Research Article

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Solutions for Fractional Schrödinger Equation Involving Critical Exponent via Local Pohozaev Identities

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Abstract: We consider the following fractional Schrödinger equation involving critical exponent:

$$\begin{cases} (-\Delta)^s u + V(y)u = u^{2_s^*-1} & \text{in } \mathbb{R}^N, \\ u > 0, & y \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$ and $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent. Under some suitable assumptions of the potential function $V(y)$, by using a finite-dimensional reduction method, combined with various local Pohazaev identities, we prove the existence of infinitely many solutions. Due to the nonlocality of the fractional Laplacian operator, we need to study the corresponding harmonic extension problem.

Keywords: Fractional Schrödinger Equation, Critical Exponent, Local Pohozaev Identities, Infinitely many Solutions

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1 Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} (-\Delta)^s u + V(y)u = u^{2_s^*-1} & \text{in } \mathbb{R}^N, \\ u > 0, & y \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$ and $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent. For any $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian in $\mathbb{R}^N$, which is a nonlocal operator defined as

$$(-\Delta)^s u(y) = c(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\,dx = c(N,s)\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\,dx, \tag{1.2}$$

where P.V. denotes the Cauchy principal value and $c(N,s)$ is a constant depending on N and s. This operator is well defined in $C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_s$, where $\mathcal{L}_s = \{u \in L^1_{\mathrm{loc}} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\,dx < \infty\}$. For more details on the fractional Laplacian, we referee to [16, 22] and the references therein.

The fractional Laplacian operator appears in divers areas, including biological modeling, physics and mathematical finances, and can be regarded as the infinitesimal generator of a stable Lévy process (see, for

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example, [5]). From the view point of mathematics, an important feature of the fractional Laplacian operator is its nonlocal property, which makes it more challenging than the classical Laplacian operator. Thus, problems with the fractional Laplacian have been extensively studied, both for the pure mathematical research and in view of concrete real world applications, see, for example, [1–4, 7–11, 17, 19, 23, 29, 37, 39, 40, 44] and the references therein.

Solutions of (1.1) are related to the existence of standing wave solutions to the following fractional Schrödinger equation:

$$\begin{cases} i\partial_t \Psi + (-\Delta)^s \Psi = F(x, \Psi) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} |\Psi(x, t)| = 0 & \text{for all } t > 0. \end{cases}$$

That is, solutions with the form $\Psi(x, t) = e^{-ict}u(x)$, where c is a constant.

In this paper, under a weaker symmetry condition for $V(y)$, we will construct multi-bump solutions for (1.1) through a finite-dimensional reduction method, combined with various local Pohozaev identities. More precisely, we consider $V(y) = V(|y'|, y'') = V(r, y'')$, $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and assume that:

(V) $V(y) \geq 0$ is a bounded function that belongs to $C^2(\mathbb{R}^N)$, and $r^{2s}V(r, y'')$ has a critical point (r_0, y_0'') satisfying $r_0 > 0$, $V(r_0, y_0'') > 0$ and $\deg(\nabla(r^{2s}V(r, y'')), (r_0, y_0'')) \neq 0$.

Since the fractional operator is nonlocal, we have to overcome more difficulties than the Laplace equation. Such as, we need to study the corresponding harmonic extension problem and deal with $(-\Delta)^s(\phi\varphi)$ in an appropriate process. Hence, some new ideas and techniques are needed. We will explain these later.

Before state the main results, let us first introduce some notations. Denote $D^s(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}$, where $\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}$ is defined by $(\int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{G}u(\xi)|^2 \, d\xi)^{\frac{1}{2}}$, and $\mathcal{G}u$ is the Fourier transformation of u:

$$\mathcal{G}u(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int\limits_{\mathbb{R}^N} e^{-i\xi\cdot x} u(x)\, dx.$$

We will construct the solutions in the following the energy space:

$$H^s(\mathbb{R}^N) = \Big\{u \in D^s(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(y)u^2 \, dy < +\infty\Big\},$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} V(y)u^2 \, dy\Big)^{\frac{1}{2}}.$$

We define the functional I on $H^s(\mathbb{R}^N)$ by

$$I(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \, dy + \frac{1}{2}\int\limits_{\mathbb{R}^N} V(|y'|, y'')u^2 \, dy - \frac{1}{2_s^*}\int\limits_{\mathbb{R}^N} (u)_+^{2_s^*} \, dy,$$

where $(u)_+ = \max(u, 0)$. Then the solutions of problem (1.1) correspond to the critical points of the functional I.

It is well known that the functions

$$U_{x,\lambda}(y) = C(N, s)\Big(\frac{\lambda}{1 + \lambda^2|y - x|^2}\Big)^{\frac{N-2s}{2}}, \quad \lambda > 0,\ x \in \mathbb{R}^N,$$

where $C(N, s) = 2^{\frac{N-2s}{2}}\Gamma(\frac{N+2s}{2})/\Gamma(\frac{N-2s}{2})$, are the only solutions of the problem (see [32])

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}}, \quad u > 0 \text{ in } \mathbb{R}^N.$$

Define

$$H_s = \Big\{u : u \in H^s(\mathbb{R}^N), u(y_1, y_2, y'') = u(y_1, -y_2, y''), \\ u\Big(r\cos\Big(\theta + \frac{2\pi j}{k}\Big), r\sin\Big(\theta + \frac{2\pi j}{k}\Big), y''\Big) = u(r\cos\theta, r\sin\theta, y'')\Big\}.$$

Let

$$x_j = \Big(\bar{r}\cos\frac{2(j-1)\pi}{k}, \bar{r}\sin\frac{2(j-1)\pi}{k}, \bar{y}''\Big), \quad j = 1, \ldots, k.$$

To construct the solution of (1.1), we hope to use $U_{x,\lambda}(y)$ as an approximation solution. However, the decay of $U_{x_j,\lambda}$ is not fast enough for us when $N \le 6s$. So, we need to cut off this function. Let $\delta > 0$ be a small constant such that $r^{2s}V(r, y'') > 0$ if $|(r, y'') - (r_0, y_0'')| \le 10\delta$. Let $\zeta(y) = \zeta(r, y'')$ be a smooth function satisfying $\zeta = 1$ if $|(r, y'') - (r_0, y_0'')| \le \delta$, $\zeta = 0$ if $|(r, y'') - (r_0, y_0'')| \ge 2\delta$, $|\nabla\zeta| \le C$ and $0 \le \zeta \le 1$. Denote

$$Z_{x_j,\lambda} = \zeta U_{x_j,\lambda}, \quad Z^*_{\bar r,\bar y'',\lambda} = \sum_{j=1}^{k} U_{x_j,\lambda}, \quad Z_{\bar r,\bar y'',\lambda} = \sum_{j=1}^{k} Z_{x_j,\lambda}.$$

Let

$$Z_{i,1} = \frac{\partial Z_{x_i,\lambda}}{\partial\lambda}, \quad Z_{i,2} = \frac{\partial Z_{x_i,\lambda}}{\partial\bar r}, \quad Z_{i,k} = \frac{\partial Z_{x_i,\lambda}}{\partial\bar y_k''}, \quad k = 3, \dots, N.$$

Then a direct computation shows that

$$Z_{i,1} = O(\lambda^{-1}Z_{x_i,\lambda}), \quad Z_{i,l} = O(\lambda Z_{x_i,\lambda}), \quad l = 2, \dots, N.$$

We obtain the following result when $N > 4s$.

Theorem 1.1. *Suppose that $N \ge 3$ and $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$. If $V(y)$ satisfies condition* (V)*, then there is an integer $k_0 > 0$ such that for any integer $k \ge k_0$, problem* (1.1) *has a solution u_k of the form*

$$u_k = Z_{\bar r_k,\bar y_k'',\lambda_k} + \phi_k,$$

where $\phi_k \in H_s$, $\lambda_k \in [L_0 k^{\frac{N-2s}{N-4s}}, L_1 k^{\frac{N-2s}{N-4s}}]$, and as $k \to \infty$, $\lambda_k^{-\frac{N-2s}{2}}\|\phi_k\|_{L^\infty} \to 0$, $(\bar r_k, \bar y_k'') \to (r_0, y_0'')$.

Remark 1.2. (1) In this case, we always assume that $k > 0$ is a large integer, $\lambda \in [L_0 k^{\frac{N-2s}{N-4s}}, L_1 k^{\frac{N-2s}{N-4s}}]$ for some constants $L_1 > L_0 > 0$, and $|(\bar r, \bar y'') - (r_0, y_0'')| \le \theta$, with $\theta > 0$ being a small constant.

(2) Let $\tau = \frac{N-4s}{2(N-2s)}$. When $N \ge 3$, the condition $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$ is equivalent to $\tau < s < 1$ and $N > 4s + 2\tau$, which guarantee the existence of a small constant $\sigma > 0$ in the proof. Moreover, it is easy to see $\frac{1}{3} < \frac{2+N-\sqrt{4+N^2}}{4} < \frac{1}{2}$ for $N \ge 3$.

We are also interested in the case $N = 4s$. Since $s < 1$, we have $N = 3 = 4s$. This case is corresponding to the Laplace equation with $N = 4$.

Theorem 1.3. *Suppose that $N = 3 = 4s$. If $V(y)$ satisfies condition* (V)*, then there is an integer $k_0 > 0$ such that for any integer $k \ge k_0$, problem* (1.1) *has a solution u_k of the form*

$$u_k = Z_{\bar r_k,\bar y_k'',\lambda_k} + \phi_k,$$

where $\phi_k \in H_s$, $\lambda_k \in [e^{L_0 k^{2s}}, e^{L_1 k^{2s}}]$, and as $k \to \infty$, $\lambda_k^{-\frac{N-2s}{2}}\|\phi_k\|_{L^\infty} \to 0$, $(\bar r_k, \bar y_k'') \to (r_0, y_0'')$.

Remark 1.4. In this case, we assume that $k > 0$ is a large integer, $\lambda_k \in [e^{L_0 k^{2s}}, e^{L_1 k^{2s}}]$, and $|(\bar r, \bar y'') - (r_0, y_0'')| \le \theta$.

When $N > 4s$, we introduce the following norms:

$$\|u\|_* = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^{-1}\lambda^{-\frac{N-2s}{2}}|u(y)|$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}\Big)^{-1}\lambda^{-\frac{N+2s}{2}}|f(y)|,$$

where $\tau = \frac{N-4s}{2(N-2s)}$. When $N = 3 = 4s$, we use the following norms:

$$\|u\|_* = \sup_{y\in\mathbb{R}^3}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}}}\Big)^{-1}\lambda^{-\frac{N-2s}{2}}|u(y)|$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^3}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}}}\Big)^{-1}\lambda^{-\frac{N+2s}{2}}|f(y)|,$$

where $\frac{N-2s}{2} = \frac{3}{4}$ and $\frac{N+2s}{2} = \frac{9}{4}$.

We will prove Theorems 1.1 and 1.3 by a finite-dimensional reduction method, combined with various local Pohozaev identities. The finite-dimensional reduction method has been extensively used to construct solutions for equations with critical growth. We refer to [6, 12–15, 18, 21, 24–26, 28, 30, 31, 33, 34, 36, 41–43] and the references therein. Roughly speaking, the outline to carry out the reduction argument is as follows: We first construct a good enough approximation solution and linearize the original problem around the approximation solution. Then we solve the corresponding finite-dimensional problem to obtain a true solution.

To finish the second step, we have to obtain some good enough estimates in the first step. In our case, since the fractional operator is nonlocal, we have to overcome more difficulties than the Laplace equation. One of them is that we will use $Z_{\bar{r},\bar{y}'',\lambda}$ as an approximate solution, but we have to deal with $(-\Delta)^s(\zeta(y)U_{x_j,\lambda}(y))$ and $(-\Delta)^s(\zeta(y)Z_{j,t}(y))$. By (1.2), we can deduce that

$$
\begin{aligned}
(-\Delta)^s(\zeta(y)U_{x_j,\lambda}(y)) &= \zeta(y)U_{x_j,\lambda}^{2_s^*-1}(y) + c(N,s)\lim_{\epsilon\to 0^+}\int\limits_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{(\zeta(y)-\zeta(x))U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\,dx \\
&=: \zeta(y)U_{x_j,\lambda}^{2_s^*-1}(y) + J.
\end{aligned}
$$

In order to obtain a good enough estimate with $\|\cdot\|_{**}$, we need to deal with some concrete difficulties, and devote ourselves to calculate the last principal value very carefully (see Lemma 2.5). More precisely, when $N > 4s$, we need to show that

$$
|J| \le \frac{C}{\lambda^{s+\sigma}}\frac{\lambda^{\frac{N+2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.
$$

Before the end of this introduction, we briefly outline the proof for the case of $N > 4s$ and point out some other difficulties (the idea of the proof for the other case is similar but with different estimates). We first use $Z_{\bar{r},\bar{y}'',\lambda}$ as an approximate solution to obtain a unique function $\phi(\bar{r},\bar{y}'',\lambda)$. Then the problem of finding critical points for $I(u)$ can be reduced to that of finding critical points of $F(\bar{r},\bar{y}'',\lambda) = I(Z_{\bar{r},\bar{y}'',\lambda}+\phi(\bar{r},\bar{y}'',\lambda))$. In the second step, we solve the corresponding finite-dimensional problem to obtain a solution. However, we can only obtain $\|\phi\|_* \le \frac{C}{\lambda^{s+\sigma}}$ (see Proposition 2.3). From Lemmas B.3 and B.5, we know that

$$
\frac{\partial F}{\partial\lambda} = \frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\lambda} + O(k\lambda^{-1}\|\phi\|_*^2) = k\Big(-\frac{2sB_1}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \frac{B_3k^{N-2s}}{\lambda^{N-2s+1}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big), \tag{1.3}
$$

$$
\frac{\partial F}{\partial\bar{r}} = \frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\bar{r}} + O(k\lambda\|\phi\|_*^2) = k\Big(\frac{B_1}{\lambda^{2s}}\frac{\partial V(\bar{r},\bar{y}'')}{\partial\bar{r}} + \sum_{j=2}^{k}\frac{B_2}{\bar{r}\lambda^{N-2s}|x_1-x_j|^{N-2s}} + O\Big(\frac{1}{\lambda^{s+\sigma}}\Big)\Big) \tag{1.4}
$$

and

$$
\frac{\partial F}{\partial\bar{y}_j''} = \frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\bar{y}_j''} + O(k\lambda\|\phi\|_*^2) = k\Big(\frac{B_1}{\lambda^{2s}}\frac{\partial V(\bar{r},\bar{y}'')}{\partial\bar{y}_j''} + O\Big(\frac{1}{\lambda^{s+\sigma}}\Big)\Big). \tag{1.5}
$$

