Research Article

Pierluigi Benevieri* and Antonio Iannizzotto

Eigenvalue Problems for Fredholm Operators with Set-Valued Perturbations

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Abstract: By means of a suitable degree theory, we prove persistence of eigenvalues and eigenvectors for set-valued perturbations of a Fredholm linear operator. As a consequence, we prove existence of a bifurcation point for a non-linear inclusion problem in abstract Banach spaces. Finally, we provide applications to differential inclusions.

Keywords: Fredholm Operators, Eigenvalue Problems, Set-Valued Maps

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1 Introduction

The present paper is devoted to the study of the following eigenvalue problem with a set-valued perturbation:

$$\begin{cases} Lx - \lambda Cx + \varepsilon\phi(x) \ni 0, \\ \qquad\qquad\qquad x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $L : E \to F$ is a Fredholm linear operator of index 0 between two real Banach spaces E and F such that $\ker L \neq 0$, C is another bounded linear operator, Ω is an open subset of E not necessarily bounded and containing 0, $\phi : \overline{\Omega} \to 2^F$ is a locally compact, upper semicontinuous (u.s.c. for short) set-valued map of CJ-type (see Section 4 for a precise definition), and $\lambda, \varepsilon \in \mathbb{R}$ are parameters.

Problem (1.1) can be seen as a set-valued perturbation of a linear eigenvalue problem (which is retrieved for $\varepsilon = 0$):

$$\begin{cases} Lx - \lambda Cx = 0, \\ \qquad\quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

So, it is reasonable to expect that, under suitable assumptions, solutions of (1.1) appear in a neighborhood of the eigenpairs (x, λ) of (1.2). In fact, we show that this is the case for the trivial eigenpairs $(x, 0)$, provided $\dim(\ker L)$ is odd, the set $\overline{\Omega} \cap \ker L$ is compact, and the following transversality condition holds:

$$\operatorname{im} L + C(\ker L) = F. \tag{1.3}$$

More precisely, we denote $\mathcal{S}_0 = \partial\Omega \cap \ker L$ the set of trivial solutions of (1.2). We prove that there exist a rectangle $\mathcal{R} = [-a, a] \times [-b, b]$ $(a, b > 0)$ and $c > 0$ such that for all $\varepsilon \in [-a, a]$ the set of real parameters $\lambda \in [-b, b]$ for which (1.1) admits a nontrivial solution $x \in E$ with $\operatorname{dist}(x, \mathcal{S}_0) < c$ is nonempty and

***Corresponding author: Pierluigi Benevieri** Instituto de Matemática e Estatística, Universidade de São Paulo, rua do Matão 1010, 05508-090 São Paulo, Brazil, e-mail: pluigi@ime.usp.br. https://orcid.org/0000-0002-4365-5379
Antonio Iannizzotto Department of Mathematics and Computer Science, University of Cagliari, Via Ospedale 72, 09124 Cagliari, Italy, e-mail: antonio.iannizzotto@unica.it. https://orcid.org/0000-0002-8505-3085

depends on ε by means of an u.s.c. set-valued map. Similarly, for all $\varepsilon \in [-a, a]$ the set of vectors $x \in E$ with $\operatorname{dist}(x, \mathcal{S}_0) < c$ that solve (1.1) for some $\lambda \in [-b, b]$ is nonempty and depends on ε by means of an u.s.c. set-valued map. This is usually referred to as a *persistence result* for eigenpairs. Using such persistence, we prove that $\mathcal{S}_0$ contains at least one bifurcation point, i.e., a trivial solution x_0 such that any neighborhood of x_0 in E contains a nontrivial solution.

The origin of this type of investigation of nonlinear eigenvalue problems goes back to a work of Chiappinelli [11], in which the author investigates a persistence property of the eigenvalues and eigenvectors of the system

$$\begin{cases} Lx + \varepsilon N(x) = \lambda x, \\ \|x\| = 1, \end{cases} \tag{1.4}$$

where L is a self-adjoint operator defined on a real Hilbert space H, $N\colon H \to H$ is a nonlinear continuous (single-valued) map, ε, λ still are real parameters. Under the assumptions that $\lambda_0 \in \mathbb{R}$ is an isolated *simple* eigenvalue of L and that N is Lipschitz continuous, Chiappinelli proves that there exist two H-valued Lipschitz curves, $\varepsilon \mapsto x_\varepsilon^1$ and $\varepsilon \mapsto x_\varepsilon^2$, defined in a neighborhood V of 0 in $\mathbb{R}$, as well as two real Lipschitz functions, $\varepsilon \mapsto \lambda_\varepsilon^1$ and $\varepsilon \mapsto \lambda_\varepsilon^2$, such that for $i = 1, 2$ and $\varepsilon \in V$ one has

$$Lx_\varepsilon^i + \varepsilon N(x_\varepsilon^i) = \lambda_\varepsilon^i x_\varepsilon^i, \quad \|x_\varepsilon^i\| = 1,$$

i.e., the triples $(x_\varepsilon^i, \varepsilon, \lambda_\varepsilon^i)$ solve (1.4) for all $\varepsilon \in V$. In particular, when $\varepsilon = 0$, these four functions satisfy $x_0^i = x^i$, $\lambda_0^i = \lambda_0$, where x^1 and x^2 are the two unit eigenvectors of L corresponding to the simple eigenvalue λ_0. After the result of Chiappinelli, in a series of papers [12–15] the above property of local persistence of the eigenvalues and eigenvectors was extended to the case in which the multiplicity of the eigenvalue λ_0 is bigger than one.

In particular, in [4] the first author, with Calamai, Furi, and Pera, proved a persistence result for (1.2) under a single-valued nonlinear map in general Banach spaces. The approach in [4] is topological, based on a concept of degree, developed in [2, 3], for a class of noncompact (single-valued) perturbations of Fredholm maps of index zero between Banach spaces.

We proceed here in the general spirit of [4], extending the result to the case of a set-valued perturbation. Such an extension requires a more general degree theory for set-valued maps, which extends Brouwer's degree for nonlinear maps on C^1-manifolds. Such a degree theory has been introduced in [30] and redefined in [9] by a precise notion of orientation for set-valued perturbations of nonlinear Fredholm maps between Banach spaces. The concept of orientation used in [9] (and reproduced here) is a natural extension of a notion of orientation for nonlinear Fredholm maps in Banach spaces presented in [5, 6] and on which is also based the approach in [4]. This orientation actually simplifies the method followed to define the degree in [30], based on the so called concept of oriented Fredholm structure, introduced by Elworty and Tromba in [19, 20] (where an orientation is constructed on the source and targets Banach spaces and manifolds).

Our abstract results find a natural application to *differential inclusions*. This type of problems, arising from control theory and differential equations with discontinuous nonlinearities (see [1, 10, 21]), extends classical differential equations by means of set-valued terms usually representing some degree of uncertainty of the problem. Problem (1.1) can model several differential inclusions, with L being a Fredholm differential operator of index 0 between two function spaces, C being some linear operator, $\partial\Omega$ representing some constraint, and ϕ being a set-valued mapping satisfying convenient conditions.

To fix ideas, we will consider the following ordinary differential inclusion with Neumann boundary conditions and an integral constraint:

$$\begin{cases} u'' + u' - \lambda u + \varepsilon \Phi(u) \ni 0 & \text{in } [0, 1], \\ u'(0) = u'(1) = 0, \\ \|u\|_1 = 1. \end{cases}$$

Here $\Phi(u) : [0, 1] \to 2^{\mathbb{R}}$ is a set-valued map depending on u, to be chosen according to several requirements (three different examples will be presented). We shall prove that the transversality condition (1.3) holds, and hence the above problem admits at least one bifurcation point. This is by no means the only possible application of our method, for instance one could define the operator

$$Lu = -\Delta u - \lambda_1 u,$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the negative Laplacian on a domain with homogeneous Dirichlet conditions. Then $\ker L$ has dimension 1, hence, by appropriately choosing the function spaces E and F, one can rephrase L as a Fredholm operator of index 0, and accordingly define C and ϕ in (1.1). Other examples for the dynamic part of the problem are shown in [4]. Nevertheless, since the novelty of our work lies in the set-valued term, we will restrict our attention on the above inclusion problem, focusing on the possible choices of Φ.

For the convenience of the reader, most of our paper (Sections 2–5) is devoted to the construction of the orientation and degree for the set-valued perturbations of Fredholm maps. Then, in Section 6, we prove our persistence and bifurcation results. Finally, in the large Section 7 we will prove bifurcation results for differential inclusions.

Notation. Whenever E, F are Banach spaces, we denote by $\mathcal{L}(E, F)$ the space of bounded linear operators from E into F (in particular, $\mathcal{L}(E) = \mathcal{L}(E, E)$). We shall use the term *operator* for linear functions, and *map* for nonlinear ones.

2 A Remark on Orientation and Transversality

In this preliminary section we recall some facts regarding the classical notions of orientation and transversality in finite dimension. We assume that the reader is familiar with the notion of orientation for finite-dimensional Banach manifolds and spaces. Let M be a real C^1-manifold, and let F be a real vector space such that

$$\dim(M) = \dim(F) < \infty.$$

Definition 2.1. A subspace $F_1 \subseteq F$ and a map $g \in C^1(M, F)$ are *transverse* if for all $x \in M$,

$$\operatorname{im} Dg(x) + F_1 = F.$$

In the situation described above, the map g is backward orientation-preserving on F_1:

Lemma 2.2. *Let M, F be oriented, and let $F_1 \subseteq F$, $g \in C^1(M, F)$ be transverse. Then any orientation of F_1 induces an orientation of $M_1 = g^{-1}(F_1)$.*

Proof. Since g is C^1, M_1 is a C^1-submanifold of M with

$$\dim(M_1) = \dim(F_1).$$

Fix $x \in M_1$, and let $T_x(M)$, $T_x(M_1)$ be the tangent spaces to M, M_1, respectively, at x. Then we have

$$T_x(M_1) = (Dg(x))^{-1}(F_1).$$

Let E_0 be a direct complement to $T_x(M_1)$ in $T_x(M)$, and $F_0 = Dg(x)(E_0)$. The restriction $Dg(x)|_{E_0} \in \mathcal{L}(E_0, F_0)$ is an isomorphism and $F_0 \oplus F_1 = F$. Now let F_1 be oriented so that any two positively oriented bases of F_0, F_1 (in this order) form a positively oriented basis of F. Thus, we can orient E_0 so that $Dg(x)|_{E_0}$ is orientation-preserving.

Similarly, we can orient $T_x(M_1)$ so that any two positively oriented bases of E_0, $T_x(M_1)$ (in this order) form a positively oriented basis of $T_x(M)$. Then this pointwise choice induces a global orientation on M_1 (see [24, p. 100] for details). □

By Lemma 2.2 we have a natural way to orient M_1:

Definition 2.3. Let M, F, $F_1 \subseteq F$ be oriented and $g \in C^1(M, F)$ transverse to F_1. The manifold $M_1 = g^{-1}(F_1)$, with the orientation induced by that of F_1 is an *oriented g-preimage* of F_1.

