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Research Article

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Periodic Solutions of Non-autonomous Allen–Cahn Equations Involving Fractional Laplacian

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Abstract: We consider periodic solutions of the following problem associated with the fractional Laplacian: $(-\partial_{xx})^s u(x) + \partial_u F(x, u(x)) = 0$ in $\mathbb{R}$. The smooth function $F(x, u)$ is periodic about x and is a double-well potential with respect to u with wells at $+1$ and -1 for any $x \in \mathbb{R}$. We prove the existence of periodic solutions whose periods are large integer multiples of the period of F about the variable x by using variational methods. An estimate of the energy functional, Hamiltonian identity and Modica-type inequality for periodic solutions are also established.

Keywords: Fractional Laplacian, Periodic Solutions, Non-autonomous, Variational Method

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1 Introduction

We consider periodic solutions of the non-autonomous Allen–Cahn equation

$$(-\partial_{xx})^s u(x) + \partial_u F(x, u(x)) = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $(-\partial_{xx})^s$, $0 < s < 1$, denotes the usual fractional Laplace operator. The function $F(x, u)$ is periodic about x; for simplicity of notation, we assume the period is 1, namely, for any given $u \in \mathbb{R}$,

$$F(x + 1, u) = F(x, u) \quad \text{for all } x \in \mathbb{R}. \tag{1.2}$$

F is also a smooth double-well potential with respect to u with wells at $+1$ and -1, namely, for any given $x \in \mathbb{R}$, it satisfies

$$\begin{cases} F(x, 1) = F(x, -1) = 0 < F(x, u) & \text{for all } -1 < u < 1, \\ \partial_u F(x, 1) = \partial_u F(x, -1) = 0. \end{cases} \tag{1.3}$$

We also assume that, for any x,

$$F \text{ is nondecreasing in } (-1, 0) \text{ and nonincreasing in } (0, 1) \text{ with respect to } u. \tag{1.4}$$

Note that conditions (1.3), (1.4) mean that, for fixed x,

$$F(x, 0) = \max_{-1 \le u \le 1} F(x, u) > 0.$$

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We may assume that F is even with respect to u and x respectively, that is to say

$$\begin{cases} \text{for any given } x, \quad F(x,u) = F(x,-u), \quad u \in \mathbb{R}, \\ \text{for any given } u, \quad F(x,u) = F(-x,u), \quad x \in \mathbb{R}. \end{cases} \tag{1.5}$$

For the corresponding autonomous equation of (1.1),

$$(-\partial_{xx})^s u + F'(u) = 0, \quad x \in \mathbb{R}, \tag{1.6}$$

while conditions (1.3), (1.4) turn into

$$\begin{cases} F(1) = F(-1) = 0 < F(u) \quad \text{for all } -1 < u < 1, \\ F'(1) = F'(-1) = 0, \end{cases} \tag{1.7}$$

and

$$F \text{ is nondecreasing in } (-1,0) \text{ and nonincreasing in } (0,1). \tag{1.8}$$

Under conditions (1.7), (1.8), the authors in [14] proved that if F is even about u, then (1.6) admits an odd periodic solution, which is a minimizer of the corresponding energy functional J. If F is not necessarily even, the existence of mountain passing periodic solutions are also established, and the solutions are not necessarily odd. In [10], an upper bound of the least positive period is given, and Hamiltonian identity, Modica-type inequality and an estimate of the energy functional for periodic solutions are also established. Recently, the existence of periodic solutions to an Allen–Cahn system with the fractional Laplacian is obtained in [11]. Existence and multiplicity of periodic solutions to so-called pesudo-relativistic Schrödinger equations are also established in [1–3]. The authors in [4] obtain some existence results of periodic solutions to some fractional Laplace equations by using variational method and bifurcation theory. In [18], the authors establish interior and boundary Harnack inequalities for nonnegative solutions to $(-\Delta)^s u = 0$ with periodic boundary conditions, and they also obtain regularity properties of the fractional Laplacian with periodic boundary conditions and the pointwise integro-differential formula for the operator.

The authors in [6, 7] proved that conditions (1.7) are both necessary and sufficient for the existence of a layer solution to problem (1.6). Moreover, they prove that such layer solution is unique and establish asymptotic behavior of the layer solution. These results also have been proven with different techniques in [17]. The author in [16] studied layer solutions of the non-autonomous Allen–Cahn-type fractional Laplace equation $(-\partial_{xx})^s u = b(x)F'(u)$, $x \in \mathbb{R}$, where F satisfies (1.7), (1.8) and $b\colon \mathbb{R} \to \mathbb{R}$ is a positive even periodic function, and obtained the existence of layer solutions for $s \geq \frac{1}{2}$.

A natural question is that whether there exist periodic solutions of such non-autonomous Allen–Cahn-type fractional Laplace equation. Our interest in the present work is to find periodic solutions of the more general non-autonomous Allen–Cahn equation (1.1).

Plainly, equation (1.1) possesses three constant periodic solutions $u = 1, -1, 0$, under conditions (1.3) and (1.4). We will try to find nonconstant periodic solutions. We investigate the existence of odd periodic solutions to equation (1.1) when F satisfies conditions (1.2)–(1.5). If $F(x,u)$ only satisfies condition (1.2)–(1.4), namely, it is not necessarily an even function with respect to u and x, in this case, we will obtain the existence of periodic solutions (not necessarily odd) to equation (1.1). We will also establish Hamiltonian identity and Modica-type inequality for periodic solutions of (1.1).

Our main results are the followings.