Note that the estimate of ϕ is only good enough for the expansion (1.3), but it destroys the main terms in the expansions of (1.4) and (1.5). To overcome this difficulty, following the idea in [35], instead of studying (1.4) and (1.5), we turn to study the following local Pohozaev identities:

$$
-\int\limits_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde{u}_k}{\partial\nu}\frac{\partial\tilde{u}_k}{\partial y_i} + \frac{1}{2}\int\limits_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\tilde{u}_k|^2\nu_i = \int\limits_{B_\rho}\big(-V(r,y'')u_k + (u_k)_+^{2_s^*-1}\big)\frac{\partial u_k}{\partial y_i}, \quad i=3,\dots,N, \tag{1.6}
$$

and

$$
\begin{aligned}
-\int\limits_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle\nabla\tilde{u}_k, Y\rangle\frac{\partial\tilde{u}_k}{\partial\nu} + \frac{1}{2}\int\limits_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\tilde{u}_k|^2\langle Y,\nu\rangle + \frac{2s-N}{2}\int\limits_{\partial\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde{u}_k}{\partial\nu}\tilde{u}_k \\
= \int\limits_{B_\rho}\big(-V(r,y'')u_k + (u_k)_+^{2_s^*-1}\big)\langle y, u_k\rangle,
\end{aligned} \tag{1.7}
$$

where $u_k = Z_{\overline{r},\overline{y}'',\lambda} + \phi$, $\tilde{u}_k$ is the extension of u_k (see below (1.8)),

$$\begin{aligned}
\mathcal{B}_\rho^+ &= \{Y = (y, t) : |Y - (r_0, y_0'', 0)| \le \rho \text{ and } t > 0\} \subseteq \mathbb{R}_+^{N+1},\\
\partial'\mathcal{B}_\rho^+ &= \{Y = (y, t) : |y - (r_0, y_0'')| \le \rho,\, t = 0\} \subseteq \mathbb{R}^N,\\
\partial''\mathcal{B}_\rho^+ &= \{Y = (y, t) : |Y - (r_0, y_0'', 0)| = \rho,\, t > 0\} \subseteq \mathbb{R}_+^{N+1},\\
\partial\mathcal{B}_\rho^+ &= \partial'\mathcal{B}_\rho^+ \cup \partial''\mathcal{B}_\rho^+,\\
B_\rho &= \{y : |y - (r_0, y_0'')| \le \rho\} \subseteq \mathbb{R}^N.
\end{aligned}$$

For any $u \in D^s(\mathbb{R}^N)$, $\tilde{u}$ is defined by

$$\tilde{u}(y, t) = \mathcal{P}_s[u] := \int_{\mathbb{R}^N} P_s(y - \xi, t)u(\xi)\, d\xi, \quad (y, t) \in \mathbb{R}_+^{N+1} := \mathbb{R}^N \times (0, +\infty), \tag{1.8}$$

where

$$P_s(x, t) = \beta(N, s)\frac{t^{2s}}{(|x|^2 + t^2)^{\frac{N+2s}{2}}},$$

with constant $\beta(N, s)$ such that $\int_{\mathbb{R}^N} P_s(x, 1)\, dx = 1$. Moreover, $\tilde{u}$ satisfies (see [11])

$$\operatorname{div}(t^{1-2s}\nabla\tilde{u}) = 0 \quad \text{in } \mathbb{R}_+^{N+1} \tag{1.9}$$

and

$$-\lim_{t\to 0} t^{1-2s}\partial_t\tilde{u}(y, t) = \omega_s(-\Delta)^s u(y) \quad \text{on } \mathbb{R}^N, \tag{1.10}$$

where $\omega_s = 2^{1-2s}\Gamma(1 - s)/\Gamma(s)$.

Due to the nonlocality of the fractional Laplacian operator, we can not built a local Pohozaev identity for problem (1.1). So, we need to study the corresponding harmonic extension problem (1.9) and (1.10). The relationship between u and $\tilde{u}$ is (1.8). Hence, we have to give some estimates for this kind of integrals. The local Pohozaev identities (1.6) and (1.7) are much more complicated. We have to integrate one more time than the Laplacian operator case. This is very difficult when we derive some sharp estimates for each term in (1.6) and (1.7). We need a lot of preliminary lemmas. For example, some suitable estimates on $\nabla\tilde{Z}_{x_i,\lambda}$ and $\nabla\tilde{\phi}$ are established in Lemmas A.5 and A.6.

Our paper is organized as follows. In Section 2, we perform a finite-dimensional reduction. We prove Theorem 1.1 in Section 3. Theorem 1.3 is proved in Section 4. In Appendix A, we give some essential estimates. We put the energy expansions for $\langle I'(Z_{\overline{r},\overline{y}'',\lambda} + \phi(\overline{r}, \overline{y}'', \lambda)), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\rangle$, $\langle I'(Z_{\overline{r},\overline{y}'',\lambda} + \phi(\overline{r}, \overline{y}'', \lambda)), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\overline{r}}\rangle$ and $\langle I'(Z_{\overline{r},\overline{y}'',\lambda} + \phi(\overline{r}, \overline{y}'', \lambda)), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\overline{y}''}\rangle$ in Appendix B.

2 Finite-Dimensional Reduction

In this section, we perform a finite-dimensional reduction by using $Z_{\overline{r},\overline{y}'',\lambda}$ as an approximation solution. We consider the following linearized problem:

$$\begin{cases}
(-\Delta)^s\phi + V(r, y'')\phi - (2_s^* - 1)Z_{\overline{r},\overline{y}'',\lambda}^{2_s^*-2}\phi = h + \displaystyle\sum_{l=1}^{N} c_l \sum_{i=1}^{k} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l},\\
u \in H_s, \quad \displaystyle\sum_{i=1}^{k}\int_{\mathbb{R}^N} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l}\phi = 0, \quad l = 1, 2, \dots, N,
\end{cases} \tag{2.1}$$

for some numbers c_l.

Lemma 2.1. *Suppose that* $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$ *or* $N = 3 = 4s$ *and* ϕ_k *solves problem* (2.1). *If* $\|h_k\|_{**} \to 0$ *as* $k \to \infty$, *then* $\|\phi_k\|_* \to 0$ *as* $k \to \infty$.

Proof. We prove this lemma by contradiction. We first consider the case $N > 4s$. We assume that there exist h_k with $\|h_k\|_{**} \to 0$ as $k \to \infty$, $\|\phi_k\|_* \geq c > 0$ with $\lambda = \lambda_k$, $\lambda_k \in [L_0 k^{\frac{N-2s}{N-4s}}, L_1 k^{\frac{N-2s}{N-4s}}]$ and $(\bar{r}_k, \bar{y}_k'') \to (r_0, y_0'')$. Without loss of generality, we can assume that $\|\phi_k\|_* \equiv 1$. For simplicity, we drop the subscript k.

Firstly, we have

$$|\phi(y)| \leq C \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}} Z^{2_s^*-2}_{\bar{r},\bar{y}'',\lambda} |\phi| \, dz + C \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}} \Big[|h| + \Big| \sum_{l=1}^{N} c_l \sum_{i=1}^{k} Z^{2_s^*-2}_{x_i,\lambda} Z_{i,l} \Big| \Big] \, dz =: A_1 + A_2.$$

For the first term A_1, by Lemmas A.1 and A.2, we can deduce that

$$|A_1| \leq C\|\phi\|_* \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}} Z^{2_s^*-2}_{\bar{r},\bar{y}'',\lambda} \sum_{i=1}^{k} \frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|z-x_i|)^{\frac{N-2s}{2}+\tau}} \, dz \leq C\|\phi\|_* \lambda^{\frac{N-2s}{2}} \sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau+\theta}},$$

where θ is a small constant. For the second term A_2, we make use of Lemma A.2, so that

$$\begin{aligned} |A_2| &\leq C\|h\|_{**} \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{\lambda^{\frac{N+2s}{2}}}{|y-z|^{N-2s}(1+\lambda|z-x_i|)^{\frac{N+2s}{2}+\tau}} \, dz + C \sum_{l=1}^{N} |c_l| \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{\lambda^{\frac{N+2s}{2}+n_l}}{|y-z|^{N-2s}(1+\lambda|z-x_i|)^{N+2s}} \, dz \\ &\leq C\|h\|_{**} \lambda^{\frac{N-2s}{2}} \sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}} + C \sum_{l=1}^{N} |c_l| \lambda^{\frac{N-2s}{2}+n_l} \sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}}, \end{aligned}$$

where $n_1 = -1$, $n_l = 1$ for $l = 2, \dots, N$. Then we have

$$\Big(\sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}} \Big)^{-1} \lambda^{-\frac{N-2s}{2}} |\phi| \leq C\|\phi\|_* \frac{\sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau+\theta}}}{\sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}}} + C\|h\|_{**} + C \sum_{l=1}^{N} |c_l| \lambda^{n_l}. \tag{2.2}$$

Multiplying both sides of (2.1) by $Z_{1,t}$, we have

$$\sum_{l=1}^{N} c_l \sum_{i=1}^{k} \int_{\mathbb{R}^N} Z^{2_s^*-2}_{x_i,\lambda} Z_{i,l} Z_{1,t} = \big\langle (-\Delta)^s \phi - V(r,y'')\phi - (2_s^*-1) Z^{2_s^*-2}_{\bar{r},\bar{y}'',\lambda} \phi, Z_{1,t} \big\rangle - \langle h, Z_{1,t} \rangle. \tag{2.3}$$

First of all, there exists a constant $\bar{c} > 0$ such that

$$\sum_{i=1}^{k} \int_{\mathbb{R}^N} Z^{2_s^*-2}_{x_i,\lambda} Z_{i,l} Z_{1,t} \begin{cases} = (\bar{c} + o(1))\lambda^{2n_t}, & l = t, \\ \leq \frac{\bar{c}\lambda^{n_t}\lambda^{n_l}}{\lambda^N}, & l \neq t. \end{cases} \tag{2.4}$$

Since $\tau < s$ and $\frac{N-2s}{2} - \tau > s$, we have

$$\begin{aligned} |\langle V(r,y'')\phi, Z_{1,t} \rangle| &\leq C\|\phi\|_* \int_{\mathbb{R}^N} \frac{\zeta \lambda^{N-2s+n_t}}{(1+\lambda|y-x_1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}} \\ &\leq C\|\phi\|_* \lambda^{N-2s+n_t} \Big[\int_{\mathbb{R}^N} \frac{\zeta}{(1+\lambda|y-x_1|)^{\frac{3N-6s}{2}+\tau}} \\ &\quad + \sum_{i=2}^{k} \frac{1}{(\lambda|x_1-x_i|)^{\tau}} \int_{\mathbb{R}^N} \zeta \Big(\frac{1}{(1+\lambda|y-x_1|)^{\frac{3N-6s}{2}}} + \frac{1}{(1+\lambda|y-x_i|)^{\frac{3N-6s}{2}}} \Big) \Big] \\ &\leq C\|\phi\|_* \int_{\mathbb{R}^N} \zeta \frac{\lambda^{N-2s+\tau+n_t}}{(1+\lambda|y-x_1|)^{\frac{3N-6s}{2}}} \leq \frac{C\lambda^{n_t}\|\phi\|_* \log\lambda}{\lambda^{\min(2s-\tau, \frac{N-2s}{2}-\tau)}} \leq \frac{C\lambda^{n_t}\|\phi\|_*}{\lambda^{s+\sigma}} \end{aligned} \tag{2.5}$$

and

$$|\langle h, Z_{1,t} \rangle| \leq C\|h\|_{**} \int_{\mathbb{R}^N} \frac{\lambda^{N+n_t}}{(1+\lambda|y-x_1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1+\lambda|y-x_i|)^{\frac{N+2s}{2}+\tau}} \leq C\lambda^{n_t}\|h\|_{**}. \tag{2.6}$$

Moreover, one has

$$\big| \big\langle (-\Delta)^s \phi - (2_s^*-1) Z^{2_s^*-2}_{\bar{r},\bar{y}'',\lambda} \phi, Z_{1,t} \big\rangle \big| \leq \frac{C\lambda^{n_t}\|\phi\|_*}{\lambda^{s+\sigma}}. \tag{2.7}$$

Combining (2.3), (2.4), (2.5), (2.6) and (2.7), we have

$$|c_t| \le \frac{C}{\lambda^{n_t}}\Big(\frac{\|\phi\|_*}{\lambda^{\sigma}} + \|h\|_{**}\Big) + \frac{C}{\lambda^{n_t}}\sum_{l\neq t}\frac{\lambda^{n_l}|c_l|}{\lambda^N}.$$

This implies that

$$\sum_{l=1}^{N}|c_l|\lambda^{n_l} \le C\Big(\frac{\|\phi\|_*}{\lambda^{\sigma}} + \|h\|_{**}\Big).$$

Thus, by (2.2) and $\|\phi\|_* = 1$, there exists $R > 0$ such that

$$\|\lambda^{-\frac{N-2s}{2}}\phi(y)\|_{L^\infty(B_{R/\lambda}(x_i))} \ge a > 0 \tag{2.8}$$

for some i. As a result, we have that $\widetilde{\phi} = \lambda^{-\frac{N-2s}{2}}\phi(\frac{y}{\lambda} + x_i)$ converges uniformly, in any compact set, to a solution u of the following equation:

$$(-\Delta)^s u - (2_s^* - 1)U_{0,\Lambda}^{2_s^*-2}u = 0 \quad \text{in } \mathbb{R}^N,$$

for some $0 < \Lambda_1 \le \Lambda \le \Lambda_2$. Since u is perpendicular to the kernel of this equation, $u = 0$. This is a contradiction to (2.8).