Now let $f \in C(M, F)$, and choose $y \in F$ such that $f^{-1}(y) \subset M$ is compact. Brouwer's degree for the triple (f, M, y) is defined and denoted by

$$\deg_B(f, M, y) \in \mathbb{Z}.$$

For the definition and properties of Brouwer's degree (both on open sets and manifolds) we refer to [28, 29]. We only need to add the following reduction property:

Proposition 2.4. *Let $M, F, F_1 \subset F$ be oriented, let $g \in C^1(M, F)$ be transverse to F_1, and let M_1 be the oriented g-preimage of F_1. In addition, let $f \in C(M, F)$, $y \in F_1$ be such that $f^{-1}(y)$ is compact and*

$$(f - g)(M) \subseteq F_1.$$

Finally, let $f_1 = f|_{M_1}$. Then

$$\deg_B(f, M, y) = \deg_B(f_1, M_1, y).$$

Proof. First we note that for all $x \in M_1$,

$$f(x) = g(x) + (f - g)(x) \in F_1,$$

so $f_1 \in C(M_1, F_1)$. In particular, $f_1^{-1}(y) = f^{-1}(y)$ is a compact subset of M_1. We orient M_1 and F_1 as in Lemma 2.2, so we can define Brouwer's degree for the triple (f_1, M_1, y). Now, the conclusion follows from [28, Lemma 4.2.3]. □

3 Orientation for Fredholm Maps

In order to develop a degree theory, we need a precise notion of *orientability* for Fredholm operators and maps. The one we are going to recall here was introduced in [5, 6].

Let E, F be two (possibly, infinite-dimensional) real Banach spaces. We first recall a basic definition:

Definition 3.1. A bounded linear operator $L \in \mathcal{L}(E, F)$ is a *Fredholm operator* of index $k \in \mathbb{Z}$ if
(i) $\dim(\ker L), \dim(\operatorname{coker} L) < \infty$,
(ii) $\dim(\ker L) - \dim(\operatorname{coker} L) = k$.
The set of such operators is denoted $\Phi_k(E, F)$.

It is known that $\Phi_k(E, F) \subset \mathcal{L}(E, F)$ is open for all $k \in \mathbb{Z}$. We are mainly interested in $\Phi_0(E, F)$, the set of Fredholm operators of index 0, also denoted Φ_0-operators. The following construction leads to a notion of orientation for such operators:

Definition 3.2. Let $L \in \Phi_0(E, F)$, $A \in \mathcal{L}(E, F)$. Then A is a *corrector* of L if
(i) $\dim(\operatorname{im} A) < \infty$ (finite rank),
(ii) $L + A \in \mathcal{L}(E, F)$ is an isomorphism.
The set of correctors of L is denoted $\mathcal{C}(L)$.

Clearly, $\mathcal{C}(L) \neq \emptyset$ for all $L \in \Phi_0(E, F)$. Following [5], we define an equivalence relation in $\mathcal{C}(L)$. Let $A, B \in \mathcal{C}(L)$, and set

$$T = (L + B)^{-1}(L + A), \quad K = I - T = (L + B)^{-1}(B - A).$$

By Definition 3.2, $T \in \mathcal{L}(E)$ is an automorphism, and $K \in \mathcal{L}(E)$ has finite rank. Let $E_0 \subseteq E$ be a non-trivial finite-dimensional subspace such that $\operatorname{im} K \subseteq E_0$, and set $T_0 = T|_{E_0}$. We note that $T_0 \in \mathcal{L}(E_0)$ and is an automorphism as well. Indeed, T_0 is injective by injectivity of T, and for all $x \in E_0$ we have

$$T_0(x) = x - K(x) \in E_0,$$

so T_0 is surjective as well (recall that $\dim(E_0) < \infty$). Thus, as soon as we fix a basis for E_0, the determinant of T_0 is well defined and denoted $\det T_0 \in \mathbb{R} \setminus \{0\}$. A remarkable fact is that $\det T_0$ does not depend on the choice of E_0 (by choosing the same basis in E_0 both as the domain and as the codomain of T_0), so we can provide T with a uniquely defined determinant by setting

$$\det T = \det T_0.$$

The above notion of determinant for linear operators between (possibly) infinite dimensional spaces can be found in [27].

Definition 3.3. Let $L \in \Phi_0(E, F)$. Two correctors $A, B \in \mathcal{C}(L)$ are *L-equivalent* if

$$\det((L + B)^{-1}(L + A)) > 0.$$

It is easily seen that L-equivalence is actually an equivalence relation, splitting $\mathcal{C}(L)$ into two equivalence classes. Now we can define a notion of orientation for Φ_0-operators:

Definition 3.4. Let $L \in \Phi_0(E, F)$.
(i) An *orientation* of L is any L-equivalence class $\alpha \subset \mathcal{C}(L)$, then the pair (L, α) is an *oriented* Φ_0-operator, and a corrector $A \in \mathcal{C}(L)$ is *positive* for (L, α) if $A \in \alpha$, *negative* if $A \in \mathcal{C}(L) \setminus \alpha$.
(ii) If L is an isomorphism, then $\alpha \subset \mathcal{C}(L)$ is the *natural orientation* of L if $0 \in \alpha$, and in such case (L, α) is *naturally oriented*.
(iii) If (L, α) is an oriented Φ_0-operator, its *sign* is defined as follows:

$$\operatorname{sign}(L, \alpha) = \begin{cases} +1 & \text{if } (L, \alpha) \text{ is a naturally oriented isomorphism,} \\ -1 & \text{if } (L, \alpha) \text{ is a non-naturally oriented isomorphism,} \\ 0 & \text{if } (L, \alpha) \text{ is not an isomorphism.} \end{cases}$$

Let (L, α) be an oriented Φ_0-operator, and let $A \in \alpha$ be a positive corrector. Since the set of isomorphisms is open in $\mathcal{L}(E, F)$, we can find a neighborhood $\mathcal{U} \subset \Phi_0(E, F)$ of L such that $A \in \mathcal{C}(T)$ for all $T \in \mathcal{U}$. Then any operator $T \in \mathcal{U}$ can be oriented so that $A \in \mathcal{C}(T)$ is a positive corrector. In such a way, any orientation of L induces orientations of nearby Φ_0-operators, which allows us to define orientability of $\Phi_0(E, F)$-valued maps:

Definition 3.5. Let X be a topological space, $h \in C(X, \Phi_0(E, F))$. An *orientation* of h is a map α that associates to every $x \in X$ an orientation, say $\alpha(x)$, of $h(x) \in \Phi_0(E, F)$ satisfying the following continuity condition: there exist $A \in \alpha(x)$ and a neighborhood $V \subset X$ of x such that $A \in \alpha(y)$ for all $y \in V$. The map h is *orientable* if it admits an orientation, and in such case (h, α) is an *oriented* $\Phi_0(E, F)$-valued map.

Now we can consider (nonlinear) Fredholm maps:

Definition 3.6. Let $\Omega \subseteq E$ be open. A map $g \in C^1(\Omega, F)$ is a Φ_0*-map* if $Dg(x) \in \Phi_0(E, F)$ for all $x \in \Omega$.

For instance, any Fredholm operator $L \in \Phi_0(E, F)$ is a Φ_0-map, since $DL(x) = L$ for all $x \in E$.

Definition 3.7. Let $\Omega \subseteq E$ be open, and let $g \in C^1(\Omega, F)$ be a Φ_0-map.
(i) An *orientation* of g is any orientation of $Dg \in C(\Omega, \Phi_0(E, F))$ (Definition 3.5).
(ii) The map g is *orientable* if it admits an orientation α, and in such case (g, α) is an *oriented* Φ_0-map.

The existence (and number) of orientations of a Φ_0-map depend mainly on the topology of its domain (see [5] for the proof):

Proposition 3.8. *Let $\Omega \subseteq E$ be open, and let $g \in C^1(\Omega, F)$ be a Φ_0-map.*
(i) *If g is orientable, then it admits at least two orientations.*
(ii) *If g is orientable and Ω is connected, then g admits exactly two orientations.*
(iii) *If Ω is simply connected, then g is orientable.*

Another important use of Definition 3.5 is towards orientation of Fredholm homotopies:

Definition 3.9. Let $\Omega \subseteq E$ be open. A map $h \in C(\Omega \times [0, 1], F)$ is a Φ_0*-homotopy* if
(i) $h(\cdot, t)$ is a Φ_0-map for all $t \in [0, 1]$,
(ii) the map $(x, t) \mapsto D_x h(x, t)$ is continuous from $\Omega \times [0, 1]$ into $\Phi_0(E, F)$, where we denote by $D_x h(x, t)$ the derivative of $h(\cdot, t)$ at x.

Note that no differentiability in t is required. Condition (ii) here is crucial, as it allows us to apply Definition 3.5 to the map $(x, t) \mapsto D_x h(x, t)$, and thus define a notion of orientation for Φ_0-homotopies:

Definition 3.10. Let $\Omega \subseteq E$ be open, and let $h \in C(\Omega \times [0, 1], F)$ be a Φ_0-homotopy.
(i) An *orientation* of h is any orientation of $D_x h \in C(\Omega \times [0, 1], \Phi_0(E, F))$ (Definition 3.5).
(ii) The homotopy h is *orientable* if it admits an orientation α, and in such case (h, α) is an *oriented* Φ_0-homotopy.

Let (h, α) be an oriented Φ_0-homotopy. Clearly, α induces an orientation α_t of the Φ_0-map $h(\cdot, t)$, for all $t \in [0, 1]$. Remarkably, the converse is also true, as shown by the following result on continuous transportation of orientations (see [5, Theorem 3.14]):

Proposition 3.11. *Let $\Omega \subseteq E$ be open, let $h \in C(\Omega \times [0, 1], F)$ be a Φ_0-homotopy, and let $t \in [0, 1]$ be such that $h(\cdot, t) \in C^1(\Omega, F)$ admits an orientation α_t. Then there exists a unique orientation α of h which induces α_t.*

We conclude this section by establishing a link between the orientation of Fredholm maps and that of manifolds:

Proposition 3.12. *Let $\Omega \subseteq E$ be open, let $g \in C^1(\Omega, F)$ be an orientable Φ_0-map, let $F_1 \subseteq F$ be a finite-dimensional subspace, transverse to g, and let $M_1 = g^{-1}(F_1)$. Then:*
(i) *$M_1 \subseteq E$ is a C^1-manifold with* $\dim(M_1) = \dim(F_1)$.
(ii) *M_1 is orientable.*
(iii) *Any orientation of g and any orientation of F_1 induce an orientation of M_1.*

Proof. Assertion (i) is obvious (see Section 2). Assertion (ii) follows from [5, Remark 2.5, Lemma 3.1].

We prove (iii). Let α be an orientation of g, and $x \in M_1$. By Definition 3.7, $\alpha(x)$ is an orientation of $Dg(x) \in \Phi_0(E, F)$. By transversality (Definition 2.1), we can find $A \in \alpha(x)$ such that $\operatorname{im} A \subseteq F_1$. Indeed, since $Dg(x) \in \Phi_0(E, F)$, we can split both Banach spaces as follows:

$$E = Dg(x)^{-1}(F_1) \oplus E_2, \quad F = F_1 \oplus F_2,$$

where E_2 is any direct complement of $Dg(x)^{-1}(F_1)$ and $F_2 := Dg(x)\,(E_2)$. Observe that $\ker Dg(x) \subseteq Dg(x)^{-1}(F_1)$ and the latter has the same dimension as F_1. So we rephrase $Dg(x)$ as

$$Dg(x) = \begin{bmatrix} L_{1,1} & 0 \\ 0 & L_{2,2} \end{bmatrix},$$

where $L_{2,2} \in \mathcal{L}(E_2, F_2)$ is an isomorphism. We may choose $A \in \mathcal{L}(E, F)$ with the structure

$$A = \begin{bmatrix} A_{1,1} & 0 \\ 0 & 0 \end{bmatrix},$$

where $A_{1,1} + L_{1,1} \in \mathcal{L}(L^{-1}(F_1), F_1)$ is an isomorphism. So $A \in \mathcal{C}(Dg(x))$ and $\operatorname{im} A \subseteq F_1$. Choosing $A_{1,1}$ in such a way that $A \in \alpha(x)$ and assigning an orientation to F_1, we orient the tangent space $T_x(M_1) \subset E$ so that the isomorphism

$$(Dg(x) + A)_{|T_x(M_1)} \in \mathcal{L}(T_x(M_1), F_1)$$

is orientation-preserving. As proved in [7], such orientation of $T_x(M_1)$ does not depend on A. This pointwise choice induces a global orientation on M_1. □

We can now give a Fredholm analogue of Definition 2.3:

Definition 3.13. Let $\Omega \subseteq E$ be open, let (g, α) be an oriented Φ_0-map, let $F_1 \subseteq F$ be a finite-dimensional subspace, transverse to g, and let $M_1 = g^{-1}(F_1)$. With the orientation induced by α and the orientation of F_1, M_1 is an *oriented* (Φ_0, g)*-preimage* of F_1.