Theorem 1.1. *Let $s \in (0,1)$. Assume $F(x,u)$ satisfies conditions* (1.2)–(1.5). *Then there exists $T_1 > 0$, for any integer T with $T > T_1$, equation* (1.1) *admits an odd periodic solution u_T with period T, and $u_T(x) \in (0,1)$ for $x \in (0, \frac{T}{2})$. Moreover, we have*

$$J(U_T, \Omega_T) \leq \begin{cases} CT^{1-2s}, & s \in (0, \frac{1}{2}), \\ C \ln T, & s = \frac{1}{2}, \\ C, & s \in (\frac{1}{2}, 1). \end{cases} \tag{1.9}$$

Here U_T is the s-harmonic extension of u_T, and $J(U_T, \Omega_T)$ is the energy functional of equation (3.1), *which will be given in Section 3.*

Theorem 1.2. *Let $s \in (0,1)$. Assume $F(x,u)$ satisfies conditions (1.2)–(1.4) and satisfies $\partial_{uu}F(x,\pm 1) > 0$ for all $x \in \mathbb{R}$. Then there exists $T_2 > 0$, for any integer with $T > T_2$, equation (1.1) admits a periodic solution u_T with period T, $|u_T(x)| < 1$ in $\mathbb{R}$, and it changes its sign at least once in a period. Moreover, for any $\sigma \in (0, \frac{1}{2})$, there exists $T_\sigma \geq T_2$ such that, for any integer T with $T > T_\sigma$, we have*

$$J(U_T, \Omega_T) < \sigma \max_{x \in [-\frac{T}{2}, \frac{T}{2}]} F(x,0)T. \tag{1.10}$$

Theorem 1.3 (Hamiltonian Identity). *Assume $U(x,y)$ is the s-harmonic extension of a periodic solution $u(x)$ of (1.1). Then, for all $x \in \mathbb{R}$, we have*

$$\int_0^\infty \frac{y^a}{2}\{U_x^2 - U_y^2\}\,dy + \int_0^x \partial_x F(\tau, U(\tau,0))\,d\tau - F(x, U(x,0)) \equiv C_T,$$

where $a = 1 - 2s$.

Theorem 1.4 (Modica-Type Inequality). *Assume U is the s-harmonic extension of an even periodic solution u of (1.1). Then, for all $y \geq 0$ and $x \in \mathbb{R}$, we have*

$$\int_0^y \frac{\tau^a}{2}\{U_x^2(x,\tau) - U_y^2(x,\tau)\}\,d\tau + \int_0^x \partial_x F(\tau, U(\tau,0))\,d\tau - F(x,U(x,0)) - C_T \leq \hat{C},$$

where $\hat{C} = \sup_{x\in\mathbb{R}}\{\int_0^x \partial_x F(\tau, U(\tau,0))\,d\tau - F(x,U(x,0)) - C_T\} > 0$ and C_T is the constant given in Theorem 1.3.

A similar Hamiltonian identity and Modica-type inequality can be found in [6, 15]. We want to point out that the Hamiltonian identity and Modica-type inequality established in [6] correspond to our particular case, namely the case that $\partial_x F(x,u) \equiv 0$.

2 Some Basic Properties

We introduce some useful lemmas which will be used in the sequel.

Lemma 2.1 (Weighted Poincaré Inequality [13]). *Assume D belongs to the class S which is defined as*

$$\begin{aligned} S = \{D \subset \mathbb{R}^n :\ & D \text{ is an open bounded set, and there exists } \alpha, \rho_0 > 0 \\ & \text{such that, for all } \hat{x} \in \partial D,\ \rho < \rho_0,\ |B_\rho(\hat{x}) \setminus D| \geq \alpha |B_\rho(\hat{x})|\}. \end{aligned}$$

Then, for all $\hat{x} \in \partial D$, $0 < \rho < \rho_0$, and all $u \in C^1(\overline{B_\rho(\hat{x}) \cap D})$ vanishing on $B_\rho(\hat{x}) \cap \partial D$,

$$\int_{B_\rho(\hat{x})\cap D} x_n^a u^2(x)\,dx \leq C(\rho, n, a) \int_{B_\rho(\hat{x})\cap D} x_n^a |\nabla u(x)|^2\,dx,$$

where $a \in (-1,1)$.

Lemma 2.2 (Hopf Principle [6]). *Assume that $a \in (-1,1)$, and consider the cylinder*

$$C_{R,1} := \Gamma_R^0 \times (0,1) \subset \mathbb{R}_+^{n+1},$$

where $\Gamma_R^0 = \{(x,0) \in \partial\mathbb{R}_+^{n+1} : |x| < R\}$. Let $u \in C(\overline{C_{R,1}}) \cap H^1(C_{R,1}, y^a)$ satisfy

$$\begin{cases} L_a u := \operatorname{div}(y^a \nabla w) \leq 0 & \text{in } C_{R,1}, \\ u > 0 & \text{in } C_{R,1}, \\ u(0,0) = 0. \end{cases}$$

Then

$$\limsup_{y\to 0^+} -y^a \frac{u(0,y)}{y} < 0.$$

In addition, if $y^a u_y \in C(\overline{C_{R,1}})$, then

$$-\lim_{y\to 0^+} y^a \frac{\partial u}{\partial y} < 0.$$

Lemma 2.3 (Strong Maximum Principles [6]). *Assume that $a \in (-1, 1)$. Let $v \in H^1(y^a, C_{R,1}) \cap C(\overline{C_{R,1}})$ satisfy*

$$\begin{cases} \operatorname{div}(y^a \nabla v) \le 0, & (x, y) \in C_{R,1}, \\ -\lim\limits_{y \to 0^+} y^a \dfrac{\partial v}{\partial y} \ge 0, & (x, y) \in \Gamma_R^0, \\ v \ge 0, & (x, y) \in C_{R,1}. \end{cases}$$

Then either $v > 0$ or $v \equiv 0$ in $C_{R,1} \cup \Gamma_R^0$.

Equation (1.1) is related to a degenerate elliptic problem (3.1) on $\mathbb{R}_+^2$ (see [8]). For more properties of fractional Laplacian and solutions of degenerate elliptic equations, readers can see [5, 9, 12, 13, 20, 21].

3 Proof of Existence

We follow the methods in [10, 14] to prove Theorem 1.1.

