$$

$$\int_{\mathbb{R}^3} u_n^2(x)\,dx = \int_{\mathbb{R}^3} v_n^2(x)\,dx + \int_{\mathbb{R}^3} w_n^2(x)\,dx + o_n(1), \tag{3.20}$$

$$\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx = \int_{\mathbb{R}^3} |v_n|^{p+1}\,dx + \int_{\mathbb{R}^3} |w_n|^{p+1}\,dx + o_n(1), \tag{3.21}$$

$$\int_{\mathbb{R}^3} |u_n|^6\,dx = \int_{\mathbb{R}^3} |v_n|^6\,dx + \int_{\mathbb{R}^3} |w_n|^6\,dx + o_n(1), \tag{3.22}$$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \ge \int \phi_{v_n} v_n^2\,dx + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2\,dx + o_n(1), \tag{3.23}$$

$$\int_{\mathbb{R}^3} \psi_{u_n} u_n^2\,dx \ge \int \psi_{v_n} v_n^2\,dx + \int_{\mathbb{R}^3} \psi_{w_n} w_n^2\,dx + o_n(1). \tag{3.24}$$

Hence, by (3.19), (3.20) and (3.22)–(3.24), we get

$$J(u_n) \geq J(v_n) + J(w_n) + o_n(1)$$

$n \to \infty$. Then

$$c^\infty = \lim_{n\to\infty} J(u_n) \geq \liminf_{n\to\infty} J(v_n) + \liminf_{n\to\infty} J(w_n) \geq \alpha + c^\infty - \alpha = c^\infty.$$

Therefore,

$$\lim_{n\to\infty} J(v_n) = \alpha, \quad \lim_{n\to\infty} J(w_n) = c^\infty - \alpha. \tag{3.25}$$

Now, $\mathcal{G}_\mu^\infty(u_n) = 0$, since $u_n \in \mathcal{M}_\mu^\infty$. By (3.19)–(3.24) we have

$$0 = \mathcal{G}_\mu^\infty(u_n) \geq \mathcal{G}_\mu^\infty(v_n) + \mathcal{G}_\mu^\infty(w_n) + o_n(1). \tag{3.26}$$

We distinguish the following two cases.

Case 1. Up to a subsequence, we may assume that either $\mathcal{G}_\mu^\infty(v_n) \leq 0$ or $\mathcal{G}_\mu^\infty(w_n) \leq 0$. Without loss of generality, we suppose that $\mathcal{G}_\mu^\infty(v_n) \leq 0$. Then

$$\begin{aligned}\frac{3}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |v_n|^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2\,dx \\ - \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_{v_n} v_n^2\,dx - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3} |v_n|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3} |v_n|^6\,dx \leq 0.\end{aligned}$$

By Lemma 3.1 for any n there exists $\theta_n > 0$ such that $\theta_n^2 v_n(\theta x) \in \mathcal{M}_\mu^\infty$ and so $\mathcal{G}_\mu^\infty(\theta_n^2 v_n(\theta x)) = 0$. Thus, (2.5) yields that

$$\begin{aligned}&\frac{3}{2}(\theta_n^{2(p-1)} - \theta_n^2)\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx + \frac{1}{2}(\theta_n^{2(p-1)} - 1)\int_{\mathbb{R}^3} V_\infty |v_n|^2\,dx \\ &\quad + \frac{3q^2}{4}(\theta_n^{2(p-1)} - \theta_n^2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{\theta_n a}}}{|x-y|} u^2(x)u^2(y)\,dx\,dy \\ &\quad - \frac{q^2}{4a}(\theta_n^{2(p-1)} - \theta_n^2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{\theta_n a}} u^2(x)u^2(y)\,dx\,dy + \frac{3}{2}(\theta_n^8 - \theta_n^{2(p-1)})\int_{\mathbb{R}^3} |v_n|^{2_s^*}\,dx \\ &= \frac{3}{2}(\theta_n^{2(p-1)} - \theta_n^2)\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx + \frac{1}{2}(\theta_n^{2(p-1)} - 1)\int_{\mathbb{R}^3} V_\infty |v_n|^2\,dx \\ &\quad + \frac{q^2}{4}(\theta_n^{2(p-1)} - \theta_n^2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{3(1 - e^{-\frac{|x-y|}{\theta_n a}}) - \frac{|x-y|}{\theta_n a} e^{-\frac{|x-y|}{\theta a}}}{|x-y|} u^2(x)u^2(y)\,dx\,dy \\ &\quad + \frac{3}{2}(\theta_n^8 - \theta_n^{2(p-1)})\int_{\mathbb{R}^3} |v_n|^{2_s^*}\,dx \leq 0,\end{aligned}$$

which implies that $\theta_n \leq 1$. Then by (3.25), we have

$$c^\infty \leq \mathcal{I}_\mu^\infty(\theta_n^2 v_n(\theta x)) = J(\theta_n^2 v_n(\theta x)) \leq J(v_n) \to \alpha < c^\infty, \tag{3.27}$$

which is the desired contradiction.

Case 2. Up to a subsequence, we may assume that $\mathcal{G}_\mu^\infty(v_n) > 0$ and $\mathcal{G}_\mu^\infty(w_n) > 0$. By formula (3.26), we see that $\mathcal{G}_\mu^\infty(v_n) = o_n(1)$ and $\mathcal{G}_\mu^\infty(w_n) = o_n(1)$. Repeating the argument of Case 1, we obtain a contradiction of type (3.27). Thus we suppose that

$$\lim_{n\to\infty} \theta_n = \theta_0 > 1.$$

Now, as $n \to \infty$

$$\begin{aligned} o_n(1) &= \mathcal{G}_\mu^\infty(v_n) \\ &= \frac{3}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |v_n|^2\,\mathrm{d}x + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2\,\mathrm{d}x - \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_{v_n} v_n^2\,\mathrm{d}x \\ &\quad - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3} |v_n|^{p+1}\,\mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}^3} |v_n|^6\,\mathrm{d}x \\ &\le \frac{3}{2}\Big(1 - \frac{1}{\theta_n^{2p-4}}\Big)\int_{\mathbb{R}^3} |\nabla v_n|^2\,\mathrm{d}x + \frac{1}{2}\Big(1 - \frac{1}{\theta_n^{2(p-1)}}\Big)\int_{\mathbb{R}^3} V_\infty |v_n|^2\,\mathrm{d}x \\ &\quad + \frac{3q^2}{4}\Big(1 - \frac{1}{\theta_n^{2p-4}}\Big)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{\theta_n a}}}{|x-y|} u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{q^2}{4a}\Big(1 - \frac{1}{\theta_n^{2p-4}}\Big)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{\theta_n a}} u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{3}{2}(\theta_n^{2(5-p)} - 1)\int_{\mathbb{R}^3} |v_n|^6\,\mathrm{d}x. \end{aligned}$$

This implies that $v_n \to 0$ in $H^1(\mathbb{R}^3)$, which is impossible by (3.25). Hence, we conclude that dichotomy cannot occur.