Remark 3.14. In what follows, we will denote an oriented Φ_0-operator (L, α) simply by L, as long as no confusion arises. We will do the same for oriented Φ_0-maps, Φ_0-homotopies, and so on.

4 Topological Properties of Set-Valued Maps

In this section, for the reader's convenience, we recall some definitions and properties of set-valued maps between metric spaces, referring to [22] for details. Let X, Y be metric spaces with distance functions d_X, d_Y, respectively. Then $X \times Y$ is a metric space under the distance

$$d((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}.$$

For all $A \subset X$, $x \in X$ we set

$$\operatorname{dist}(x,A) = \inf_{z \in A} d_X(x,z),$$

and for all $\varepsilon > 0$ we set

$$B_\varepsilon(A) = \{x \in X : \operatorname{dist}(x,A) < \varepsilon\}$$

(if $A = \{x\}$, then we set $B_\varepsilon(A) = B_\varepsilon(x)$). A set-valued map $\phi : X \to 2^Y$ is a map from X to the set of all parts of Y. *We will always assume that ϕ is compact-valued, i.e., that $\phi(x) \subseteq Y$ is either $\emptyset$ or compact, for all $x \in X$.* The graph of ϕ is defined by

$$\operatorname{graph}\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}.$$

We also recall a classical definition:

Definition 4.1. A set-valued map $\phi : X \to 2^Y$ is *upper semicontinuous (u.s.c.)* if for all open $V \subseteq Y$ the set

$$\phi^+(V) = \{x \in X : \phi(x) \subseteq V\}$$

is open.

Any (single-valued) map $f : X \to Y$ coincides with the set-valued map $\phi(x) = \{f(x)\}$, in such case ϕ is u.s.c. iff f is continuous. A remarkable property of u.s.c. set-valued maps is that they preserve compactness, in the following sense: if $\phi : X \to 2^Y$ is a u.s.c. set-valued map and C is a compact subset of X, then $\phi(C)$ is compact.

Remark 4.2. Any compact subset of a product space can be seen as the graph of a u.s.c. set-valued mapping. Precisely, if $\mathcal{K} \subset X \times Y$ is compact, define for all $x \in X$,

$$\phi(x) = \{y \in Y : (x,y) \in \mathcal{K}\}.$$

Then $\phi : X \to 2^Y$ is u.s.c. (see [22, Proposition 14.5] or [26, Theorem 1.1.5]).

We introduce the notion of *approximability*:

Definition 4.3. Let $\phi : X \to 2^Y$.
(i) For all $\varepsilon > 0$, $f \in C(X,Y)$ is an *ε-approximation* of ϕ if for all $x \in X$ there exists $x' \in B_\varepsilon(x)$ such that $f(x) \in B_\varepsilon(\phi(x'))$ (the set of ε-approximations of ϕ is denoted $B_\varepsilon(\phi)$).
(ii) ϕ is *approximable* if $B_\varepsilon(\phi) \neq \emptyset$ for all $\varepsilon > 0$.

Note that all approximations of a set-valued map are required to be *continuous*. A characterization (whose proof is an obvious consequence of Definition 4.3):

Lemma 4.4. *Let $\phi : X \to 2^Y$, $\varepsilon > 0$, $f \in C(X,Y)$. Then the following are equivalent:*
(i) $f \in B_\varepsilon(\phi)$.
(ii) $f(x) \in B_\varepsilon(\phi(B_\varepsilon(x)))$ *for all* $x \in X$.
(iii) $\operatorname{graph} f \subseteq B_\varepsilon(\operatorname{graph}\phi)$.

Approximation of an u.s.c. set-valued map is a special case, enjoying several properties (see [22, Proposition 22.3]):

Proposition 4.5. *Let $\phi : X \to 2^Y$ be u.s.c. Then:*
(i) *For all compact $X_1 \subseteq X$, $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in B_\delta(\phi)$ we have $f|_{X_1} \in B_\varepsilon(\phi|_{X_1})$.*
(ii) *If X is compact, then for any metric space Z, $g \in C(Y,Z)$, and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in B_\delta(\phi)$ we have $g \circ f \in B_\varepsilon(g \circ \phi)$.*

(iii) *If X is compact, then for any u.s.c. set-valued map $\psi : X \times [0,1] \to 2^Y$, $\varepsilon > 0$, and $t \in [0,1]$ there exists $\delta > 0$ such that for all $f \in B_\delta(\psi)$ we have $f(\cdot, t) \in B_\varepsilon(\psi(\cdot, t))$.*
(iv) *For any metric space Z, any u.s.c. set-valued map $\psi : X \to 2^Z$, and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in B_\delta(\phi)$, $g \in B_\delta(\psi)$ we have $(f, g) \in B_\varepsilon(\phi \times \psi)$.*

Approximability of a set-valued map is strongly influenced by the topology of its values, the easiest case being in general that of convex-valued maps between Banach spaces. In the general case of a metric space, convexity makes no sense and it must be replaced by a more general notion, of topological nature. We recall from [22] some definitions and properties (here $\mathbb{S}^{n-1}$, $\mathbb{B}^n$ denote the unit sphere and closed ball, respectively, in $\mathbb{R}^n$):

Definition 4.6. A set $A \subset Y$ is *aspheric* if for any $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for all $n \in \mathbb{N}$ and all $g \in C(\mathbb{S}^{n-1}, B_\delta(A))$ there is $\tilde{g} \in C(\mathbb{B}^n, B_\varepsilon(A))$ such that $\tilde{g}|_{\mathbb{S}^{n-1}} = g$.

The following characterization of aspheric sets holds in ANR-spaces (*absolute neighborhood retracts*, see [22, Definition 1.7]):

Proposition 4.7. *Let Y be an* ANR*-space, $A \subseteq Y$. Then the following are equivalent:*
(i) *A is aspheric.*
(ii) *There exists a decreasing sequence (A_n) of compact, contractible subsets of Y such that $\bigcap_{n=1}^{\infty} A_n = A$ (R_δ-set).*

We go back to set-valued maps:

Definition 4.8. A set-valued map $\phi : X \to 2^Y$ is a *J-map* if ϕ is u.s.c. and $\phi(x)$ is aspheric for all $x \in X$. The set of J-maps from X to Y is denoted by $J(X, Y)$.

Some sufficient conditions (see [22, Definition 1.7] the definition of AR-set):

Lemma 4.9. *Let Y be an* ANR*-space, let $\phi : X \to 2^Y$ be u.s.c., and let one of the following hold:*
(i) *$\phi(x)$ an R_δ-set for all $x \in X$;*
(ii) *$\phi(x)$ is an* AR*-set (*absolute retract*) for all $x \in X$.*
Then $\phi \in J(X, Y)$.

By Lemma 4.9, in particular, if either ϕ has contractible values, or Y is a Banach space and ϕ has convex values, then $\phi \in J(X, Y)$.

In our results, we shall need a slightly more general class of set-valued maps:

Definition 4.10. A set-valued map $\phi : X \to 2^Y$ is a *CJ-map* if there exist a metric space Z, $\psi \in J(X, Z)$, and $k \in C(Z, Y)$ such that $\phi = k \circ \psi$. The set of CJ-maps from X to Y is denoted by $CJ(X, Y)$.

The following result ensures that CJ-maps are approximable (the proof is easily deduced from [22, Theorems 23.8, 23.9, and Section 26]):

Proposition 4.11. *Let X be a compact* ANR*-space, $\phi \in CJ(X, Y)$. Then:*
(i) *ϕ is approximable.*
(ii) *For all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $\delta \in (0, \delta_\varepsilon)$ and all $f, g \in B_\delta(\phi)$ we can find a homotopy $h \in C(X \times [0,1], Y)$ such that $h(\cdot, 0) = f$, $h(\cdot, 1) = g$, and $h(\cdot, t) \in B_\varepsilon(\phi)$ for all $t \in [0,1]$.*

5 Degree for Multitriples

In this section we develop a degree theory for set-valued maps, extending Brouwer's degree. This degree has been presented in [9] and its construction basically follows [30], except for the notion of orientation. In fact, our approach is based on the notion of orientation for Fredholm maps, introduced in [6, 7] and recalled here in Section 3, while the construction in [30] makes use of the concept of oriented Fredholm structures, introduced in [19, 20]. For a comprehensive presentation of degree theory for set-valued maps the reader can see the very rich textbook of Väth [32]. Throughout this section E, F are real Banach spaces and $\Omega \subseteq E$ is an open set.

Definition 5.1. Let $g \in C^1(\Omega, F)$ be an oriented Φ_0-map, let $U \subseteq \Omega$ be open, and let $\phi \in CJ(\Omega, F)$ be locally compact. Then (g, U, ϕ) is an *admissible (multi)triple* if the coincidence set

$$C(g, U, \phi) = \{x \in U : g(x) \in \phi(x)\}$$

is compact.

We construct our degree as an integer-valued function defined on the set of admissible triples. First we assume

$$\dim(\phi(U)) < \infty. \tag{5.1}$$

Since $C(g, U, \phi)$ is compact, we can find an open neighborhood $W \subset U$ of $C(g, U, \phi)$ and a subspace $F_1 \subseteq F$ such that $\dim(F_1) = m < \infty$, $\phi(U) \subset F_1$ (by virtue of (5.1)), and F_1 is transverse to g in W (Definition 2.1), as it can be seen as follows: given any $x \in C(g, U, \phi)$, take a finite-dimensional subspace F_x of F containing $\phi(U)$ and transverse to g at x. This is possible since $Dg(x)$ is Fredholm. By the continuity of $z \mapsto Dg(z)$, there exists a neighborhood W_x of x in U such that g is transverse to F_x at any $z \in W_x$. Then F_1 and W as above are obtained by the compactness of $C(g, U, \phi)$.

We orient F_1 and set $M = g^{-1}(F_1)$, hence M is an orientable C^1-manifold in E with $\dim(M) = m$. We then orient M so that it is an oriented (Φ_0, g)-preimage of F_1 (Definition 3.13). Then $C(g, U, \phi) \subset M$ is compact even as a subset of M, and the following open covering of $C(g, U, \phi)$ exists:

Lemma 5.2. *Let* (g, U, ϕ) *be an admissible triple satisfying* (5.1), *and let* W, F_1, M *be defined as above. Then there exist* $k \in \mathbb{N}$, *and bounded open sets* $V_1, \dots, V_k \subset M$ *such that*
(i) $\overline{V}_j \subset M$, $j = 1, \dots, k$ *(by* $\overline{V}_j$ *we denote the closure of* V_j *in E),*
(ii) $C(g, U, \phi) \subset V := \bigcup_{j=1}^k V_j$,
(iii) $\overline{V}_j$ *is diffeomorphic to a closed convex subset of* $\mathbb{R}^m$, $j = 1, \dots, k$.

