Research Article

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New Results About the Lambda Constant and Ground States of the W-Functional

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Abstract: In this paper, we study properties of the lambda constants and the existence of ground states of Perelman's famous W-functional from a variational formulation. We have two kinds of results. One is about the estimation of the lambda constant of G. Perelman, and the other is about the existence of ground states of his W-functional, both on a complete non-compact Riemannian manifold (M, g). One consequence of our estimation is that, on an ALE (or asymptotic flat) manifold (M, g), if the scalar curvature s of (M, g) is non-negative and has quadratical decay at infinity, then M is scalar flat, i.e., $s = 0$ in M. We also introduce a new constant $d(M, g)$. For the existence of the ground states, we use Lions' concentration-compactness method.

Keywords: Lambda Constants, Ground States, W-functional, Lions' Concentration-Compactness Method

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1 Introduction

This paper has two parts. One is about the estimation of the lambda (λ) constant of G. Perelman [21] (see also [9] and [25, Chapter 2]), and the other is about the existence of ground states of his F-functional and W-functional on a complete non-compact Riemannian manifold. See also [7, 23] for the closely related results about logarithmic Sobolev inequalities, and in [23], it has been shown that there exists a smooth minimizer of the W-functional on any compact Riemannian manifold (see [4, Proposition 17.24]). We remark that the understanding of the λ-constant has important geometric applications in [22, Section 8], where Perelman gives estimates of the λ-constants before and after the surgery of the Ricci flow. As functionals defined on the space of Riemannian metrics on a closed manifold, the λ-constant and the μ-constant have been studied in detail in the book [4, Chapter 17]. The existence problem of a minimizer of the F-functional (and W-functional) on a complete non-compact Riemannian manifold remains largely open.

Our main results are Theorems 3, 5, 6 and 1. We now mention one of our results below. Assume that (M, g) is a complete non-compact manifold with bounded Ricci curvature and with polynomial volume growth. Then the λ-constant of g cannot be positive provided the scalar curvature function s can be written as a divergence of some bounded smooth vector field, saying, $s = \operatorname{div}(Y)$ for some Y being a bounded vector field on M. One consequence of our estimation is that, on an asymptotic flat manifold (M, g), if the scalar curvature s of (M, g) is non-negative and has quadratical decay at infinity, then $s = 0$ in M. For the existence of the ground states of the Euler–Lagrange equations of F or W-functionals introduced by G. Perelman [21], we use Lions' concentration-compactness method [11, 12].

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Critical values of F-functionals are of a similar characterization as the eigenvalues of the Laplacian operators on Riemannian manifolds [1, 25]. Then we can expect that there is some relation between the λ-constants and minimal surfaces in compact mm-spaces. Recall that the λ-constants on any compact Riemannian manifold (with smooth boundaries) can be defined, and they depend smoothly on the metrics, just like the ordinary Laplacian cases. However, this fact may not be true on complete non-compact Riemannian manifolds. One even does not know if the λ-constants correspond to the energies achieved by corresponding ground states. We could give a partial answer to this question in this paper. For more ambitious purpose, we need to know the analytical dependence of the λ-constant on Riemannian metrics. Considering the λ-constant as a functional on the space of Riemannian metrics on manifolds, we expect that it has good analytical properties such as some kind of continuity property depending on Riemannian metrics. Just like the scalar curvature functional, the study of the F-functional and the W-functional on complete and non-compact Riemannian manifolds is a difficult topic. Let us now mention one property of the λ-constant under the Cheeger–Gromov convergence of Riemannian manifolds [2]. Given an n-dimensional complete Riemannian manifold (M, g) with scalar curvature bounded below, for $u \in H^1 := H^1(M, g)$, we define the F-functional by

$$I_0(u) = \int_M (4|\nabla u|^2 + Ru^2)\, dv_g.$$

The λ-constant $\lambda(M, g)$ is then well-defined by

$$\lambda(M, g) = \inf_{\{u \in H^1;\, \int_M u^2\, dv_g = 1\}} I_0(u).$$

From this very definition, we can see that the following fact is true. Suppose that $(M_j^n, g_j, x_j) \to (M_\infty^n, g_\infty, x_\infty)$ in the C^∞ Cheeger–Gromov sense. Then we may prove that

$$\lambda(M_\infty^n, g_\infty) \geq \overline{\lim_{j\to\infty}}\, \lambda(M_j^n, g_j).$$

Note that a similar result for the μ-constant ($\mu(g, \tau)$ had been proved in [3, Lemma 6.28]. We remark that the assumption that $(M_j^n, g_j, x_j) \to (M_\infty^n, g_\infty, x_\infty)$ in the C^∞ Cheeger–Gromov sense is highly nontrivial, and it is not so easy to verify without the hard analysis estimates about curvatures.