As a result, compactness holds for the sequence $(\rho_n)_n$ by Lemma 2.9, i.e., there exists $(y_n)_n \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$ there is an $R > 0$ satisfying

$$\liminf_{n\to\infty} \int_{B_R(y_n)} \rho_n(x)\,\mathrm{d}x \ge c^\infty - \varepsilon.$$

Since $\lim_{n\to\infty} \mathcal{I}_\mu^\infty(u_n) = \lim_{n\to\infty} J(u_n) = c^\infty$, it follows that

$$\varepsilon > c^\infty - (c^\infty - \varepsilon) \ge \lim_{n\to\infty} J(u_n) - \liminf_{n\to\infty} \int_{B_R(y_n)} \rho_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R(y_n)} \rho_n(x)\,\mathrm{d}x,$$

which implies that the conclusion holds true for all n sufficiently large. □

Proof of Theorem 1.2. Let $(u_n)_n \subset \mathcal{M}_\mu^\infty$ be a minimizing sequence for c^∞. By Lemma 3.4 there exists $(y_n)_n \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$ there exists an $R > 0$ satisfying

$$\int_{\mathbb{R}^3\setminus B_R(y_n)} (|\nabla u_n|^2 + V_\infty u_n^2)\,\mathrm{d}x \le \varepsilon. \tag{3.28}$$

Define $\widetilde{u_n}(x) = u_n(x - y_n) \in H^1(\mathbb{R}^3)$; then we have $\phi_{\widetilde{u_n}} = \phi_{u_n}(\cdot - y_n)$ by Lemma 2.4 and thus $\widetilde{u_n} \in \mathcal{M}_\mu^\infty$ and also $\mathcal{I}_\mu^\infty(u_n) = \mathcal{I}_\mu^\infty(\widetilde{u_n})$. This means that $(\widetilde{u_n})_n$ is also a minimizing sequence for c^∞. Hence, by (3.28) for any $\varepsilon > 0$ there exists an $R > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_R(0)} (|\nabla \widetilde{u_n}|^2 + V_\infty \widetilde{u_n}^2)\,\mathrm{d}x \le \varepsilon. \tag{3.29}$$

Since $(\widetilde{u_n})_n$ is bounded in $H^1(\mathbb{R}^3)$, up to a subsequence, we may assume that there exists $\tilde{u} \in H^1(\mathbb{R}^3)$ such that

$$\begin{cases} \widetilde{u_n} \rightharpoonup \tilde{u} & \text{in } H^1(\mathbb{R}^3), \\ \widetilde{u_n} \to \tilde{u} & \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^3), \text{ with } 1 \le r < 6, \\ \widetilde{u_n} \to \tilde{u} & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.30}$$

By Fatou's lemma and (3.29) we get

$$\int_{\mathbb{R}^3\setminus B_R(0)} (|\nabla \tilde{u}|^2 + V_\infty \tilde{u}^2)\,\mathrm{d}x \le \varepsilon. \tag{3.31}$$

By (3.29)–(3.31), and the Sobolev embedding theorem, we have that for any $r \in [2, 6)$ and any $\varepsilon > 0$ there exists a $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} |\widetilde{u_n} - \tilde{u}|^r \, dx &= \int_{B_R(0)} |\widetilde{u_n} - \tilde{u}|^r \, dx + \int_{\mathbb{R}^3 \setminus B_R(0)} |\widetilde{u_n} - \tilde{u}|^r \, dx \\ &\le \varepsilon + C\big(\|\widetilde{u_n}\|_{H^1(\mathbb{R}^3 \setminus B_R(0))} + \|\tilde{u}\|_{H^1(\mathbb{R}^3 \setminus B_R(0))}\big) \\ &\le (1 + 2C)\varepsilon,\end{aligned}$$

where $C > 0$ is the constant of the embedding $H^1(B_R(0)) \hookrightarrow L^r(B_R(0))$. Hence, we have proved that

$$\widetilde{u_n} \to \tilde{u} \quad \text{in } L^r(\mathbb{R}^3) \quad \text{for any } r \in [2, 6). \tag{3.32}$$

Since $\widetilde{u_n} \in \mathcal{M}_\mu^\infty$, Lemma 3.1 yields that $\|\widetilde{u_n}\|_{p+1} \ge C > 0$. Hence $\|\tilde{u}\|_{p+1} \ge C > 0$, and as a result $\tilde{u} \ne 0$.

Finally, we show that $\widetilde{u_n} \to \tilde{u}$ in $H^1(\mathbb{R}^3)$. From Lemma 2.4 and (3.32) we deduce that

$$\phi_{\widetilde{u_n}} \to \phi_{\tilde{u}} \quad \text{in } \mathcal{D}(\mathbb{R}^3),$$

and thus

$$\int_{\mathbb{R}^3} \phi_{\widetilde{u_n}} \widetilde{u_n}^2 \, dx \to \int_{\mathbb{R}^3} \phi_{\tilde{u}} \tilde{u}^2 \, dx. \tag{3.33}$$

Set $\widetilde{v_n} = \widetilde{u_n} - \tilde{u}$. By (3.30), we have as $n \to \infty$

$$\|\tilde{u}_n\|^2 - \|\tilde{u}\|^2 = \|\widetilde{v_n}\|^2 + o_n(1),$$

which implies that

$$\|\nabla \widetilde{u_n}\|_2^2 - \|\nabla \tilde{u}\|_2^2 = \|\nabla \tilde{v}_n\|_2^2 + o_n(1). \tag{3.34}$$

The Brézis–Lieb lemma and (3.30) give as $n \to \infty$

$$\|\tilde{u}_n\|_6^6 - \|\tilde{u}\|_6^6 = \|\tilde{v}_n\|_6^6 + o_n(1). \tag{3.35}$$

Hence, from (3.30), (3.33), (3.34) and (3.35) it follows as $n \to \infty$ that

$$c^\infty - \mathcal{J}_\mu^\infty(\tilde{u}) = \mathcal{J}_\mu^\infty(\widetilde{u_n}) - \mathcal{J}_\mu^\infty(\tilde{u}) + o_n(1) = \frac{1}{2}\|\nabla \tilde{v}_n\|_2^2 - \frac{1}{6}\|\widetilde{v_n}\|_6^6 + o_n(1). \tag{3.36}$$

Next, we claim that $\mathcal{J}_\mu^\infty(\tilde{u}) \ge 0$. We first prove that $\mathcal{G}_\mu^\infty(\tilde{u}) \ge 0$. Suppose by contradiction that $\mathcal{G}_\mu^\infty(\tilde{u}) < 0$ and so there exists $\theta \in (0, 1)$ such that $\tilde{u}_\theta \in \mathcal{M}_\mu^\infty$. By (3.3) and (3.30), we deduce that