Proof of Theorem 1.1. For given positive integer T, we denote $\Omega_T := [0, \frac{T}{2}] \times [0, +\infty)$. The solution of equation (1.1) is related to the following equation (see [8]):

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R},\ y > 0\}, \\ \dfrac{\partial U}{\partial v^a} = -\partial_U F(x, U(x, 0)) & \text{on } \mathbb{R}, \end{cases} \tag{3.1}$$

where $a = 1 - 2s$, $\frac{\partial U}{\partial v^a} = -\lim_{y \to 0} y^a \partial_y U$. In other words, if U is a solution of (3.1), then a positive constant multiple of $u(x) := U(x, 0)$ satisfies (1.1). Problem (3.1) corresponds to an energy functional

$$J(U, \Omega_T) = \frac{1}{2} \int\limits_{\Omega_T} y^a |\nabla U(x, y)|^2 \, dx \, dy + \int\limits_0^{\frac{T}{2}} F(x, U(x, 0)) \, dx.$$

We denote the admissible set of the energy J as

$$\Lambda_T := \{U : U \ge 0,\ U(0, y) = 0 = U(\tfrac{T}{2}, y) \text{ for all } y \ge 0,\ U \in H^1(\Omega_T, y^a)\}.$$

Here

$$H^1(\Omega_T, y^a) := \{U : y^a(U^2 + |\nabla U|^2) \in L^1(\Omega_T)\}.$$

Note that $J(U, \Omega_T) \ge 0$. On the other hand, we have that $0 \in \Lambda_T$ and $J(0, \Omega_T) < +\infty$. Hence there exists a minimizing sequence $U_k \in \Lambda_T$ of J, namely

$$\lim_{k \to \infty} J(U_k, \Omega_T) = m_T = \inf_{U \in \Lambda_T} J(U, \Omega_T).$$

Clearly, we have $0 \le U_k \le 1$ on Ω_T since $F(x, u)$ achieves its minimum value 0 at $u = 1$ for any x. Then, for sufficiently large k, we have

$$\int\limits_{\Omega_T} y^a |\nabla U_k(x, y)|^2 \, dx \, dy \le 2m_T + 1. \tag{3.2}$$

From the weighted Poincaré inequality, we obtain

$$\int\limits_{\Omega_T} y^a U_k^2(x, y) \, dx \, dy \le C < +\infty. \tag{3.3}$$

From (3.2) and (3.3), we deduce that there exists a subsequence of $\{U_k\}$ still denoted as $\{U_k\}$, converging weakly in $H^1(\Omega_T, y^a)$ to a function $U_T \in H^1(\Omega_T, y^a)$. Due to the weak lower semi-continuity of the norm, we obtain

$$\int\limits_{\Omega_T} y^a |\nabla U_T(x, y)|^2 \, dx \, dy \le \liminf_{k \to +\infty} \int\limits_{\Omega_T} y^a |\nabla U_k(x, y)|^2 \, dx \, dy.$$

By Fatou's lemma, we also have

$$\int_0^{\frac{T}{2}} F(x, U_T(x,0))\,dx \le \liminf_{k\to+\infty} \int_0^{\frac{T}{2}} F(x, U_k(x,0))\,dx.$$

Hence $J(U_T, \Omega_T) \le m_T$. Note that Λ_T is weakly closed; then $J(U_T, \Omega_T) = m_T$. For any given $\eta \in \Lambda_T$, $\tau > 0$, then $U_T + \tau\eta \in \Lambda_T$. We construct a real-valued function $i(\tau) := J(U_T + \tau\eta, \Omega_T)$. Then

$$\begin{aligned} 0 \le i'(0^+) &= \int_{\Omega_T} y^a \nabla U_T(x,y)\nabla\eta(x,y)\,dx\,dy + \int_0^{\frac{T}{2}} \partial_u F(x, U_T(x,0))\eta(x,0)\,dx \\ &= -\int_{\Omega_T} \operatorname{div}(y^a\nabla U_T(x,y))\eta(x,y)\,dx\,dy + \int_0^{\frac{T}{2}} \Big[\frac{\partial U_T}{\partial \nu^a} + \partial_u F(x, U_T(x,0))\Big]\eta(x,0)\,dx. \end{aligned}$$

Hence, by the arbitrariness of η, we obtain

$$\begin{cases} \operatorname{div}(y^a\nabla U_T) \le 0 & \text{in } \Omega_T, \\ \dfrac{\partial U_T}{\partial \nu^a} \ge -\partial_u F(x, U_T) & \text{on } (\partial\Omega)_0 = [0, \frac{T}{2}]. \end{cases}$$

Next, our task is to prove that $U_T \not\equiv 0$. For $\sigma \in (0,1)$, we define the continuous function

$$h(x) = \begin{cases} \frac{4}{\sigma T}x & \text{if } x \in [0, \frac{\sigma T}{4}], \\ 1 & \text{if } x \in [\frac{\sigma T}{4}, \frac{T}{2} - \frac{\sigma T}{4}], \\ \frac{2}{\sigma} - \frac{4}{\sigma T}x & \text{if } x \in [\frac{T}{2} - \frac{\sigma T}{4}, \frac{T}{2}]. \end{cases}$$

Note that $0 \le h \le 1$; then we construct a function $\psi \in \Lambda_T$ as follows:

$$\psi(x,y) = \exp\Big\{\frac{-y}{2^{b+1}}\Big\}h(x),$$

where the parameter b will be determined later. We next compute the energy $J(\psi, \Omega_T)$. From conditions (1.3) and (1.4) of F, we have

$$\int_0^{\frac{T}{2}} F(x, \psi(x,0))\,dx = \int_0^{\frac{T}{2}} F(x, h(x))\,dx < \int_0^{\frac{\sigma T}{4}} F(x,0)\,dx + \int_{\frac{T}{2}-\frac{\sigma T}{4}}^{\frac{T}{2}} F(x,0)\,dx. \tag{3.4}$$