By (iii), $\overline{V}_1, \dots, \overline{V}_k, \overline{V}$ are compact ANR-spaces. So, Lemma 4.9 implies that $\phi|_{\overline{V}} \in CJ(\overline{V}, F_1)$. Thus, by Proposition 4.11, $\phi|_{\overline{V}}$ is approximable. In addition, observe that, by the construction of V, one has

$$g(\partial V) \cap \phi(\partial V) = \emptyset.$$

Recalling that $g(\partial V)$ and $\phi(\partial V)$ are compact sets (since ∂V is compact and ϕ is u.s.c.), they have positive distance, say $d > 0$. Hence, taking (for example) $\varepsilon \in (0, \frac{d}{2})$, every $f \in B_\varepsilon(\phi|_{\overline{V}})$ satisfies

$$\operatorname{dist}(0, (g - f)(\partial V)) > 0.$$

So, Brouwer's degree for the triple $(g|_{\overline{V}} - f, V, 0)$ is well defined and it enjoys the reduction property displayed in Proposition 2.4. Now we prove that such degree is invariant:

Lemma 5.3. *Let* (g, U, ϕ) *be an admissible triple satisfying* (5.1), *and let* F_1, V, f *be defined as above. Then* $\deg_B(g|_{\overline{V}} - f, V, 0)$ *does not depend on* F_1, V, *and* f.

Proof. We prove our assertion in three steps (backward):

(a) Let F_1, V be fixed, and let $f', f'' \in B_\varepsilon(\phi|_{\overline{V}})$ be two approximations of ϕ. By homotopy invariance of Brouwer's degree and Proposition 4.11, by reducing $\varepsilon > 0$ if necessary we can apply [30, Lemma 3.4] and get

$$\deg_B(g|_{\overline{V}} - f', V, 0) = \deg_B(g|_{\overline{V}} - f'', V, 0).$$

(b) Let F_1 be fixed, and let $V', V'' \subset M$ be open such that $C(g, U, \phi) \subset V' \cap V''$ and $\overline{V'}$, $\overline{V''}$ are compact ANR-spaces. Without loss of generality we may assume $V' \subset V''$. By Proposition 4.5 (i), by reducing $\varepsilon > 0$ if necessary we can find $f \in B_\varepsilon(\phi|_{\overline{V''}})$ such that $f|_{\overline{V'}} \in B_\varepsilon(\phi|_{\overline{V'}})$. So, by the excision property of Brouwer's degree, we have

$$\deg_B(g|_{\overline{V'}} - f|_{\overline{V'}}, V', 0) = \deg_B(g|_{\overline{V''}} - f, V'', 0).$$

(c) Finally, let F_1', F_1'' be finite-dimensional subspaces of F, transverse to g in W, such that $\phi(U) \subset F_1' \cap F_1''$. Then, by Proposition 2.4, we have for any choice of V, f the same $\deg_B(g|_{\overline{V}} - f, V, 0)$.

So, $\deg_B(g|_{\overline{V}} - f, V, 0)$ is independent of F_1, V, and f. □

By virtue of Lemma 5.3, we can define a degree for the triple (g, U, ϕ):

Definition 5.4. Let (g, U, ϕ) be an admissible triple satisfying (5.1), and let F_1, V, f be defined as above. The *degree* of (g, U, ϕ) is defined by

$$\deg(g, U, \phi) = \deg_B(g|_{\overline{V}} - f, V, 0).$$

The following is a special homotopy invariance result, which will be useful in the forthcoming construction:

Lemma 5.5. *Let $U \subseteq E$ be open, let $h : U \times [0, 1] \to F$ be an oriented Φ_0-homotopy, and let $\phi \in CJ(U \times [0, 1], F)$ be locally compact such that*
(i) *the coincidence set*

$$C(h, U \times [0, 1], \phi) = \{(x, t) \in U \times [0, 1] : h(x, t) \in \phi(x, t)\}$$

is compact,
(ii) $\dim(\phi(U \times [0, 1])) < \infty$.
Then the map $t \mapsto \deg(h(\cdot, t), U, \phi(\cdot, t))$ is constant in $[0, 1]$.

Proof. By (i) and (ii) we can find an open neighborhood $W \subset U \times [0, 1]$ of $C(h, U \times [0, 1], \phi)$ and a subspace $F_1 \subseteq F$ such that $\dim(F_1) = m < \infty$, $\phi(U \times [0, 1]) \subset F_1$, and for all $t \in [0, 1]$, F_1 is transverse to $h(\cdot, t)$ in the set

$$W_t := \{x \in U : (x, t) \in W\}.$$

Set $M_1 = h^{-1}(F_1) \cap W$; then M_1 is an $(m + 1)$-dimensional C^1-manifold in $E \times \mathbb{R}$ with boundary

$$\partial M_1 = \{(x, t) \in M_1 : t = 0, 1\}.$$

We orient F_1, so that the orientations of h, F_1 induce an orientation of M_1 in a unique way (Proposition 3.11). Now let $V \subset M_1$ be an open (in M_1) neighborhood of $C(h, U \times [0, 1], \phi)$ such that $\overline{V} \subset M_1$ is a compact ANR-space (the construction is analogous to that of Lemma 5.2). By Propositions 4.5 and 4.11, the restriction $\phi|_{\overline{V}} \in CJ(\overline{V}, F_1)$ is approximable, and for all $\varepsilon > 0$ small enough we can find $f \in B_\varepsilon(\phi|_{\overline{V}})$ such that for all $t \in [0, 1]$,

$$\deg(h(\cdot, t), U, \phi(\cdot, t)) = \deg_B(h(\cdot, t)|_{\overline{V}} - f(\cdot, t), V, 0)$$

(Definition 5.4). By homotopy invariance of Brouwer's degree, the latter does not depend on $t \in [0, 1]$, which concludes the proof. □

Now we remove assumption (5.1). Let (g, U, ϕ) be an admissible triple, not necessarily satisfying (5.1). Since g is locally proper, ϕ is locally compact, and $C(g, U, \phi)$ is compact (Definition 5.1), we can find a bounded open neighborhood $U_1 \subset U$ of $C(g, U, \phi)$ such that $g|_{\overline{U}_1}$ is proper and $\phi|_{\overline{U}_1}$ is compact. Recalling that ϕ has closed graph, being u.s.c., one can see that $g - \phi : \overline{U}_1 \to 2^F$ is a closed set-valued map, and $0 \notin (g - \phi)(\partial U_1)$. Since $(g - \phi)(\partial U_1)$ is closed, there exists $\delta > 0$ such that

$$B_\delta(0) \cap (g - \phi)(\partial U_1) = \emptyset.$$

The set $K = \overline{\phi(\overline{U}_1)}$ is compact. So we can find a finite-dimensional subspace $F_1 \subset F$ and a (single-valued) map $j_\delta \in C(K, F_1)$ such that for all $x \in K$,

$$\|j_\delta(x) - x\|_F < \frac{\delta}{2}$$

(this is a classical result in nonlinear functional analysis, see e.g. [17, Proposition 8.1]). Set

$$\phi_1 = j_\delta \circ \phi \in CJ(\overline{U}_1, F)$$

(Definition 4.10); then it satisfies

$$B_{\delta/2}(0) \cap \phi_1(\partial U_1) = \emptyset,$$

and $C(g, U_1, \phi_1)$ is compact. So, (g, U_1, ϕ_1) is an admissible triple satisfying (5.1). Definition 5.4 then applies, and produces a degree $\deg(g, U_1, \phi_1)$. Moreover, such a degree is invariant:

Lemma 5.6. *Let (g, U, ϕ) be an admissible triple, and let U_1, j_δ be defined as above. Then $\deg(g, U_1, \phi_1)$ does not depend on U_1, j_δ.*

Proof. Just as in Lemma 5.3, we divide the proof in two steps backward:

(a) Let U_1 be fixed, let F_1, K, δ be defined as above, and let $j'_\delta, j''_\delta \in C(K, F_1)$ be such that for all $x \in K$,

$$\|j'_\delta(x) - x\|_F,\ \|j''_\delta(x) - x\|_F < \frac{\delta}{2}.$$

Set for all $(x, t) \in U_1 \times [0, 1]$,

$$h(x, t) = g(x),\ \ \tilde{\phi}(x, t) = (1 - t)j'_\delta(\phi(x)) + tj''_\delta(\phi(x)).$$

Then the map $h : U_1 \times [0, 1] \to F$ is a Φ_0-homotopy (Definition 3.9). A more delicate question is proving that $\tilde{\phi} \in CJ(U_1 \times [0, 1], F)$, since this map is not explicitly defined as a composition of a J-map and a continuous single-valued function (Definition 4.10). Since $\phi \in CJ(U, F)$, there exist a metric space Z, $\psi \in J(U_1, Z)$, and $k \in C(Z, F)$ such that $\phi = k \circ \psi$. Set for all $(x, t) \in U_1 \times [0, 1]$,

$$\tilde{\psi}(x, t) = \psi(x) \times \{t\},$$

so clearly $\tilde{\psi} \in J(U_1 \times [0, 1], Z \times [0, 1])$; and set for all $(z, t) \in Z \times [0, 1]$,

$$\tilde{k}(z, t) = (1 - t)j'_\delta(k(z)) + tj''_\delta(k(z)),$$

so $\tilde{k} \in C(Z \times [0, 1], F)$. Then

$$\tilde{\phi} = \tilde{k} \circ \tilde{\psi} \in CJ(U_1 \times [0, 1], F).$$

Now we prove that the coincidence set

$$C(h, U_1 \times [0, 1], \tilde{\phi}) = \{(x, t) \in U_1 \times [0, 1] : h(x, t) \in \tilde{\phi}(x, t)\}$$

is compact. Let (x_n, t_n) be a sequence in $C(h, U_1 \times [0, 1], \tilde{\phi})$. Passing to a subsequence, we have $t_n \to t$. For all $n \in \mathbb{N}$ there exist $y'_n, y''_n \in \phi(x_n)$ such that

$$g(x_n) = (1 - t_n)j'_\delta(y'_n) + t_nj''_\delta(y''_n).$$

By compactness of $\phi|_{\overline{U}_1}$, passing again to a subsequence we have $y'_n \to y'$, $y''_n \to y''$, which implies

$$g(x_n) \to (1 - t)j'_\delta(y') + tj''_\delta(y'').$$

By properness of $g|_{\overline{U}_1}$, we can find $x \in \overline{U}_1$ such that up to a further subsequence $x_n \to x$. We need to prove that $x \in U_1$. Arguing by contradiction, let $x \in \partial U_1$. Then, by the choice of $\delta > 0$, we have

$$\operatorname{dist}(g(x), \phi(x)) \geqslant \delta.$$

Besides, since ϕ is u.s.c., we have $y', y'' \in \phi(x)$, hence by the metric properties of the maps j'_δ, j''_δ we have

$$\begin{aligned}\operatorname{dist}(g(x), \phi(x)) &\leqslant (1 - t)\operatorname{dist}(j'_\delta(y'), \phi(x)) + t\operatorname{dist}(j''_\delta(y''), \phi(x))\\ &\leqslant (1 - t)\|j'_\delta(y') - y'\|_F + t\|j''_\delta(y'') - y''\|_F \leqslant \frac{\delta}{2},\end{aligned}$$

a contradiction. So, $x \in U_1$ and we deduce that $C(h, U_1 \times [0, 1], \tilde{\phi})$ is compact. In addition, $\tilde{\phi}$ has a finite-dimensional rank. Then, by Lemma 5.5, $\deg(h(\cdot\,, t), U_1, \tilde{\phi}(\cdot\,, t))$ is independent of $t \in [0, 1]$. In particular, taking $t = 0, 1$ we get

$$\deg(g, U_1, j'_\delta \circ \phi) = \deg(g, U_1, j''_\delta \circ \phi).$$

(b) Let $U'_1, U''_1 \subset U$ be open neighborhoods of $C(g, U, \phi)$ such that the restrictions of g to both $\overline{U'_1}$, $\overline{U''_1}$ are proper and the restrictions of ϕ to $\overline{U'_1}$, $\overline{U''_1}$ are compact, respectively. Without loss of generality we may assume $U'_1 \subseteq U''_1$, hence we continue the construction in U''_1. Then independence of the degree follows from the excision property of Brouwer's degree.