$$\begin{aligned}\mathcal{J}_\mu^\infty(\tilde{u}_\theta) &= \frac{p-2}{2p-1}\int_{\mathbb{R}^3} |\nabla \tilde{u}_\theta|^2 \, dx + \frac{p-1}{2p-1}\int_{\mathbb{R}^3} V_\infty |\tilde{u}_\theta|^2 \, dx + \frac{p-2}{4p-2}\int_{\mathbb{R}^3} q^2 \phi_{\tilde{u}_\theta} |\tilde{u}_\theta|^2 \, dx \\ &\quad + \frac{q^2}{4a(2p-1)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} \tilde{u}_\theta^2(x)\tilde{u}_\theta^2(y) \, dx \, dy + \frac{5-p}{6p-3}\int_{\mathbb{R}^3} |\tilde{u}_\theta|^6 \, dx \\ &= \frac{p-2}{2p-1}\theta^3 \int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \, dx + \frac{p-1}{2p-1}\theta \int_{\mathbb{R}^3} V_\infty |\tilde{u}|^2 \, dx + \frac{p-2}{4p-2}\theta^3 \int_{\mathbb{R}^3} q^2 \phi_{\tilde{u}} |\tilde{u}|^2 \, dx \\ &\quad + \frac{\theta^2 q^2}{4a(2p-1)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{\theta a}} \tilde{u}^2(x)\tilde{u}^2(y) \, dx \, dy + \frac{5-p}{6p-3}\theta^9 \int_{\mathbb{R}^3} |\tilde{u}|^6 \, dx \\ &< \frac{p-2}{2p-1}\int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \, dx + \frac{p-1}{2p-1}\int_{\mathbb{R}^3} V_\infty |\tilde{u}|^2 \, dx + \frac{p-2}{4p-2}\int_{\mathbb{R}^3} \phi_{\tilde{u}} |\tilde{u}|^2 \, dx \\ &\quad + \frac{q^2}{4a(2p-1)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} \tilde{u}^2(x)\tilde{u}^2(y) \, dx \, dy + \frac{5-p}{6p-3}\int_{\mathbb{R}^3} |\tilde{u}|^6 \, dx \\ &\le \liminf_{n\to\infty}\bigg[\frac{p-2}{2p-1}\int_{\mathbb{R}^3} |\nabla \widetilde{u_n}|^2 \, dx + \frac{p-1}{2p-1}\int_{\mathbb{R}^3} V_\infty |\widetilde{u_n}|^2 \, dx + \frac{p-2}{4p-2}\int_{\mathbb{R}^3} q^2 \phi_{\tilde{u}_n} |\tilde{u}|^2 \, dx \\ &\quad + \frac{q^2}{4a(2p-1)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} \tilde{u}_n^2(x)\tilde{u}_n^2(y) \, dx \, dy + \frac{5-p}{6p-3}\int_{\mathbb{R}^3} |\tilde{u}_n|^6 \, dx\bigg] \\ &\le \liminf_{n\to\infty} \mathcal{J}_\mu^\infty(\widetilde{u_n}) = c^\infty,\end{aligned}$$

which contradicts the fact that $\mathcal{J}_\mu^\infty(\tilde{u}_\theta) \geq c^\infty$. Therefore, the claim is proved and we can infer that

$$\begin{aligned}(2p-1)\mathcal{J}_\mu^\infty(\tilde{u}) &\geq (2p-1)\mathcal{J}_\mu^\infty(\tilde{u}) - \mathcal{G}_\mu^\infty(\tilde{u})\\ &= (p-2)\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2\,dx + (p-1)\int_{\mathbb{R}^3}V_\infty|\tilde{u}|^2\,dx + \frac{p-2}{2}\int_{\mathbb{R}^3}q^2\phi_{\tilde{u}}|\tilde{u}|^2\,dx + \frac{5-p}{3}\int_{\mathbb{R}^3}|\tilde{u}|^6\,dx\\ &\quad + \frac{q^2}{4a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\frac{|x-y|}{a}}\tilde{u}^2(x)\tilde{u}^2(y)\,dx\,dy > 0.\end{aligned}$$

Hence, Lemma 3.3 and (3.36) yield as $n \to \infty$

$$\frac{1}{2}\|\nabla\tilde{v}_n\|_2^2 - \frac{1}{6}\|\tilde{v}_n\|_6^6 + o_n(1) = c^\infty - \mathcal{J}_\mu^\infty(\tilde{u}) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}. \tag{3.37}$$

On the other hand, it follows from (3.30) that

$$(\mathcal{J}_\mu^\infty)'(\tilde{v}_n)[\varphi] = \int_{\mathbb{R}^3}(\nabla\tilde{v}_n\nabla\varphi + V_\infty\tilde{v}_n\varphi)\,dx + \int_{\mathbb{R}^3}\phi_{\tilde{v}_n}\tilde{v}_n\varphi\,dx - \mu\int_{\mathbb{R}^3}|\tilde{v}_n|^{p-1}v_n\varphi\,dx - \int_{\mathbb{R}^3}|\widetilde{v_n}|^4 v_n\varphi\,dx \to 0$$

for any $\varphi \in H^1(\mathbb{R}^3)$. We take $\varphi = \tilde{v}_n$, using (3.30) and the boundedness of $(\widetilde{v_n})_n$, we have as $n \to \infty$

$$\|\nabla\widetilde{v_n}\|_2^2 - \|\widetilde{v_n}\|_6^6 = o_n(1). \tag{3.38}$$

We may assume that $\lim_{n\to\infty}\|\nabla\tilde{v}_n\|_2^2 = l \geq 0$. Thus, (3.38) gives that $\lim_{n\to\infty}\|\widetilde{v_n}\|_6^6 = l$. If $l > 0$, from the definition of $\mathcal{S}$, we have that

$$\mathcal{S} \leq \frac{\|\nabla\tilde{v}_n\|_2^2}{\|\widetilde{v_n}\|_6^6},$$

which implies that $l \geq \mathcal{S}^{\frac{3}{2}}$. Therefore, we get

$$\lim_{n\to\infty}\Big[\frac{1}{2}\|\nabla\widetilde{v_n}\|_2^2 - \frac{1}{6}\|\widetilde{v_n}\|_6^6\Big] = \frac{1}{3}l \geq \frac{1}{3}\mathcal{S}^{\frac{3}{2}}.$$

This contradicts (3.37). Hence, we have $l = 0$, that is $\widetilde{u_n} \to \tilde{u}$ in $H^1(\mathbb{R}^3)$ and so we conclude that $\tilde{u} \in \mathcal{M}_\mu^\infty$ and $\mathcal{J}_\mu^\infty(\tilde{u}) = c^\infty$. □

4 The Nonconstant Potential Case

Let $\Lambda = [\frac{1}{2}, 1]$. We consider a family of functionals $\mathcal{I}_{V,\lambda} : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{I}_{V,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)\,dx + \frac{q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx - \frac{\mu\lambda}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx - \frac{\lambda}{6}\int_{\mathbb{R}^3}|u|^6\,dx.$$

Let $\mathcal{I}_{V,\lambda}(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)\,dx + \frac{q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx \to \infty \quad \text{as } \|u\| \to \infty$$

and

$$B(u) = \frac{\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx + \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\,dx.$$

Let us first show that the family $\{\mathcal{I}_{V,\lambda}\}_{\lambda\in\Lambda}$ verifies all the assumptions of Theorem 2.10.

Lemma 4.1. *Suppose that* (V_1) *and* (V_2) *hold and that* $2 < p < 5$. *Then:*

(i) *There exists a* $v_0 \in H^1(\mathbb{R}^3) \setminus \{0\}$ *such that* $\mathcal{I}_{V,\lambda}(v_0) < 0$ *for any* $\lambda \in \Lambda$.