For the other part of energy, we have

$$\begin{aligned} \int_0^{\frac{T}{2}}\int_0^{+\infty} y^a|\nabla\psi(x,y)|^2\,dx\,dy &= \int_0^{\infty} y^a \exp\Big\{-\frac{y}{2^b}\Big\}\,dy \int_0^{\frac{T}{2}} \Big[\frac{h^2(x)}{2^{2b+2}} + (h'(x))^2\Big]\,dx \\ &\le \Big[\frac{1}{2^{2b}} + \frac{8}{\sigma T}\Big]\int_0^{\infty} y^a \exp\Big\{-\frac{y}{2^b}\Big\}\,dy \\ &= 2^{b(a+1)}\Big[\frac{1}{2^{2b}} + \frac{8}{\sigma T}\Big]\int_0^{\infty} z^a e^{-z}\,dz \\ &\le \Gamma(a+1)2^{b(a-1)}\Big(\frac{T}{8} + 2^{2b}\frac{8}{\sigma T}\Big). \end{aligned}$$

Note that $a - 1 < 0$. For the purpose that the term $2^{b(a-1)}\Gamma(a+1)$ is small, we can choose sufficiently large b. The other term $2^{2b}\frac{8}{\sigma T}$ is also small provided that T is large enough. Hence there exists $T_1 > 0$ such that, for any $T > T_1$, the following estimate holds true:

$$\frac{1}{2}\int_0^{\frac{T}{2}}\int_0^{+\infty} y^a|\nabla\psi(x,y)|^2\,dx\,dy < \int_{\frac{\sigma T}{4}}^{\frac{T}{2}-\frac{\sigma T}{4}} F(x,0)\,dx. \tag{3.5}$$

From (3.4) and (3.5), we have

$$J(0, \Omega_T) > J(\psi, \Omega_T) \ge J(U_T, \Omega_T),$$

which shows that $U_T \not\equiv 0$.

In view of $U_T \ge 0$ and $U_T \not\equiv 0$, by the strong maximum principle of a strictly elliptical operator and the Hopf principle (Lemma 2.2), we have that $U_T > 0$ in $(0, \frac{T}{2}) \times [0, +\infty)$. Choosing any function

$$\eta_0 \in C_c^\infty((0, \tfrac{T}{2}) \times [0, +\infty)),$$

if $|\tau|$ is sufficiently small, then one has $U_T + \tau\eta_0 \in \Lambda_T$. Then

$$0 = i'(0) = -\int_{\Omega_T} \operatorname{div}(y^a \nabla U_T)\eta_0 \, dx\, dy + \int_0^{\frac{T}{2}} \frac{\partial U_T}{\partial \nu^a}\eta_0 + \partial_u F(x, U_T)\eta_0 \, dx,$$

which yields

$$\begin{cases} \operatorname{div}(y^a \nabla U_T) = 0 & \text{in } \Omega_T, \\ \dfrac{\partial U_T}{\partial \nu^a} = -\partial_u F(x, U_T) & \text{on } (\partial\Omega)_0 = [0, \frac{T}{2}]. \end{cases}$$

Now we extend U_T oddly (in x) from Ω_T to $[-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty)$. Furthermore, we extend it periodically (in x again) from $[-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty)$ to the whole half space $\mathbb{R}^2_+$, and we still denote it as U_T. For any integer $T > T_1$, we claim that U_T is a weak solution of (3.1).

We first prove that the odd function U_T in x is a weak solution of

$$\begin{cases} \operatorname{div}(y^a \nabla U_T) = 0 & \text{in } [-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty), \\ \dfrac{\partial U_T}{\partial \nu^a} = -\partial_u F(x, U_T) & \text{on } [-\frac{T}{2}, \frac{T}{2}]. \end{cases}$$

Namely, for any $\eta \in C_c^\infty((-\frac{T}{2}, \frac{T}{2}) \times [0, +\infty))$, the following equality holds:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_0^\infty y^a \nabla U_T(x, y) \nabla \eta(x, y) \, dx\, dy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \partial_u F(x, U_T(x, 0))\eta(x, 0) \, dx = 0. \tag{3.6}$$

We compute

$$\begin{aligned} &\int_{-\frac{T}{2}}^{0} \int_0^\infty y^a \nabla U_T(x, y) \nabla \eta(x, y) \, dx\, dy + \int_{-\frac{T}{2}}^{0} \partial_u F(x, U_T(x, 0))\eta(x, 0) \, dx \\ &\quad = \int_0^{\frac{T}{2}} \int_0^\infty y^a \nabla U_T(-x, y) \nabla \eta(-x, y) \, dx\, dy + \int_0^{\frac{T}{2}} \partial_u F(-x, U_T(-x, 0))\eta(-x, 0) \, dx \\ &\quad = \int_0^{\frac{T}{2}} \int_0^\infty y^a \nabla U_T(x, y)(-\partial_x \eta(-x, y), -\partial_y \eta(-x, y)) \, dx\, dy - \int_0^{\frac{T}{2}} \partial_u F(x, U_T(x, 0))\eta(-x, 0) \, dx, \end{aligned}$$

where we used the facts that $F(x, u)$ is even in x and u, respectively, and U_T is odd in x. Hence we have

$$\begin{aligned} &\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_0^\infty y^a \nabla U_T(x, y) \nabla \eta(x, y) \, dx\, dy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \partial_u F(x, U_T(x, 0))\eta(x, 0) \, dx \\ &\quad = \int_0^{\frac{T}{2}} \int_0^\infty y^a \nabla U_T(x, y) \nabla \varphi(x, y) \, dx\, dy + \int_0^{\frac{T}{2}} \partial_u F(x, U_T(x, 0))\varphi(x, 0) \, dx, \end{aligned}$$

where we have set $\varphi(x, y) = \eta(x, y) - \eta(-x, y)$. Clearly, this function is admissible since it vanishes on $x = 0$, $x = \frac{T}{2}$ for any $y \ge 0$. Hence (3.6) holds true. A similar argument shows that, for any integer $T > T_1$, the periodic function U_T in x is a weak solution of (3.1), where we need to use the periodic condition (1.2) and the assumption that T is an integer.