When $N = 3 = 4s$, we take $\lambda_k \in [e^{L_0k^{2s}}, e^{L_1k^{2s}}]$ and $\tau = 0$ in the above proofs. We also need to alter (2.5) as follows:

$$|\langle V(r, y'')\phi, Z_{1,t}\rangle| \le Ck\|\phi\|_*\lambda^{3-2s+n_t}\int_{\mathbb{R}^3}\frac{\zeta}{(1+\lambda|y-x_1|)^{\frac{9}{4}}} \le \frac{Ck\lambda^{n_t}\|\phi\|_*}{\lambda^{\frac{3}{4}}} = \frac{Ck\lambda^{n_t}\|\phi\|_*}{\lambda^{s}}.$$

The proof is complete. □

Using the same argument as in the proof of [20, Proposition 4.1], we can obtain the following proposition.

Proposition 2.2. *There exist $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \ge k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (2.1) has a unique solution $\phi = L_k(h)$. Besides,*

$$\|L_k(h)\|_* \le C\|h\|_{**}, \quad |c_l| \le \frac{C}{\lambda^{n_l}}\|h\|_{**}.$$

Now we consider the following problem:

$$\begin{cases} (-\Delta)^s(Z_{\overline{r},\overline{y}'',\lambda} + \phi) + V(r, y'')(Z_{\overline{r},\overline{y}'',\lambda} + \phi) = (Z_{\overline{r},\overline{y}'',\lambda} + \phi)^{2_s^*-1} + \displaystyle\sum_{l=1}^{N}c_l\sum_{i=1}^{k}Z_{x_i,\lambda}^{2_s^*-2}Z_{i,l} & \text{in } \mathbb{R}^N, \\ \phi \in H_s, \quad \displaystyle\sum_{i=1}^{k}\int_{\mathbb{R}^N}Z_{x_i,\lambda}^{2_s^*-2}Z_{i,l}\phi = 0, \quad l = 1,\dots,N. \end{cases} \tag{2.9}$$

In the rest of this section, we devote ourselves to the proof of the following proposition by using the contraction mapping theorem.

Proposition 2.3. *There exist $k_0 > 0$ and a constant $C > 0$, independent of k, such that the following hold:*

(a) *When $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$ for all $k \ge k_0$, $L_0k^{\frac{N-2s}{N-4s}} \le \lambda \le L_1k^{\frac{N-2s}{N-4s}}$, $|(\overline{r}, \overline{y}'') - (r_0, y_0'')| \le \theta$, problem (2.9) has a unique solution $\phi = \phi(\overline{r}, \overline{y}'', \lambda)$ satisfying*

$$\|\phi\|_* \le \frac{C}{\lambda^{s+\sigma}}, \quad |c_l| \le \frac{C}{\lambda^{s+\sigma}},$$

where $\sigma > 0$ is a small constant.

(b) *When $N = 3 = 4s$ for all $k \ge k_0$, $e^{L_0k^{2s}} \le \lambda \le e^{L_1k^{2s}}$, $|(\overline{r}, \overline{y}'') - (r_0, y_0'')| \le \theta$, problem (2.9) has a unique solution $\phi = \phi(\overline{r}, \overline{y}'', \lambda)$ satisfying*

$$\|\phi\|_* \le \frac{C}{\lambda^{s}}, \quad |c_l| \le \frac{C}{\lambda^{s}}.$$

We rewrite (2.9) as

$$\begin{cases} (-\Delta)^s\phi + V(r,y'')\phi - (2_s^*-1)(Z_{\bar{r},\bar{y}'',\lambda})^{2_s^*-2}\phi = \mathcal{F}(\phi) + l_k(y) + \sum\limits_{l=1}^{N} c_l \sum\limits_{i=1}^{k} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l} \quad \text{in } \mathbb{R}^N, \\ \phi \in H_s, \quad \sum\limits_{i=1}^{k} \int\limits_{\mathbb{R}^N} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l}\phi = 0, \quad l = 1,\dots,N, \end{cases}$$

where

$$\mathcal{F}(\phi) = (Z_{\bar{r},\bar{y}'',\lambda} + \phi)_+^{2_s^*-1} - Z_{\bar{r},\bar{y}'',\lambda}^{2_s^*-1} - (2_s^*-1)Z_{\bar{r},\bar{y}'',\lambda}^{2_s^*-2}\phi$$

and

$$\begin{aligned} l_k(y) &= \Big(Z_{\bar{r},\bar{y}'',\lambda}^{2_s^*-1}(y) - \zeta(y)\sum_{j=1}^{k} U_{x_j,\lambda}^{2_s^*-1}(y)\Big) - V(r,y'')Z_{\bar{r},\bar{y}'',\lambda}(y) - \sum_{j=1}^{k} c(N,s)\lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{(\zeta(y)-\zeta(x))U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\,dx \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

In order to use the contraction mapping theorem to prove Proposition 2.3, we need to estimate $\mathcal{F}(\phi)$ and $l_k(y)$. In the following, we assume that $\|\phi\|_*$ is small.

Lemma 2.4. *There exists a constant $C > 0$, independent of k, such that:*
(a) *when $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$, we have $\|\mathcal{F}(\phi)\|_{**} \le C\lambda^{\frac{2s(N-4s)}{(N-2s)^2}}\|\phi\|_*^{\min(2,2_s^*-1)}$,*
(b) *when $N = 3 = 4s$, we have $\|\mathcal{F}(\phi)\|_{**} \le C(\ln\lambda)^{\frac{1}{s}}\|\phi\|_*^2$.*

Proof. We first prove (a). If $2_s^* \le 3$, then using the Hölder inequality, we obtain

$$\begin{aligned} |\mathcal{F}(\phi)| &\le C\|\phi\|_*^{2_s^*-1}\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^{2_s^*-1} \\ &\le C\|\phi\|_*^{2_s^*-1}\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\tau}}\Big)^{\frac{4s}{N-2s}} \\ &\le C\lambda^{\frac{2s(N-4s)}{(N-2s)^2}}\|\phi\|_*^{2_s^*-1}\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}. \end{aligned}$$

When $2_s^* > 3$, we have

$$\begin{aligned} |\mathcal{F}(\phi)| &\le C\|\phi\|_*^2\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^2\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-3} \\ &\quad + C\|\phi\|_*^{2_s^*-1}\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^{2_s^*-1} \\ &\le C(\|\phi\|_*^2 + \|\phi\|_*^{2_s^*-1})\lambda^{\frac{N+2s}{2}}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^{2_s^*-1} \\ &\le C\lambda^{\frac{2s(N-4s)}{(N-2s)^2}}\|\phi\|_*^2\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}. \end{aligned}$$

Hence, we obtain $\|\mathcal{F}(\phi)\|_{**} \le C\lambda^{\frac{2s(N-4s)}{(N-2s)^2}}\|\phi\|_*^{\min(2,2_s^*-1)}$.

Now, we prove (b):

$$\begin{aligned} |\mathcal{F}(\phi)| &\le C\|\phi\|_*^2\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}}}\Big)^2\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-3} + C\|\phi\|_*^{2_s^*-1}\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}}}\Big)^{2_s^*-1} \\ &\le C(\|\phi\|_*^2 + \|\phi\|_*^3)\lambda^{\frac{N+2s}{2}}\Big(\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{3}{4}}}\Big)^3 \le C(\ln\lambda)^{\frac{1}{s}}\|\phi\|_*^2\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}. \end{aligned}$$

So, we have $\|\mathcal{F}(\phi)\|_{**} \le C(\ln\lambda)^{\frac{1}{s}}\|\phi\|_*^2$. ∎

Next, we estimate $l_k(y)$.

Lemma 2.5. *There exists a constant $C > 0$, independent of k, such that the following hold:*
(i) *If $\frac{2+N-\sqrt{4+N^2}}{4} < s < \min(\frac{N}{4}, 1)$, then there exists a small $\sigma > 0$ such that $\|l_k\|_{**} \le \frac{C}{\lambda^{s+\sigma}}$.*
(ii) *If $N = 3 = 4s$, then $\|l_k\|_{**} \le \frac{C}{\lambda^s}$.*

Proof. We first prove (a). By symmetry, we can assume that $y \in \Omega_1$. Then $|y - x_j| \ge |y - x_1|$. We first estimate the term J_1. We have

$$|J_1| \le C\Big[\Big(\sum_{j=2}^{k} U_{x_j,\lambda}\Big)^{2_s^*-1} + U_{x_1,\lambda}^{2_s^*-2}\sum_{j=2}^{k} U_{x_j,\lambda} + \sum_{j=2}^{k} U_{x_j,\lambda}^{2_s^*-1}\Big]$$
$$\le C\lambda^{\frac{N+2s}{2}}\Big(\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-1} + \frac{C\lambda^{\frac{N+2s}{2}}}{(1+\lambda|y-x_1|)^{4s}}\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}}.$$

If $N - 2s \ge \frac{N+2s}{2} - \tau$, then we have

$$\frac{1}{(1+\lambda|y-x_1|)^{4s}}\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}} \le \frac{1}{(1+\lambda|y-x_1|)^{\frac{N+2s}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}-\tau}}$$
$$\le \frac{1}{(1+\lambda|y-x_1|)^{\frac{N+2s}{2}+\tau}}\Big(\frac{k}{\lambda}\Big)^{\frac{N+2s}{2}-\tau}$$
$$\le \frac{1}{\lambda^{s+\sigma}}\frac{1}{(1+\lambda|y-x_1|)^{\frac{N+2s}{2}+\tau}}.$$

If $N - 2s < \frac{N+2s}{2} - \tau$, then $4s > \frac{N+2s}{2} + \tau$, and we obtain that

$$\frac{1}{(1+\lambda|y-x_1|)^{4s}}\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}} \le \frac{1}{(1+\lambda|y-x_1|)^{\frac{N+2s}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{(\lambda|x_1-x_j|)^{N-2s}}$$
$$\le \frac{1}{(1+\lambda|y-x_1|)^{\frac{N+2s}{2}+\tau}}\Big(\frac{k}{\lambda}\Big)^{N-2s}.$$

Using the Hölder inequality, we have

$$\Big(\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-1} \le \sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{4s}(\frac{N-2s}{2}-\frac{N-2s}{N+2s}\tau)}}\Big)^{\frac{4s}{N-2s}}$$
$$\le C\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}\Big(\frac{k}{\lambda}\Big)^{\frac{N+2s}{N-2s}(\frac{N-2s}{2}-\frac{N-2s}{N+2s}\tau)}$$
$$\le C\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}\Big(\frac{1}{\lambda}\Big)^{s+\sigma}.$$

Thus,

$$\|J_1\|_{**} \le C\Big(\frac{1}{\lambda}\Big)^{s+\sigma}.$$

Now, we estimate J_2. Note that $\zeta = 0$ when $|(r, y'') - (r_0, y_0'')| \ge 2\delta$ and $\frac{1}{\lambda} \le \frac{C}{1+\lambda|y-x_j|}$ when $|(r, y'') - (r_0, y_0'')| < 2\delta$. We have

$$|J_2| \le \frac{C}{\lambda^{2s}}\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{\zeta}{(1+\lambda|y-x_j|)^{N-2s}} \le \frac{C}{\lambda^{\min(2s, N-\frac{N+2s}{2}-\tau)}}\lambda^{\frac{N+2s}{2}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.$$

If $N > 4s + 2\tau$, then $\|J_2\|_{**} \le \frac{C}{\lambda^{s+\sigma}}$.

We have

$$J_3 = \sum_{j=1}^{k} c(N,s)\Big(\lim_{\epsilon\to 0^+}\int_{B_{\delta/4}(y)\setminus B_\epsilon(y)}\frac{(\zeta(y)-\zeta(x))U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\,dx + \int_{\mathbb{R}^N\setminus B_{\delta/4}(y)}\frac{(\zeta(y)-\zeta(x))U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\,dx\Big)$$
$$=: \sum_{j=1}^{k} c(N,s)(J_{31} + J_{32}).$$

We first estimate J_{31}. From the definition of function ζ, we have $\zeta(y)-\zeta(x)=0$ for $x,y\in B_\delta(x_j)$ or $x,y\in\mathbb{R}^N\setminus\overline{B_{2\delta}(x_j)}$. So, $J_{31}\neq 0$ only if $B_{\delta/4}(y)\subset B_{5/2\delta}(x_j)\setminus B_{1/2\delta}(x_j)$. We have

$$\frac{3}{4}\delta\le|y-x_j|\le|x-y|+|x-x_j|\le\frac{\delta}{4}+|x-x_j|\le\frac{3}{2}|x-x_j|\le\frac{15}{4}\delta\quad\text{for } B_{\delta/4}(y)\subset B_{5/2\delta}(x_j)\setminus B_{1/2\delta}(x_j).$$

Furthermore, we divide J_{31} as follows:

$$J_{31}=\lim_{\epsilon\to0^+}\int\limits_{B_{\delta/4}(y)\setminus B_\epsilon(y)}\frac{\nabla\zeta(y)\cdot(y-x)U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\,dx+O\Big(\lim_{\epsilon\to0^+}\int\limits_{B_{\delta/4}(y)\setminus B_\epsilon(y)}\frac{U_{x_j,\lambda}(x)}{|x-y|^{N+2s-2}}\,dx\Big)=:J_{311}+J_{312}.$$