(ii) $c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}_{V,\lambda}(\gamma(t)) > \max\{\mathcal{I}_{V,\lambda}(0), \mathcal{I}_{V,\lambda}(v_0)\} = 0$ *for all* $\lambda \in \Lambda$, *where*

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(1) = v_0\}.$$

Proof. (i) Fix $v \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $\lambda \in \Lambda$. Then

$$\mathcal{I}_{V,\lambda}(v) \le \mathcal{I}_\lambda^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla v|^2 + V_\infty v^2)\,dx + \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_v v^2\,dx - \frac{\mu\lambda}{p+1}\int_{\mathbb{R}^3} |v|^{p+1}\,dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} |v|^6\,dx.$$

Set $v_\theta = \theta^2 v(\theta x)$ for all $\theta > 0$. Lemma 3.2 gives

$$\mathcal{I}_\lambda^\infty(v_\theta) \to -\infty \quad \text{as } \theta \to \infty.$$

Hence, take $v_0 = v_\theta$ for θ large, so that $\mathcal{I}_{V,\lambda}(v_0) \le \mathcal{I}_\lambda^\infty(v_0) < 0$.

(ii) The Sobolev embedding theorem and Lemma 2.4 yield

$$\mathcal{I}_{V,\lambda}(v) \ge \frac{1}{2}\|v\|^2 - \frac{\mu\lambda}{p+1}\int_{\mathbb{R}^3} |v|^{p+1}\,dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} |v|^6\,dx \ge \frac{1}{2}\|v\|^2 - C_1\|v\|^{p+1} - C_2\|v\|^6.$$

Since $p > 2$, we see that there exist $\beta > 0$ and $r_0 > 0$ such that

$$\mathcal{I}_{V,\lambda}(v) \ge \beta > 0 \quad \text{for all } v, \text{ with } \|v\| = r_0 \text{ and for any } \lambda \in \Lambda.$$

Therefore, for any $\gamma \in \Gamma$, there exists a $t_0 \in (0,1)$ such that $\|\gamma(t_0)\| = r_0$ and so

$$\max_{t\in[0,1]} \mathcal{I}_{V,\lambda}(\gamma(t)) \ge \mathcal{I}_{V,\lambda}(\gamma(t_0)) \ge \beta > \max\{\mathcal{I}_{V,\lambda}(0),\ \mathcal{I}_{V,\lambda}(v_0)\} = 0,$$

which implies that $c_\lambda > 0$. □

Thanks to Lemma 4.1 we can apply Theorem 2.10 and get that for a.e. $\lambda \in \Lambda$, there exists a bounded sequence $(u_n)_n \subset H^1(\mathbb{R}^3)$, which, for simplicity, we denote by $(u_n)_n$ instead of $(u_n(\lambda))_n$, such that

$$\mathcal{I}_{V,\lambda}(u_n) \to c_\lambda, \quad \mathcal{I}'_{V,\lambda}(u_n) \to 0$$

as $n \to \infty$. Next, we aim to prove the strong convergence of the above sequence $(u_n)_n$ in $H^1(\mathbb{R}^3)$.

Now, for any $\lambda \in \Lambda$ the limit problem corresponding to (1.4) is

$$\begin{cases} -\Delta u + V_\infty u + \phi(x)u = \mu\lambda|u|^{p-1}u + \lambda|u|^4 u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

with $2 < p < 5$, and has a ground state solution in $H^1(\mathbb{R}^3)$ by Theorem 1.2, i.e. for any $\lambda \in \Lambda$,

$$c_\lambda^\infty = \inf_{u\in\mathcal{M}_\lambda^\infty} \mathcal{I}_\lambda^\infty(u)$$

can be achieved at some $u_\lambda^\infty \in \mathcal{M}_\lambda^\infty$ and $(\mathcal{I}_\lambda^\infty)'(u_\lambda^\infty) = 0$, where

$$\mathcal{I}_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2)\,dx + \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{\mu\lambda}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} |u|^6\,dx,$$

$$\mathcal{M}_\lambda^\infty = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{G}_\lambda^\infty(u) = 0\}$$

and

$$\begin{aligned}\mathcal{G}_\lambda^\infty(u) = {} & \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty|u|^2\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx \\ & - \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_u u^2\,dx - \lambda\mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{3\lambda}{2}\int_{\mathbb{R}^3} |u|^6\,dx.\end{aligned}$$

Lemma 4.2. *Let $(u_n)_n$ be a bounded sequence in $H^1(\mathbb{R}^3)$ such that $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $r \in [1,6)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then, for any $p \in [1,5]$, there holds*

$$\int_{\mathbb{R}^3} (|u_n|^{p-1}u_n - |u|^{p-1}u)\varphi\,dx \to 0 \quad \textit{for all } \varphi \in C_0^\infty(\mathbb{R}^3).$$

Proof. By Hölder's inequality and, for any $m \geq 1$, by the elementary inequality $|a^m - b^m| \leq L|a-b|^m$ valid for all $a, b \geq 0$, and some appropriate $L = L(m) \geq 1$, we have for any $p \in [1,5]$,

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(|u_n|^{p-1}u_n - |u|^{p-1}u)\varphi\,dx\right| &\leq \int_{\mathbb{R}^3}|u_n|^{p-1}|u_n-u||\varphi|\,dx + \int_{\mathbb{R}^3}\big||u_n|^{p-1}-|u|^{p-1}\big||u\varphi|\,dx\\ &\leq \|\varphi\|_\infty\|u_n\|_p^{p-1}\bigg(\int_{\operatorname{supp}\varphi}|u_n-u|^p\,dx\bigg)^{\frac{1}{p}}\\ &\quad+\|\varphi\|_\infty\|u\|_p\bigg(\int_{\operatorname{supp}\varphi}\big||u_n|^{p-1}-|u|^{p-1}\big|^{\frac{p}{p-1}}\,dx\bigg)^{\frac{p-1}{p}}\\ &\leq \|\varphi\|_\infty\|u_n\|_p^{p-1}\bigg(\int_{\operatorname{supp}\varphi}|u_n-u|^p\,dx\bigg)^{\frac{1}{p}}\\ &\quad+C\|\varphi\|_\infty\|u\|_p\bigg(\int_{\operatorname{supp}\varphi}|u_n-u|^p\,dx\bigg)^{\frac{p-1}{p}}\to 0.\end{aligned}$$
□

Lemma 4.3 (Global Compactness Lemma). *Assume that* (V_1)–(V_3) *hold and let* $(u_n)_n$ *be a bounded* $(PS)_{c_\lambda}$ *sequence for the functional* $\mathcal{J}_{V,\lambda}$, *with* $c_\lambda < \frac{S^{3/2}}{3\sqrt{\lambda}}$ *for any* $\lambda \in \Lambda$. *Then there exist a subsequence of* $(u_n)_n$, *still denoted by* $(u_n)_n$, *an integer* $l \in \mathbb{N}\cup\{0\}$, *sequences* $(y_n^k) \subset \mathbb{R}^3$, *functions* $w^k \in H^1(\mathbb{R}^3)$ *for* $1 \leq k \leq l$ *such that*

(i) $u_n \rightharpoonup u_0$ *and* $\mathcal{J}'_{V,\lambda}(u_0) = 0$,

(ii) $|y_n^k| \to \infty$ *and* $|y_n^k - y_n^{k'}| \to \infty$ *for* $k \neq k'$,

(iii) $w^k \neq 0$ *and* $(\mathcal{J}_\lambda^\infty)'(w^k) = 0$ *for* $1 \leq k \leq l$,

(iv) $\|u_n - u_0 - \sum_{k=1}^l w^k(\cdot - y_n^k)\| \to 0$,

(v) $\mathcal{J}_{V,\lambda}(u_n) \to \mathcal{J}_{V,\lambda}(u_0) + \sum_{k=1}^l \mathcal{J}_\lambda^\infty(w^k)$.

