Research Article

Joshua Flynn*

Sharp Caffarelli–Kohn–Nirenberg-Type Inequalities on Carnot Groups

https://doi.org/10.1515/ans-2019-2065
Received May 15, 2019; revised September 12, 2019; accepted September 15, 2019

Abstract: The main purpose of this paper is to establish several general Caffarelli–Kohn–Nirenberg (CKN) inequalities on Carnot groups G (also known as stratified groups). These CKN inequalities are sharp for certain parameter values. In case G is an Iwasawa group, it is shown here that the L^2-CKN inequalities are sharp for all parameter values except one exceptional case. To show this, generalized Kelvin transforms K_σ are introduced and shown to be isometries for certain weighted Sobolev spaces. An interesting transformation formula for the sub-Laplacian with respect to K_σ is also derived. Lastly, these techniques are shown to be valid for establishing CKN-type inequalities with monomial and horizontal norm weights.

Keywords: Caffarelli–Kohn–Nirenberg Inequalities, Carnot Groups, Stratified Groups, Heisenberg Group, Kelvin Transform, Iwasawa Groups

MSC 2010: 35R45, 35R03, 35A23

Communicated by: Guozhen Lu

1 Introduction

It was observed by Costa in [7] that the L^2-Caffarelli–Kohn–Nirenberg (CKN) inequalities [3] on $\mathbb{R}^N$ may be obtained by the elementary fact that $As^2 + Bs + C \geq 0$ holding for all $s \in \mathbb{R}$ implies $B^2 - 4AC \leq 0$. Indeed, by an application of integration by parts, Costa showed that

$$\int_{\mathbb{R}^N} \left| \frac{\nabla u}{|x|^b} + s \frac{\nabla |x|}{|x|^a} u \right|^2 dx \geq 0 \tag{1.1}$$

is equivalent to

$$As^2 - Bs + C \geq 0$$

when

$$A = \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx, \quad B = [N - (a + b + 1)] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx, \quad C = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx,$$

where $a, b \in \mathbb{R}$ are arbitrary, and $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$. (The integrand of (1.1) is written differently than that given in [7] to motivate the generalization given below; note also $|\nabla |x||^2 = 1$.) Consequently, by the elementary fact, the L^2-CKN inequalities follow:

$$\frac{|N - (a + b + 1)|}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx \leq \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx \right)^{\frac{1}{2}}, \tag{1.2}$$

***Corresponding author: Joshua Flynn,** Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA, e-mail: joshua.flynn@uconn.edu

where the constant $\frac{1}{2}|N-(a+b+1)|$ is sharp for a suitable range of $(a,b)\in\mathbb{R}^2$. From this inequality one may obtain continuous embeddings of weighted Sobolev spaces into weighted L^2-spaces. Moreover, by specific choices of $(a,b)\in\mathbb{R}^2$, one may obtain classical inequalities; e.g., Hardy's inequality follows since, if $a=1$, $b=0$, then

$$\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\,dx\leqslant\int_{\mathbb{R}^N}|\nabla u|^2\,dx.$$

It is important to point out that this proof technique cannot directly establish all sharp $L^rL^pL^q$-interpolation inequalities given in [3] by Caffarelli, Kohn, and Nirenberg (see also [20] for sharp constants in some cases):

$$\||x|^\gamma u\|_{L^r}\leq C\||x|^\alpha|\nabla u|\|^a_{L^p}\||x|^\beta u\|^{1-a}_{L^q}. \tag{1.3}$$

Indeed, the sharp constants of such inequalities are not such simple rational expressions of (a,b). However, as presented here, this proof technique combined with Hölder's inequality can establish such general non-L^2-interpolation inequalities for a specific range of r,p,q, and this range seems sharp for this proof technique (see Theorem 1.1).

It is important to note that the L^2-CKN inequalities of Costa comprise all L^2 inequalities obtainable from (1.3). This may be verified by taking $r=p=q=2$ in the constraints on the parameters (see [3]) and deriving (1.2).