Now we set $u_T(x) = U_T(x,0)$. Then u_T is an odd periodic solution of (1.1) with period T. Using the Hopf principle, we get $U_T(x,0) = u_T(x) \in (0,1)$ in $(0, \frac{T}{2})$, where we used the fact that $\partial_u F(x,1) = \partial_u F(x,-1) = 0$. So $u_T(x) \in (0,1)$ for $x \in (0, \frac{T}{2})$.

Next let us calculate the estimate of (1.9). From [10], we know that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}^+} y^a |\nabla U(x,y)|^2 \,dx\,dy = \frac{C(s)}{2d_s} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx.$$

We construct the continuous function

$$\phi(x) = \begin{cases} \frac{x}{d}, & x \in [0,d], \\ 1, & x \in [d, \frac{T}{2}-d], \\ -\frac{1}{d}(x-\frac{T}{2}), & x \in [\frac{T}{2}-d, \frac{T}{2}] \end{cases}$$

for some constant d. We extend ϕ oddly from $[0, \frac{T}{2}]$ to $[-\frac{T}{2}, \frac{T}{2}]$. Further, we extend it periodically with period T. We still denote it as ϕ. It is easy to verify that there exists a function $\Phi(x,y) \in \Lambda_T$ such that $\phi(x) = \Phi(x,0)$. Therefore, to prove (1.9), it is enough to show that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx + \int_{-\frac{T}{2}}^{\frac{T}{2}} F(x,\Phi(x,0))\,dx \le \begin{cases} CT^{1-2s}, & s \in (0,\frac{1}{2}), \\ C\ln T, & s = \frac{1}{2}, \\ C, & s \in (\frac{1}{2},1). \end{cases} \tag{3.7}$$

To obtain (3.7), we only need to prove that

$$\begin{aligned} &\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{|x-\bar{x}|\ge\frac{T}{2}} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx + \int_{-\frac{T}{2}+d}^{-d} \int_{d}^{\frac{T}{2}-d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx \\ &\quad + \int_{-\frac{T}{2}+d}^{-d} \int_{-d}^{d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx + \int_{-d}^{d}\int_{-d}^{d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx \le \begin{cases} CT^{1-2s}, & s \in (0,\frac{1}{2}), \\ C\ln T, & s = \frac{1}{2}, \\ C, & s \in (\frac{1}{2},1). \end{cases} \end{aligned} \tag{3.8}$$

For the first integral, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{|x-\bar{x}|\ge\frac{T}{2}} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx \le CT^{1-2s}. \tag{3.9}$$

For the second integral, we have

$$\int_{-\frac{T}{2}+d}^{-d} \int_{d}^{\frac{T}{2}-d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx \le \frac{4}{2s} \int_{-\frac{T}{2}+d}^{-d} |d-x|^{-2s}\,dx \le \begin{cases} CT^{1-2s}, & s \in (0,\frac{1}{2}), \\ C\ln T, & s = \frac{1}{2}, \\ C, & s \in (\frac{1}{2},1). \end{cases} \tag{3.10}$$

For the third integral, we have

$$\begin{aligned} \int_{-\frac{T}{2}+d}^{-d} \int_{-d}^{d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx &= d^{-2} \int_{-\frac{T}{2}+d}^{-d} \int_{-d}^{d} \frac{|d+\bar{x}|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx = d^{-2} \int_{-d}^{d} \bigg(\int_{-\frac{T}{2}+d}^{-d} \frac{|d+\bar{x}|^2}{|x-\bar{x}|^{1+2s}}\,dx \bigg) d\bar{x} \\ &\le Cd^{-2} \int_{-d}^{d} |d+\bar{x}|^2 |d+\bar{x}|^{-2s}\,d\bar{x} \le C. \end{aligned} \tag{3.11}$$

For the last integral, we have

$$\int_{-d}^{d}\int_{-d}^{d} \frac{|\phi(x)-\phi(\bar{x})|^2}{|x-\bar{x}|^{1+2s}} \,d\bar{x}\,dx \le d^{-2} \int_{-d}^{d}\int_{-d}^{d} |x-\bar{x}|^{1-2s}\,dx\,d\bar{x} \le Cd^{1-2s} \le C. \tag{3.12}$$

From inequalities (3.8)–(3.12), we obtain (3.7). □

Remark 3.1. By adjusting the admissible set and using an argument similar to Theorem 1.1, we can obtain the existence of even periodic solutions of (1.1). Precisely, we consider the energy functional $J(U,\Omega_T)$ in the admissible set

$$\Lambda_T=\{U\in H^1(\Omega_T,y^a):U(-x,y)=U(x,y),\ U(0,y)\le 0\le U(\tfrac{T}{2},y)\},$$

where $\Omega_T=[-\frac{T}{2},\frac{T}{2}]\times[0,+\infty)$. We can find a minimizer U_T of the energy J in Λ_T and prove that $U_T\not\equiv 0$. From the even symmetry of U_T (in x) and F, we know that U_T is also a minimizer of the energy $J(U,\hat{\Omega}_T)$, where $\hat{\Omega}_T=[0,\frac{T}{2}]\times[0,+\infty)$. Hence U_T satisfies

$$\begin{cases}\operatorname{div}(y^a\nabla U_T)=0 & \text{in } \hat{\Omega}_T,\\ \dfrac{\partial U_T}{\partial\nu^a}=-\partial_u F(x,U_T) & \text{on } [0,\frac{T}{2}].\end{cases}$$

From this and the facts that $\partial_x(U_T)=0$ on $x=0$ and $\varphi(\frac{T}{2},y):=\eta(\frac{T}{2},y)+\eta(-\frac{T}{2},y)=0$ on $x=\frac{T}{2}$ for any $y\ge 0$, we obtain (3.6) for any $\eta\in C_c^\infty((-\frac{T}{2},\frac{T}{2})\times[0,+\infty))$. We extend U_T periodically (in x) from $[-\frac{T}{2},\frac{T}{2}]\times[0,+\infty)$ to the whole half space $\mathbb{R}^2_+$ (still denoted as U_T). An argument similar to that of (3.6) shows that U_T is a weak solution of (3.1), where we need to use the assumptions (1.2) and that T is an integer.