Here we agree that in the case $l = 0$ *the above properties* (iv) *and* (v) *hold without* w^k *and* y_n^k.

Proof. We divide the proof into three steps.

Step 1. Since $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, we may assume that, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, $u_n \to u_0$ in $L^r_{\text{loc}}(\mathbb{R}^3)$ for $1 \leq r < 6$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$. Let us prove that $\mathcal{J}'_{V,\lambda}(u_0) = 0$. Noting that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, it is sufficient to check that $\mathcal{J}'_{V,\lambda}(u_0)[\varphi] = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Observe that

$$\begin{aligned}\mathcal{J}'_{V,\lambda}(u_n)[\varphi] - \mathcal{J}'_{V,\lambda}(u_0)[\varphi] &= \int_{\mathbb{R}^3}(\nabla(u_n-u_0)\nabla\varphi + V(x)(u_n-u_0)\varphi)\,dx + \int_{\mathbb{R}^3}(\phi_{u_n}u_n - \phi_{u_0}u_0)\varphi\,dx\\ &\quad-\mu\lambda\int_{\mathbb{R}^3}(|u_n|^{p-1}u_n - |u|^{p-1}u)\varphi\,dx - \lambda\int_{\mathbb{R}^3}(|u_n|^4u_n - |u_0|^4u_0)\varphi\,dx.\end{aligned} \tag{4.1}$$

Since $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, we have $\langle u_n - u_0, \varphi\rangle \to 0$.

By (ii) of Lemma 2.5 and Hölder's inequality we deduce that

$$\int_{\mathbb{R}^3}\phi_{u_n}u_n\varphi\,dx - \int_{\mathbb{R}^3}\phi_{u_0}u_0\varphi\,dx \to 0.$$

The definition of weak convergence, Lemma 4.2 and (4.1) give that

$$\mathcal{J}'_{V,\lambda}(u_n)[\varphi] - \mathcal{J}'_{V,\lambda}(u_0)[\varphi] \to 0.$$

This implies that $\mathcal{J}'_{V,\lambda}(u_0) = 0$. On the other hand, $\mathcal{J}_{V,\lambda}(u_0) \geq 0$. Indeed, put

$$a_0 = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2\,dx, \qquad b_0 = \frac{1}{2}\int_{\mathbb{R}^3}V(x)|u_0|^2\,dx, \quad c_0 = \frac{q^2}{4}\int_{\mathbb{R}^3}\phi_{u_0}u_0^2\,dx, \quad d_0 = \frac{1}{4}\int_{\mathbb{R}^3}\frac{q^2}{a}\psi_{u_0}u_0^2\,dx,$$

$$e_0 = \frac{\lambda\mu}{p+1}\int_{\mathbb{R}^3}|u_0|^{p+1}\,dx, \quad f_0 = \frac{\lambda}{6}\int_{\mathbb{R}^3}|u_0|^6\,dx, \qquad g_0 = \frac{1}{2}\int_{\mathbb{R}^3}(x, \nabla V(x))|u_0|^2\,dx.$$

Then we get the following linear system in $a_0, b_0, c_0, d_0, e_0, f_0, g_0$:

$$\begin{cases} a_0 + b_0 + c_0 - e_0 - f_0 = \mathcal{J}_{V,\lambda}(u_0), \\ 2a_0 + 2b_0 + 4c_0 - (p+1)e_0 - 6f_0 = 0, \\ a_0 + 3b_0 + 5c_0 + d_0 - 3e_0 - 3f_0 + g_0 = 0, \end{cases} \tag{4.2}$$

where the first equation comes from the definition of $\mathcal{J}_{V,\lambda}(u_0)$, the second is $(\mathcal{J}_{V,\lambda})'(u_0)[u_0] = 0$, and the last comes from the Pohozaev identity in Proposition 2.6. System (4.2) and assumption (V_1) yield that

$$3\mathcal{J}_{V,\lambda}(u_0) = (2b_0 + g_0) + d_0 + 2(p-2)e_0 + 6f_0 \geq 0.$$

Step 2. Set $v_n^1 = u_n - u_0$; then $v_n^1 \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$. By (i) of Lemma 2.4, we have

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx - \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \, dx - \int_{\mathbb{R}^3} \phi_{u_n - u_0} (u_n - u_0)^2 \, dx \to 0 \tag{4.3}$$

as $n \to \infty$. Moreover, it follows from the Brézis–Lieb lemma that

$$\begin{cases} \|v_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1), \\ \|v_n^1\|_{p+1}^{p+1} = \|u_n\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1} + o_n(1), \\ \|v_n^1\|_6^6 = \|u_n\|_6^6 - \|u_0\|_6^6 + o_n(1). \end{cases} \tag{4.4}$$

By (4.3) and (4.4), it is easy to check that

$$\mathcal{J}_{V,\lambda}(v_n^1) = \mathcal{J}_{V,\lambda}(u_n) - \mathcal{J}_{V,\lambda}(u_0) + o_n(1) \tag{4.5}$$

and as $n \to \infty$

$$\mathcal{J}'_{V,\lambda}(v_n^1)[(v_n)^1] = \mathcal{J}'_{V,\lambda}(u_n), \quad [u_n] - \mathcal{J}'_{V,\lambda}(u_0)[u_0] + o_n(1) = o_n(1). \tag{4.6}$$

Recalling that $\mathcal{J}_{V,\lambda}(u_0) \geq 0$, we have that

$$\mathcal{J}_{V,\lambda}(v_n^1) \leq \mathcal{J}_{V,\lambda}(u_n) < \frac{1}{3\sqrt{\lambda}} \mathcal{S}^{\frac{3}{2}}.$$

Let us introduce

$$\delta := \lim_{n\to\infty} \sup \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^1|^2 \, dx.$$

Case 1: $\delta = 0$. Namely,

$$\sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^1|^2 \, dx \to 0.$$

Using the Vanishing Lemma 2.7, we get $v_n^1 \to 0$ in $L^r(\mathbb{R}^3)$ for r, with $2 < r < 2_s^*$. Lemma 2.4 yields

$$\int_{\mathbb{R}^3} \phi_{v_n^1} (v_n^1)^2 \, dx \leq C \|v_n^1\|_{\frac{12}{5}}^4 \to 0. \tag{4.7}$$

Hence, (4.6) and (4.7) give as $n \to \infty$

$$\|v_n^1\|^2 = \lambda \|v_n^1\|_6^6 + o_n(1).$$

Up to a subsequence, we may assume that $\|v_n^1\|^2 \to \eta$ and $\lambda\|v_n^1\|_6^6 \to \eta$, with $\eta \geq 0$. Suppose that $\eta > 0$. By (4.5) and (4.7), we have

$$\mathcal{J}_{V,\lambda}(v_n^1) = c_\lambda - \mathcal{J}_{V,\lambda}(u_0) + o_n(1) \to \frac{\eta}{3}$$

and thus

$$\eta = 3(c_\lambda - \mathcal{J}_{V,\lambda}(u_0)) < \frac{1}{\sqrt{\lambda}} \mathcal{S}^{\frac{3}{2}}. \tag{4.8}$$

But, by (2.1)

$$\|v_n^1\|^2 \geq \int_{\mathbb{R}^3} |\nabla v_n^1|^2 \, dx \geq \mathcal{S} \Big(\int_{\mathbb{R}^3} |v_n^1|^6 \, dx \Big)^{\frac{1}{3}}.$$

This implies that

$$\eta \geq \frac{1}{\sqrt{\lambda}} \mathcal{S}^{\frac{3}{2}} \quad \text{for all } \lambda \in \Lambda.$$

This contradicts (4.8). Hence, $\lim_{n\to\infty} \|v_n^1\| = 0$.