There are also the higher order CKN inequalities

$$\||x|^\gamma\nabla^j u\|_{L^r}\leq C\||x|^\alpha\nabla^m u\|^a_{L^p}\||x|^\beta u\|^{1-a}_{L^q}$$

established by Lin [21], which seem possible to obtain by using this method for a suitable matrix replacement for the vectors ∇u and x, and the fractional CKN inequalities

$$\||x|^\gamma u\|_{L^r}\leq C|u|^a_{W^{s,p,\alpha}}\||x|^\beta u\|^{1-a}_{L^q}$$
$$|u|^p_{W^{s,p,\alpha}}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy,\qquad \alpha=\alpha_1+\alpha_2,$$

recently established by Nguyen and Squassina in [26], which seem unlikely to be obtained by using Costa's method since there is no suitable Leibniz rule for $(-\Delta)^{\frac{s}{2}}$. Recently, in [9], Dong established the existence of extremizers of the higher order CKN inequalities for a certain range of the parameters.

Another direction, which is explored in this paper, is to generalize these classical inequalities to analogous inequalities with different weights. In particular, there has been recent interest in studying inequalities with the monomial weights

$$x^A=|x_1|^{A_1}\cdots|x_N|^{A_N},\quad\text{where } A=(A_1,\dots,A_N)\in\mathbb{R}^N.$$

See for example [2, 4, 18, 19] and references therein. For example, one may try to establish a CKN-type inequality of the form

$$\|x^A u\|_{L^r}\leq C\|x^B|\nabla u|\|^a_{L^p}\|x^C u\|^{1-a}_{L^q},$$

where the exponents $A,B,C\in\mathbb{R}^N$.

Now, since the proof technique used in [7] does not decidedly rely on the Euclidean structure, it is natural to investigate this proof technique and resulting inequalities in other settings. For example, although approaching the problem differently, Garofalo and Lanconelli established in [12, Corollaries 2.1 and 2.2] the following Hardy-like inequality for the Heisenberg group H^n:

$$\left(\frac{Q-2}{2}\right)^2\int_{H^n}|u|^2\frac{\psi}{d^2}\,dH\leq\int_{H^n}|\nabla_H u|^2\,dH, \tag{1.4}$$

where $u\in C_0^\infty(H^n\backslash\{0\})$, Q is the homogeneous dimension, ∇_H is the horizontal gradient,

$$d=(|x|^4+t^2)^{\frac{1}{4}}$$

is the homogeneous norm of $z = (x, t)$, and

$$\psi = \frac{|x|^2}{d^2} = |\nabla_H d|^2 \tag{1.5}$$

is a geometrically significant density function. Recall that, for $\mathbb{R}^N$ with standard norm d, the density $\psi = |\nabla d|^2 = 1$. Using the Cauchy–Schwarz inequality, the uncertainty principle for H^n follows:

$$\frac{Q-2}{2} \int_{H^n} |u|^2 \psi \, dH \leq \Big(\int_{H^n} |\nabla_H u|^2 \, dH \Big)^{\frac{1}{2}} \Big(\int_{H^n} u \psi d^2 \, dH \Big)^{\frac{1}{2}}. \tag{1.6}$$

For more general Carnot groups G (see e.g., [12, 14, 16, 27, 28] and the references therein), the L^p-Hardy inequality takes the form

$$\frac{Q-p}{p} \|d^{-1} |\nabla d| u\|_{L^p(G)} \leq \|\nabla_H u\|_{L^p(G)}, \quad Q \geq 3,\ 1 < p < Q, \tag{1.7}$$

where Q is the homogeneous dimension, ∇_H the horizontal gradient, and d is an $\mathcal{L}$-gauge. More generally, Kombe established in [16] the following weighted Hardy-type inequalities on G:

$$\Big(\frac{Q+\alpha-2}{2} \Big)^2 \int_G d^{\alpha-2} |\nabla_H d|^2 u^2 \, dx \leq \int_G d^{\alpha} |\nabla_H u|^2 \, dx, \tag{1.8}$$

where $Q \geq 3$, $\alpha \in \mathbb{R}$, $u \in C_0^\infty(G \backslash \{0\})$.