Note that $B_{\delta/4}(y)\setminus B_\epsilon(y)$ is a symmetric set. Then, by the mean value theorem, we get that

$$\begin{aligned}|J_{311}|&=\Big|\lim_{\epsilon\to0^+}\int\limits_{B_{\delta/4}(y)\setminus B_\epsilon(y)}\frac{\nabla\zeta(y)\cdot(y-x)U_{x_j,\lambda}(x)}{|x-y|^{N+2s}}\Big|\\
&=\Big|C(N,s)\lim_{\epsilon\to0^+}\int\limits_{B_{\delta/4}(0)\setminus B_\epsilon(0)}\frac{\nabla\zeta(y)\cdot z}{|z|^{N+2s}}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda^2|z+y-x_j|^2)^{\frac{N-2s}{2}}}\Big|\\
&=\Big|\frac{C(N,s)\lambda^{\frac{N-2s}{2}}}{2}\lim_{\epsilon\to0^+}\int\limits_{B_{\delta/4}(0)\setminus B_\epsilon(0)}\frac{\nabla\zeta(y)\cdot z}{|z|^{N+2s}}\Big(\frac{1}{(1+\lambda^2|z+y-x_j|^2)^{\frac{N-2s}{2}}}-\frac{1}{(1+\lambda^2|-z+y-x_j|^2)^{\frac{N-2s}{2}}}\Big)\Big|\\
&\le C\lambda^{\frac{N-2s}{2}+1}\int\limits_{B_{\delta/4}(0)}\frac{|\nabla\zeta(y)|}{|z|^{N+2s-2}}\frac{1}{(1+\lambda|(2\vartheta-1)z+y-x_j|)^{N-2s+1}}\\
&\le\frac{C}{\lambda^{s+\sigma}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}},\end{aligned}$$

for $0<\vartheta<1$ and since $|(2\vartheta-1)z+y-x_j|\ge|y-x_j|-|(2\vartheta-1)z|\ge\frac{2}{3}|y-x_j|$ for $z\in B_{\delta/4}(0)$. Similarly, we can obtain

$$|J_{312}|\le\frac{C}{\lambda^{s+\sigma}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.$$

For the term J_{32}, we divide three cases:

Case 1: If $y\in B_\delta(x_j)$, then

$$\begin{aligned}|J_{32}|&\le\int\limits_{\mathbb{R}^N\setminus(B_{\delta/4}(y)\cup B_\delta(x_j))}\frac{1}{|x-y|^{N+2s}}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|x-x_j|)^{N-2s}}\\
&\le\frac{C}{\lambda^{2s}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}}\int\limits_{\mathbb{R}^N\setminus B_{\delta/4}(y)}\frac{1}{|x-y|^{N+2s}}\\
&\le\frac{C}{\lambda^{s+\sigma}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.\end{aligned}$$

Case 2: If $\delta\le|y-x_j|\le3\delta$, then, by Lemma A.3,

$$\begin{aligned}|J_{32}|&\le\int\limits_{\mathbb{R}^N\setminus B_{\delta/4}(y)}\frac{1}{|x-y|^{N+2s}}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|x-x_j|)^{N-2s}}\\
&\le C\lambda^{\frac{N+2s}{2}}\int\limits_{\mathbb{R}^N\setminus B_{\delta\lambda/4}(\lambda y)}\frac{1}{|z-\lambda y|^{N+2s}}\frac{1}{(1+|z-\lambda x_j|)^{N-2s}}\\
&\le C\lambda^{\frac{N+2s}{2}}\Big(\frac{1}{(\lambda|y-x_j|)^N}+\frac{1}{\lambda^{2s}}\frac{1}{(\lambda|y-x_j|)^{N-2s}}\Big)\\
&\le\frac{C}{\lambda^{s+\sigma}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.\end{aligned}$$

Case 3: Suppose that $|y-x_j|>3\delta$. Note that $|x-y|\geq|y-x_j|-|x-x_j|\geq\frac{1}{3}|y-x_j|$ when $|y-x_j|\geq 3\delta$ and $|x-x_j|\leq 2\delta$. Then we have

$$\begin{aligned}|J_{32}| &\leq \int\limits_{B_{2\delta}(x_j)} \frac{1}{|x-y|^{N+2s}}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|x-x_j|)^{N-2s}}\\ &\leq \frac{C}{\lambda^{\frac{N-2s}{2}}}\int\limits_{B_{2\delta}(x_j)}\frac{1}{|x-y|^{N+2s}}\frac{1}{|x-x_j|^{N-2s}}\\ &\leq \frac{C\lambda^{\frac{N+2s}{2}}}{\lambda^N}\frac{1}{|y-x_j|^{\frac{N+2s}{2}+\tau}}\int\limits_{B_{2\delta}(x_j)}\frac{1}{|x-x_j|^{N-2s}}\\ &\leq \frac{C}{\lambda^{s+\sigma}}\lambda^{\frac{N+2s}{2}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{2}+\tau}}.\end{aligned}$$

Then we obtain $\|J_3\|_{**}\leq\frac{C}{\lambda^{s+\sigma}}$.

As a result, we have proved that $\|l_k\|_{**}\leq\frac{C}{\lambda^{s+\sigma}}$.

Now, we prove (b). Since $4s>\frac{9}{4}$, we have

$$\begin{aligned}\frac{1}{(1+\lambda|y-x_1|)^{4s}}\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}} &\leq \frac{1}{(1+\lambda|y-x_1|)^{\frac{9}{4}}}\sum_{j=2}^{k}\frac{1}{(\lambda|x_1-x_j|)^{N-2s}}\\ &\leq \frac{1}{(1+\lambda|y-x_1|)^{\frac{9}{4}}}\Big(\frac{k}{\lambda}\Big)^{\frac{3}{2}}.\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}\Big(\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-1} &\leq \sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}\Big(\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2s}{4s}\frac{3}{4}}}\Big)^{\frac{4s}{N-2s}}\\ &\leq C\sum_{j=2}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}\Big(\frac{k}{\lambda}\Big)^{\frac{9}{4}}.\end{aligned}$$

Thus,

$$\|J_1\|_{**}\leq C\Big(\frac{1}{\lambda}\Big)^{s}.$$

For the term J_2, we have

$$|J_2|\leq\frac{C}{\lambda^s}\lambda^{\frac{9}{4}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{N-s}}=\frac{C}{\lambda^s}\lambda^{\frac{9}{4}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}.$$

Now, we estimate the term J_3:

$$\begin{aligned}|J_{31}| &\leq \frac{C}{\lambda^{2s-1}}\lambda^{\frac{9}{4}}\int\limits_{B_{\delta/4}(0)}\frac{|\nabla\zeta(y)|}{|z|^{N+2s-2}}\frac{1}{(1+\lambda|(2\vartheta-1)z+y-x_j|)^{2s+1}}\\ &\quad+\frac{C}{\lambda^{2s}}\lambda^{\frac{9}{4}}\int\limits_{B_{\delta/4}(0)}\frac{1}{|z|^{N+2s-2}}\frac{1}{(1+\lambda|z+y-x_j|)^{N-2s}}\\ &\leq \frac{C}{\lambda^s}\lambda^{\frac{9}{4}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}.\end{aligned}$$

Similar to (a), we can deduce that

$$|J_{32}|\leq\frac{C}{\lambda^s}\lambda^{\frac{9}{4}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{9}{4}}}.$$

So, we obtain $\|J_3\|_{**}\leq\frac{C}{\lambda^s}$.

As a result, we have proved that $\|l_k\|_{**}\leq\frac{C}{\lambda^s}$. □

Proof of Proposition 2.3. Let $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. If $N > 4s$, we set

$$E = \Big\{u : u \in C(\mathbb{R}^N) \cap H_s,\ \|u\|_* \le \frac{1}{\lambda^s},\ \sum_{i=1}^{k} \int_{\mathbb{R}^N} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l} u = 0,\ l = 1, \dots, N\Big\}.$$

By Proposition 2.2, the solution ϕ of (2.9) is equivalent to the following fixed point problem:

$$\phi = A(\phi) =: L_k(\mathcal{F}(\phi)) + L_k(l_k).$$

Hence, it is sufficient to prove that the operator A is a contraction map from the complete space E to itself. In fact, if $\phi \in L^\infty(\mathbb{R}^N)$, then, by [38, Proposition 2.9], we can obtain $\phi \in C(\mathbb{R}^N)$. For any $\phi \in E$, by Proposition 2.2, Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}\|A(\phi)\|_* &\le C\|L_k(\mathcal{F}(\phi))\|_* + C\|L_k(l_k)\|_* \le C[\|\mathcal{F}(\phi)\|_{**} + \|l_k\|_{**}]\\ &\le C\Big[\frac{\lambda^{\frac{2s(N-4s)}{(N-2s)^2}}}{\lambda^{s\times\min(2,2_s^*-1)}} + \frac{1}{\lambda^{s+\sigma}}\Big] \le \frac{C}{\lambda^s},\end{aligned}$$

since $\frac{2s(N-4s)}{(N-2s)^2} < s \times \min(1, 2_s^* - 2)$. This shows that A maps E to E itself and E is invariant under A operator.

If $2_s^* \le 3$, then for all $\phi_1, \phi_2 \in E$, we have

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\|_* &= \|L_k(\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2))\|_* \le C\|\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)\|_{**}\\ &\le C\|(|\phi_1| + |\phi_2|)^{2_s^*-2}|\phi_1 - \phi_2|\|_{**} \le \frac{1}{2}\|\phi_1 - \phi_2\|_*.\end{aligned}$$

The case $2_s^* > 3$ can be discussed in a similar way.

Hence, A is a contraction map. The Banach fixed point theorem tells us that there exists a unique solution $\phi \in E$ for problem (2.9).

Finally, by Proposition 2.2, we have

$$\|\phi\|_* \le C\Big(\frac{1}{\lambda}\Big)^{s+\sigma} \quad \text{and} \quad |c_l| \le C\|\mathcal{F}(\phi) + l_k\|_{**} \le C\Big(\frac{1}{\lambda}\Big)^{s+\sigma}.$$

When $N = 3 = 4s$, we set

$$E = \Big\{u : u \in C(\mathbb{R}^N) \cap H_s,\ \|u\|_* \le \frac{C_0}{\lambda^s},\ \sum_{i=1}^{k} \int_{\mathbb{R}^N} Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l} u = 0,\ l = 1, \dots, N\Big\},$$

where $C_0 >$ is a large constant such that

$$\|A(\phi)\|_* \le C\|L_k(\mathcal{F}(\phi))\|_* + C\|L_k(l_k)\|_* \le C\Big[C(\ln\lambda)^{\frac{1}{s}}\|\phi\|_*^2 + \frac{C}{\lambda^s}\Big] \le \frac{C_0}{\lambda^s}.$$

By the process of the case $N > 4s$, we can obtain the result. □

3 Proof of the Main Theorem: The Case $N > 4s$

Let ϕ be the function obtained in Proposition 2.3 and $u_k = Z_{\bar{r},\bar{y}'',\lambda} + \phi$. In order to use local Pohozaev identities, we quote the extension of u_k, that is, $\tilde{u}_k = \tilde{Z}_{\bar{r},\bar{y}'',\lambda} + \tilde{\phi}$. $\tilde{Z}_{\bar{r},\bar{y}''}$ and $\tilde{\phi}$ are extensions of $Z_{\bar{r},\bar{y}''}$ and ϕ, respectively. Then we have

$$\begin{cases}\operatorname{div}(t^{1-2s}\nabla\tilde{u}_k) = 0 & \text{in } \mathbb{R}_+^{N+1},\\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t\tilde{u}_k = \omega_s\Big(-V(r,y'')u_k + (u_k)_+^{2_s^*-1} + \sum\limits_{l=1}^{N} c_l \sum\limits_{j=1}^{k} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l}\Big) & \text{on } \mathbb{R}^N.\end{cases} \tag{3.1}$$

Without loss of generality, we may assume $\omega_s = 1$. Multiplying (3.1) by $\frac{\partial \tilde{u}_k}{\partial y_i}$ $(i = 3, \dots, N)$ and $\langle \nabla \tilde{u}_k, Y \rangle$, respectively, and then integrating by parts, we have the following two Pohozaev identities:

$$
\begin{aligned}
&-\int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} \frac{\partial \tilde{u}_k}{\partial \nu} \frac{\partial \tilde{u}_k}{\partial y_i} + \frac{1}{2} \int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} |\nabla \tilde{u}_k|^2 \nu_i \\
&\quad = \int_{B_\rho} \Big(-V(r, y'') u_k + (u_k)_+^{2_s^*-1} + \sum_{l=1}^N c_l \sum_{j=1}^k Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} \Big) \frac{\partial u_k}{\partial y_i}, \quad i = 3, \dots, N,
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&-\int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} \langle \nabla \tilde{u}_k, Y \rangle \frac{\partial \tilde{u}_k}{\partial \nu} + \frac{1}{2} \int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} |\nabla \tilde{u}_k|^2 \langle Y, \nu \rangle + \frac{2s - N}{2} \int_{\partial \mathcal{B}_\rho^+} t^{1-2s} \frac{\partial \tilde{u}_k}{\partial \nu} \tilde{u}_k \\
&\quad = \int_{B_\rho} \Big(-V(r, y'') u_k + (u_k)_+^{2_s^*-1} + \sum_{l=1}^N c_l \sum_{j=1}^k Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} \Big) \langle y, u_k \rangle.
\end{aligned}
\tag{3.3}
$$

In the following, assume $\rho \in (2\delta, 5\delta)$. We have the following lemma.

Lemma 3.1. *Suppose that $(\bar{r}, \bar{y}'', \lambda)$ satisfies*

$$
-\int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} \frac{\partial \tilde{u}_k}{\partial \nu} \frac{\partial \tilde{u}_k}{\partial y_i} + \frac{1}{2} \int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} |\nabla \tilde{u}_k|^2 \nu_i = \int_{B_\rho} (-V(r, y'') u_k + (u_k)_+^{2_s^*-1}) \frac{\partial u_k}{\partial y_i}, \quad i = 3, \dots, N, \tag{3.4}
$$

$$
\begin{aligned}
&-\int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} \langle \nabla \tilde{u}_k, Y \rangle \frac{\partial \tilde{u}_k}{\partial \nu} + \frac{1}{2} \int_{\partial'' \mathcal{B}_\rho^+} t^{1-2s} |\nabla \tilde{u}_k|^2 \langle Y, \nu \rangle + \frac{2s - N}{2} \int_{\partial \mathcal{B}_\rho^+} t^{1-2s} \frac{\partial \tilde{u}_k}{\partial \nu} \tilde{u}_k \\
&\quad = \int_{B_\rho} (-V(r, y'') u_k + (u_k)_+^{2_s^*-1}) \langle y, u_k \rangle
\end{aligned}
\tag{3.5}
$$

and

$$
\int_{\mathbb{R}^N} ((-\Delta)^s u_k + V(r, y'') u_k - (u_k)_+^{2_s^*-1}) \frac{\partial Z_{\bar{r}, \bar{y}'', \lambda}}{\partial \lambda} = 0. \tag{3.6}
$$

Then we have $c_l = 0$, $l = 1, \dots, N$.