Case 2: $\delta > 0$. We may assume that there exists $y_n^1 \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |v_n^1|^2 \, dx > \frac{\delta}{2} > 0,$$

Let us define $\widetilde{v_n^1}(\cdot) := v_n^1(\cdot + y_n^1)$. Then $(\widetilde{v_n^1})_n$ is bounded in $H^1(\mathbb{R}^3)$ and we may assume that $\widetilde{v_n^1} \rightharpoonup w^1$ in $H^1(\mathbb{R}^3)$ and $\widetilde{v_n^1} \to w^1$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $1 \leq r < 6$ and $\widetilde{v_n^1} \to w^1$ a.e. in $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |\widetilde{v_n^1}|^2 \, dx > \frac{\delta}{2},$$

we have

$$\int_{B_1(0)} |w^1|^2 \, dx > \frac{\delta}{2},$$

and $w^1 \neq 0$. But, since $v_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, it follows that $(y_n^1)_n$ must be unbounded. Up to a subsequence, we suppose that $|y_n^1| \to \infty$.

We claim that $(\mathcal{J}_\lambda^\infty)'(w^1) = 0$. Similar to the proof of (4.1), we see that

$$(\mathcal{J}_\lambda^\infty)'(\widetilde{v_n^1})[\varphi] - (\mathcal{J}_\lambda^\infty)'(w^1)[\varphi] \to 0$$

for any fixed $\varphi \in C_0^\infty(\mathbb{R}^3)$. Since $v_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, similar as the proof of (4.1), we obtain that

$$\mathcal{J}'_{V,\lambda}(v_n^1), [\varphi(\cdot - y_n^1)] - \mathcal{J}'_{V,\lambda}(0)[\varphi(\cdot - y_n^1)] \to 0,$$

which implies that

$$\mathcal{J}'_{V,\lambda}(v_n^1)[\varphi(\cdot - y_n^1)] \to 0. \tag{4.9}$$

By (V_2), for n large enough, we have

$$\int_{\mathbb{R}^3} V(x + y_n^1) v_n^1(x) \varphi(x) \, dx - \int_{\mathbb{R}^3} V_\infty \tilde{v}_n^1(x) \varphi(x) \, dx \to 0. \tag{4.10}$$

Thus, we use (4.9) minus $(\mathcal{J}_\lambda^\infty)'(\widetilde{v_n^1})[\varphi]$ and (4.10) to deduce that

$$(\mathcal{J}_\lambda^\infty)'(\widetilde{v_n^1})[\varphi] \to 0.$$

In view of (V_2) and the locally compactness of Sobolev embedding, we have

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)(u_n - u_0)^2 \, dx \to 0.$$

Thus, by (4.3), (4.4) and (4.5), we conclude that

$$\mathcal{J}_{V,\lambda}(u_n) - \mathcal{J}_{V,\lambda}(u_0) - \mathcal{J}_\lambda^\infty(v_n^1) \to 0. \tag{4.11}$$

Step 3. Let us set $v_n^2(\cdot) := v_n^1(\cdot) - w^1(\cdot - y_n^1)$; then $v_n^2 \to 0$ in $H^s(\mathbb{R}^3)$. From the Brézis–Lieb lemma and Lemma 2.5 we get as $n \to \infty$

$$\|v_n^2\|^2 = \|u_n\|^2 - \|u_0\|^2 - \|w^1(\cdot - y_n^1)\|^2 + o_n(1), \tag{4.12}$$

$$\|v_n^2\|_6^6 = \|u_n\|_6^6 - \|u_0\|_6^6 - \|w^1\|_6^6 + o_n(1), \tag{4.13}$$

$$\|v_n^2\|_{p+1}^{p+1} = \|u_n\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1} - \|w^1\|_{p+1}^{p+1} + o_n(1), \tag{4.14}$$

$$\int_{\mathbb{R}^3} \phi_{v_n^2} (v_n^2)^2 \, dx = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx - \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \, dx - \int_{\mathbb{R}^3} \phi_{w^1} (w^1)^2 \, dx + o_n(1), \tag{4.15}$$

$$\int_{\mathbb{R}^3} \phi_{v_n^2} v_n^2 \varphi \, dx = \int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi \, dx - \int_{\mathbb{R}^3} \phi_{u_0} u_0 \varphi \, dx - \int_{\mathbb{R}^3} \phi_{w^1(x - y_n^1)} w^1(x - y_n^1) \varphi \, dx + o_n(1), \tag{4.16}$$

for any $\varphi \in (H^1(\mathbb{R}^3))'$ and

$$\int V(x)|v_n^2|^2\,dx = \int V(x)|u_n|^2\,dx - \int V(x)|u_0|^2\,dx - \int_{\mathbb{R}^3} V(x)|w^1(x-y_n^1)|^2\,dx + o_n(1). \tag{4.17}$$

By (4.12)–(4.17), we can similarly check that

$$\begin{cases} \mathcal{J}_{V,\lambda}(v_n^2) = \mathcal{J}_{V,\lambda}(u_n) - \mathcal{J}_{V,\lambda}(u_0) - \mathcal{J}_\lambda^\infty(w^1) + o_n(1), \\ \mathcal{J}_\lambda^\infty(v_n^2) = \mathcal{J}_{V,\lambda}(v_n^1) - \mathcal{J}_\lambda^\infty(w^1) + o_n(1), \\ \mathcal{J}'_{V,\lambda}(v_n^2)[\varphi] = \mathcal{J}'_{V,\lambda}(u_n)[\varphi] - \mathcal{J}'_{V,\lambda}(u_0)[\varphi] - (\mathcal{J}_\lambda^\infty)'(w^1)[\varphi] + o_n(1) = o_n(1). \end{cases} \tag{4.18}$$

Hence, (4.11) gives as $n \to \infty$

$$\mathcal{J}_{V,\lambda}(u_n) = \mathcal{J}_{V,\lambda}(u_0) + \mathcal{J}_\lambda^\infty(v_n^1) + o_n(1) = \mathcal{J}_{V,\lambda}(u_0) + \mathcal{J}_\lambda^\infty(w^1) + \mathcal{J}_\lambda^\infty(v_n^2) + o_n(1).$$

Recalling that any critical point of $\mathcal{J}_\lambda^\infty$ is at nonnegative level, then $\mathcal{J}_\lambda^\infty(w^1) \geq 0$, and from Step 1, we know that $\mathcal{J}_{V,\lambda}(u_0) \geq 0$. Consequently,

$$\mathcal{J}_{V,\lambda}(v_n^2) = \mathcal{J}_{V,\lambda}(u_n) - \mathcal{J}_{V,\lambda}(u_0) - \mathcal{J}_\lambda^\infty(w^1) + o_n(1) \leq c_\lambda < \frac{1}{3\sqrt{\lambda}}\mathcal{S}^{\frac{3}{2}}.$$