We should mention, in higher generality, Lu established in [24, 25] higher order interpolation inequalities with general weights of a pairs $v \in A_p$ (A_p is the pth-order Muckenhoupt class), and w doubling for which a weighted Poincaré inequality for vector fields holds (see [11, 22, 23]). On a stratified group G with Hörmander basis $X_1, \ldots, X_N$, these inequalities take the form

$$\|X^i f\|_{L^q_w(G)} \leq C \|f\|_{L^p_v(G)}^{1-\frac{\lambda+i}{k}} \|X^k f\|_{L^p_v(G)}^{\frac{\lambda+i}{k}},$$

where $X^k u = (X^I u)_{\{I : d(I)=m\}}$, $I = (i_1, \ldots, i_N) \in \mathbb{N}^N$, $X^I = X_1^{i_1} \cdots X_N^{i_N}$, and L^p_v, L^q_w are weighted Lebesgue spaces. By taking special weights of v and w as power weights in terms of the homogeneous norm on the group G or some other type of special weights satisfying the Muckenhoupt condition, one can derive many Sobolev interpolation inequalities on the stratified group G as particular examples.

Therefore, this sub-elliptic setting presents a natural setting to investigate CKN-type inequalities which desirably subsume (1.4), (1.6), (1.7), and (1.8). Indeed, in [29], Dou, Han, and Zhang, generalized the CKN inequalities to Heisenberg-type groups:

$$\|\psi^{\frac{\alpha-\gamma}{2}} d^{\gamma} u\|_{L^r} \leq C \|d^{\alpha} X u\|_{L^p}^{a} \|\psi^{\frac{\alpha-\beta}{2}} d^{\beta} u\|_{L^q}^{1-a}, \tag{1.9}$$

where $\psi = \frac{|x|^2}{d^2}$ is as above. Note that, as is the case for (1.3), C is not generally explicit, and that restrictions are imposed on the parameters so that $u \in C_0^\infty(G)$ is permitted. Similar to what was done in [7], some of restrictions will be lifted in this paper by assuming $u \in C_0^\infty(G \backslash \{|\nabla_H d|^{q'} d \neq 0\})$ (q' defined below).

In this paper, it is shown by the methods in [5] and [7] that general and relatively explicit CKN-type inequalities may be established for Carnot groups G when $|x|$ and ∇ are replaced by an $\mathcal{L}$-gauge d and the horizontal gradient ∇_H of G, respectively. Written out, they take the form

$$\hat{C} \|u |\nabla_H d|^{\frac{2}{q+q'+1}} d^{-\frac{a+b+1}{q+q'+1}}\|_{L^{q+q'+1}(G)} \leq \|u |\nabla_H d|^{\frac{1}{q}} d^{-\frac{a}{q}}\|_{L^{2q}(G)}^{\frac{q+q'}{q+q'+1}} \||\nabla_H u| |\nabla_H d|^{-\frac{q'}{q}} d^{-b+\frac{aq'}{q}}\|_{L^{\frac{2q}{q-q'}}(G)}^{\frac{1}{q+q'+1}},$$

where $q > q' \geq 0$, $a, b, \in \mathbb{R}$, and $u \in C_0^\infty(G \backslash \{|\nabla_H d|^{q'} d \neq 0\})$ (see Theorem 1.1). Observe that these inequalities define a family of inequalities different than that of (1.9) – in fact, the families intersect only at $q' = 0$. These CKN-type inequalities play a role analogous to the role CKN inequalities in $\mathbb{R}^N$ play; i.e., they subsume certain classical inequalities such as Hardy's, and the corresponding uncertainty principle. Explicit CKN-type inequalities are then established for the Heisenberg-type groups introduced by Kaplan [15], including, in particular, the Heisenberg group. This is done by making use of explicit formulas for the $\mathcal{L}$-gauges which can be obtained from [15] by application of [1, Proposition 5.4.2].