The basic question about the λ-constant and the μ-constant (see the definition below) is the existence of the ground states of them, which is the main topic of this paper. Still, speaking about the λ-constant, the existence of ground states of the λ-constant in complete non-compact Riemannian manifolds is a highly nontrivial question, and one cannot make sure that there always exists a minimizer of the λ-constant in a complete non-compact Riemannian manifold without curvature assumptions. We are interested in the existence of ground states of the λ-constant in complete non-compact Riemannian manifolds under suitable geometric or analytic assumptions. This is a principal eigenvalue problem, and our understanding of this topic is still limited. Motivated by Lions' concentration-compactness method, we can define the λ-constant at infinity λ_∞ (see Section 3). Roughly speaking, we can show in Section 3 that if we have the strict inequality $\lambda(M, g) < \lambda_\infty$, then there is a positive function $u \in H^1$ such that $I_0(u) = \lambda$. A similar and deep result has been proved by Zhang [26] for Perelman's μ-constant on the complete Riemannian manifold with bounded geometry, which is a strong assumption. We mention that N. Hirano [20] first gave a condition about the scalar curvature on the ALE manifold (M, g), which satisfies $d < d_\infty$. See also the related result of Zhang [26], and as an application, he can prove a no-breathers theorem for some non-compact Ricci flows [8, 27]. In this kind of manifolds, the L^2 Sobolev inequality is true, which is a key tool in the study of nonlinear problems in Riemannian geometry. We also obtain the ground states for Perelman's μ-constant on the complete Riemannian manifold with Ricci curvature bounded from below and with positive injectivity radius. Our method here is different from previous studies on this topic, and our goal is to consider this kind of problems from another angle. Our research follows our works [13, 15–17], and we consider the existence problems of minimizers (or ground states) on complete non-compact Riemannian manifolds. This paper can also be considered as a continuation of our research made in [18]. Partial results are announced in our paper [14].

To introduce one of our main results, we recall a few definitions. For $u \in H^1(M,g)$ with $\int_M u^2\,dv = 1$, G. Perelman [2, 21] defines the W-functional (with the parameter τ being fixed and normalized) by

$$W(u,g) = \int_M (4|\nabla u|^2 + Ru^2 - u^2\log u^2)\,dv$$

and the μ-constant by

$$\mu(g) = \inf\Big\{W(u,g);\ \int_M u^2 = 1\Big\}.$$

One can easily show that $\mu(g)$ is well-defined, and the Euler–Lagrange equation of the W-functional is

$$4\Delta u - Ru + 2u\log u + \mu(g)u = 0,\ M. \tag{1.1}$$

One can define two related functionals to (1.1) by

$$I(u) := I(u,g) := \int_M (4|\nabla u|^2 + Ru^2 - u^2\log u^2 + u^2)\,dv,$$

$$N(u) \ = \int_M (4|\nabla u|^2 + Ru^2 - u^2\log u^2)\,dv,$$

where $u \in H^1(M,g)$. Define the Nehari manifold by

$$\mathbb{N} = \{u \in H^1(M) - \{0\};\ N(u) = 0\} \quad \text{and} \quad d := d(g) = \inf\{I(u);\ u \in \mathbb{N}\}.$$

In [26], Zhang defines the log-Sobolev constant of (M,g) at infinity as the quantity

$$\mu_\infty = \lim_{r\to\infty} \inf\{W(u,g);\ u \in C_0^\infty(M - B_r(0)),\ |u|_2 = 1\}.$$

Similarly, we can define

$$d_\infty = \lim_{r\to\infty} \inf\Big\{\int_{M-B_r(0)} u^2\,dv;\ u \in C_0^\infty(M - B_r(0));\ u \neq 0,\ N(u) = 0\Big\}.$$

Then, using Lions' variational principle at infinity [11, 12], we can prove the below.

Theorem 1. *Assume that (M,g) is a complete non-compact Riemannian manifold with its Ricci curvature bounded from below and $\inf_{x\in M}\operatorname{vol}(B_1(x)) > 0$, where $\operatorname{vol}(B_1(x))$ is the volume of the unit ball at $x \in M$. Assume that $d < d_\infty$. Then d is attained at some $u \in \mathbb{N}$ such that $-4\Delta u + Ru = u\log u$ and $u > 0$ such that $d = \int_M u^2$.*

The proof of Theorem 1 will be given in Section 4.

The plan of this paper is below. In Section 2, we recall the definitions of Perelman's F-functional, the modified scalar curvature and the λ-constant. We consider the estimation of the λ-constant and the existence of the ground states of it in Section 3. In Section 4, we introduce another constant $d(M,g)$ on the Nehari manifold and discuss the existence of the ground states of this minimization problem.

2 Perelman's Modified Scalar Curvature, λ-Constant and Lions' Lemma

We prefer to recall some background about the F-functional and the λ-constant.