Similar to the arguments in Step 2, let

$$\delta_1 = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^2|^2\,dx.$$

If $\delta_1 = 0$, then $\|v_n^2\| \to 0$, i.e. $\|u_n - u_0 - w^1(\cdot - y_n^1)\| \to 0$ and so Lemma 4.3 applies, with $k = 1$. If $\delta_1 > 0$, then there exists a sequence $(y_n^2)_n \subset \mathbb{R}^3$ and $w^2 \in H^1(\mathbb{R}^3)$ such that the sequence $\widetilde{v_n^2}(x) := v_n^2(x + y_n^2) \to w^2$ in $H^1(\mathbb{R}^3)$. Hence, $(\mathcal{J}_\lambda^\infty)'(w^2) = 0$ by (4.18). Furthermore, $v_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $|y_n^2| \to \infty$ and $|y_n^1 - y_n^2| \to \infty$. By iterating this procedure we obtain sequences of points $(y_n^k)_n \subset \mathbb{R}^3$ such that $|y_n^k| \to \infty$ and $|y_n^k - y_n^{k'}| \to \infty$ for $k \neq k'$ and $v_n^k = v_n^{k-1} - w^{k-1}(\cdot - y_n^{k-1})$, with $k \geq 2$, such that

$$v_n^k \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^3) \qquad \text{and} \qquad (\mathcal{J}_\lambda^\infty)'(w^k) = 0$$

and

$$\begin{cases} \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|w^j(\cdot - y_n^j)\|^2 = \|u_n - u_0 - \sum_{j=1}^{k-1} w^j(\cdot - y_n^j)\|^2, \\ \mathcal{J}_{V,\lambda}(u_n) - \mathcal{J}_{V,\lambda}(u_0) - \sum_{j=1}^{k-1} \mathcal{J}_\lambda^\infty(w^j) - \mathcal{J}_\lambda^\infty(v_n^k) = o_n(1). \end{cases} \tag{4.19}$$

Since $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, system (4.19) implies that the iteration stops at some finite index $l + 1$. Therefore, $v_n^{l+1} \to 0$ in $H^1(\mathbb{R}^3)$ by (4.19), and it is easy to verify that conclusions (iv) and (v) hold. The proof is completed. □

Based on Global Compactness Lemma 4.3, we can prove that the functional $\mathcal{J}_{V,\lambda}$ verifies the $(PS)_{c_\lambda}$ condition. That is, we have the following result.

Lemma 4.4. *Assume that* (V_1)–(V_3) *hold and* $2 < p < 5$, *and let* $(u_n)_n$ *be a bounded* $(PS)_{c_\lambda}$ *sequence for* $\mathcal{J}_{V,\lambda}$. *Then, up to a subsequence,* $(u_n)_n$ *converges to a nontrivial critical point* u_λ *of* $\mathcal{J}_{V,\lambda}$, *with* $\mathcal{J}_{V,\lambda}(u_\lambda) = c_\lambda$ *for any* $\lambda \in \Lambda$.

Proof. Fix any $\lambda \in \Lambda$. Let u_λ^∞ be the minimizer of c_λ^∞, so that $\mathcal{J}_\lambda^\infty(u_\lambda^\infty) = \max_{\theta\geq 0} \mathcal{J}_\lambda^\infty(\theta^2 u_\lambda^\infty(\theta x))$ by Lemma 3.1. Then choosing $v(x) = \theta^2 u_\lambda^\infty(\theta x)$ for θ large enough in Lemma 4.1, by (V_2) we have

$$c_\lambda \leq \max_{\theta\geq 0} \mathcal{J}_{V,\lambda}(\theta^2 u_\lambda^\infty(\theta x)) < \max_{\theta\geq 0} \mathcal{J}_\lambda^\infty(\theta^2 u_\lambda^\infty(\theta x)) = \mathcal{J}_\lambda^\infty(u_\lambda^\infty) = c_\lambda^\infty < \frac{1}{3\sqrt{\lambda}}\mathcal{S}^{\frac{3}{2}}. \tag{4.20}$$

By Lemma 4.3 there exist $l \in \mathbb{N} \cup \{0\}$ and $(y_n^k)_n \subset \mathbb{R}^3$, with $|y_n^k| \to \infty$ for each $1 \leq k \leq l$, and $u_\lambda \in H^1(\mathbb{R}^3)$, $w^k \in H^1(\mathbb{R}^3)$ such that

$$\mathcal{J}'_{V,\lambda}(u_\lambda) = 0, \quad u_n \rightharpoonup u_\lambda, \quad \mathcal{J}_{V,\lambda}(u_n) \to \mathcal{J}_{V,\lambda}(u_\lambda) + \sum_{k=1}^{l} \mathcal{J}_\lambda^\infty(w^k),$$

where w^j is a critical point of $\mathcal{I}_\lambda^\infty$ for $1 \le k \le l$. Set

$$a_\lambda = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\lambda|^2\,dx, \quad b_\lambda = \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_\lambda|^2\,dx, \quad c_\lambda = \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_{u_\lambda}u_\lambda^2\,dx, \quad d_\lambda = \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_{u_\lambda}u_\lambda^2\,dx,$$
$$e_\lambda = \frac{\mu\lambda}{p+1}\int_{\mathbb{R}^3} |u_\lambda|^{p+1}\,dx, \quad f_\lambda = \frac{\lambda}{6}\int_{\mathbb{R}^3} |u_\lambda|^6\,dx, \quad g_\lambda = \frac{1}{2}\int_{\mathbb{R}^3} (x, \nabla V(x))|u_\lambda|^2\,dx.$$

Then

$$\begin{cases} a_\lambda + b_\lambda + c_\lambda - e_\lambda - f_\lambda = \mathcal{I}_{V,\lambda}(u_\lambda), \\ 2a_\lambda + 2b_\lambda + 4c_\lambda - (p+1)e_\lambda - 6f_\lambda = 0, \\ a_\lambda + 3b_\lambda + 5c_\lambda + d_\lambda - 3e_\lambda - 3f_\lambda + g_\lambda = 0. \end{cases}$$

Similarly to the arguments used for handling system (4.2), we get

$$3\mathcal{I}_{V,\lambda}(u_\lambda) = (2b_\lambda + g_\lambda) + d_\lambda + (2p-4)e_\lambda + 6f_\lambda \ge 0.$$

Thus, if $l \ne 0$, we have

$$c_\lambda = \lim_{n\to\infty} \mathcal{I}_{V,\lambda}(u_n) = \mathcal{I}_{V,\lambda}(u_\lambda) + \sum_{k=1}^{l} \mathcal{I}_\lambda^\infty(w^k) \ge c_\lambda^\infty,$$

which contradicts (4.20). Hence, $l = 0$ and Lemma 4.3 yields that $u_n \to u_\lambda$ and $c_\lambda = \mathcal{I}_{V,\lambda}(u_\lambda)$. □