Concerning the sharpness of these CKN inequalities, the best constant is given as

$$\inf_{u\in C_0^\infty(G\setminus\{0\})} \frac{\| u|\nabla_H d|^{\frac{1}{q}} d^{-\frac{a}{q}} \|_{L^{2q}(G)}^{\frac{q+q'}{q+q'+1}} \| |\nabla_H u| |\nabla_H d|^{-\frac{q'}{q}} d^{-b+\frac{aq'}{q}} \|_{L^{\frac{2q}{q-q'}}(G)}^{\frac{1}{q+q'+1}}}{\| u|\nabla_H d|^{\frac{2}{q+q'+1}} d^{-\frac{a+b+1}{q+q'+1}} \|_{L^{q+q'+1}(G)}}$$

and one should look for nonzero minimizers in appropriate weighted Sobolev spaces $H^{1,q,q'}_{a,b}(G)$ defined as the completion of $C_0^\infty(G\setminus\{0\})$ with respect to the norm

$$\|u\|_{H^{1,q,q'}_{a,b}} = \Big(\int_G u^{2q} |\nabla_H d|^2 d^{-2a}\, dH \Big)^{\frac{1}{2q}} + \Big(\int_G |\nabla_H u|^{\frac{2q}{q-q'}} |\nabla_H d|^{-\frac{2q'}{q-q'}} d^{\frac{2(-bq+aq')}{q-q'}}\, dH \Big)^{\frac{q-q'}{2q}}.$$

In case $q = 1$, $q' = 0$ (i.e., the L^2 case), the spaces $H^{1,q,q'}_{a,b}(G)$ will be denoted by $H^1_{a,b}(G)$; the norms are given by

$$\|u\|_{H^1_{a,b}} = \Big(\int_G u^2 |\nabla_H d|^2 d^{-2a} + |\nabla_H u|^2 d^{-2b}\, dH \Big)^{\frac{1}{2}}.$$

In the Euclidean L^2 case (1.2), it was pointed out and corrected by Catrina and Costa in [5] that the method in [7] does not actually establish a sharp inequality for all values of the parameters $(a, b) \in \mathbb{R}^2$. (For a heuristic as to why the argument does not yield a sharp constant, see the remark at the end of the paper.) To remedy this in [5], Catrina and Costa first established the sharp constants for radial functions by proving that a 1-dimensional Kelvin-like transform is an isometry of certain weighted Sobolev spaces, and then reduced the general case to the radial case by use of spherical harmonics. Unfortunately, an appropriate spherical harmonic theory is not available for general Carnot groups, and, therefore, to extend the results of [5], a spherical harmonic-free proof is necessary. In an attempt to establish such a proof, it is shown here that generalized Kelvin-like transforms are isometries for certain weighted Sobolev spaces on Iwasawa groups, even in general dimensions. This is inspired by Catrina and Costa's 1-dimensional proof, and Garofalo and Vassilev's result from [13] that the CR Kelvin transform is an isometry of certain Sobolev spaces on Iwasawa groups. To generalize the latter result to weighted Sobolev spaces, a result of Korányi from [17], and Cowling, Dooley, Korányi, and Ricci from [8], which shows that the Kelvin transform on Iwasawa groups intertwines with their sub-Laplacians, is needed. Unfortunately and quite interestingly, for Heisenberg-type groups, this intertwining property characterizes Iwasawa groups (see [8, Theorem 4.2]), and therefore this limits some of the results to Iwasawa groups. This suggests that solving the sharpness problem for all Carnot groups may be fairly nontrivial.