Let (M,g) be an n-dimensional Riemannian manifold. Let R be the scalar curvature of the metric g. Given a smooth function f with $\int_M e^{-f}\,dv_g = 1$, set $dm = e^{-f}\,dv_g$. G. Perelman [21] defines the modified scalar curvature related to the measure dm by $R^m = 2\Delta f - |\nabla f|^2 + R$ and the F-functional

$$\mathbb{F}(g,f) = \int_M R^m\,dm.$$

The properties of this functional are very useful in the understanding of the properties of the W-functional.

One natural question related to the Yamabe problem is to find a smooth function f with $\int_M e^{-f}\,dv_g = 1$ such that R^m is a constant. This problem is easy to solve when M is compact. Here is a solution in case that M is compact. Using integration by parts, one has

$$\mathbb{F}(g,f) = \int_M (R + |\nabla f|^2)\,dm,$$

which is the original definition of the F-functional. The λ-constant is defined by

$$\lambda(g) = \inf\Big\{\mathbb{F}(g,f);\ f \in C^\infty(M),\ \int_M e^{-f}\,dv_g = 1\Big\}.$$

Let $u = e^{-f/2}$ and

$$I_0(u) = \int_M (4|\nabla u|^2 + Ru^2)\,dv_g.$$

Note that $\int_M u^2\,dv_g = 1$. Clearly, we have the minimization problem

$$\lambda(g) = \inf\Big\{I_0(u);\ u \in C_0^\infty(M),\ u > 0,\ \int_M u^2\,dv_g = 1\Big\}, \tag{2.1}$$

and one always has a positive minimizer u on M by the direct method. In this case, we have $\int_M u^2\,dv_g = 1$ and $-4\Delta u + Ru = \lambda(g)u$. Let $f = -2\log u$. By direct computation, we have $\lambda(g) = 2\Delta f - |\nabla f|^2 + R = R^m$ with $dm = e^{-f}\,dv_g = u^2\,dv_g$.

Minimization problem (2.1) is nontrivial when the Riemannian manifold (M,g) is complete and non-compact. The main purpose of this paper is to present some consideration of this topic. We always assume that (M,g) is a complete non-compact manifold with its Ricci curvature bounded from below by some real constant K and with uniform lower bound of the injectivity radius. In this case, we have the L^2 Sobolev inequality [19], which says that there is a constant $C > 0$ such that

$$C\Big(\int_M |u|^{p+1}\Big)^{2/(p+1)} \le \int_M (|\nabla u|^2 + u^2)$$

for all $u \in C_0^\infty(M)$ and $p = \frac{n+2}{n-2}$ when $n \ge 3$. From the very definition of $\lambda(g)$, we have $\lambda(g) \ge \inf_M R(x)$. Hence minimization problem (2.1) is meaningful. With the help of the lower bound of the Ricci curvature and the L^2 Sobolev inequality, we can set-up the Lions lemma as follows.

Lemma 2. *Assume that $(u_j) \subset H^1$ is a bounded sequence satisfying*

$$\limsup_{j\to\infty}\ \sup_{z\in M} \int_{B_r(z)} |u_j|^2\,dv = 0 \quad \text{for some } r > 0,$$

where $B_r(z)$ denotes the open geodesic ball of radius r centered at $z \in M$. Then $u_j \to 0$ strongly in $L^q(M,g)$ for all $2 < q < 2^* := \frac{2n}{(n-2)_+}$.

Here is the proof for $n \ge 3$.

Proof. By interpolation, we only need to prove that $u_j \to 0$ strongly in $L^q(M,g)$ for some $2 < q < 2^*$. By Hölder's inequality, it is clear that

$$\limsup_{j\to\infty}\ \sup_{z\in M} \int_{B_r(z)} |u_j|^s\,dv = 0 \quad \text{for any } s \in (2, 2^*).$$

Choose $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{2^*}$. By Hölder's inequality and the L^2 Sobolev inequality, we have

$$|u|_{L^q(B_r(z))} \le |u|^{1-\theta}_{L^2(B_r(z))}|u|^{\theta}_{L^{2^*}(B_r(z))} \le C|u|^{1-\theta}_{L^2(B_r(z))}|u|^{\theta}_{H^1(B_r(z))}.$$

Then we have

$$|u|^{2/\theta}_{L^q(B_r(z))} \le C^q|u|^{2/\theta(1-\theta)}_{L^2(B_r(z))}|u|^2_{H^1(B_r(z))}.$$

Choose $q = \frac{2(n+2)}{n}$ and $\theta = \frac{n}{(n+2)}$. Then we have

$$|u|^q_{L^q(B_r(z))} \le C^q |u|^{2/\theta(1-\theta)}_{L^2(B_r(z))} |u|^2_{H^1(B_r(z))}.$$

We now take a cover of M by balls $\{B_r(y)\}$ of radius r as above so that each point of M is contained in at most $N = N(r, n)$ such balls. Then we have

$$|u|^q_{L^q(M)} \le NC^q \sup_{z\in M} |u|^{q(1-\theta)}_{L^2(B_r(z))} |u|^2_{H^1(M)},$$

which shows the conclusion of the lemma. □

This lemma will be used in Section 4.