Proof of Theorem 1.1. From Lemma 4.1, it follows that for a.e. $\lambda \in \Lambda$ there exists a nontrivial critical point $u_\lambda \in H^1(\mathbb{R}^3)$ for $\mathcal{I}_{V,\lambda}$ and $\mathcal{I}_{V,\lambda}(u_\lambda) = c_\lambda$. Let us choose a sequence $\lambda_n \in [\frac{1}{2}, 1]$, with $\lambda_n \to 1$. Then there exists a sequence of nontrivial critical points $u_{\lambda_n})_n$ of $\mathcal{I}_{V,\lambda_n}$ and $\mathcal{I}_{V,\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$. Next we claim that $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^3)$. Set

$$a_{\lambda_n} = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx, \quad b_{\lambda_n} = \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_{\lambda_n}|^2\,dx, \quad c_{\lambda_n} = \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_{u_{\lambda_n}}u_{\lambda_n}^2\,dx, \quad d_{\lambda_n} = \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_{u_{\lambda_n}}u_{\lambda_n}^2\,dx,$$
$$e_{\lambda_n} = \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3} |u_{\lambda_n}|^{p+1}\,dx, \quad f_{\lambda_n} = \frac{\lambda_n}{6}\int_{\mathbb{R}^3} |u_{\lambda_n}|^6\,dx, \quad g_{\lambda_n} = \int_{\mathbb{R}^3} (x, \nabla V(x))|u_{\lambda_n}|^2\,dx.$$

Then

$$\begin{cases} a_{\lambda_n} + b_{\lambda_n} + c_{\lambda_n} - e_{\lambda_n} - f_{\lambda_n} = \mathcal{I}_{V,\lambda_n}(u_{\lambda_n}), \\ 2a_{\lambda_n} + 2b_{\lambda_n} + 4c_{\lambda_n} - (p+1)e_{\lambda_n} - 6f_{\lambda_n} = 0, \\ a_{\lambda_n} + 3b_{\lambda_n} + 5c_{\lambda_n} + d_{\lambda_n} - 3e_{\lambda_n} - 3f_{\lambda_n} + g_{\lambda_n} = 0. \end{cases} \tag{4.21}$$

Similarly to the arguments used for (4.2), we get

$$(2b_{\lambda_n} + g_{\lambda_n}) + d_{\lambda_n} + 2(p-2)e_{\lambda_n} + 6f_{\lambda_n} = 3\mathcal{I}_{V,\lambda_n}(u_{\lambda_n}) \le 3c_{\frac{1}{2}}$$

and

$$2(a_{\lambda_n} + b_{\lambda_n}) + (p-3)e_{\lambda_n} + 2f_{\lambda_n} = 4c_{\lambda_n} \le 4c_{\frac{1}{2}}.$$

In view of (V_1) we deduce that $(a_{\lambda_n} + b_{\lambda_n})_n$ is bounded, that is, $(u_{\lambda_n})_n$ is bounded in $H^1(\mathbb{R}^3)$. Therefore, using the fact that the map $\lambda \to c_\lambda$ is left continuous, we have

$$\lim_{n\to\infty} \mathcal{I}_{V,1}(u_{\lambda_n}) = \lim_{n\to\infty}\left(\mathcal{I}_{V,\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^3}\left(\frac{\mu}{p+1}|u_{\lambda_n}|^{p+1} + \frac{1}{6}|u_{\lambda_n}|^6\right)dx\right) = \lim_{n\to\infty} c_{\lambda_n} = c_1$$

and

$$\lim_{n\to\infty} \mathcal{I}'_{V,1}(u_{\lambda_n})[\varphi] = \lim_{n\to\infty}\left(\mathcal{I}'_{V,\lambda_n}(u_{\lambda_n})[\varphi] + (\lambda_n - 1)\int_{\mathbb{R}^3}(\mu|u_{\lambda_n}|^{p-1}u_{\lambda_n} + |u_{\lambda_n}|^4u_{\lambda_n})\varphi\,dx\right) = 0.$$

These show that $(u_{\lambda_n})_n$ is a bounded $(PS)_{c_1}$ sequence for $\mathcal{I}_V := \mathcal{I}_{V,1}$. Then by Lemma 4.4 there exists a nontrivial critical point $u \in H^1(\mathbb{R}^3)$ for $\mathcal{I}_V$ and $\mathcal{I}_V(u) = c_1$.

Finally, we prove the existence of a ground state solution for problem (1.4). Set

$$m = \inf\{\mathcal{I}_V(u) : u \neq 0,\ \mathcal{I}'_V(u) = 0\}.$$

Then $0 \le m \le \mathcal{I}_V(u) = c_1 < \infty$. We rule out the case $m = 0$. By contradiction, let $(u_n)_n$ be a $(\mathrm{PS})_0$ sequence for $\mathcal{I}_V$. Hence,

$$0 = \mathcal{I}'_V(u_n)[u_n] \ge \frac{1}{2}\|u_n\|^2 - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6\,dx,$$

which implies that

$$\|u_n\| \ge C > 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.22}$$

Since $\mathcal{I}'_V(u_n) = 0$ for any $n \in \mathbb{N}$, Proposition 2.6 and (V_1) give

$$\begin{aligned} 3\mathcal{I}_V(u_n) &= 3\mathcal{I}_V(u_n) - \mathcal{G}_V(u_n) \\ &= \mu\frac{2(p-2)}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx + \int_{\mathbb{R}^3} |u_n|^6\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \frac{q^2}{a}\psi_{u_n}u_n^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} (2V(x) + (x, \nabla V(x))u_n^2\,dx \\ &\ge \mu\frac{2(p-2)}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx + \int_{\mathbb{R}^3} |u_n|^6\,dx, \end{aligned}$$

where $\mathcal{G}_V$ is defined by

$$\begin{aligned} \mathcal{G}_V(u) = {} & \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} V(x)u^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2(x\cdot\nabla V(x))\,dx + \frac{3q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{q^2}{4a}\int_{\mathbb{R}^3} \psi_u u^2\,dx \\ & - \mu\frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx - \frac{3}{2}\int_{\mathbb{R}^3} |u|^6\,dx. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\|u_n\|_{p+1} = 0$ and $\lim_{n\to\infty}\|u_n\|_6 = 0$. Combining them with $\mathcal{I}'_V(u_n)[u_n] = 0$, it is easy to verify that $\lim_{n\to\infty}\|u_n\| = 0$. This contradicts (4.22).

In order to complete the proof, it suffices to prove that $\mathcal{I}_V$ can be achieved in $H^1(\mathbb{R}^3)$. Let $(u_n)_n$ be a sequence of nontrivial critical points of $\mathcal{I}_V$ satisfying $\mathcal{I}'_V(u_n) = 0$ and $\mathcal{I}_V(u_n) \to m < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}$. Since $(\mathcal{I}_V(u_n))_n$ is bounded, by similar arguments used as in (4.21), we conclude that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$ and so $(u_n)_n$ is a $(\mathrm{PS})_m$ sequence of $\mathcal{I}_V$. Arguing as the proof of Lemma 4.4, we show that there exists a nontrivial critical point $u \in H^1(\mathbb{R}^3)$ of $\mathcal{I}_V$, with $\mathcal{I}_V(u) = m$. □

Funding: Lin Li is supported by Research Fund of National Natural Science Foundation of China (No. 11861046), China Postdoctoral Science Foundation (No. 2019M662796), Chongqing Municipal Education Commission (No. KJQN20190081). Patrizia Pucci was partly supported by the *Fondo Ricerca di Base di Ateneo – Esercizio 2017–2019* of the University of Perugia, named *PDEs and Nonlinear Analysis*. Patrizia Pucci is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM) and the manuscript was realized within the auspices of the INdAM–GNAMPA Projects *Equazioni alle derivate parziali: problemi e modelli* (Prot_U-UFMBAZ-2020-000761). Xianhua Tang is supported by the National Natural Science Foundation of China (No. 11971485).

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