Proof. By (3.2), (3.3), (3.4) and (3.5), we have

$$
\sum_{l=1}^N c_l \sum_{j=1}^k \int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} \frac{\partial u_k}{\partial y_i} = 0, \quad i = 3, \dots, N, \qquad \sum_{l=1}^N c_l \sum_{j=1}^k \int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} \langle y, \nabla u_k \rangle = 0. \tag{3.7}
$$

Note that $\zeta = 0$ in $\mathbb{R}^N \setminus B_\rho$. By (3.6) and (3.7), we have

$$
\sum_{l=1}^N c_l \sum_{j=1}^k \int_{\mathbb{R}^N} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} v = \sum_{l=1}^N c_l \sum_{j=1}^k \int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} v = 0 \tag{3.8}
$$

for $v = \frac{\partial u_k}{\partial y_i}$, $v = \langle \nabla u_k, y \rangle$ and $v = \frac{\partial Z_{\bar{r}, \bar{y}'', \lambda}}{\partial \lambda}$.

By direct calculations, we have

$$
\sum_{j=1}^k \int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,2} \langle y', \nabla_{y'} Z_{\bar{r}, \bar{y}'', \lambda} \rangle = k\lambda^2 (a_1 + o(1)),
$$

$$
\sum_{j=1}^k \int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,i} \frac{\partial Z_{\bar{r}, \bar{y}'', \lambda}}{\partial y_i} = k\lambda^2 (a_2 + o(1)), \quad i = 3, \dots, N,
$$

and

$$\sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,1}\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda} = \frac{k}{\lambda^2}(a_3+o(1)), \tag{3.9}$$

where $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$.

Furthermore, we have

$$\begin{aligned}\sum_{l=1}^{N} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l}\langle y,\nabla Z_{\bar r,\bar y'',\lambda}\rangle &= \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,2}\langle y',\nabla_{y'} Z_{\bar r,\bar y'',\lambda}\rangle c_2 + O\Big(\frac{k}{\lambda^{N-2}}|c_2|\Big) + o\Big(k\lambda^2\sum_{l=3}^{N}|c_l|\Big) + o(k|c_1|)\\ &= k\lambda^2(a_1+o(1))c_2 + o\Big(k\lambda^2\sum_{l=3}^{N}|c_l|\Big) + o(k|c_1|)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\sum_{l=1}^{N} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l}\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial y_i} &= \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,i}\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial y_i} c_i + o\Big(k\lambda^2\sum_{l\neq 1,i}|c_l|\Big) + o(k|c_1|)\\ &= k\lambda^2(a_2+o(1))c_i + o\Big(k\lambda^2\sum_{l\neq 1,i}|c_l|\Big) + o(k|c_1|), \quad i = 3,\dots,N.\end{aligned} \tag{3.11}$$

Since ϕ is a solution to (2.9), by fractional elliptical equation estimates (see, for example, [38, Proposition 2.9] and [16, Theorem 12.2.1]), we can obtain $\phi \in C^1(B_\rho)$. By integrating by parts and using $\|\phi\|_* \le \frac{C}{\lambda^{s+\sigma}}$, we have

$$\sum_{l=1}^{N} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} v = o\Big(k\lambda^2\sum_{l=2}^{N}|c_l|\Big) + o(k|c_1|),$$

for $v = \langle y,\nabla\phi_{\bar r,\bar y'',\lambda}\rangle$ and $v = \frac{\partial\phi_{\bar r,\bar y'',\lambda}}{\partial y_i}$.

It follows from (3.8) that

$$\sum_{l=1}^{N} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} v = o\Big(k\lambda^2\sum_{l=2}^{N}|c_l|\Big) + o(k|c_1|), \tag{3.12}$$

for $v = \langle\nabla y, Z_{\bar r,\bar y'',\lambda}\rangle$ and $v = \frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial y_i}$.

By (3.10), (3.11) and (3.12), we have

$$c_l = o\Big(\frac{1}{\lambda^2}|c_1|\Big), \quad l = 2,\dots,N. \tag{3.13}$$

From (3.8), (3.9) and (3.13), we deduce that

$$0 = \sum_{l=1}^{N} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,1}\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda} = \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,1}\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda} c_1 + o\Big(\frac{k}{\lambda^2}\Big)c_1 = k(a_3+o(1))c_1 + o\Big(\frac{k}{\lambda^2}\Big)c_1,$$

which implies that $c_1 = 0$. We also have $c_l = 0$, $l = 2,\dots,N$. □

Note that

$$\frac{2s-N}{2}\int_{\partial\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde u_k}{\partial\nu}\tilde u_k = \frac{2s-N}{2}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde u_k}{\partial\nu}\tilde u_k + \frac{2s-N}{2}\int_{B_\rho}\Big(-V(r,y'')u_k + (u_k)_+^{2_s^*-1} + \sum_{l=1}^{N} c_l\sum_{j=1}^{k} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l}\Big)u_k,$$

$$\begin{aligned}\int_{B_\rho}(-V(r,y'')u_k + (u_k)_+^{2_s^*-1})\langle y,\nabla u_k\rangle &= \int_{B_\rho}\Big(-\frac{1}{2}V(r,y'')\langle y,\nabla u_k^2\rangle + \frac{1}{2_s^*}\langle y,\nabla(u_k)_+^{2_s^*}\rangle\Big)\\ &= -\frac{1}{2}\int_{\partial B_\rho} V(r,y'')u_k^2\langle y,\nu\rangle + \frac{1}{2}\int_{B_\rho}(NV(r,y'') + \langle\nabla V(r,y''),y\rangle)u_k^2\\ &\quad + \frac{1}{2_s^*}\int_{\partial B_\rho}(u_k)_+^{2_s^*}\langle y,\nu\rangle + \frac{2s-N}{2}\int_{B_\rho}(u_k)_+^{2_s^*}\end{aligned}$$

and $\sum_{l=1}^{N} c_l \int_{B_\rho} \sum_{j=1}^{k} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l}\phi = 0$. We find that (3.5) is equivalent to

$$\begin{aligned}\int_{B_\rho} \Big(sV(r,y'') + \frac{1}{2}\langle \nabla V(r,y''), y\rangle\Big)u_k^2 &= -\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \tilde{u}_k, Y\rangle \frac{\partial \tilde{u}_k}{\partial \nu} + \frac{1}{2}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla \tilde{u}_k|^2\langle Y,\nu\rangle + \frac{2s-N}{2}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial \tilde{u}_k}{\partial \nu}\tilde{u}_k \\ &\quad + \frac{1}{2}\int_{\partial B_\rho} V(r,y'')u_k^2\langle y,\nu\rangle - \frac{1}{2_s^*}\int_{\partial B_\rho} (u_k)_+^{2_s^*}\langle y,\nu\rangle + \frac{2s-N}{2}\sum_{l=1}^{N} c_l \int_{B_\rho}\sum_{j=1}^{k} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} Z_{\bar r,\bar y'',\lambda}.\end{aligned} \tag{3.14}$$

Similarly, (3.4) is equivalent to

$$\begin{aligned}\frac{1}{2}\int_{B_\rho} \frac{\partial V(r,y'')}{\partial y_i''}u_k^2 &= \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial \tilde{u}_k}{\partial \nu}\frac{\partial \tilde{u}_k}{\partial y_i} - \frac{1}{2}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla \tilde{u}_k|^2\nu_i \\ &\quad + \frac{1}{2}\int_{\partial B_\rho} V(r,y'')u_k^2\nu_i + \frac{1}{2_s^*}\int_{\partial B_\rho} u_k^{2_s^*}\nu_i, \quad i = 3,\dots,N.\end{aligned} \tag{3.15}$$

Lemma 3.2. *Relations* (3.14) *and* (3.15) *are, respectively, equivalent to*

$$\int_{B_\rho} \Big(sV(r,y'') + \frac{1}{2}\langle \nabla V(r,y''), y\rangle\Big)u_k^2 = O\Big(\frac{k}{\lambda^{2s+\sigma}}\Big) \tag{3.16}$$

and

$$\int_{B_\rho} \frac{\partial V(r,y'')}{\partial y_i}u_k^2 = O\Big(\frac{k}{\lambda^{2s+\sigma}}\Big), \quad i = 3,\dots,N. \tag{3.17}$$

Proof. We only give the proof for (3.16). The proof of (3.17) is similar.

Note that $\tilde{u}_k = \tilde{Z}_{\bar r,\bar y'',\lambda} + \tilde{\phi}$. We have

$$\begin{aligned}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \tilde{u}_k, Y\rangle \frac{\partial \tilde{u}_k}{\partial \nu} &= \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{Z}_{\bar r,\bar y'',\lambda}, Y\rangle \frac{\partial \widetilde{Z}_{\bar r,\bar y'',\lambda}}{\partial \nu} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{\phi}, Y\rangle \frac{\partial \widetilde{\phi}}{\partial \nu} \\ &\quad + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{Z}_{\bar r,\bar y'',\lambda}, Y\rangle \frac{\partial \widetilde{\phi}}{\partial \nu} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{\phi}, Y\rangle \frac{\partial \widetilde{Z}_{\bar r,\bar y'',\lambda}}{\partial \nu}.\end{aligned}$$

Using Lemma A.5, we obtain

$$\begin{aligned}\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{Z}_{\bar r,\bar y'',\lambda}, Y\rangle \frac{\partial \widetilde{Z}_{\bar r,\bar y'',\lambda}}{\partial \nu}\Big| &\le \frac{C}{\lambda^{N-2s}}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\Big(\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N-2s+1}}\Big)^2 \\ &\le \frac{Ck^2}{\lambda^{N-2s}}\int_{\partial''\mathcal{B}_\rho^+} \frac{t^{1-2s}}{(1+|y-x_1|)^{2N-4s+2}} \le \frac{Ck^2}{\lambda^{N-2s}}.\end{aligned} \tag{3.18}$$

By (A.2) in Lemma A.6,

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{\phi}, Y\rangle \frac{\partial \widetilde{\phi}}{\partial \nu}\Big| \le C\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla \widetilde{\phi}|^2 \le \frac{Ck\|\phi\|_*^2}{\lambda^\tau}. \tag{3.19}$$

By the process of the proof of (3.18) and (3.19), we also have

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{Z}_{\bar r,\bar y'',\lambda}, Y\rangle \frac{\partial \widetilde{\phi}}{\partial \nu} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle \nabla \widetilde{\phi}, Y\rangle \frac{\partial \widetilde{Z}_{\bar r,\bar y'',\lambda}}{\partial \nu}\Big| \le \frac{Ck\|\phi\|_*}{\lambda^{\frac{N-2s}{2}}}.$$

Note that $N > 4s$. So we have proved that

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\langle\nabla\tilde{u}_k, Y\rangle\frac{\partial\tilde{u}_k}{\partial\nu}\Big| \le \frac{Ck}{\lambda^{2s+\sigma}}.$$

Similarly, we can prove

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\tilde{u}_k|^2\langle Y,\nu\rangle\Big| \le \frac{Ck}{\lambda^{2s+\sigma}}.$$

Next, we estimate the term $\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde{u}_k}{\partial\nu}\tilde{u}_k$:

$$\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde{u}_k}{\partial\nu}\tilde{u}_k = \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{Z}_{r'',\bar{y}'',\lambda}}{\partial\nu}\widetilde{Z}_{r'',\bar{y}'',\lambda} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{\phi}}{\partial\nu}\widetilde{\phi}$$
$$+ \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{Z}_{r'',\bar{y}'',\lambda}}{\partial\nu}\widetilde{\phi} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{\phi}}{\partial\nu}\widetilde{Z}_{r'',\bar{y}'',\lambda}.$$

By Lemma A.5,

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{Z}_{r'',\bar{y}'',\lambda}}{\partial\nu}\widetilde{Z}_{r'',\bar{y}'',\lambda}\Big| \le \frac{C}{\lambda^{N-2s}}\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N-2s+1}}\times\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N-2s}}$$
$$\le \frac{Ck^2}{\lambda^{N-2s}}\int_{\partial''\mathcal{B}_\rho^+}\frac{t^{1-2s}}{(1+|y-x_1|)^{2N-4s+1}} \le \frac{Ck^2}{\lambda^{N-2s}}.$$

It follows from (A.4) that

$$\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\widetilde{\phi}|^2 \le \frac{Ck\|\phi\|_*^2}{\lambda^\tau}. \tag{3.20}$$

By (3.20) and (A.2) in Lemma A.6, one has

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{\phi}}{\partial\nu}\widetilde{\phi}\,dS\Big| \le \Big(\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\widetilde{\phi}|^2\Big)^{\frac{1}{2}}\Big(\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\widetilde{\phi}^2\Big)^{\frac{1}{2}} \le \frac{Ck\|\phi\|_*^2}{\lambda^\tau}.$$

Similarly,

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{Z}_{r'',\bar{y}'',\lambda}}{\partial\nu}\widetilde{\phi} + \int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\widetilde{\phi}}{\partial\nu}\widetilde{Z}_{r'',\bar{y}'',\lambda}\Big| \le \frac{Ck\|\phi\|_*^2}{\lambda^\tau}.$$