We shall denote by C the various uniform constants which may change from line to line.

3 Property of the λ-Constant

We now introduce the scalar curvature potential function on (M, g) with bounded geometry. Assume $f: M \to \mathbb{R}$ to be a smooth function on a complete non-parabolic manifold (M, g) with bounded geometry. Denote by $G(x, y)$ the minimal Green function on M, and let $s = R$ be the scalar curvature of g in this section. Assume that (M, g) has its Ricci curvature bounded from below by $-K$, where $K \ge 0$ is a constant, and assume that $f \in L^1(M)$. Then the function defined by $u(x) = \int_M G(x, y)f(y)\, dv_g$ satisfies in weak sense that $-\Delta u = f$ on M. Furthermore, with some locally uniform L^p bound about f, by the Moser iteration argument and L^p theory of uniform elliptic equations (see also [24, proof of Lemma b.3 in the appendix]), we have $|\nabla u| \in L^\infty$ provided M is an ALE manifold. When $f = s$, we call u the potential function of the scalar curvature s.

We can prove the following result even for more general Riemannian manifolds (see [5] for related result for the Laplace operator).

Theorem 3. *Assume that (M, g) is a complete non-compact manifold with bounded Ricci curvature and with polynomial volume growth. Suppose that the scalar curvature s has a potential function with uniformly bounded gradient. Then $\lambda(g) \le 0$, and furthermore, when $s \ge 0$, we have $\lambda(g) = 0$.*

Proof. By the definition of the polynomial volume growth, we mean that there exist constants $C > 0$ and $k \ge 1$, and a fixed point $p \in M$ such that the volume $V_p(R)$ of the geodesic ball $B_R(p)$ satisfies $V_p(R) \le CR^k$, $R > 0$.

By our assumption, we have that $s(x) = -\Delta u(x)$ on M, and u has uniform bounded gradient, i.e., $|\nabla u| \le C$ for some uniform constant C.

Assume $\lambda(g) > 0$. Take $R > 1$, and take $\phi(x)$ to be the cut-off function on $B_{2R}(p)$ such that $\phi = 1$ on $B_R(p)$. By the definition of $\lambda(g)$, we know that

$$\lambda(g) \int \phi^2 \le \int_M (4|\nabla\phi|^2 + s\phi^2). \tag{3.1}$$

Note that

$$\int_M s\phi^2 = \int_M 2\phi\nabla u \cdot \nabla\phi,$$

which is bounded by $CR^{-1}V_p(2R) \le CR^{k-1}$. By (3.1), we have

$$\lambda(g)V_p(R) \le C(R^{k-2} + R^{k-1}) \le CR^{k-1}.$$

We iterate this relation k times to get $V_p(R) \le CR^{-1} \to 0$ as $R \to \infty$. This is impossible. Thus, we have $\lambda(g) \le 0$.

When $s \ge 0$, by definition, we have $\lambda(g) \ge 0$, and then $\lambda(g) = 0$. □

We remark that, in the above argument, we need only assume that $s = \mathrm{div}(Y)$ for some smooth vector field Y in $L^p(M)$ with some $1 < p \le \infty$.

One interesting consequence of Theorem 3 is below, and we refer to [20] for the definition of ALE manifolds (or sometimes called asymptotic flat manifolds).

Theorem 4. *Assume that (M, g) is an n-dimensional ALE manifold ($n \geq 3$). Assume that the scalar curvature s of the metric g is non-negative with some decay property that $s(x) = d(x_0, x)^{-\beta}$ for some $\beta > 2$. Then the λ-constant of (M, g) is zero, i.e., $\lambda(g) = 0$.*

Proof. Clearly, (M, g) has the volume growth as in the n-dimensional Euclidean space R^n. Note that (M, g) is a non-parabolic manifold with its minimal Green function $G(x, y)$ asymptotic to the standard Green function on R^n. We define $u(x) = \int_M G(x, y)s(y)\,dv_y$. It is easy to verify that $u(x)$ is well-defined and satisfies $-\Delta u(x) = s(x)$ and $|\nabla u(x)|$ is bounded. Applying the above result to s, we conclude that $\lambda(g) = 0$. □

This result implies that the λ-constant may not have close relationship with the ADM mass on an ALE manifold. We remark that the general results from Li–Schoen [10] cannot be directly used to obtain Theorem 4 since our potential function $u(x)$ may not be in the class Lp.