So, $\deg(g, U_1, \phi_1)$ does not depend on the choice of U_1, j_δ. □

By virtue of Lemma 5.6, we can define a degree for the triple (g, U, ϕ) extending Definition 5.4:

Definition 5.7. Let (g, U, ϕ) be an admissible triple, and let U_1, ϕ_1 be defined as above. The *degree* of (g, U, ϕ) is defined by

$$\deg(g, U, \phi) = \deg(g, U_1, \phi_1).$$

The degree theory we just introduced enjoys some classical properties:

Proposition 5.8. *The following properties hold:*

(i) (Normalization) *If $U \subset E$ is open such that $0 \in U$, and I is the naturally oriented identity of E, then*

$$\deg(I, U, 0) = 1.$$

(ii) (Domain additivity) *If (g, U, ϕ) is an admissible triple, $U_1, U_2 \subset U$ are open such that $U_1 \cap U_2 = \emptyset$, $C(g, U, \phi) \subset U_1 \cup U_2$, then (g, U_1, ϕ), (g, U_2, ϕ) are admissible triples and*

$$\deg(g, U, \phi) = \deg(g, U_1, \phi) + \deg(g, U_2, \phi).$$

(iii) (Homotopy invariance) *If $U \subset E$ is open, $h : U \times [0, 1] \to F$ is an oriented Φ_0-homotopy, $\phi \in CJ(U \times [0, 1], F)$ is locally compact such that $C(h, U \times [0, 1], \phi)$ is compact, then for all $t \in [0, 1]$, $(h(\cdot, t), U, \phi(\cdot, t))$ is an admissible triple and the function*

$$t \mapsto \deg(h(\cdot, t), U, \phi(\cdot, t))$$

is constant in $[0, 1]$.

Proof. Properties (i) and (ii) follow from Definition 5.7 and the corresponding properties of Brouwer's degree (the proof is straightforward, so we omit it).

To prove (iii), we first fix $t \in [0, 1]$. By compactness, we can find $\sigma > 0$ and a bounded open neighborhood $W \subset U$ of the section

$$C_t = \{x \in U : (x, t) \in C(h, U \times [0, 1], \phi)\}$$

such that $h|_{\overline{W} \times I_\sigma}$ is proper and $\phi|_{\overline{W} \times I_\sigma}$ is compact, where we have set

$$I_\sigma = [t - \sigma, t + \sigma] \cap [0, 1].$$

We also introduce a finite rank map $j \in C(K, F)$, close enough to the identity of $K = \overline{\phi(\overline{W} \times I_\sigma)}$. By the excision property of Brouwer's degree and the construction above, for all $s \in I_\sigma$ we have

$$\deg(h(\cdot, s), U, \phi(\cdot, s)) = \deg(h(\cdot, s), W, j \circ \phi(\cdot, s)).$$

Besides, by Lemma 5.5 the function

$$s \mapsto \deg(h(\cdot, s), W, j \circ \phi(\cdot, s))$$

is constant in I_σ, hence $\deg(h(\cdot, s), U, \phi(\cdot, s))$ turns out to be locally constant in $[0, 1]$. Since $[0, 1]$ is connected, we get the conclusion. □

Remark 5.9. Proposition 5.8 (iii) holds in a stronger form, i.e., for subsets of $E \times \mathbb{R}$ which are not necessarily products, as it can be seen from the proof. In fact, consider the case in which the domain of h and ϕ is an open subset U of the product $E \times [0, 1]$. Fix $t \in [0, 1]$ and call

$$U_t := \{x \in E : (x, t) \in U\}.$$

By the compactness of $C(h, U, \phi)$ we can find an open subset W_t of E and a positive σ such that

(a) $C_t(h, U, \phi) \subset W_t \subset \overline{W}_t \subset U_t$,

(b) $\overline{W}_t \times I_\sigma \subset U$,

(c) $C_s(h, U, \phi) \subset W_t$ for every $s \in I_\sigma$,

where $C_s(h, U, \phi) := \{x \in E : (x, t) \in C(h, U, \phi)\}$ and I_σ is as in the proof of (iii). Now, item (iii) of Proposition 5.8 implies that

$$s \mapsto \deg(h(\cdot, s), W_t, \phi(\cdot, s))$$

is constant in I_σ. The constance of $\deg(h(\cdot, s), U_s, \phi(\cdot, s))$ follows from the excision property of the degree, which is straightforward (we omit the proof). This property is usually called *generalized homotopy invariance*.

6 Persistence Results and Bifurcation Points

We can now prove the main results of the present paper, as announced in the Introduction. Throughout this section, E and F are two real Banach spaces, $\Omega \subset E$ is an open (not necessarily bounded) set such that $0 \in \Omega$, $L \in \Phi_0(E, F)$ satisfy $\ker L \neq 0$, $C \in \mathcal{L}(E, F)$ is another bounded linear operator, and $\phi \in CJ(\overline{\Omega}, F)$ is locally compact. The linear operators L, C satisfy the transversality condition (1.3). For all $\varepsilon, \lambda \in \mathbb{R}$ we consider the perturbed problem (1.1), whose solutions are meant in the following sense:

Definition 6.1. A *solution* of (1.1) is a triple $(x, \varepsilon, \lambda) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}$ such that

$$Lx - \lambda Cx + \varepsilon\phi(x) \ni 0.$$

The set of solutions is denoted by $\mathcal{S}$. A solution $(x, \varepsilon, \lambda) \in \mathcal{S}$ is a *trivial* solution if $\varepsilon = \lambda = 0$. Finally, we say that $x_0 \in \partial\Omega$ is a *bifurcation point* if $(x_0, 0, 0) \in \mathcal{S}$ and any neighborhood of $(x_0, 0, 0)$ in $E \times \mathbb{R} \times \mathbb{R}$ contains at least one non-trivial solution.

Clearly, any trivial solution $(x, 0, 0)$ of (1.1) identifies with its first component x. The set of such vectors is

$$\mathcal{S}_0 = \partial\Omega \cap \ker L.$$

Regarding our definition of a bifurcation point, we note that it is analogous to that of [4], and fits in the very general definition given in [16, p. 2]. Finally, we note that, whenever $(x, 0, \lambda) \in \mathcal{S}$, (x, λ) is an eigenpair of the eigenvalue problem (1.2): thus, we keep the names *eigenvector* for x and *eigenvalue* for λ, respectively, for any triple $(x, \varepsilon, \lambda) \in \mathcal{S}$.

As observed in [4, Remark 5.1], transversality condition (1.3) is in fact equivalent to

$$\operatorname{im} L \oplus C(\ker L) = F.$$

Thus, we can find $b > 0$ such that $L - \lambda C \in \Phi_0(E, F)$ is invertible for all $0 < |\lambda| \leqslant b$. Besides, since $0 \notin \partial\Omega$, for any bifurcation point $x_0 \in \mathcal{S}_0$ we can find a neighborhood $W \subset E \times \mathbb{R} \times \mathbb{R}$ of $(x_0, 0, 0)$ such that any triple $(x, \varepsilon, \lambda) \in \mathcal{S} \cap W$ actually must have $\varepsilon \neq 0$.

The map $\lambda \mapsto L - \lambda C$ (which is orientable according to Definition 3.5 since its domain is simply connected, see Proposition 3.8 (iii)) exhibits a sign jump property (a special case of [8, Corollary 5.1]):

Lemma 6.2. *Let $b > 0$ be defined as above, and let $h \in C([-b, b], \Phi_0(E, F))$ be defined by*

$$h(\lambda) = L - \lambda C,$$

and oriented. Then:
(i) *The map $\lambda \mapsto \operatorname{sign} h(\lambda)$ is constant in both $[-b, 0)$ and $(0, b]$.*
(ii) *$\operatorname{sign} h(b) \neq \operatorname{sign} h(-b)$ iff $\dim(\ker L)$ is odd.*

Lemma 6.2 above is the reason why the assumption that $\dim(\ker L)$ is odd is so important in our theory. Now we prove an existence result on *bounded* subdomains, which is the core of our argument:

Proposition 6.3. *Let $\dim(\ker L)$ be odd, let (1.3) hold, and let $U \subseteq \Omega$ be an open, bounded set such that $0 \in U$ and $\phi|_{\overline{U}} \in CJ(\overline{U}, F)$ is compact. Then there exist $a, b > 0$ such that for all $\varepsilon \in [-a, a]$ there exist $\lambda \in [-b, b]$, $x \in \partial U$ such that*

$$Lx - \lambda Cx + \varepsilon\phi(x) \ni 0.$$

Proof. Let $b > 0$ be as in Lemma 6.2, and fix $a > 0$ (to be better determined later). Set

$$\mathcal{R} = [-a, a] \times [-b, b],$$

and define the set

$$\mathcal{K} = \{(x, \varepsilon, \lambda) \in \partial U \times \mathcal{R} : Lx - \lambda Cx + \varepsilon\phi(x) \ni 0\}. \tag{6.1}$$

The set $\mathcal{K} \subset E \times \mathbb{R} \times \mathbb{R}$ is compact. Indeed, let $(x_n, \varepsilon_n, \lambda_n)$ be a sequence in $\mathcal{K}$. Then $(\varepsilon_n, \lambda_n)$ is a bounded sequence in $\mathcal{R}$, hence passing to a subsequence we have $(\varepsilon_n, \lambda_n) \to (\varepsilon, \lambda)$ for some $(\varepsilon, \lambda) \in \mathcal{R}$. As seen above, we have eventually $\varepsilon_n \neq 0$. Now set for all $n \in \mathbb{N}$

$$y_n = -\frac{1}{\varepsilon_n}(Lx_n - \lambda_n Cx_n) \in \phi(x_n).$$

Since $\overline{\phi(\overline{U})}$ is compact, passing if necessary to a further subsequence, we have $y_n \to y$ for some $y \in F$, which implies

$$\lim_n (Lx_n - \lambda Cx_n) = \lim_n (Lx_n - \lambda_n Cx_n) + \lim_n (\lambda_n - \lambda)Cx_n = -\varepsilon y$$

(recall that (x_n) is a bounded sequence and C is a bounded operator). The operator $L - \lambda C \in \Phi_0(E, F)$ is proper on closed and bounded subsets of E, hence passing again to a subsequence if necessary we have $x_n \to x$ for some $x \in \partial U$. Thus, $(x_n, \varepsilon_n, \lambda_n) \to (x, \varepsilon, \lambda)$ for some $(x, \varepsilon, \lambda) \in \mathcal{K}$.

Clearly, the projection of $\mathcal{K}$ onto $\mathcal{R}$, namely the set

$$\Gamma = \{(\varepsilon, \lambda) \in \mathcal{R} : (x, \varepsilon, \lambda) \in \mathcal{K} \text{ for some } x \in \partial U\},$$

is compact as well. Now we choose an orientation of $L \in \Phi_0(E, F)$ (Definition 3.4), and fix $(\varepsilon, \lambda) \in \mathcal{R} \setminus \Gamma$. Then $(L - \lambda C, U, -\varepsilon\phi)$ is an admissible triple (Definition 5.1), since the coincidence set

$$C(L - \lambda C, U, -\varepsilon\phi) = \{x \in U : Lx - \lambda Cx + \varepsilon\phi(x) \ni 0\}$$

is compact. Indeed, arguing as above, for any sequence (x_n) in $C(L - \lambda C, U, -\varepsilon\phi)$ we can find a relabeled subsequence such that $x_n \to x$ for some $x \in \overline{U}$. It remains to prove that $x \in U$. Otherwise, we would have $x \in \partial U$, hence $(\varepsilon, \lambda) \in \Gamma$, a contradiction.