We have proved that

$$\Big|\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}\frac{\partial\tilde{u}_k}{\partial\nu}\tilde{u}_k\Big| \le \frac{Ck}{\lambda^{2s+\sigma}}.$$

Since $\zeta = 0$ on ∂B_ρ, $u_k = \phi$ on ∂B_ρ, we deduce that

$$\Big|\int_{\partial B_\rho} V(r,y'')u_k^2\langle y,\nu\rangle\Big| \le C\|\phi\|_*^2\int_{\partial B_\rho}\Big(\sum_{j=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}}\Big)^2 \le \frac{Ck^2\|\phi\|_*^2}{\lambda^{2\tau}} \le \frac{Ck}{\lambda^{2s+\tau}}$$

and

$$\Big|\int_{\partial B_\rho}(u_k)_+^{2_s^*}\langle y,\nu\rangle\Big| \le \frac{Ck^{2_s^*}\|\phi\|_*^{2_s^*}}{\lambda^{2_s^*\tau}} \le \frac{Ck}{\lambda^{2s+\tau}}.$$

From Proposition 2.3, we know the following estimate for c_l:

$$|c_l| \le C\Big(\frac{1}{\lambda}\Big)^{s+\sigma}.$$

On the other hand,

$$\sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} Z_{\bar r,\bar y'',\lambda} = \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-1} Z_{j,l} + \sum_{j=1}^{k}\int_{B_\rho}\sum_{i\neq j} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} Z_{x_i,\lambda} = O\Big(\frac{1}{\lambda^N}\Big) + O\Big(\frac{k}{\lambda^{2s}}\Big).$$

These imply that

$$\Big|\sum_{l=1} c_l \sum_{j=1}^{k}\int_{B_\rho} Z_{x_j,\lambda}^{2_s^*-2} Z_{j,l} Z_{\bar r,\bar y'',\lambda}\Big| \le \frac{Ck}{\lambda^{2s+\sigma}}.$$

Combining the above estimates, we find that (3.14) is equivalent to

$$\int_{B_\rho}\Big(sV(r,y'') + \frac{1}{2}\langle\nabla V(r,y''), y\rangle\Big)u_k^2 = O\Big(\frac{k}{\lambda^{2s+\sigma}}\Big).$$

The proof is complete. □

Lemma 3.3. *For any function $g(r,y'') \in C^1(\mathbb{R}^N)$, we have*

$$\int_{B_\rho} g(r,y'')u_k^2 = k\Big(\frac{1}{\lambda^{2s}} g(\bar r,\bar y'')\int_{\mathbb{R}^N} U_{0,1}^2 + o\Big(\frac{1}{\lambda^{2s}}\Big)\Big).$$

Proof. We have

$$\int_{B_\rho} g(r,y'')u_k^2 = \int_{D_\rho} g(r,y'')Z_{\bar r,\bar y'',\lambda}^2 + 2\int_{D_\rho} g(r,y'')Z_{\bar r,\bar y'',\lambda}\phi + \int_{D_\rho} g(r,y'')\phi^2.$$

Note that

$$\begin{aligned}\Big|2\int_{B_\rho} g(r,y'')Z_{\bar r,\bar y'',\lambda}\phi + \int_{B_\rho} g(r,y'')\phi^2\Big| &\le C\Big(\|\phi\|_*\int_{B_\rho}\sum_{i=1}^{k}\frac{\zeta\lambda^{N-2s}}{(1+\lambda|y-x_i|)^{N-2s}}\sum_{j=1}^{k}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}+\tau}} \\ &\quad + \|\phi\|_*^2\int_{B_\rho}\Big(\sum_{i=1}^{k}\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}}\Big)^2\Big) \\ &\le \frac{Ck\|\phi\|_*}{\lambda^s} + \frac{Ck\|\phi\|_*^2}{\lambda^{2\tau}} \le \frac{Ck}{\lambda^{2s+\sigma}}\end{aligned}$$

and

$$\int_{B_\rho} g(r,y'')Z_{\bar r,\bar y'',\lambda}^2 = \sum_{j=1}^{k}\Big(\int_{B_\rho} g(r,y'')Z_{x_j,\lambda}^2 + \sum_{i\neq j}\int_{B_\rho} g(r,y'')Z_{x_i,\lambda}Z_{x_j,\lambda}\Big) = k\Big(\frac{1}{\lambda^{2s}} g(\bar r,\bar y'')\int_{\mathbb{R}^N} U_{0,1}^2 + o\Big(\frac{1}{\lambda^{2s}}\Big)\Big),$$

and we get the result. □

Proof of Theorem 1.1. By (3.16) and (3.17), we deduce that

$$\int_{B_\rho}\Big(sV(r,y'') + \frac{1}{2}r\frac{\partial V(r,y'')}{\partial r}\Big)u_k^2 = O\Big(\frac{k}{\lambda^{2s+\sigma}}\Big).$$

That is

$$\int_{B_\rho}\frac{1}{r^{2s-1}}\frac{\partial(r^{2s}V(r,y''))}{\partial r}u_k^2 = O\Big(\frac{k}{\lambda^{2s+\sigma}}\Big). \tag{3.21}$$

Applying Lemma 3.3 to (3.17) and (3.21), we obtain

$$k\Big(\frac{1}{\lambda^{2s}}\frac{\partial V(\bar r,\bar y'')}{\partial \bar y_i}\int_{\mathbb{R}^N} U_{0,1}^2 + o\Big(\frac{1}{\lambda^{2s}}\Big)\Big) = o\Big(\frac{k}{\lambda^{2s}}\Big)$$

and

$$k\Big(\frac{1}{\lambda^{2s}}\frac{1}{\bar r^{2s-1}}\frac{\partial(\bar r^{2s}V(\bar r,\bar y''))}{\partial\bar r}\int\limits_{\mathbb{R}^N}U_{0,1}^2+o\Big(\frac{1}{\lambda^{2s}}\Big)\Big)=o\Big(\frac{k}{\lambda^{2s}}\Big).$$

Therefore, the equations to determine $(\bar r,\bar y'')$ are

$$\frac{\partial(\bar r^{2s}V(\bar r,\bar y''))}{\partial\bar y_i}=o(1),\quad i=3,\dots,N, \tag{3.22}$$

and

$$\frac{\partial(\bar r^{2s}V(\bar r,\bar y''))}{\partial\bar r}=o(1).$$

From (3.6) and (B.1), the equation to determine λ is

$$-\frac{B_1}{\lambda^{2s+1}}V(\bar r,\bar y'')+\frac{B_3k^{N-2s}}{\lambda^{N-2s+1}}=O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big). \tag{3.23}$$

Let $\lambda=tk^{\frac{N-2s}{N-4s}}$. Then $t\in[L_0,L_1]$. It follows from (3.23) that

$$-\frac{B_1}{t^{2s+1}}V(\bar r,\bar y'')+\frac{B_3}{t^{N-2s+1}}=o(1),\quad t\in[L_0,L_1]. \tag{3.24}$$

Define

$$H(t,\bar r,\bar y'')=\Big(\nabla_{\bar r,\bar y''}(\bar r^{2s}V(\bar r,\bar y'')),-\frac{B_1}{t^{2s+1}}V(\bar r,\bar y'')+\frac{B_3}{t^{N-2s+1}}\Big).$$

Then

$$\deg(H(t,\bar r,\bar y''),[L_0,L_1]\times B_\theta((r_0,y_0'')))=-\deg(\nabla_{\bar r,\bar y''}(\bar r^{2s}V(\bar r,\bar y'')),B_\theta((r_0,y_0''))\neq 0.$$

Hence, (3.22), (3.22) and (3.24) have a solution $t_k\in[L_0,L_1]$ and $(\bar r_k,\bar y_k'')\in B_\theta((r_0,y_0''))$. □

4 Proof of the Main Theorem: The Case $N=3=4s$

Lemma 4.1. *When $N=3=4s$, relations* (3.14) *and* (3.15) *are, respectively, equivalent to*

$$\int\limits_{B_\rho}\Big(sV(r,y'')+\frac{1}{2}\langle\nabla V(r,y''),y\rangle\Big)u_k^2=O\Big(\frac{k^2}{\lambda^{2s}}\Big) \tag{4.1}$$

and

$$\int\limits_{B_\rho}\frac{\partial V(r,y'')}{\partial y_i}u_k^2=O\Big(\frac{k^2}{\lambda^{2s}}\Big),\quad i=3. \tag{4.2}$$

Proof. By using (A.3) in Lemma A.6, we can prove this result as Lemma 3.2. □

Lemma 4.2. *For any function $g(r,y'')\in C^1(\mathbb{R}^3)$, we have*

$$\int\limits_{B_\rho}g(r,y'')u_k^2=k\Big(4\pi C^2(N,s)g(\bar r,\bar y'')\frac{\ln\lambda}{\lambda^{2s}}+o\Big(\frac{\ln\lambda}{\lambda^{2s}}\Big)\Big).$$

Proof. We have

$$\begin{aligned}\Big|2\int\limits_{B_\rho}g(r,y'')Z_{\bar r,\bar y'',\lambda}\phi+\int\limits_{B_\rho}g(r,y'')\phi^2\Big|&\le C\Big(\|\phi\|_*\int\limits_{B_\rho}\sum_{i=1}^k\frac{\zeta\lambda^{N-2s}}{(1+\lambda|y-x_i|)^{N-2s}}\sum_{j=1}^k\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2s}{2}}}\\&\quad+\|\phi\|_*^2\int\limits_{B_\rho}\Big(\sum_{i=1}^k\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}}}\Big)^2\Big)\\&\le\frac{Ck^2\|\phi\|_*}{\lambda^s}+Ck^2\|\phi\|_*^2\le\frac{Ck^2}{\lambda^{2s}}\end{aligned}$$

and

$$\int_{B_\rho} g(r,y'')Z^2_{\bar{r},\bar{y}'',\lambda} = \sum_{j=1}^{k}\Big(\int_{B_\rho} g(r,y'')Z^2_{x_j,\lambda} + \sum_{i\neq j}\int_{B_\rho} g(r,y'')Z_{x_i,\lambda}Z_{x_j,\lambda}\Big)$$
$$= k\Big(g(\bar{r},\bar{y}'')\int_{B_\rho(x_j)} U^2_{x_j,\lambda} + O\Big(\frac{k}{\lambda^{2s}}\Big)\Big)$$
$$= k\Big(g(\bar{r},\bar{y}'')\frac{4\pi C^2(N,s)}{\lambda^{2s}}\int_0^{\lambda\rho}\frac{r^2}{(1+r^2)^{\frac{3}{2}}}\,dr + O\Big(\frac{k}{\lambda^{2s}}\Big)\Big)$$
$$= k\Big(4\pi C^2(N,s)g(\bar{r},\bar{y}'')\frac{\ln\lambda}{\lambda^{2s}} + o\Big(\frac{\ln\lambda}{\lambda^{2s}}\Big)\Big).$$

So, we get the result. □

Proof of Theorem 1.3. By (4.1) and (4.2), we can deduce that

$$\int_{B_\rho}\frac{1}{r^{2s-1}}\frac{\partial(r^{2s}V(r,y''))}{\partial r}u_k^2 = O\Big(\frac{k^2}{\lambda^{2s}}\Big). \tag{4.3}$$

Applying Lemma 4.2 to (4.2) and (4.3), we obtain

$$k\Big(\frac{\ln\lambda}{\lambda^{2s}}4\pi C^2(N,s)\frac{\partial V(\bar{r},\bar{y}'')}{\partial\bar{y}_i} + o\Big(\frac{\ln\lambda}{\lambda^{2s}}\Big)\Big) = O\Big(\frac{k^2}{\lambda^{2s}}\Big)$$

and

$$k\Big(\frac{\ln\lambda}{\lambda^{2s}}4\pi C^2(N,s)\frac{1}{\bar{r}^{2s-1}}\frac{\partial(\bar{r}^{2s}V(\bar{r},\bar{y}''))}{\partial\bar{r}} + o\Big(\frac{\ln\lambda}{\lambda^{2s}}\Big)\Big) = O\Big(\frac{k^2}{\lambda^{2s}}\Big).$$

This is

$$\frac{\partial(\bar{r}^{2s}V(\bar{r},\bar{y}''))}{\partial\bar{y}_i} = o(1), \quad i=3,$$

and

$$\frac{\partial(\bar{r}^{2s}V(\bar{r},\bar{y}''))}{\partial\bar{r}} = o(1).$$

By (B.2), we have

$$-\frac{D_1\ln\lambda}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \frac{D_3k^{2s}}{\lambda^{2s+1}} = o\Big(\frac{\ln\lambda}{\lambda^{2s+1}}\Big).$$

Similar to the proof of Theorem 1.1, we can prove Theorem 1.3. □

A Some Estimates

In this section, we give some essential estimates. For $x_i, x_j, y \in \mathbb{R}^N$, define $g_{ij}(y) = \frac{1}{(1+|y-x_i|)^\alpha(1+|y-x_j|)^\beta}$, where $x_i \neq x_j$, $\alpha > 0$ and $\beta > 0$ are two constants.