Using the log-trick (see the book [6] for related references), we can prove the below.

Theorem 5. *Assume that (M, g) is a complete non-compact Riemannian manifold with non-positive scalar curvature s and quadratic polynomial volume growth. If $\lambda(g) = 0$, then (M, g) is scalar flat, i.e., $s = 0$.*

Proof. By the property of the quadratic polynomial volume growth, we know that there exists constants $C > 0$ such that the volume $V_p(R)$ of the geodesic ball $B_R(p)$ satisfies $V_p(R) \leq CR^2$. Using the domain exhaustion and the direct method, we can find a positive function u on M such that $-4\Delta u + su = \lambda(g)u = 0$.

Let $w = \log u$ and $q = -s$. Then we have $-\Delta w = q + |\nabla w|^2$. Then, for a compactly supported function ϕ, we have

$$\int_M (q + |\nabla w|^2)\phi^2 = -2 \int \phi \nabla\phi \cdot \nabla w,$$

and the right side is bounded by

$$\epsilon \int_M \phi^2 |\nabla w|^2 + \epsilon^{-1} \int_M |\nabla \phi|^2 \quad \text{for } \epsilon = \frac{1}{2}.$$

Then we have

$$\int_M \Big(q + \frac{1}{2}|\nabla w|^2\Big)\phi^2 \leq 2 \int_M |\nabla\phi|^2.$$

Let $r = d(x, p)$ be the distance function, and let $\phi = 1$ for $r \leq \sqrt{R}$, $\phi = 0$ for $r > R$, and $\phi = 2 - 2\frac{\log r}{\log R}$ for $\sqrt{R} < r \leq\leq R$. By a direct computation, we know from the quadratic polynomial volume growth that

$$\int_M |\nabla\phi|^2 \leq \frac{C}{\log R} \to 0 \quad \text{as } R \to \infty.$$

Hence we get

$$\int_{B_{\sqrt{R}}(p)} \Big(q + \frac{1}{2}|\nabla w|^2\Big) \to 0 \quad \text{as } R \to \infty.$$

Hence $q = 0$ and $|\nabla w|^2 = 0$ on M, which implies that $s = 0$. □

This result gives some rigidity result for open manifolds $M = \Sigma \times N$, where Σ is an open surface and N is compact.

As for the existence of minimizers of minimization problem (2.1) is nontrivial when the Riemannian manifold (M, g) is complete and non-compact with scalar curvature bounded below, we have the following result by using P. L. Lions' concentration-compactness method (see [11, p. 115 ff] and [12]). Recall

$$I_0(u, g) := \int_M (4|\nabla u|^2 + Ru^2)\,dv$$

on the manifold

$$\Sigma := \Big\{u \in H^1(M, g); \int_M u^2\,dv = 1\Big\}.$$

Fix $0 \in M$. Assume that $\lim_{d(x,0)\to\infty} R(x) = R_\infty$ for some real constant R_∞. We define the λ-constant of (M, g) at infinity as the quantity

$$\lambda_\infty = \inf\{I_0^\infty(u, g);\ u \in C_0^\infty(M),\ |u|_2 = 1\},$$

where

$$I_0^\infty(u, g) = \int_M (4|\nabla u|^2 + R_\infty u^2)\, dv.$$

Theorem 6. *Assume that (M, g) is a complete non-compact Riemannian manifold with its Ricci curvature bounded from below. Assume $\lim_{d(x,0)\to\infty} R(x) = R_\infty$ for some real constant R_∞, and assume $\lambda(g) < \lambda_\infty$. Then there is a positive function $u \in H^1$ such that $I_0(u, g) = \lambda(g)$.*

Proof. Let $(u_j) \subset \Sigma$ be the minimizing sequence of minimization problem (2.1). Then it is easy to see that (u_j) is a bounded sequence in H^1. Then we may assume that (u_j) converges weakly in H^1 to a function $u \in H^1$ and converges in $L^2_{\mathrm{loc}}(M, g)$. Let $v_j = u_j - u$. Then we have

$$I_0(u_j) = I_0(v_j) + I_0(u) + \circ(1) \to \lambda(g) \quad \text{and} \quad 1 = \int_M u_j^2 = \int_M v_j^2 + \int_M u^2 + \circ(1).$$

We also have $I_0(v_j) = I_0^\infty(v_j) + \circ(1)$. Hence $I_0^\infty(v_j) + I_0(u) + \circ(1) \to \lambda(g)$.

Let $m = \int_M u^2$. Then $m \in [0, 1]$. If $m = 0$, then we have $u_j = v_j$ and

$$\int_{B_R(0)} u_j^2 = \circ(1) \quad \text{for any fixed } R > 1.$$

Then $\lambda_\infty \le I_0^\infty(v_j, g) \to \lambda(g)$, which is a contradiction to the assumption $\lambda(g) < \lambda_\infty$.