In preparation of stating the sharp constants and main results, we define the following subregions $\mathscr{A}, \mathscr{B} \subset \mathbb{R}^2$:

$$\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2, \quad \mathscr{A}_1 = \Big\{(a,b) : a < b+1,\ b \le \frac{Q-2}{2}\Big\}, \quad \mathscr{A}_2 = \Big\{(a,b) : a > b+1,\ b \ge \frac{Q-2}{2}\Big\},$$

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2, \quad \mathscr{B}_1 = \Big\{(a,b) : a > b+1,\ b \le \frac{Q-2}{2}\Big\}, \quad \mathscr{B}_2 = \Big\{(a,b) : a < b+1,\ b \ge \frac{Q-2}{2}\Big\},$$

and for $(a, b) \in \mathscr{A}$, let $\mathscr{Q}_{a,b} \subset \mathbb{R}^2$ be given by

$$\mathscr{Q}_{a,b} = \Big\{(q, q') \in \mathbb{R}^2 : \frac{2q}{1+q'-q} < \frac{2a-Q}{1+b-a}\Big\}.$$

The main results proved here may now be stated.

Theorem 1.1. *Let $+\infty > q > q' \ge 0$, let $a, b \in \mathbb{R}$ be arbitrary, let $\beta = b - a + 1$, and let d be a $\mathcal{L}$-gauge on G. Then there holds for $u \in C_0^\infty(G\setminus\{|\nabla_H d|^{q'} d \ne 0\})$ the inequality*

$$\hat{C} \| u|\nabla_H d|^{\frac{2}{q+q'+1}} d^{-\frac{a+b+1}{q+q'+1}} \|_{L^{q+q'+1}(G)} \le \| u|\nabla_H d|^{\frac{1}{q}} d^{-\frac{a}{q}} \|_{L^{2q}(G)}^{\frac{q+q'}{q+q'+1}} \| |\nabla_H u| |\nabla_H d|^{-\frac{q'}{q}} d^{-b+\frac{aq'}{q}} \|_{L^{\frac{2q}{q-q'}}(G)}^{\frac{1}{q+q'+1}}, \tag{1.10}$$

where

$$\hat{C} = \Big(\frac{|Q - (a+b+1)|}{q+q'+1} \Big)^{\frac{1}{q+q'+1}}.$$

If $(a, b) \in \mathscr{A}$, then the following hold:

(1) *If $1 + q' - q > 0$, then $\hat{C}$ is sharp for all $(q, q') \in \mathbb{R}^2$, and it is achieved by the functions $u(g)$ defined as follows:*
   - *if $\beta > 0$, $s > 0$, $c_1 > 0$, then*

$$u(g) = \begin{cases} (1 + q' - q)^{\frac{1}{1+q'-q}} (c_1 - \frac{s}{\beta} d(g)^{\beta})^{\frac{1}{1+q'-q}} & \text{if } c_1 - \frac{s}{\beta} d(g)^{\beta} > 0, \\ 0 & \text{if } c_1 - \frac{s}{\beta} d(g)^{\beta} \le 0, \end{cases} \tag{1.11}$$

   - *if $\beta < 0$, $s < 0$, $c_2 > 0$, then*

$$u(g) = \begin{cases} (1 + q' - q)^{\frac{1}{1+q'-q}} (c_2 - \frac{s}{\beta} d(g)^{\beta})^{\frac{1}{1+q'-q}} & \text{if } c_2 - \frac{s}{\beta} d(g)^{\beta} > 0, \\ 0 & \text{if } c_2 - \frac{s}{\beta} d(g)^{\beta} \le 0. \end{cases} \tag{1.12}$$

(2) *If $1 + q' - q < 0$, $\beta \neq 0$, then $\hat{C}$ is sharp for $(q, q') \in \mathscr{Q}_{a,b}$; moreover, $\hat{C}$ is achieved by the functions*

$$u(g) = (q - q' - 1)^{\frac{1}{1+q'-q}} \Big(c_3 + \frac{s}{\beta} d(g)^{\beta} \Big)^{\frac{1}{1+q'-q}}, \tag{1.13}$$

where $c_3 > 0$ and $\operatorname{sgn}(s) = \operatorname{sgn}(\beta)$.