So, the integer-valued map

$$(\varepsilon, \lambda) \mapsto \deg(L - \lambda C, U, -\varepsilon\phi)$$

is well defined in the relatively open set $\mathcal{R} \setminus \Gamma$ (Definition 5.7), and constant on any connected component of $\mathcal{R} \setminus \Gamma$ by homotopy invariance (Proposition 5.8 (iii)).

By the choice of $b > 0$, both operators $L \pm bC$ are invertible. Hence, $(0, \pm b) \in \mathcal{R} \setminus \Gamma$ (recall that $0 \notin \partial U$). We claim that

$$\deg(L + bC, U, 0) \neq \deg(L - bC, U, 0). \tag{6.2}$$

Indeed, let $L + bC \in \Phi_0(E, F)$ be naturally oriented; then

$$\operatorname{sign}(L + bC) = 1$$

(Definition 3.4 (ii), (iii)). We fix a non-trivial, finite-dimensional subspace $F_1 \subset F$ and set $E_1 = (L + bC)^{-1}(F_1)$, then we orient F_1 and E_1 so that E_1 is the oriented $(L + bC)$-preimage of F_1. With such an orientation of the involved spaces and maps, recalling Definitions 5.7 and 5.4, we have

$$\deg(L + bC, U, 0) = \deg_B((L + bC)|_{E_1}, U \cap E_1, 0) = 1.$$

The second of the above two equalities is consequence of classical properties in Brouwer degree (see e.g. [25, Chapter 5, Section 1]). Since $\dim(\ker L)$ is odd, by Lemma 6.2 (ii) we have

$$\operatorname{sign}(L - bC) = -1,$$

which, repeating the construction above with the same orientations, leads to

$$\deg(L - bC, U, 0) = -1.$$

Similar arguments can be developed if different orientations are chosen, so (6.2) holds in any case.

By (6.2), we deduce that $(0, \pm b)$ lie in different connected components of $\mathcal{R} \setminus \Gamma$. By reducing further $a > 0$ if necessary, we may assume that $(\varepsilon, \pm b)$ lie in different connected components of $\mathcal{R} \setminus \Gamma$, for all $\varepsilon \in [-a, a]$ (the situation is depicted in Figure 1). So, for all $\varepsilon \in [-a, a]$ we can find $\lambda \in [-b, b]$ such that $(\varepsilon, \lambda) \in \Gamma$, which concludes the proof. □

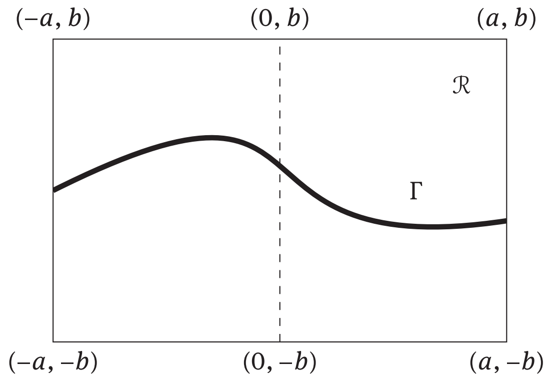

Figure 1: The set Γ cutting the rectangle ℛ.

Proposition 6.3 is the main tool for proving persistence of the eigenpairs under a set-valued perturbation, with the additional assumption that the set $\Omega_0 := \overline{\Omega} \cap \ker L$ is *compact* (note that $\Omega_0 \neq \emptyset$ as $0 \in \Omega$). We begin with eigenvalues:

Theorem 6.4. *Let* $\dim(\ker L)$ *be odd, let* (1.3) *hold, and let* $\Omega_0 := \overline{\Omega} \cap \ker L$ *be non-empty and compact. Then, for all* $c > 0$ *small enough, there exist* $a, b > 0$ *such that the set-valued map* $\Gamma : [-a, a] \to 2^{[-b,b]}$ *defined by*

$$\Gamma(\varepsilon) = \{\lambda \in [-b, b] : (x, \varepsilon, \lambda) \in \mathcal{S} \text{ for some } x \in \partial\Omega \cap B_c(\mathcal{S}_0)\}$$

has the following properties:
(i) $\Gamma(\varepsilon) \neq \emptyset$ *for all* $\varepsilon \in [-a, a]$,
(ii) Γ *is u.s.c.*

Proof. Since the set Ω_0 is compact, we can find a bounded open neighborhood $W \subset E$ of Ω_0 such that $\phi|_{\overline{U}}$ is compact, where we have set $U = W \cap \Omega$. Clearly, U is a bounded open set such that $0 \in U$. Then we can apply Proposition 6.3 and thus find a rectangle

$$\mathcal{R} = [-a, a] \times [-b, b] \quad (a, b > 0)$$

such that for all $\varepsilon \in [-a, a]$ there exist $\lambda \in [-b, b]$, $x \in \partial U$ such that

$$Lx - \lambda Cx + \varepsilon\phi(x) \ni 0.$$

Besides, let $c > 0$ be small enough that $B_c(\mathcal{S}_0) \subset W$ (recall that $\mathcal{S}_0 = \partial\Omega \cap \ker L$). We define $\mathcal{K} \subset E \times \mathcal{R}$ as in (6.1). As in the proof of Proposition 6.3, we see that $\mathcal{K}$ is compact. We define a set-valued map $\psi : \mathcal{R} \to 2^F$ by setting

$$\psi(\varepsilon, \lambda) = \{x \in \partial U : (x, \varepsilon, \lambda) \in \mathcal{K}\}.$$

We claim that

$$\psi(0, 0) = \mathcal{S}_0 \subset B_c(\mathcal{S}_0).$$

Indeed, for all $x \in \mathcal{S}_0$ we have $x \in \Omega_0 \subset W$, which along with $x \in \partial\Omega$ implies $x \in \partial U$, while $(x, 0, 0) \in \mathcal{S}$, so $(x, 0, 0) \in \mathcal{K}$. Conversely, if $x \in \psi(0, 0)$, then $x \in \partial U \subseteq \partial\Omega \cup \partial W$, while $x \in \Omega_0 \subset W$, so we deduce $x \in \partial\Omega$ and since $(x, 0, 0) \in \mathcal{K}$ we have $x \in \mathcal{S}_0$.

Plus, the set graph $\psi \subset \mathcal{R} \times E$ is obtained as a continuous image of $\mathcal{K}$ (by a swap of coordinates) and hence is compact. So, recalling Remark 4.2, ψ is u.s.c.

Therefore by reducing $a, b > 0$ if necessary we have for all $(\varepsilon, \lambda) \in \mathcal{R}$,

$$\psi(\varepsilon, \lambda) \subset B_c(\mathcal{S}_0). \tag{6.3}$$

Now we can prove both assertions. Fix $\varepsilon \in [-a, a]$. By Proposition 6.3 there exist $\lambda \in [-b, b]$ and $x \in \partial U$ such that $x \in \psi(\varepsilon, \lambda)$, so by (6.3) we have $x \in B_c(\mathcal{S}_0)$. Then $x \in \partial U \cap W \subset \partial\Omega$, so $(x, \varepsilon, \lambda) \in \mathcal{S}$. Thus $\lambda \in \Gamma(\varepsilon)$, which proves (i).

To prove (ii), we just need to note that

$$\operatorname{graph}\Gamma = \{(\varepsilon,\lambda) \in \mathcal{R} : (x,\varepsilon,\lambda) \in \mathcal{K} \text{ for some } x \in \partial\Omega \cap B_c(\mathcal{S}_0)\}$$

is but the projection of $\mathcal{K}$ onto $\mathcal{R}$, hence compact. As above, $\Gamma : [-a,a] \to 2^{[-b,b]}$ is u.s.c. □

A similar persistence result holds for the eigenvectors:

Theorem 6.5. *Let* $\dim(\ker L)$ *be odd, let* (1.3) *hold, and let* Ω_0 *be compact. Then, for all* $c > 0$ *small enough, there exist* $a, b > 0$ *such that the set-valued map* $\Sigma : [-a,a] \to 2^E$ *defined by*

$$\Sigma(\varepsilon) = \{x \in \partial\Omega \cap B_c(\mathcal{S}_0) : (x,\varepsilon,\lambda) \in \mathcal{S} \text{ for some } \lambda \in [-b,b]\}$$

has the following properties:
(i) $\Sigma(\varepsilon) \neq \emptyset$ *for all* $\varepsilon \in [-a,a]$,
(ii) Σ *is u.s.c.*

Proof. As in the proof of Theorem 6.4, for all $c > 0$ small enough we find a rectangle $\mathcal{R} = [-a,a] \times [-b,b]$ ($a, b > 0$) and an open neighborhood $W \subset E$ of Ω_0 such that, setting $U = W \cap \Omega$, the set $\mathcal{K}$ defined by (6.1) is compact, and in addition $x \in B_c(\mathcal{S}_0)$ whenever $(x,\varepsilon,\lambda) \in \mathcal{K}$ (see (6.3)).

In particular, for all $(x,\varepsilon,\lambda) \in \mathcal{K}$ we have $x \in \Sigma(\varepsilon)$. Then Proposition 6.3 implies (i). Besides, since

$$\operatorname{graph}\Sigma = \{(\varepsilon,x) \in [-a,a] \times (\partial\Omega \cap B_c(\mathcal{S}_0)) : (x,\varepsilon,\lambda) \in \mathcal{K} \text{ for some } \lambda \in [-b,b]\}$$

is the projection of $\mathcal{K}$ onto $[-a,a] \times E$, hence compact, we also deduce (ii) (recall Remark 4.2). □

As a consequence, we prove that the set $\mathcal{S}_0$ contains at least one bifurcation point (Definition 6.1):

Theorem 6.6. *Let* $\dim(\ker L)$ *be odd, let* (1.3) *hold, and let* Ω_0 *be compact. Then problem* (1.1) *has at least one bifurcation point.*

Proof. We argue by contradiction: assume that $\mathcal{S}_0$ contains no bifurcation points, i.e., for all $x \in \mathcal{S}_0$ there exists an open neighborhood $\mathcal{U}_x \subset E \times \mathbb{R} \times \mathbb{R}$ of $(x,0,0)$ such that for all $(x,\varepsilon,\lambda) \in \mathcal{S} \cap \mathcal{U}_x$ we have $(\varepsilon,\lambda) = (0,0)$. The family $(\mathcal{U}_x)_{x \in \mathcal{S}_0}$ is an open covering of the compact set $\mathcal{S}_0 \times \{(0,0)\}$ in $E \times \mathbb{R} \times \mathbb{R}$, so we can find a finite sub-covering, which we relabel as $(\mathcal{U}_i)_{i=1}^m$.

Let $a, b, c > 0$ be such that

$$B_c(\mathcal{S}_0) \times \mathcal{R} \subset \bigcup_{i=1}^m \mathcal{U}_i,$$

where as usual $\mathcal{R} = [-a,a] \times [-b,b]$. Thus, we have

$$\mathcal{S} \cap (B_c(\mathcal{S}_0) \times \mathcal{R}) = \mathcal{S}_0 \times \{(0,0)\}$$

(i.e., there are no solutions in $B_c(\mathcal{S}_0) \times \mathcal{R}$ except the trivial ones). By reducing $a, b, c > 0$ if necessary, Theorem 6.5 applies. So, for all $\varepsilon \in [-a,a] \setminus \{0\}$ there exist $x \in \partial\Omega \cap B_c(\mathcal{S}_0)$, $\lambda \in [-b,b]$ such that $(x,\varepsilon,\lambda) \in \mathcal{S}$, a contradiction. □

Remark 6.7. Since $\ker L$ has finite dimension, compactness of Ω_0 (which is assumed in the statements of the last theorems) is clearly verified as long as Ω is bounded. On the other hand, trivial examples in Euclidean spaces show that, if Ω is unbounded, then Ω_0 may fail to be compact. We want to present a special type of (possibly unbounded) domains which satisfy our assumption: let $\gamma : E \to \mathbb{R}$ be a continuous norm (not necessarily coinciding with the norm $\|\cdot\|$ of E) and set

$$\Omega = \{x \in E : \gamma(x) < 1\}.$$

Let (x_n) be a sequence in $\Omega_0 = \overline{\Omega} \cap \ker L$. Without loss of generality we may assume $x_n \neq 0$ for all $n \in \mathbb{N}$. Setting $y_n = x_n/\|x_n\|$, we define a bounded sequence (y_n) in the finite-dimensional space $\ker L$, so passing to a subsequence if necessary we have $y_n \to y$, $\|y\| = 1$. By continuity, $\gamma(y_n) \to \gamma(y) > 0$, so

$$\|x_n\| = \frac{\gamma(x_n)}{\gamma(y_n)} \leqslant \frac{1}{\gamma(y_n)}$$

is bounded. Passing to a further subsequence, we have $\|x_n\| \to \mu \geqslant 0$, and hence $x_n \to \mu y$. So, Ω_0 is compact.