We now have $m > 0$. If $m < 1$, then $\int_M v_j^2 \to 1 - m > 0$. Then we have

$$\lambda(g) \le I_0\Big(\frac{u}{\sqrt{m}}\Big) = \frac{I_0(u)}{m} \quad \text{and} \quad \lambda_\infty(g) \le I_0^\infty\Big(\frac{v_j}{\sqrt{1-m}}\Big) = \frac{\lambda(g) - I_0(u)}{1 - m}.$$

Hence we have

$$\lambda(g) = I_0(u) + (\lambda(g) - I_0(u)) \ge m\lambda(g) + (1 - m)\lambda_\infty(g),$$

which implies a contrary conclusion, again that $\lambda(g) \ge \lambda_\infty(g)$. Hence $\int_M u^2 = 1$, and the minimizing sequence (u_j) converges strongly in H^1 to the limit u. By the maximum principle, we know that $u > 0$ in M. □

In general, we can define

$$\lambda_\infty = \lim_{r\to\infty} \inf\Big\{\int_{M-B_r(0)} (4|\nabla u|^2 + Ru^2)\, dv;\ u \in C_0^\infty(M - B_r(0));\ u \ne 0, \int_M u^2\, dv = 1\Big\}.$$

We have the following result.

Theorem 7. *Assume that (M, g) is a complete non-compact Riemannian manifold with its Ricci curvature bounded from below. Assume that $\lambda(g) < \lambda_\infty$. Then there is a positive function $u \in H^1$ such that $I_0(u, g) = \lambda(g)$.*

Proof. Let $(u_j) \subset \Sigma$ be the minimizing sequence of minimization problem (2.1). Then it is easy to see that (u_j) is a bounded sequence in H^1, and we may assume that $u_j \ge 0$, (u_j) converges to u weakly in H^1 and strongly in $L^2_{\mathrm{loc}}(M)$. Apply Lions' concentration-compactness method to the measures $\rho(u_j) = |\nabla u_j|^2 + u_j^2$.

In the vanishing case, we have

$$\overline{\lim_{j\to\infty}} \sup_{y\in M} \int_{B_r(y)} (|\nabla u_j|^2 + u_j^2) = 0 \quad \text{for all } r > 0.$$

Fix $y \in M$. Choose the cut-off function ξ_R such that $\xi_R(x) = 0$ in $B_r(y)$ and $\xi_r(x) = 1$ outside $B_{2r}(y)$. Then we have

$$\int_M (|\nabla(\xi_r u_j)|^2 + (\xi_r u_j)^2)\, dv = \int_M (|\nabla u_j|^2 + u_j^2)\, dv + \circ(1).$$

Recall that $u_j \in \Sigma$,

$$I_0(u_j, g) = \int_M (4|\nabla u_j|^2 + R_j u^2)\, dv \to \lambda.$$

Since $\int_M (\xi_r u_j)^2 = 1 + \circ(1)$, we have

$$\int_{M-B_r(0)} (4|\nabla(\xi_r u_j)|^2 + R(\xi_r u_j)^2)\, dv \geq \lambda_\infty.$$

However, we have

$$\int_{M-B_r(0)} (4|\nabla(\xi_r u_j)|^2 + R(\xi_r u_j)^2)\, dv = \int_M (4|\nabla u_j|^2 + Ru_j^2)\, dv \to \lambda,$$

which gives us that $\lambda \geq \lambda_\infty$, a contradiction to our assumption.

In the dichotomy case, we have

$$\lambda \leftarrow I_0(u_j, g) = I_0(u_j^1, g) + I_0(u_j^2, g) + \circ(1) \tag{3.2}$$

for $u_j = u_j^1 + u_j^2$, for two sequences $u_j^1 \in H^1$ and $u_j^2 \in H^1$ and $\operatorname{dist}(\operatorname{supp}(u_j^1), \operatorname{supp}(u_j^2)) \to \infty$. We may assume that $\operatorname{dist}(y, \operatorname{supp}(u_j^2)) \to \infty$ and $\int_M (u_j^2)^2 \to \alpha > 0$. Clearly, $\alpha \leq 1$. Then we have

$$I_0\Big(\frac{u_j^2}{\sqrt{\alpha}}, g\Big) \geq \lambda_\infty,$$

that is, $I_0(u_j^2, g) \geq \alpha\lambda_\infty$. Similarly, we have $I_0(u_j^1, g) \geq (1-\alpha)\lambda$. Then we have

$$I_0(u_j^1, g) + I_0(u_j^2, g) \geq (1-\alpha)\lambda + \alpha\lambda_\infty.$$

Combining this with (3.2), we obtain that $\lambda \geq (1-\alpha)\lambda + \alpha\lambda_\infty$, which gives a contradiction to our assumption that $\lambda < \lambda_\infty$.