Note that similar energy estimates are obtained in [17] for minimizers of the functional in a finite interval $[a,b]$ with a homogeneous condition outside the interval instead of a periodic condition, and higher-dimensional estimates have been subsequently obtained in [19].

Proof of Theorem 1.2. We borrow the idea in [14] to prove this theorem. Now define the Hilbert space as

$$\mathcal{H}:=\Bigg\{U(x,y):|U(x,y)|\le 1,\ U(-\tfrac{T}{2},y)=U(\tfrac{T}{2},y)\text{ for all } y\ge 0,\ \|U\|^2_{\mathcal{H}}=\int_{\overline{\Omega_T}}y^a|\nabla U(x,y)|^2\,dx\,dy+\int_{-\frac{T}{2}}^{\frac{T}{2}}U^2(x,0)\,dx<+\infty\Bigg\},$$

where $\overline{\Omega_T}:=[-\frac{T}{2},\frac{T}{2}]\times[0,+\infty)$. We consider the corresponding energy functional

$$J(U,\overline{\Omega_T})=\frac{1}{2}\int_{\overline{\Omega_T}}y^a|\nabla U(x,y)|^2\,dx\,dy+\int_{-\frac{T}{2}}^{\frac{T}{2}}F(x,U(x,0))\,dx.$$

Since $F(x,u)$ is a smooth function, we can obtain $J\in C^1(\mathcal{H},\mathbb{R})$. Next we verify the Palais–Smale condition. Namely, for any sequence $\{U_k\}\subset\mathcal{H}$ with $J(U_k,\overline{\Omega_T})$ bounded and $J'(U_k,\overline{\Omega_T})\to 0$ in $\mathcal{H}$, it contains a convergent subsequence of $\{U_k\}$. Estimates similar to (3.2) and (3.3) yield that there exists a subsequence, still denoted as $\{U_k\}$, converging weakly to a function $\overline{U}$ in $\mathcal{H}$. In view of $\mathcal{H}(\overline{\Omega_T})\hookrightarrow H^s(-\frac{T}{2},\frac{T}{2})\hookrightarrow\hookrightarrow L^2(-\frac{T}{2},\frac{T}{2})$, we have

$$U_k(x,0)\to\overline{U}(x,0)\quad\text{in } L^2(-\tfrac{T}{2},\tfrac{T}{2}).\tag{3.13}$$

Note that

$$\begin{aligned}\int_{\overline{\Omega_T}}y^a|\nabla U_k(x,y)-\nabla\overline{U}(x,y)|^2\,dx\,dy &=\langle J'(U_k,\overline{\Omega_T})-J'(\overline{U},\overline{\Omega_T}),U_k-\overline{U}\rangle\\ &\quad-\int_{-\frac{T}{2}}^{\frac{T}{2}}[\partial_u F(x,U_k(x,0))-\partial_u F(x,\overline{U}(x,0))](U_k(x,0)-\overline{U}(x,0))\,dx.\end{aligned}$$

Clearly, $\langle J'(U_k,\overline{\Omega_T})-J'(\overline{U},\overline{\Omega_T}),U_k-\overline{U}\rangle\to 0$. We also have

$$\Bigg|\int_{-\frac{T}{2}}^{\frac{T}{2}}[\partial_u F(x,U_k(x,0))-\partial_u F(x,\overline{U}(x,0))](U_k(x,0)-\overline{U}(x,0))\,dx\Bigg|\le C\int_{-\frac{T}{2}}^{\frac{T}{2}}|U_k(x,0)-\overline{U}(x,0)|^2\,dx\to 0,$$

where the convergence result follows from (3.13). Hence

$$\int_{\overline{\Omega_T}} y^a |\nabla U_k(x,y) - \nabla \overline{U}(x,y)|^2\,dx\,dy \to 0.$$

This and (3.13) give that $U_k \to \overline{U}$ in $\mathcal{H}$. We have obtained the Palais–Smale condition.

We set $\Gamma := \{g \in C([0,1];\mathcal{H}) : g(0) = -1,\ g(1) = 1\}$. Note that

$$J(1,\overline{\Omega_T}) = J(-1,\overline{\Omega_T}) = \int_{-\frac{T}{2}}^{\frac{T}{2}} F(x,\pm 1)\,dx = 0 \le J(v,\overline{\Omega_T}) \quad \text{for all } v \in \mathcal{H}.$$

and J is stable at 1 and -1, namely

$$\int_{\overline{\Omega_T}} y^a |\nabla \varphi(x,y)|^2\,dx\,dy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \partial_{uu}F(x,\pm 1)\varphi^2(x,0)\,dx > 0 \quad \text{for all } \varphi \ne 0 \in \mathcal{H}.$$

Hence we have

$$\delta_T = \inf_{g\in\Gamma} \sup_{t\in[0,1]} J(g(t),\overline{\Omega_T}) > 0.$$

We set $J(U_T,\overline{\Omega_T}) = \delta_T$, where $U_T = g(t_0)$ for some $g \in \Gamma$ and some $t_0 \in (0,1)$. Choose T as an integer. We extend U_T periodically (in x) to the whole half space $\mathbb{R}^2_+$ (still denoted it as U_T). An argument similar to that of (3.6) shows that U_T is a solution of (3.1).