(3) *If $1 + q' - q = 0$, then $\hat{C}$ is sharp for all $(q, q') \in \mathbb{R}^2$, and it is achieved by the functions*

$$u(g) = c_4 \exp\Big(-\frac{s}{\beta} d(g)^{\beta} \Big), \tag{1.14}$$

where $c_4 \in \mathbb{R}$, and $\operatorname{sgn}(s) = \operatorname{sgn}(\beta)$.

Some remarks concerning the parameters are in order. It is interesting to point out that in case $q - q' = 1$, the three L^p norms in (1.10) agree with the L^{2q} norm, and this is the only case when the extremizer for $(a, b) \in \mathscr{A}$ is known to be an exponential. It is also worth pointing out that, in case $1 + q' - q < 0$, q and q' are “far away” from each other, and this somehow presents a difficulty in proving sharpness. Next, since the proof for a sharp inequality in the case that $(a, b) \in \mathscr{B}$, $q = 1$, $q' = 0$, relies decidedly an L^2 theory (see [5] or the proof of Theorem 1.2), it seems that showing sharpness for general $(a, b) \in \mathscr{B}$, $(q, q') \in \mathbb{R}^2$ may be fairly nontrivial. Lastly, it will be shown that

$$s = \frac{[Q - (a + b + 1)]}{q + q' + 1} \frac{\int_G u^{q+q'+1} \frac{|\nabla_H d|^2}{d^{a+b+1}} \, dH}{\int_G u^{2q} \frac{|\nabla_H d|^2}{d^{2a}} \, dH}$$

when defined. See the proof of Theorem 1.1 to see where s comes from.

Note that in the Euclidean case, $|\nabla d| = 1$, and so the Euclidean CKN inequalities (1.10) are obtained from the general CKN inequalities (1.3) by specifying

$$\gamma = -\frac{a + b + 1}{q + q' + 1}, \quad \alpha = -b + \frac{aq'}{q}, \quad \beta = -\frac{a}{q},$$
$$r = q + q' + 1, \quad p = \frac{2q}{q - q'}, \quad q = 2q,$$
$$a = \frac{1}{q + q' + 1},$$

where the parameters on the left-hand side of the equalities are those of (1.3), and those on the right-hand side of the equalities are those of (1.10). One may verify by direct computation that, even in the general Carnot group case, these parameters satisfy the dimensional balance

$$\frac{1}{r} + \frac{\gamma}{Q} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{Q} \Big) + (1 - a)\Big(\frac{1}{q} + \frac{\beta}{Q} \Big)$$

as given in [3] and [29]. Lastly, the L^2-CKN inequalities (1.2) follow by taking $q = 1$, $q' = 0$.

Suppose now that G is a group of Heisenberg type. Then a general point $g \in G$ may be written as $g = (x, t)$, where x and t are respectively the horizontal and central coordinates. In this case, one has $|\nabla_H d|^2 = |x|^2 d^{-2}$

and d explicit, and so the corresponding CKN inequalities are explicit. If G is moreover an Iwasawa group, then sharp inequalities can be proved for all parameters values $(a, b) \in \mathscr{A} \cup \mathscr{B}$.

Theorem 1.2. *Let G be a Heisenberg-type group. Then, for $u \in C_0^\infty(G\backslash\{0\})$, and all $a, b \in \mathbb{R}$, there holds the inequality*

$$\hat{C} \int_G |u|^2 \frac{|x|^2}{d^{a+b+3}}\, dH \leq \Big(\int_G \frac{|\nabla_H u|^2}{d^{2b}}\, dH \Big)^{\frac{1}{2}} \Big(\int_G |u|^2 \frac{|x|^2}{d^{2a+2}}\, dH \Big)^{\frac{1}{2}}, \tag{1.15}$$

where

$$\hat{C} = \frac{|Q - (a + b + 1)|}{2}.$$

If G is an Iwasawa group, then the following hold:

(i) *For $(a, b) \in \mathscr{A}$, the sharp constant is*

$$C(Q, a, b) = \frac{|Q - (a + b + 1)|}{2},$$

and it is achieved by the functions

$$u(x) = D \exp\Big(-\frac{s d^{b+1-a}}{b + 1 - a}\Big),$$

where $\operatorname{sgn}(s) = \operatorname{sgn}(b - a + 1)$.