So we are left the compactness case. In this case, it is standard to get the H^1-convergence of the sequence (u_j) to the function u, and so we omit the detail. □

We remark that all results above can be extended to the operator $L_\beta = -4\Delta + \beta R$ (replacing $-4\Delta + R$), where $\beta > 0$ is any constant.

4 Ground States of the *W*-Functional

The purpose of this section is to prove Theorem 1. Recall that, for $u \in H^1(M, g)$ with $\int_M u^2\, dv = 1$, G. Perelman [2, 21] defines the W-functional (with the parameter τ being fixed and normalized) by

$$W(u, g) = \int_M (4|\nabla u|^2 + Ru^2 - u^2 \log u^2)\, dv$$

and the μ-constant by

$$\mu(g) = \inf\Big\{W(u, g); \int_M u^2 = 1\Big\}.$$

Note that the Euler–Lagrange equation of the W-functional is

$$4\Delta u - Ru + 2u \log u + \mu(g)u = 0, M. \tag{4.1}$$

To understand (4.1) well, we recall that

$$I(u) := I(u, g) := \int_M (4|\nabla u|^2 + Ru^2 - u^2 \log u^2 + u^2)\, dv,$$

$$N(u) \;= \int_M (4|\nabla u|^2 + Ru^2 - u^2 \log u^2)\, dv,$$

where $u \in H^1(M, g)$.

We use Nehari's method and define the Nehari manifold by $\mathbb{N} = \{u \in H^1(M) - \{0\};\ N(u) = 0\}$. Then we can define the value of the ground states by $d(g) = \inf\{I(u);\ u \in \mathbb{N}\}$. For $u \in \mathbb{N}$, we have $I(u) = \int_M u^2\,dv$. One can show that $d := d(g) > 0$. The two constants $\mu(g)$ and d are related by the relation $\mu(g) = \log(2\sqrt{d})$. In fact, for $u \in H^1(M)$, $|u|_2 = 1$, we have

$$\mu(g) + 2\int_M u^2 \log|u|\,dv \le \int_M (4|\nabla u|^2 + Ru^2)\,dv.$$

For $f \in H^1(M) - \{0\}$, let $u = \frac{f}{|f|_2}$. Then we have

$$|f|_2^2(\mu(g) - 2\log|f|_2) + 2\int_M f^2 \log|f|\,dv \le \int_M (4|\nabla f|^2 + Rf^2)\,dv.$$

For $w \in \mathbb{N}$ and $a > 0$, putting $f = aw$, we have

$$|w|_2^2(\mu(g) - 2\log|w|_2 - 2\log a) + 2\int_M w^2(\log|w| + \log a)\,dv \le \int_M (4|\nabla w|^2 + Rw^2)\,dv.$$

Using the relation $N(w) = 0$, we know that $\mu(g) - 2\log|w|_2 \le 0$. Then $\mu(g) \le \log(2\sqrt{d})$. Conversely, we can also prove that $2\sqrt{d} \le e^{\mu(g)}$. Hence we have the following result.

Proposition 8. *For the complete non-compact Riemannian manifold (M, g) with the constants $d(g)$ and $\mu(g)$ defined above, we have $\mu(g) = \log(2\sqrt{d(g)})$.*

For readers' convenience, let us recall that the log-Sobolev constant of (M, g) at infinity as introduced by Zhang in [27] is

$$\mu_\infty = \lim_{r\to\infty} \inf\{W(u, g);\ u \in C_0^\infty(M - B_r(0)),\ |u|_2 = 1\}.$$

Similarly, we can define

$$d_\infty = \lim_{r\to\infty} \inf\Big\{\int_{M-B_r(0)} u^2\,dv;\ u \in C_0^\infty(M - B_r(0));\ u \ne 0,\ N(u) = 0\Big\}.$$

We remark that, in our case, the L^2-Sobolev inequality in (M, g) is true [19].

The proof of Theorem 1 is similar to that of P. L. Lions [11, p. 115 ff.], [12] and [24, Theorem 4.3]. Here we choose $\rho(u) = |\nabla u|^2 + u^2$. Because of the reason just mentioned, we only give an outline of the proof of Theorem 1 below.