Lemma A.1. *For any constant $\gamma \in (0, \min(\alpha,\beta)]$, we have*

$$g_{ij}(y) \le \frac{C}{(1+|x_i-x_j|)^\gamma}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\gamma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\gamma}}\Big).$$

Proof. See the proof of [42, Lemma A.1]. □

Lemma A.2. *For any constant $0 < \vartheta < N-2s$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2s}}\frac{1}{(1+|z|)^{2s+\vartheta}}\,dz \le \frac{C}{(1+|y|)^\vartheta}.$$

Proof. See the proof of [26, Lemma 2.1]. □

Lemma A.3. *Let $\mu > 0$. For any constants $0 < \beta < N$, there exists a constant $C > 0$, independent of μ, such that*

$$\int_{\mathbb{R}^N\setminus B_\mu(y)} \frac{1}{|y-z|^{N+2s}}\frac{1}{(1+|z|)^\beta}\,dz \le C\Big(\frac{1}{(1+|y|)^{\beta+2s}} + \frac{1}{\mu^{2s}}\frac{1}{(1+|y|)^\beta}\Big).$$

Proof. Without loss of generality, we set $|y| \ge 2$, and let $d = \frac{|y|}{2}$. Then we have

$$\int_{\mathbb{R}^N\setminus B_\mu(y)} \frac{1}{|y-z|^{N+2s}}\frac{1}{(1+|z|)^\beta}\,dz \le \int_{B_d(0)} + \int_{B_d(y)\setminus B_\mu(y)} + \int_{\mathbb{R}^N\setminus(B_d(0)\cup B_d(y))} \frac{1}{|y-z|^{N+2s}}\frac{1}{(1+|z|)^\beta}\,dz.$$

By direct computation, we have

$$\int_{B_d(0)} \frac{dz}{|y-z|^{N+2s}(1+|z|)^\beta} \le \frac{C}{d^{N+2s}}\int_0^d \frac{r^{N-1}\,dr}{(1+r)^\beta} \le \frac{C}{d^{\beta+2s}}$$

and

$$\int_{B_d(y)\setminus B_\mu(y)} \frac{dz}{|y-z|^{N+2s}(1+|z|)^\beta} \le \frac{C}{d^\beta}\int_{B_d(y)\setminus B_\mu(y)} \frac{dz}{|y-z|^{N+2s}} \le \frac{C}{\mu^{2s}d^\beta}.$$

For $z \in \mathbb{R}^N \setminus (B_d(0)\cup B_d(y))$, we have $|y-z| \ge \frac{|y|}{2}$ and $|z| \ge \frac{|y|}{2}$. If $|z| \ge 2|y|$, then $|y-z| \ge |z| - |y| \ge \frac{|z|}{2}$, and if $|z| < 2|y|$, then $|y-z| \ge \frac{|y|}{2} > \frac{|z|}{4}$. Thus, we have

$$\int_{\mathbb{R}^N\setminus(B_d(0)\cup B_d(y))} \frac{dz}{|y-z|^{N+2s}(1+|z|)^\beta} \le C\int_{\mathbb{R}^N\setminus B_d(0)} \frac{dz}{(1+|z|)^\beta|z|^{N+2s}} \le \frac{C}{d^{\beta+2s}}.$$

The proof is complete. □

Lemma A.4. *Let $\rho > 0$. Suppose that $(y-x)^2 + t^2 = \rho^2$, $t > 0$ and $\alpha > N$. Then, for $0 < \beta < N$, we have*

$$\int_{\mathbb{R}^N} \frac{1}{(t+|z|)^\alpha}\frac{1}{|y-z-x|^\beta}\,dz \le C\Big(\frac{1}{(1+|y-x|)^\beta}\frac{1}{t^{\alpha-N}} + \frac{1}{(1+|y-x|)^{\alpha+\beta-N}}\Big). \tag{A.1}$$

Proof. The proof is similar to that of [27, Lemma A.3]. □

Lemma A.5. *Suppose that $(y-x)^2 + t^2 = \rho^2$. Then there exists a constant $C > 0$ such that*

$$|\tilde{Z}_{x_i,\lambda}| \le \frac{C}{\lambda^{\frac{N-2s}{2}}}\frac{1}{(1+|y-x_i|)^{N-2s}} \quad \text{and} \quad |\nabla\tilde{Z}_{x_i,\lambda}| \le \frac{C}{\lambda^{\frac{N-2s}{2}}}\frac{1}{(1+|y-x_i|)^{N-2s+1}}.$$

Proof. By Lemma A.4, we have

$$\begin{aligned}
|\tilde{Z}_{x_i,\lambda}(y,t)| &= \Big|\beta(N,s)\int_{\mathbb{R}^N} \frac{t^{2s}}{(|y-\xi|^2+t^2)^{\frac{N+2s}{2}}}\zeta(\xi)U_{x_i,\lambda}(\xi)\,d\xi\Big| \\
&= \Big|\beta(N,s)C(N,s)\int_{\mathbb{R}^N} \frac{t^{2s}}{(|y-\xi|^2+t^2)^{\frac{N+2s}{2}}}\zeta(\xi)\Big(\frac{\lambda}{1+\lambda^2|\xi-x_i|^2}\Big)^{\frac{N-2s}{2}}\,d\xi\Big| \\
&\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N} \frac{1}{(1+|z|)^{N+2s}}\frac{1}{(\lambda^{-1}+|y-tz-x_i|)^{N-2s}}\,dz \\
&\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N} \frac{t^{2s}}{(t+|z|)^{N+2s}}\frac{1}{(\lambda^{-1}+|y-z-x_i|)^{N-2s}}\,dz \\
&\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\frac{1}{(1+|y-x_i|)^{N-2s}}.
\end{aligned}$$

Note that, for $l = 1, \dots, N$,

$$\begin{aligned}
&\frac{\partial}{\partial y_l}\int_{\mathbb{R}^N}\frac{t^{2s}}{(|y-\xi|^2+t^2)^{\frac{N+2s}{2}}}\zeta(\xi)\Big(\frac{\lambda}{1+\lambda^2|\xi-x_i|^2}\Big)^{\frac{N-2s}{2}}\,d\xi\\
&\quad=\frac{1}{\lambda^{\frac{N-2s}{2}}}\frac{\partial}{\partial y_l}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\zeta(y-tz)\Big(\frac{1}{\lambda^{-2}+|y-tz-x_i|^2}\Big)^{\frac{N-2s}{2}}\,dz\\
&\quad=\frac{2s-N}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\zeta(y-tz)\frac{(y-tz-x_i)_l}{(\lambda^{-2}+|y-tz-x_i|^2)^{\frac{N-2s}{2}+1}}\,dz\\
&\qquad+\frac{1}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\frac{\partial\zeta(y-tz)}{\partial y_l}\frac{1}{(\lambda^{-2}+|y-tz-x_i|^2)^{\frac{N-2s}{2}}}\,dz,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^N}\frac{t^{2s}}{(|y-\xi|^2+t^2)^{\frac{N+2\sigma}{2}}}\Big(\frac{\lambda}{1+\lambda^2|\xi-x_i|^2}\Big)^{\frac{N-2s}{2}}\,d\xi\\
&\quad=\frac{N-2s}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\zeta(y-tz)\frac{\sum_{l=1}^N(y-tz-x_{k,L})_l z_l}{(\lambda^{-2}+|y-tz-x_i|^2)^{\frac{N-2s}{2}+1}}\,dz\\
&\qquad+\frac{1}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\frac{\nabla\zeta(y-tz)\cdot z}{(\lambda^{-2}+|y-tz-x_i|^2)^{\frac{N-2s}{2}}}\,dz.
\end{aligned}$$

Then, by the definition of ζ and (A.1), we have

$$\begin{aligned}
|\nabla\tilde{Z}_{x_i,\lambda}| &\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|)^{N+2s-1}}\frac{1}{(1+|y-tz-x_i|)^{N-2s+1}}\,dz\\
&\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\int_{\mathbb{R}^N}\frac{t^{2s-1}}{(t+|z|)^{N+2s-1}}\frac{1}{(1+|y-z-x_i|)^{N-2s+1}}\,dz\\
&\le \frac{C}{\lambda^{\frac{N-2s}{2}}}\frac{1}{(1+|y-x_i|)^{N-2s+1}}.
\end{aligned}$$

The proof is complete. □

For any $\delta > 0$, we define the following two sets

$$D_1 = \{Y = (y,t) : \delta < |Y - (r_0, y_0'', 0)| < 6\delta,\ t > 0\}$$

and

$$D_2 = \{Y = (y,t) : 2\delta < |Y - (r_0, y_0'', 0)| < 5\delta,\ t > 0\}.$$

Lemma A.6. *For any $\delta > 0$, there exists $\rho = \rho(\delta) \in (2\delta, 5\delta)$ such that when $N > 4s$,*

$$\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dS \le \frac{Ck\|\phi\|_*^2}{\lambda^\tau}, \tag{A.2}$$

and when $N = 3 = 4s$,

$$\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dS \le Ck^2\|\phi\|_*^2, \tag{A.3}$$

where C is a constant, dependent on δ.

Proof. We first consider the case $N > 4s$. By (A.1), for $(y,t) \in D_1$, we have

$$
\begin{aligned}
|\widetilde{\phi}(y,t)| &= \Big| \int_{\mathbb{R}^N} \beta(N,s) \frac{t^{2s}}{(|y-\xi|^2+t^2)^{\frac{N+2s}{2}}} \phi(\xi)\,d\xi \Big| \\
&\leq \frac{C\|\phi\|_* t^{2s}}{\lambda^\tau} \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{1}{(|z|+t)^{N+2s}} \frac{1}{|y-z-x_i|^{\frac{N-2s}{2}+\tau}}\,dz \\
&\leq \frac{C\|\phi\|_* t^{2s}}{\lambda^\tau} \sum_{i=1}^k \Big(\frac{1}{(1+|y-x_i|)^{\frac{N-2s}{2}+\tau}} \frac{1}{t^{2s}} + \frac{1}{(1+|y-x_i|)^{\frac{N+2s}{2}+\tau}} \Big) \\
&\leq \frac{C\|\phi\|_*}{\lambda^\tau} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2s}{2}+\tau}}.
\end{aligned}
\tag{A.4}
$$

Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ be a function with $\varphi(y,t)=1$ in D_2, $\varphi(y,t)=0$ in $\mathbb{R}^{N+1}\setminus D_1$ and $|\nabla\varphi| \leq C$. Note that $\widetilde{\phi}$ satisfies

$$
\begin{aligned}
&-\operatorname{div}(t^{1-2s}\nabla\widetilde{\phi}) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
-\lim_{t\to 0} t^{1-2s}\partial_t\widetilde{\phi}(y,t) &= -V(r,y'')\phi + (2_s^*-1)(Z_{\overline{r},\overline{y}'',\lambda})^{2_s^*-2}\phi + \mathcal{F}(\phi) + l_k + \sum_{l=1}^N c_l \sum_{i=1}^k Z_{x_i,\lambda}^{2_s^*-2} Z_{i,l}, \quad \text{in } \mathbb{R}^N.
\end{aligned}
$$

Multiplying $\varphi^2\widetilde{\phi}$ on the both sides of the equation and integrating by parts over D_1, we have

$$
\begin{aligned}
0 &= \int_{D_1} -\operatorname{div}(t^{1-2s}\nabla\widetilde{\phi})\varphi^2\widetilde{\phi}\,dy\,dt = \int_{D_1} t^{1-2s}\nabla\widetilde{\phi}\nabla(\varphi^2\widetilde{\phi})\,dy\,dt \\
&= \int_{D_1} t^{1-2s}\nabla\widetilde{\phi}(\varphi^2\nabla\widetilde{\phi} + 2\varphi\nabla\varphi\widetilde{\phi})\,dy\,dt.
\end{aligned}
$$

For any $\epsilon > 0$, we have

$$
\int_{D_1} t^{1-2s}\nabla\widetilde{\phi}\varphi\nabla\varphi\widetilde{\phi}\,dy\,dt \leq \epsilon \int_{D_1} t^{1-2s}|\nabla\widetilde{\phi}|^2\varphi^2\,dy\,dt + C(\epsilon)\int_{D_1} t^{1-2s}\widetilde{\phi}^2|\nabla\varphi|^2\,dy\,dt.
$$

Taking $\epsilon = \frac{1}{4}$ and using (A.4), we obtain that

$$
\begin{aligned}
\int_{D_2} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dy\,dt &\leq C\int_{D_1} t^{1-2s}\widetilde{\phi}^2|\nabla\varphi|^2 \leq \frac{C\|\phi\|_*^2}{\lambda^{2\tau}} \int_{D_1} t^{1-2s}\Big(\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2s}{2}+\tau}}\Big)^2 \\
&\leq \frac{C\|\phi\|_*^2}{\lambda^{2\tau}} \int_{D_1} \frac{t^{1-2s}k^2}{(1+|y-x_1|)^{N-2s+2\tau}} \leq \frac{Ck\|\phi\|_*^2}{\lambda^\tau}.
\end{aligned}
$$

By using the mean value theorem of integrals, there exists $\rho = \rho(\delta) \in (2\delta, 5\delta)$ such that

$$
\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dS \leq \frac{Ck\|\phi\|_*^2}{\lambda^\tau}.
$$

Now, we consider the case $N = 3 = 4s$. We have

$$
\begin{aligned}
|\widetilde{\phi}(y,t)| &\leq C\|\phi\|_* t^{2s} \sum_{i=1}^k \int_{\mathbb{R}^3} \frac{1}{(|z|+t)^{N+2s}} \frac{1}{|y-z-x_i|^{\frac{N-2s}{2}}}\,dz \\
&\leq C\|\phi\|_* t^{2s} \sum_{i=1}^k \Big(\frac{1}{(1+|y-x_i|)^{\frac{N-2s}{2}}} \frac{1}{t^{2s}} + \frac{1}{(1+|y-x_i|)^{\frac{N+2s}{2}}} \Big) \\
&\leq C\|\phi\|_* \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2s}{2}}}.
\end{aligned}
$$

This gives

$$\int_{D_2} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dy\,dt \le C\int_{D_1} t^{1-2s}\widetilde{\phi}^2|\nabla\varphi|^2 \le Ck^2\|\phi\|_*^2.$$

So, we have

$$\int_{\partial''\mathcal{B}_\rho^+} t^{1-2s}|\nabla\widetilde{\phi}|^2\,dS \le Ck^2\|\phi\|_*^2.$$

The proof is complete. □

B Energy Expansion

In this section, we give some estimates of the energy expansions for $\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi(\bar r,\bar y'',\lambda)), \frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\rangle$, $\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi(\bar r,\bar y'',\lambda)), \frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\bar r}\rangle$ and $\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi(\bar r,\bar y'',\lambda)), \frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\bar y''}\rangle$.

Lemma B.1. *If $N > 4s$, then*

$$\frac{\partial I(Z_{\bar r,\bar y'',\lambda})}{\partial\lambda} = k\Big(-\frac{2sB_1}{\lambda^{2s+1}}V(\bar r,\bar y'') + \sum_{j=2}^k \frac{B_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big),$$

where B_1 and B_2 are two positive constants.