We conclude this section by presenting a special case of Theorem 6.6:

Corollary 6.8. *Let* $\dim(\ker L)$ *be odd, let* (1.3) *hold, let* $\gamma \in C(E, \mathbb{R})$ *be a norm, and let*

$$\Omega = \{x \in E : \gamma(x) < 1\}.$$

Then problem (1.1) *has at least one bifurcation point.*

Proof. Clearly, $\Omega \subset E$ is an open set such that $0 \in \Omega$ and $\partial\Omega = \gamma^{-1}(1)$. In addition, by Remark 6.7, the set $\Omega_0 = \overline{\Omega} \cap \ker L$ is compact. Thus, we can apply Theorem 6.6 and conclude. □

7 Applications to Differential Inclusions

We devote this final section to an application of our abstract results in the field of differential inclusions. We consider the problem stated in the Introduction:

$$\begin{cases} u'' + u' - \lambda u + \varepsilon\Phi(u) \ni 0 \text{ in } [0, 1] \\ u'(0) = u'(1) = 0 \\ \|u\|_1 = 1. \end{cases} \tag{7.1}$$

We recall that $\Phi(u) : [0, 1] \to 2^{\mathbb{R}}$ is a set-valued mapping depending on u, to be defined later, while $\varepsilon, \lambda \in \mathbb{R}$ are parameters and $\| \cdot \|_1$ is the usual L^1-norm on $[0, 1]$. Problem (7.1) falls into the general pattern (1.1), with the following definitions. Set

$$E = \{u \in C^2([0, 1], \mathbb{R}) : u'(0) = u'(1) = 0\}, \quad F = C^0([0, 1], \mathbb{R}),$$

endowed with the usual norms. Then E, F are real Banach spaces, in particular E is a 2-codimensional subspace of $C^2([0, 1], \mathbb{R})$. Set for all $u \in E$,

$$Lu = u'' + u', \quad Cu = u.$$

Then $L, C \in \mathcal{L}(E, F)$. Besides, $L \in \Phi_0(E, F)$ as the composition of the embedding $E \hookrightarrow C^2([0, 1], \mathbb{R})$ (which is Fredholm of index -2) and the linear differential operator $u \mapsto u'' + u'$ (which is Fredholm of index 2 between $C^2([0, 1], \mathbb{R})$ and F). In order to check the transversality condition (1.3), we need more detailed information about L. It is easily seen that $\ker L$ is the space of constant functions, i.e.,

$$\ker L = \mathbb{R},$$

in particular $\dim(\ker L) = 1$ (odd). In addition, we have

$$\operatorname{im} L = \left\{ f \in F : \int_0^1 f(t)e^t \, dt = 0 \right\}.$$

Indeed, for all $f \in \operatorname{im} L$ there exists $u \in E$ such that $u'' + u' = f$, so integrating by parts we deduce

$$\int_0^1 f(t)e^t \, dt = \int_0^1 u''(t)e^t \, dt + \int_0^1 u'(t)e^t \, dt = 0.$$

Besides, since $L \in \Phi_0(E, F)$, we have $\dim(\operatorname{coker} L) = 1$ (Definition 3.1), so the condition above is also sufficient. Now we prove (1.3), or equivalently

$$\operatorname{im} L \oplus C(\ker L) = F.$$

Indeed, $C(\ker L) = \mathbb{R}$ is not contained in the 1-codimensional subspace $\operatorname{im} L$, hence it is a (direct) complement for it in F.

The integral constraint rephrases as $u \in \partial\Omega$, where we have set

$$\Omega = \{u \in E : \|u\|_1 < 1\}.$$

Since $\|\cdot\|_1$ is a continuous norm on E, Ω is an (unbounded) open set such that $\Omega_0 = \overline{\Omega} \cap \ker L$ is compact (Remark 6.7). Besides, from the characterization of $\ker L$ we have $\mathcal{S}_0 = \{\pm 1\}$.

The construction of Φ requires some care. We are going to consider a set-valued map $\phi \in CJ(\overline{\Omega}, F)$, and then set for all $u \in \overline{\Omega}$, $t \in [0,1]$,

$$\Phi(u)(t) = \{w(t) : w \in \phi(u)\}$$

(this can be seen as a set-valued superposition operator). Details will be given in Examples 7.2, 7.3, and 7.7 below. We can now apply our abstract results to prove existence of a bifurcation point:

Theorem 7.1. *Let E, F, L, C, Ω, and Φ be as above, being $\phi \in CJ(\overline{\Omega}, F)$ locally compact. Then there exist sequences (u_n) in $\partial\Omega$, (ε_n) in $\mathbb{R} \setminus \{0\}$, (λ_n) in $\mathbb{R}$ such that $(u_n, \varepsilon_n, \lambda_n)$ is a solution of* (7.1) *for all $n \in \mathbb{N}$, and*

$$u_n \to \pm 1, \quad \varepsilon_n \to 0, \quad \lambda_n \to 0.$$

Proof. By Remark 6.7 and Corollary 6.6, problem (1.1) has at least one bifurcation point in $\mathcal{S}_0$, that is, either the constant 1 or -1. So, we can find a sequence $(u_n, \varepsilon_n, \lambda_n)$ if non-trivial solutions of (1.1) (more precisely, with $\varepsilon_n \neq 0$) converging to either $(1,0,0)$ or $(-1,0,0)$ in $E \times \mathbb{R} \times \mathbb{R}$. By the definition of Φ, for all $n \in \mathbb{N}$ and all $t \in [0,1]$ we have

$$u_n''(t) + u_n'(t) - \lambda_n u_n(t) + \varepsilon_n \Phi(u_n)(t) \ni 0,$$

so $(u_n, \varepsilon_n, \lambda_n)$ solves (7.1). □

We present three examples of locally compact CJ-maps $\phi : \overline{\Omega} \to 2^F$. The first and second examples are quite easy, ϕ being defined by means of finite-dimensional reduction.

Example 7.2. We define a set-valued map $\phi : \overline{\Omega} \to 2^F$ whose values consist of piecewise affine functions along a decomposition of $[0,1]$, satisfying some bounds at the nodal points. Fix $m \in \mathbb{N}$, points $0 = t_0 < t_1 < \dots < t_m = 1$, and $\rho \in (0,1)$. For all $u \in \overline{\Omega}$ we define $\phi(u)$ as the set of all functions $w \in F$ such that

(a) w is affine in $[t_{j-1}, t_j]$, $j = 1, \dots, m$,

(b) $u(t_j) - \rho \leqslant w(t_j) \leqslant u(t_j) + \rho$, $j = 0, \dots, m$.

We first prove that ϕ has convex values. For any $u \in \overline{\Omega}$, $w_0, w_1 \in \phi(u)$, and $\mu \in [0,1]$, by (a) the function $w = (1-\mu)w_0 + \mu w_1$ is affine in any interval $[t_{j-1}, t_j]$ $(j = 1, \dots, m)$, while (b) implies

$$u(t_j) - \rho \leqslant w(t_j) \leqslant u(t_j) + \rho \quad (j = 0, \dots, m).$$

Then we prove that ϕ has compact values. Let $u \in \overline{\Omega}$, and let (w_n) be a sequence in $\phi(u)$. By (a) we can find $\alpha_1, \dots, \alpha_m > 0$ such that $|w_n'(t)| \leqslant \alpha_j$ for all $t \in (t_{j-1}, t_j)$, $j = 1, \dots, m$, and $n \in \mathbb{N}$. So the sequence (w_n) is uniformly bounded and equi-continuous (by (b)), hence by Ascoli's theorem we can pass to a subsequence such that $w_n \to w$ in F. Due to uniform convergence, w is piecewise affine and satisfies the bounds at t_j $(j = 0, \dots, m)$, so $w \in \phi(u)$. Thus, $\phi(u)$ is compact.

In addition, the set-valued map ϕ is locally compact. Indeed, let (u_n) be a bounded sequence in $\overline{\Omega}$ and let (w_n) be a sequence in F such that $w_n \in \phi(u_n)$ for all $n \in \mathbb{N}$. Recalling that (u_n') is uniformly bounded, we can argue as above to find a relabeled subsequence w_n such that $w_n \to w$ in F. Thus, $\overline{\phi(u_n)}$ is compact.

We prove finally that graph ϕ is closed in $\overline{\Omega} \times F$. Indeed, let (u_n, w_n) be a sequence in $\overline{\Omega} \times F$ such that $w_n \in \phi(u_n)$ for all $n \in \mathbb{N}$, and $(u_n, w_n) \to (u, w)$. Then $w \in \phi(u)$. By [22, Proposition 4.15], ϕ is u.s.c. We conclude that $\phi \in CJ(\overline{\Omega}, F)$ and is locally compact.

Example 7.3. In this second example, $\phi(u)$ depends on u in a single-valued sense, but is multiplied by an interval depending on the mean value of u (non-local dependence). Fix $f \in C^0(\mathbb{R}, \mathbb{R})$, $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ such that α is lower semicontinuous, β is upper semicontinuous, and $\alpha(s) \leqslant \beta(s)$ for all $s \in \mathbb{R}$. For all $u \in \overline{\Omega}$ set

$$\bar{u} = \int_0^1 u(\tau)\, d\tau.$$

We define $\phi(u)$ as the set of all functions $w \in F$ of the form

$$w(t) = cf(u(t)), \quad \alpha(\bar{u}) \leqslant c \leqslant \beta(\bar{u}).$$

Obviously, $\phi : \overline{\Omega} \to 2^F$ has convex values.

We prove that ϕ has compact values. Let $u \in \overline{\Omega}$, (w_n) be a sequence in $\phi(u)$. Then, for all $n \in \mathbb{N}$, there exists $c_n \in [\alpha(\bar{u}), \beta(\bar{u})]$ such that $w_n = c_n f(u)$. The sequence (c_n) is bounded, so passing to a subsequence we have $c_n \to c$ for some $c \in \mathbb{R}$. Set $w = cf(u)$, then clearly $w_n \to w$ in F and $w \in \phi(u)$. Thus, $\phi(u)$ is compact.

The map ϕ is locally compact. Indeed, let (u_n) be a bounded sequence in $\overline{\Omega}$ and let (w_n) be a sequence in F such that $w_n \in \phi(u_n)$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we can find $c_n \in [\alpha(\bar{u}_n), \beta(\bar{u}_n)]$ such that $w_n = c_n f(u_n)$. Since (u_n) is uniformly bounded and equi-continuous, passing to a subsequence we have $u_n \to u$ uniformly in $[0, 1]$ (note that $u \notin E$ in general). Hence, $\bar{u}_n \to \bar{u}$. So (c_n) turns out to be bounded, and up to a subsequence $c_n \to c$. Passing to the limit, due to the properties of α and β, we have

$$\alpha(\bar{u}) \leqslant c \leqslant \beta(\bar{u}).$$

So, setting $w = cf(u)$, we deduce $w_n \to w$ in F. Thus, $\overline{\phi(u_n)}$ is compact.