Proof. Let $(u_j) \subset \mathbb{N}$ be a minimizing sequence such that $I(u_j) \to d$ and $u_j \ge 0$. Then (u_j) is a bounded sequence in L^2 and $I'(u_j) \to 0$ weakly in $H^{-1}(M)$. Using the fact that $u_j \in \mathbb{N}$, we know that (u_j) is bounded in H^1. The trick here is that, for some $p > 1$ small, there is a positive constant C_p such that $t\log t \le C_p t^{(p+1)/2}$ for all $t > 0$, and for $u \in \mathbb{N}$,

$$N(u) = \int_M \Big(4|\nabla u|^2 + (R - 2\log A)u^2 - u^2 \log\Big(\frac{u}{A}\Big)^2\Big)\,dv,$$

where A is a uniform constant such that $R - 2\log A \ge 1$ on M. For $u \in \mathbb{N}$,

$$\Big(\int_M u^{p+1}\,dv\Big)^{2/(p+1)} \le \int_M \rho(u) \le \int_M (4|\nabla u|^2 + (R - 2\log A)u^2)\,dv \le \int_M u^2 \log\Big(\frac{u}{A}\Big)^2\,dv.$$

By the interpolation inequality and the Sobolev inequality, we have a uniform constant $C > 0$ such that

$$\Big(\int_M u^{p+1}\,dv\Big)^{2/(p+1)} \le C\int_M \rho(u),$$

$$\int_M u^2 \log u^2\,dv \le 2C_p \int_M u \cdot u^{(p+1)/2}\,dv \le 2C_p\Big(\int_M u^2\Big)^{1/2}\Big(\int_M u^{p+1}\,dv\Big)^{1/2}.$$

Then we have, for some uniform constants $C_1 > 0$ and $C_2 > 0$,

$$\Big(\int_M u^{p+1}\,dv\Big)^{2/(p+1)} \le C_1\Big(\int_M u^{p+1}\,dv\Big)^{1/2} + C_2,$$

which implies the uniform bound of the sequence (u_j).

We may now assume that $u_j \to u$ weakly in H^1 for some $u \in H^1$. If $u \neq 0$, then we are done. For $u = 0$, this is the vanishing case, and we must have $u_j \to u$ in $L^2_{\mathrm{loc}}(M)$. Then we get a contradiction with our choice of the sequence $\{u_j\} \subset \mathbb{N}$. In the dichotomy case, we can find $(x_j) \subset M$ such that $x_j \to \infty$ and $|u_j|_{L^2(B_1(x_j))} \ge \delta > 0$ for some uniform constant δ. We may assume that (M, x_j, g) converges to $(M_\infty, x_\infty, g_\infty)$ in the Cheeger–Gromov sense and $u_j(\exp_{x_j} \cdot \exp_\infty^{-1}(z)) \to u_\infty(z)$ strongly in $L^p(B_1(x_\infty))$ for any $2 \le p < \frac{2n}{n-2}$. Then we know that u_∞ satisfies the Euler–Lagrange equation of $I(\,\cdot\,, g_\infty)$ weakly in $(M_\infty, x_\infty, g_\infty)$. By this, we know that $d \ge d_\infty$, which is impossible by our assumption. We may give a little more detail.

(1) If vanishing takes place, by Lions' lemma we know that $u_j \to 0$ in $L^q(M)$ for $2 < q < \frac{2n}{(n-2)}$. Recall that, for $p > 1$ small, $t^{p+1} \le \delta t^2 + C_\delta t^{2n/(n-2)}$. Then $|u_j|^2_{H^1} \le \delta |u_j|^2_{L^2} + C_\delta |u_j|^{2n/(n-2)}_{H^1}$. By this, we get

$$|u_j|^2_{H^1} \le 2C_\delta |u_j|^{2n/(n-2)}_{H^1},$$

and this implies that if $|u_j|_{H^1}$ is small, then $|u_j|_{H^1} = 0$, which is impossible.

(2) To rule out the dichotomy, we need to verify that, for any $\alpha > 0$,

$$0 < d < d_\alpha := \inf\{I(u) - N(u);\ u \in H^1,\ N(u) = -\alpha\}.$$

In fact, for any $u \in H^1$ with $N(u) = -\alpha$, since $N(tu) = t^2 N(u) - 2t^2 \log t \int_M u^2$, there is a constant $t_0 \in (0, 1)$ such that $N(t_0 u) = 0$. Then we can show that $t_0 \le k(\alpha) < 1$ for some uniform constant $k(\alpha)$. Hence we have $0 < d < d_\alpha$. In our dichotomy case, we have a positive constant α and u_j^1, u_j^2 with large distance supports such that $\mathrm{supp}(u_j^2) \to \infty$,

$$0 = N(u_j) = N(u_j^1) + N(u_j^2) + \circ(1), \quad N(u_j^2) \to -\alpha,$$

and $I(u_j) = I(u_j^1) + I(u_j^2) + \circ(1) \to d$. Then we have

$$I(u_j^1) - N(u_j^1) + I(u_j^2) - N(u_j^2) \to d.$$

Noting that $I(u_j^1) - N(u_j^1) \ge \delta > 0$ and $I(u_j^2) - N(u_j^2) \ge d_\alpha$, we obtain $d_\alpha \le d$, which is impossible. Hence we are left the compactness case, and the argument is by now standard to get a positive ground state u, and the detail is omitted. □

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