Proof. By a direct computation, we have

$$\begin{aligned}
\frac{\partial I(Z_{\bar r,\bar y'',\lambda})}{\partial\lambda} &= \frac{\partial I(Z^*_{\bar r,\bar y'',\lambda})}{\partial\lambda} + O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big)\\
&= \int_{\mathbb{R}^N} V(y)Z^*_{\bar r,\bar y'',\lambda}\frac{\partial Z^*_{\bar r,\bar y'',\lambda}}{\partial\lambda} - \int_{\mathbb{R}^N}\Big((Z^*_{\bar r,\bar y'',\lambda})^{2_s^*-1} - \sum_{j=1}^k U^{2_s^*-1}_{x_j,\lambda}\Big)\frac{\partial Z^*_{\bar r,\bar y'',\lambda}}{\partial\lambda} + O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big)\\
&= I_1 - I_2 + O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big).
\end{aligned}$$

For the term I_1, by Lemma A.1, we can check that

$$\begin{aligned}
I_1 &= k\Big(\int_{\mathbb{R}^N} V(y)U_{x_1,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda} + O\Big(\frac{1}{\lambda}\int_{\mathbb{R}^N} U_{x_1,\lambda}\sum_{j=2}^k U_{x_j,\lambda}\Big)\Big)\\
&= k\Big(\frac{V(\bar r,\bar y'')}{2}\frac{\partial}{\partial\lambda}\int_{\mathbb{R}^N} U^2_{x_1,\lambda}\,dy + O\Big(\frac{1}{\lambda^{2s+1}}\sum_{j=2}^k\frac{1}{(\lambda|x_1-x_j|)^{N-4s-\sigma}}\Big) + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big)\\
&= k\Big(-\frac{sV(\bar r,\bar y'')}{\lambda^{2s+1}}\int_{\mathbb{R}^N} U^2_{0,1}\,dy + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big)\\
&= k\Big(-\frac{2sB_1V(\bar r,\bar y'')}{\lambda^{2s+1}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big),
\end{aligned}$$

where $B_1 = \frac{1}{2}\int_{\mathbb{R}^N} U^2_{0,1}\,dy > 0$ and $N - 4s - \sigma - \frac{N-4s}{N-2s} > 0$.

Next, we estimate I_2:

$$\begin{aligned}
I_2 &= k\int_{\Omega_1}\Big((Z^*_{\bar r,\bar y'',\lambda})^{2_s^*-1} - \sum_{j=1}^k U^{2_s^*-1}_{x_j,\lambda}\Big)\frac{\partial Z^*_{\bar r,\bar y'',\lambda}}{\partial\lambda}\\
&= k\Big(\int_{\Omega_1}(2_s^*-1)U^{2_s^*-2}_{x_1,\lambda}\sum_{j=2}^k U_{x_j,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big)\\
&= k\Big(-\sum_{j=2}^k\frac{B_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big),
\end{aligned}$$

for some constant $B_2 > 0$.

Thus, we obtain that

$$\frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\lambda} = k\Big(-\frac{2sB_1}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \sum_{j=2}^{k}\frac{B_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big).$$

The proof is complete. □

Lemma B.2. *If $N = 3 = 4s$, then*

$$\frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\lambda} = k\Big(-\frac{2sD_1\ln\lambda}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \sum_{j=2}^{k}\frac{D_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + o\Big(\frac{\ln\lambda}{\lambda^{2s+1+\sigma}}\Big)\Big),$$

where D_1 and D_2 are two positive constants.

Proof. We have

$$\begin{aligned}\frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\lambda} &= \int_{\mathbb{R}^3} V(y)Z_{\bar{r},\bar{y}'',\lambda}\frac{\partial Z_{\bar{r},\bar{y}'',\lambda}}{\partial\lambda} - \int_{\mathbb{R}^3}\Big((Z^*_{\bar{r},\bar{y}'',\lambda})^{2_s^*-1} - \sum_{j=1}^{k}U_{x_j,\lambda}^{2_s^*-1}\Big)\frac{\partial Z^*_{\bar{r},\bar{y}'',\lambda}}{\partial\lambda} + O\Big(\frac{k}{\lambda^{2s+1}}\Big)\\ &= I_1 - I_2 + O\Big(\frac{k}{\lambda^{2s+1}}\Big).\end{aligned}$$

For the term I_1, By Lemma A.1, we can check that

$$\begin{aligned}I_1 &= k\Big(\int_{B_\rho(x_1)} V(y)U_{x_1,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda} + O\Big(\frac{1}{\lambda}\int_{B_\rho(x_1)} U_{x_1,\lambda}\sum_{j=2}^{k}U_{x_j,\lambda} + \frac{1}{\lambda^{2s+1}}\Big)\Big)\\ &= k\Big(\frac{V(\bar{r},\bar{y}'')sC^2(3,s)}{\lambda^{2s+1}}\int_{B_\rho(x_1)}\frac{\lambda^3(1-\lambda^2|y-x_1|^2)}{(1+\lambda^2|y-x_1|^2)^{\frac{5}{2}}} + O\Big(\frac{k}{\lambda^{2s+1}}\Big)\Big)\\ &= k\Big(\frac{V(\bar{r},\bar{y}'')4\pi sC^2(3,s)}{\lambda^{2s+1}}\int_0^{\rho\lambda}\frac{(1-r^2)r^2}{(1+r^2)^{\frac{5}{2}}} + O\Big(\frac{k}{\lambda^{2s+1}}\Big)\Big)\\ &= k\Big(-2sD_1\frac{\ln\lambda}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + o\Big(\frac{\ln\lambda}{\lambda^{2s+1}}\Big)\Big),\end{aligned}$$

where

$$D_1 = 2\pi C^2(3,s) \quad\text{and}\quad \lim_{\lambda\to\infty}\frac{\int_0^{\rho\lambda}\frac{(1-r^2)r^2}{(1+r^2)^{5/2}}}{\ln\lambda} = -1.$$

Similar to the proof in Lemma B.1, we have

$$I_2 = k\Big(-\sum_{j=2}^{k}\frac{D_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + O\Big(\frac{1}{\lambda^{2s+1}}\Big)\Big),$$

for some constant $D_2 > 0$.

Thus, we obtain that

$$\frac{\partial I(Z_{\bar{r},\bar{y}'',\lambda})}{\partial\lambda} = k\Big(-\frac{2sD_1\ln\lambda}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \sum_{j=2}^{k}\frac{D_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + o\Big(\frac{\ln\lambda}{\lambda^{2s+1}}\Big)\Big).$$

The proof is complete. □

Lemma B.3. *If $N > 4s$, then we have*

$$\begin{aligned}\Big\langle I'(Z_{\bar{r},\bar{y}'',\lambda}+\phi), \frac{\partial Z_{\bar{r},\bar{y}'',\lambda}}{\partial\lambda}\Big\rangle &= k\Big(-\frac{2sB_1}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \sum_{j=2}^{k}\frac{B_2}{\lambda^{N-2s+1}|x_j-x_1|^{N-2s}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big)\\ &= k\Big(-\frac{2sB_1}{\lambda^{2s+1}}V(\bar{r},\bar{y}'') + \frac{B_3k^{N-2s}}{\lambda^{N-2s+1}} + O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big)\Big), \qquad \text{(B.1)}\end{aligned}$$

where B_1 and B_2 are the same constants as in Lemma B.1, $B_3 > 0$.

Proof. By symmetry, we have

$$\begin{aligned}\Big\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle &= \int_{\mathbb{R}^N}((-\Delta)^s u_k + V(r,y'')u_k-(u_k)_+^{2_s^*-1})\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\\
&= \Big\langle I'(Z_{\bar r,\bar y'',\lambda}),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle + k\Big\langle(-\Delta)^s\phi+V(r,y'')\phi-(2_s^*-1)Z_{\bar r,\bar y'',\lambda}^{2_s^*-2}\phi,\frac{\partial Z_{x_1,\lambda}}{\partial\lambda}\Big\rangle\\
&\quad-\int_{\mathbb{R}^N}((Z_{\bar r,\bar y'',\lambda}+\phi)_+^{2_s^*-1}-Z_{\bar r,\bar y'',\lambda}^{2_s^*-1}-(2_s^*-1)Z_{\bar r,\bar y'',\lambda}^{2_s^*-2}\phi)\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\\
&=: \Big\langle I'(Z_{\bar r,\bar y'',\lambda}),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle + kJ_1 - J_2.\end{aligned}$$

By (2.5) and (2.7), we have

$$J_1 = O\Big(\frac{\|\phi\|_*}{\lambda^{1+s+\sigma}}\Big) = O\Big(\frac{1}{\lambda^{2s+1+\sigma}}\Big).$$

Note that $(1+t)_+^\gamma - 1 - \gamma t = O(t^2)$ for all $t\in\mathbb{R}^N$ if $1<\gamma\le 2$, and $|(1+t)_+^\gamma-1-\gamma t|\le C(t^2+|t|^\gamma)$ for all $t\in\mathbb{R}^N$ if $\gamma>2$. So, if $2_s^*\le 3$, we have

$$\begin{aligned}|J_2| &= \Big|\int_{\mathbb{R}^N}((Z_{\bar r,\bar y'',\lambda}+\phi)_+^{2_s^*-1}-Z_{\bar r,\bar y'',\lambda}^{2_s^*-1}-(2_s^*-1)Z_{\bar r,\bar y'',\lambda}^{2_s^*-2}\phi)\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big|\\
&\le C\int_{\mathbb{R}^N} Z_{\bar r,\bar y'',\lambda}^{2_s^*-3}\phi^2\Big|\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big|\\
&\le \frac{C\|\phi\|_*^2}{\lambda}\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_j|)^{N-2s}}\Big)^{2_s^*-2}\Big(\sum_{i=1}^k\frac{\lambda^{\frac{N-2s}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2s}{2}+\tau}}\Big)^2\\
&\le \frac{C\|\phi\|_*^2}{\lambda}\int_{\mathbb{R}^N}\lambda^N\sum_{j=1}^k\frac{1}{(1+\lambda|y-x_j|)^{4s}}\sum_{i=1}^k\frac{1}{(1+\lambda|y-x_i|)^{N-2s+\tau}}\\
&\le \frac{Ck\|\phi\|_*^2}{\lambda} = O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big).\end{aligned}$$

If $2_s^*>3$, we have

$$|J_2|\le C\int_{\mathbb{R}^N}\Big(Z_{\bar r,\bar y'',\lambda}^{2_s^*-3}\phi^2\Big|\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big| + |\phi|^{2_s^*-1}\Big|\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big|\Big) = O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big).$$

Thus, we obtain

$$\Big\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle = \Big\langle I'(Z_{\bar r,\bar y'',\lambda}),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle + O\Big(\frac{k}{\lambda^{2s+1+\sigma}}\Big).$$

Combining this with Lemma B.1, we finish the proof. □

Lemma B.4. *If $N=3=4s$, then we have*

$$\begin{aligned}\Big\langle I'(Z_{\bar r,\bar y'',\lambda}+\phi),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle &= \Big\langle I'(Z_{\bar r,\bar y'',\lambda}),\frac{\partial Z_{\bar r,\bar y'',\lambda}}{\partial\lambda}\Big\rangle + O\Big(\frac{k}{\lambda^{2s+1}}\Big)\\
&= k\Big(-\frac{2sD_1\ln\lambda}{\lambda^{2s+1}}V(\bar r,\bar y'') + \frac{D_3k^{N-2s}}{\lambda^{N-2s+1}} + o\Big(\frac{\ln\lambda}{\lambda^{2s+1}}\Big)\Big),\end{aligned}\tag{B.2}$$

where D_1 is the same constants as in Lemma B.1 and $D_3>0$.

Proof. The proof is similar to that of Lemma B.3 . □

Note that $Z_{i,l} = O(\lambda Z_{x_i,\lambda})$, $l=2,\dots,N$. Similarly, we can prove the following lemma.

Lemma B.5. *If $N > 4s$, then we have*

$$\Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}+\phi), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{r}}\Big\rangle = \Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{r}}\Big\rangle + O\Big(\frac{k}{\lambda^{s+\sigma}}\Big)$$
$$= k\Big(\frac{B_1}{\lambda^{2s}}\frac{\partial V(\overline{r},\overline{y}'')}{\partial \overline{r}} + \sum_{j=2}^{k}\frac{B_2}{\overline{r}\lambda^{N-2s}|x_1-x_j|^{N-2s}} + O\Big(\frac{1}{\lambda^{s+\sigma}}\Big)\Big)$$

and

$$\Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}+\phi), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{y}_j''}\Big\rangle = \Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{y}_j''}\Big\rangle + O\Big(\frac{k}{\lambda^{s+\sigma}}\Big) = k\Big(\frac{B_1}{\lambda^{2s}}\frac{\partial V(\overline{r},\overline{y}'')}{\partial \overline{y}_j''} + O\Big(\frac{1}{\lambda^{s+\sigma}}\Big)\Big),$$

where B_1 and B_2 are the same constants as in Lemma B.1.

Lemma B.6. *If $N = 3 = 4s$, then we have*

$$\Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}+\phi), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{r}}\Big\rangle = k\Big(\frac{D_1\ln\lambda}{\lambda^{2s}}\frac{\partial V(\overline{r},\overline{y}'')}{\partial \overline{r}} + \sum_{j=2}^{k}\frac{D_2}{\overline{r}\lambda^{N-2s}|x_1-x_j|^{N-2s}} + O\Big(\frac{1}{\lambda^{s}}\Big)\Big)$$

and

$$\Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}+\phi), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial \overline{y}_j''}\Big\rangle = k\Big(\frac{D_1}{\lambda^{2s}}\frac{\partial V(\overline{r},\overline{y}'')}{\partial \overline{y}_j''} + O\Big(\frac{1}{\lambda^{s}}\Big)\Big),$$

where D_1 and D_2 are the same constants as in Lemma B.2.

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