Next we show that $U_T \not\equiv 0$. It is enough to prove that $U_T \not\equiv 0$ on $\overline{\Omega_T}$. We choose a function $\psi \in \mathcal{H}$ similar to the above section,

$$\psi(x,y) = \exp\Big\{\frac{-y}{2^{b+1}}\Big\}\tilde{h}(x),$$

where $\tilde{h}(x)$ is the odd extension of h from $(0,\frac{T}{2})$ onto $(-\frac{T}{2},\frac{T}{2})$. We construct a path as

$$\overline{g}(t) = \begin{cases} 2t\psi + (1-2t)\times(-1) & \text{for } 0 \le t \le \frac{1}{2},\\ (2-2t)\psi + (2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Clearly, $\overline{g} \in \Gamma$, and we denote $\overline{g}(t)$ as $\overline{g_t}(x,y)$. Then

$$\int_{\overline{\Omega_T}} y^a |\nabla \overline{g}_t(x,y)|^2\,dx\,dy \le \int_{\overline{\Omega_T}} y^a |\nabla \psi(x,y)|^2\,dx\,dy.$$

Then, for $0 \le t \le \frac{1}{2}$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x,\overline{g_t}(x,0))\,dx = \int_{-\frac{T}{2}}^{0} F(x,\overline{g_t}(x,0))\,dx + \int_{0}^{\frac{T}{2}} F(x,\overline{g_t}(x,0))\,dx \le \int_{-\frac{T}{2}}^{0} F(x,\psi(x,0))\,dx + \int_{0}^{\frac{T}{2}} F(x,0)\,dx.$$

Similarly, for $\frac{1}{2} \le t \le 1$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x,\overline{g_t}(x,0))\,dx \le \int_{0}^{\frac{T}{2}} F(x,\psi(x,0))\,dx + \int_{-\frac{T}{2}}^{0} F(x,0)\,dx.$$

Then a computation similar to (3.4) and (3.5) shows that there exists $T_2 > 0$ such that, for any $T > T_2$, we have $J(\overline{g_t},\overline{\Omega_T}) < J(0,\overline{\Omega_T})$ for all $t \in [0,1]$. Hence

$$J(U_T,\overline{\Omega_T}) = \delta_T \le \max_{t\in[0,1]} J(\overline{g_t},\overline{\Omega_T}) < J(0,\overline{\Omega_T}),$$

which gives that $U_T \not\equiv 0$ on $\overline{\Omega_T}$.

For large enough integer T, $u_T(x) := U_T(x,0)$ is a periodic solution of equation (1.1). Plainly, $u_T(x)$ changes its sign at least once in a period. The Hopf principle gives again that $|u_T(x)| = |U_T(x,0)| < 1$.

Finally, we show estimate (1.10). To this end, for any given integer $m > 1$, we define $2m-1$ continuous functions h_i $(1 \le i \le 2m-1)$ as follows:

$$h_i(x) = \begin{cases} -\frac{8m}{T}x + 4m & \text{for } x \in [\frac{T}{2} - \frac{T}{8m}, \frac{T}{2}], \\ 1 & \text{for } x \in [\frac{T}{2} - \frac{iT}{2m} + \frac{T}{8m}, \frac{T}{2} - \frac{T}{8m}], \\ \frac{8m}{T}x - 4(m-i) & \text{for } x \in [\frac{T}{2} - \frac{iT}{2m} - \frac{T}{8m}, \frac{T}{2} - \frac{iT}{2m} + \frac{T}{8m}], \\ -1 & \text{for } x \in [-\frac{T}{2} + \frac{T}{8m}, \frac{T}{2} - \frac{iT}{2m} - \frac{T}{8m}], \\ -\frac{8m}{T}x - 4m & \text{for } x \in [-\frac{T}{2}, -\frac{T}{2} + \frac{T}{8m}]. \end{cases}$$

Note that $h_i \in [-1,1]$. Similarly, we define $\psi_i \in \mathcal{H}$ $(1 \le i \le (2m-1))$ by

$$\psi_i(x,y) = \exp\left\{\frac{-y}{2^{b+1}}\right\} h_i(x).$$

Now we construct a path as

$$\widehat{g}(t) = \begin{cases} 2mt\psi_1 + (1-2mt)(-1) & \text{for } 0 \le t \le \frac{1}{2m}, \\ ((i+1) - 2mt)\psi_i + (2mt - i)\psi_{i+1} & \text{for } \frac{i}{2m} \le t \le \frac{i+1}{2m},\ 1 \le i \le 2m-2, \\ (2m - 2mt)\psi_{2m-1} + (2mt - (2m-1)) & \text{for } \frac{2m-1}{2m} \le t \le 1. \end{cases}$$

Clearly, $\widehat{g} \in \Gamma$. For $0 \le t \le \frac{1}{2m}$, from the definition of $\widehat{g}$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx = \int_{-\frac{T}{2}}^{-\frac{T}{2}+\frac{T}{8m}} F(x, \widehat{g_t}(x,0))\,dx + \int_{\frac{T}{2}-\frac{5T}{8m}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx \le \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{6T}{8m}.$$

Similarly, for $\frac{2m-1}{2m} \le t \le 1$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx = \int_{-\frac{T}{2}}^{\frac{T}{2}-\frac{(2m-1)}{2m}T+\frac{T}{8m}} F(x, \widehat{g_t}(x,0))\,dx + \int_{\frac{T}{2}-\frac{T}{8m}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx \le \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{6T}{8m}.$$

For the case $\frac{i}{2m} \le t \le \frac{i+1}{2m}$ $(1 \le i \le 2m-2)$, we have

$$\begin{aligned}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx &= \int_{-\frac{T}{2}}^{-\frac{T}{2}+\frac{T}{8m}} F(x, \widehat{g_t}(x,0))\,dx + \int_{\frac{T}{2}-\frac{(i+1)}{2m}T-\frac{T}{8m}}^{\frac{T}{2}-\frac{(i)}{2m}T+\frac{T}{8m}} F(x, \widehat{g_t}(x,0))\,dx + \int_{\frac{T}{2}-\frac{T}{8m}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx \\ &\le \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{T}{m}.\end{aligned}$$

Therefore,

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(x, \widehat{g_t}(x,0))\,dx \le \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{T}{m} \quad \text{for all } t \in [0,1]. \tag{3.14}$$

For the other part of the energy, we have

$$\int_{\Omega_T} y^a |\nabla \widehat{g}(x,y)|^2\,dx\,dy \le \max_{1\le i\le 2m-1} 2\int_{\Omega_T} y^a |\nabla \psi_i(x,y)|^2\,dx\,dy.$$