Alternatively, we can prove that graph ϕ is closed in $\overline{\Omega} \times F$. Indeed, let (u_n, w_n) be a sequence in $\overline{\Omega} \times F$ such that $w_n \in \phi(u_n)$ for all $n \in \mathbb{N}$, and $(u_n, w_n) \to (u, w)$. For all $n \in \mathbb{N}$ we find $c_n \in [\alpha(\bar{u}_n), \beta(\bar{u}_n)]$ such that $w_n = c_n f(u_n)$. Then $\bar{u}_n \to \bar{u}$, and $f(u_n) \to f(u)$ uniformly in $[0, 1]$. We prove now that (c_n) converges, indeed avoiding trivial cases we may assume that $f(u(t)) \neq 0$ at some $t \in [0, 1]$, then

$$\lim_n c_n = \frac{w_n(t)}{f(u_n(t))} = \frac{w(t)}{f(u(t))} = c,$$

with $c \in [\alpha(\bar{u}), \beta(\bar{u})]$. So $w \in \phi(u)$. By [22, Proposition 4.15] again, ϕ is u.s.c. We conclude that $\phi \in CJ(\overline{\Omega}, F)$ and is locally compact.

The last example is more sophisticated, since in the construction of ϕ we preserve the infinite dimension, and we apply some classical results from functional analysis to prove all required compactness properties. We recall such results, starting from a weak notion of compactness in L^1 (see [23, Definition 2.94] and [26, Definition 4.2.1]):

Definition 7.4. A sequence (v_n) in $L^1([0, 1], \mathbb{R})$ is said to be *semicompact* if
(i) it is integrably bounded, i.e., if there exists $g \in L^1([0, 1], \mathbb{R})$ such that $|v_n(t)| \leqslant g(t)$ for a.e. $t \in [0, 1]$ and all $n \in \mathbb{N}$,
(ii) the image sequence $(v_n(t))$ is relatively compact in $\mathbb{R}$ for a.e. $t \in [0, 1]$.

The following result follows from the Dunford–Pettis Theorem (see also [26, Proposition 4.21]):

Proposition 7.5. *Every semicompact sequence is weakly compact in $L^1(0, 1)$.*

We also recall Mazur's well-known theorem (see e.g. [18]):

Theorem 7.6. *Let E be a normed space, and let (x_n) be a sequence in E weakly converging to x. Then there exists a sequence of convex linear combinations*

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k, \quad a_{n,k} \in (0, 1],$$

such that $y_n \to x$ (strongly) in E.

We can now present our last example:

Example 7.7. Let $\alpha, \beta : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that

$$\alpha(t, s) \leqslant \beta(t, s) \quad \text{for all } (t, s) \in [0, 1] \times \mathbb{R},$$

and for all $u \in \overline{\Omega}$ define $\phi(u)$ as the set of all functions $w \in F$ for which there exists $v \in L^1(0, 1)$ such that for all $t \in [0, 1]$,

$$w(t) = \int_0^t v(\tau)\, d\tau,$$

and for a.e. $t \in [0, 1]$,

$$v(t) \in [\alpha(t, u(t)), \beta(t, u(t))].$$

In short, we might define $\phi(u)$ as a set-valued integral in the sense of Aumann

$$\phi(u)(t) = \int_0^t [\alpha(\tau, u(\tau)), \beta(\tau, u(\tau))]\, d\tau$$

(see [9, 26]). Clearly, for all $u \in \overline{\Omega}$, any $w \in \phi(u)$ is absolutely continuous and hence a.e. differentiable in $[0, 1]$ with derivative v. We first prove that ϕ has convex values. Let $u \in \overline{\Omega}$, $w_0, w_1 \in \phi(u)$, and $\mu \in [0, 1]$. There exist $v_0, v_1 \in L^1(0, 1)$ such that

$$w_i(t) = \int_0^t v_i(\tau)\, d\tau, \quad v_i(t) \in [\alpha(t, u(t)), \beta(t, u(t))] \text{ (a.e.)}.$$

Set $w = (1 - \mu)w_0 + \mu w_1$ and $v = (1 - \mu)v_0 + \mu v_1$; then we have in $[0, 1]$

$$w(t) = \int_0^t v(\tau)\, d\tau, \quad v(t) \in [\alpha(t, u(t)), \beta(t, u(t))] \text{ (a.e.)},$$

which implies $w \in \phi(u)$.

We prove now that ϕ has compact values (this is not immediate and will require several steps). Let $u \in \overline{\Omega}$, and let (w_n) be a sequence in $\phi(u)$; then there exists a sequence (v_n) in $L^1(0, 1)$ such that for all $n \in \mathbb{N}$,

$$w_n(t) = \int_0^t v_n(\tau)\, d\tau, \quad v_n(t) \in [\alpha(t, u(t)), \beta(t, u(t))] \text{ (a.e.)}.$$

Clearly, (w_n) is bounded in F. Also, since (v_n) is essentially bounded, (w_n) turns out to be equi-absolutely continuous. By Ascoli's theorem, passing if necessary to a subsequence, we have $w_n \to w$ in F and w is the primitive of some $v \in L^1(0, 1)$, i.e., for all $t \in [0, 1]$ we have

$$w(t) = \int_0^t v(\tau)\, d\tau. \tag{7.2}$$

On the other side, (v_n) is a semicompact sequence in $L^1(0, 1)$ (Definition 7.4), so by Proposition 7.5 we can pass to a further subsequence and have (v_n) weakly converging in $L^1(0, 1)$ to some $\hat{v} \in L^1(0, 1)$. For all $t \in [0, 1]$, the linear functional

$$f \mapsto \int_0^t f(\tau)\, d\tau$$

is bounded in $L^1(0, 1)$, so weak convergence is enough to deduce that for all $t \in [0, 1]$,

$$w_n(t) = \int_0^t v_n(\tau)\, d\tau \to \int_0^t \hat{v}(\tau)\, d\tau.$$

Comparing to (7.2), we see that $v = \hat{v}$ in $L^1(0, 1)$. So (v_n) converges weakly to v in $L^1(0, 1)$. By Theorem 7.6, we can find a sequence $(\tilde{v}_n)$ of convex linear combinations of (v_n) such that $\tilde{v}_n \to v$ in $L^1(0, 1)$ (strongly). Clearly, for all $n \in \mathbb{N}$ and a.e. $t \in [0, 1]$ we have

$$\tilde{v}_n(t) \in [\alpha(t, u(t)), \beta(t, u(t))],$$

so we can pass to the limit and deduce the same property for v. So, by (7.2), we have $w \in \phi(u)$. Thus, $\phi(u)$ is compact. By similar arguments, we prove that ϕ is locally compact.

Upper semicontinuity of ϕ can be proved as in the previous cases, applying [22, Proposition 4.1.5]. Nevertheless, in order to give the reader a more complete picture, we present a direct proof based on Definition 4.1.

Let $V \subset F$ be open, and let $\bar{u} \in \phi^+(V)$. We claim that there exists a neighborhood of $\bar{u}$ contained in $\phi^+(V)$. Indeed, by compactness of $\phi(\bar{u})$, there exist $w_1, \dots, w_n \in \phi(\bar{u})$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that
(i) $B_{\varepsilon_i}(w_i) \subseteq V$ $(i = 1, \dots, n)$,
(ii) $\phi(\bar{u}) \subset \bigcup_{i=1}^n B_{\varepsilon_i/2}(w_i)$.
Consider now the following compact subset of $\mathbb{R}^2$:

$$C = \{(t, y) \in \mathbb{R}^2 : t \in [0, 1],\ \alpha(t, \bar{u}(t)) \leqslant y \leqslant \beta(t, \bar{u}(t))\},$$

and let A be the open ball of $[0, 1] \times \mathbb{R}$ centered at C with radius 1. Then set

$$\bar{\varepsilon} := \frac{\min\{\varepsilon_1, \dots, \varepsilon_n\}}{2}.$$

Since α, β are uniformly continuous in $\overline{A}$, we can find $\delta \in (0, 1)$ such that

$$\max\{|\alpha(t, y) - \alpha(t, z)|, |\beta(t, y) - \beta(t, z)|\} < \bar{\varepsilon} \quad \text{for all } (t, y), (t, z) \in \overline{A},\ \operatorname{dist}((t, y), (t, z)) < \delta. \tag{7.3}$$

We claim that $\phi(B_\delta(\bar{u})) \subseteq V$. Indeed, fix $u \in B_\delta(\bar{u})$. Clearly, we have $|u(t) - \bar{u}(t)| < \delta$ for all $t \in [0, 1]$. Take now $w \in \phi(u)$, which can be written as

$$w(t) = \int_0^t v(\tau)\, d\tau$$

with $v \in L^1(0, 1)$ satisfying for a.e. $t \in [0, 1]$,

$$\alpha(t, u(t)) \leqslant v(t) \leqslant \beta(t, u(t)).$$

If for a.e. $t \in [0, 1]$,

$$\alpha(t, \bar{u}(t)) \leqslant v(t) \leqslant \beta(t, \bar{u}(t)),$$

then $w \in \phi(\bar{u}) \subseteq V$ and we are done. Otherwise, consider the truncated map $\bar{v} : [0, 1] \to \mathbb{R}$ defined by

$$\bar{v}(t) = \begin{cases} \alpha(t, \bar{u}(t)) & \text{if } v(t) < \alpha(t, \bar{u}(t)), \\ v(t) & \text{if } \alpha(t, \bar{u}(t)) \leqslant v(t) \leqslant \beta(t, \bar{u}(t)), \\ \beta(t, \bar{u}(t)) & \text{if } v(t) > \beta(t, \bar{u}(t)), \end{cases}$$

which is an L^1-function (see e.g. [31]), and denote

$$\bar{w}(t) = \int_0^t \bar{v}(\tau)\, d\tau,$$

so $\bar{w} \in \phi(\bar{u})$. By the bounds above we have for all $t \in [0, 1]$ that $(t, u(t)), (t, \bar{u}(t)) \in \overline{A}$ with

$$\operatorname{dist}((t, u(t)), (t, \bar{u}(t))) < \delta,$$

so by (7.3) we have $|v(t) - \bar{v}(t)| < \bar{\varepsilon}$ for a.e. $t \in [0, 1]$. This in turn implies $\|w - \bar{w}\|_\infty < \bar{\varepsilon}$. By (ii), we can find $i \in \{1, \dots, n\}$ such that $\bar{w} \in B_{\varepsilon_i/2}(w_i)$. So, recalling the definition of $\bar{\varepsilon}$, we have

$$\|w - w_i\|_\infty \leqslant \|w - \bar{w}\|_\infty + \|\bar{w} - w_i\|_\infty < \varepsilon_i,$$

hence by (i) $w \in V$. Thus, $\phi(B_\delta(\bar{u})) \subseteq V$ and ϕ turns out to be u.s.c. In conclusion, $\phi \in CJ(\overline{\Omega}, F)$ and it is locally compact.

Remark 7.8. Comparing Definition 6.1 and problem (7.1), one may be left in doubt that the non-trivial solutions ensured by Theorem 7.1 might be triples $(\pm 1, \varepsilon, 0)$ with $\varepsilon \neq 0$ (quite trivial in fact). But this case may only occur if $0 \in \phi(\pm 1)$. Easy computations show that in Example 7.2 we have $0 \notin \phi(\pm 1)$, due to the choice $\rho \in (0, 1)$. Similarly, in Example 7.3 it is enough to choose functions f, α, and β to be positive in order to have $0 \notin \phi(\pm 1)$, thus avoiding such difficulty. Also in Example 7.7, we can easily find α, β such that $0 \notin \phi(\pm 1)$.

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