Similarly, by choosing large enough b and T, we obtain

$$\max_{1\le i\le 2m-1} 2\int_{\Omega_T} y^a |\nabla \psi_i(x,y)|^2\,dx\,dy \le \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{T}{m} \quad \text{for all } T > T_m, \tag{3.15}$$

where $T_m \geq T_2$ and $\lim_{m\to 0} T_m \to +\infty$. Inequalities (3.14) and (3.15) give that

$$\max_{t\in[0,1]} J(\widehat{g_t}, \overline{\Omega_T}) \leq \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{2T}{m} \quad \text{for all } T > T_m.$$

Hence, for any $0 < \sigma < \frac{1}{2}$, we can take large $m = m(\sigma)$ such that, for any $T > T_\sigma$,

$$J(U_T, \overline{\Omega_T}) \leq \max_{t\in[0,1]} J(\widehat{g_t}, \overline{\Omega_T}) \leq \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)\frac{2T}{m} < \sigma \max_{x\in[-\frac{T}{2},\frac{T}{2}]} F(x,0)T,$$

which is the desired estimate (1.10). Here $T_\sigma := T_{m(\sigma)} \to \infty$ as $\sigma \to 0$. □

4 Hamiltonian Identity and Modica-Type Inequality

We will first prove the Hamiltonian identity for periodic solutions of (3.1).

Proof of Theorem 1.3. Similarly to [6, Lemma 5.1], we have $\int_0^\infty y^a |\nabla U(x,y)|^2\,dy < \infty$. Hence

$$\lim_{y\to+\infty} y^a U_y(x,y)U_x(x,y) = 0.$$

We introduce the function

$$w(x) := \frac{1}{2}\int_0^\infty y^a[U_x^2(x,y) - U_y^2(x,y)]\,dy.$$

The regularity result allows us to differentiate within the integral in the above equality to get

$$w'(x) = \int_0^\infty y^a[U_xU_{xx} - U_yU_{xy}](x,y)\,dy.$$

Note that $(y^aU_y)_y + y^aU_{xx} = 0$. Using integration by parts, we have

$$w'(x) = -[y^aU_y(x,y)U_x(x,y)]\big|_{y=0}^{+\infty} = \lim_{y\to0^+} y^aU_y(x,y)U_x(x,y) = \partial_uF(x,U(x,0))U_x(x,0).$$

Owing to

$$\begin{aligned}\frac{d}{dx}\left\{F(x,U(x,0)) - \int_0^x \partial_xF(\sigma,U(\sigma,0))\,d\sigma\right\} &= \partial_xF(x,U(x,0)) + \partial_uF(x,U(x,0))\cdot U_x(x,0) - \partial_xF(x,U(x,0))\\ &= \partial_uF(x,U(x,0))U_x(x,0),\end{aligned}$$

we obtain

$$\int_0^\infty \frac{y^a}{2}\{U_x^2 - U_y^2\}\,dy + \int_0^x \partial_xF(\sigma,U(\sigma,0))\,d\sigma - F(x,U(x,0)) \equiv C_T,$$

where C_T is a constant depending on T. □

Proof of Theorem 1.4. We introduce the function

$$v(x,y) := \frac{1}{2}\int_0^y [U_x^2(x,\tau) - U_y^2(x,\tau)]\tau^a\,d\tau$$

and define

$$\hat{v} := \frac{1}{2}\int_0^y [U_x^2(x,\tau) - U_y^2(x,\tau)]\tau^a\,d\tau - F(x,U(x,0)) + \int_0^x \partial_xF(\sigma,U(\sigma,0))\,d\sigma - C_T.$$

By the periodicity and even symmetry of $U(x,y)$ (in x), it suffices to prove the Modica-type inequality for every $y \geq 0$ and all $x \in [0, \frac{T}{2}]$. Note that

$$\lim_{y\to+\infty} \hat{v}(x,y) = 0 \tag{4.1}$$

and $U_x(0,y) = 0 = U_x(\frac{T}{2}, y)$ for all $y \geq 0$. Then we have

$$\hat{v}(0,y) < \hat{v}(0,0), \quad \hat{v}(\tfrac{T}{2}, y) < \hat{v}(\tfrac{T}{2}, 0). \tag{4.2}$$

Hence $\hat{v}$ is not identically constant.

Elementary calculation shows $\hat{v}_x = -y^a U_x U_y$ and

$$\operatorname{div}(y^{-a}\nabla\hat{v}) = ay^{-1}u_y^2. \tag{4.3}$$

Without loss of generality, we may assume that $U_x \leq 0$ in $(0, \frac{T}{2}) \times (0, +\infty)$. The strong maximum principle yields that U_x is strictly negative in this domain. Equation (4.3) can be written as

$$\operatorname{div}(y^{-a}\nabla\hat{v}) + ay^{-1-a}\frac{U_y}{U_x}\hat{v}_x = 0.$$

Note that the operator in the left-hand side is uniformly elliptic with continuous coefficients in compact sets of $(0, \frac{T}{2}) \times (0, +\infty)$. Since $\hat{v}$ is not identically constant, $\hat{v}$ cannot achieve its maximum in any interior point of $(0, \frac{T}{2}) \times (0, +\infty)$. This fact and (4.1), (4.2) show that $\hat{v}$ achieves its maximum in $[0, \frac{T}{2}] \times [0, +\infty)$ at $[0, \frac{T}{2}] \times \{0\}$, and we denote the maximum as $\hat{C}$. Then

$$\begin{aligned}\hat{C} &= \sup_{x\in[0,\frac{T}{2}]} \Big\{ \int_0^x \partial_x F(\sigma, U(\sigma,0))\,d\sigma - F(x, U(x,0)) - C_T \Big\} \\ &\geq \int_0^{\frac{T}{2}} \partial_x F(\sigma, U(\sigma,0))\,d\sigma - F(\tfrac{T}{2}, U(\tfrac{T}{2}, 0)) - C_T = \frac{1}{2}\int_0^{+\infty} U_y^2(\tfrac{T}{2}, \tau)\,d\tau > 0.\end{aligned}$$

□

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