(ii) *For $(a, b) \in \mathscr{B}$, the constant $\hat{C}$ may be improved, and the sharp constant is*

$$C(Q, a, b) = \frac{|Q - (3b - a + 3)|}{2},$$

and it is achieved by the functions

$$u(x) = D d^{2b-Q+2} \exp\Big(\frac{s d^{b+1-a}}{b + 1 - a}\Big)$$

where $\operatorname{sgn}(s) = -\operatorname{sgn}(b - a + 1)$, *and* $D \in \mathbb{R}$.

Note that the case $a = b + 1$ (i.e., $(a, b) \notin \mathscr{A} \cup \mathscr{B}$) is still open. In [6], Catrina and Wang show that, for $a = b + 1$ and $G = \mathbb{R}^N$, inequality (1.15) is sharp and that the best constant is not achieved. Their proof uses a cylindrical coordinate change of variables and a study of the resulting Euler-Lagrange equation for the best constant. A generalization will likely need a transformation formula for the sub-Laplacian like that of Proposition 1.4.

Proving sharpness in case $(a, b) \in \mathscr{A}$ is straightforward and only requires solving a simple system of PDEs. To prove sharpness in case $(a, b) \in \mathscr{B}$, it is first shown that the σ-Kelvin transform

$$u(z) \mapsto K_\sigma u(g) := d^\sigma u(h(g)), \quad \sigma \in \mathbb{R},$$

where $h : G \to G$ is the CR inversion (defined below), is an isometry on the space $D^{1,2}(G, d^{-\sigma-Q+2}\, dH)$, which is the closure of $C_0^\infty(G\backslash\{0\})$ with respect to the norm

$$\|u\|_{D^{1,2}} = \Big(\int_G \frac{|\nabla_H u|^2}{d^{\sigma+Q-2}}\, dH(g) \Big)^2.$$

Note that Catrina and Wang considered such a generalized Kelvin transform to define a "modified inversion" symmetry. For an appropriate σ depending on (a, b), the σ-Kelvin transformation transforms the CKN inequality with parameters $(a, b) \in \mathscr{B}$ to an equivalent CKN inequality with parameter $(a', b') \in \mathscr{A}$, thereby allowing the sharp constant to be calculated.

The following proposition is effectively an extension of [5, Lemma 1] and [13, Theorem 8.1].

Proposition 1.3. *Let G be an Iwasawa group. For any $\sigma \in \mathbb{R}$, the mapping*

$$K_\sigma : D^{1,2}(G, d^{-\sigma-Q+2}\, dH) \to D^{1,2}(G, d^{-\sigma-Q+2}\, dH)$$

defined by $v(g) = K_\sigma u(g) = d(g)^\sigma u \circ h(g)$ *is a linear isometry; i.e.,*

$$\int_G \frac{|\nabla_H u(g)|^2}{d(g)^{\sigma+Q-2}}\, dH(g) = \int_G \frac{|\nabla_H v(g')|^2}{d(g')^{\sigma+Q-2}}\, dH(g').$$

Proposition 1.3 follows from the following transformation formula which captures how much K_σ fails to intertwine with the sub-Laplacian $\mathcal{L}$ on G.

Proposition 1.4. *Let G be an Iwasawa group, and $\sigma \in \mathbb{R}$. Then for $u \in C^2(G\backslash\{0\})$, there holds*

$$K_\sigma \mathcal{L} K_\sigma u = d^4 \mathcal{L} u - 2(\sigma + Q - 2) d^3 \langle \nabla u, \nabla d\rangle + \sigma(\sigma + Q - 2)|x|^2 u.$$

Observe that, if $\sigma = -Q + 2$, then the intertwining formula obtained by Korányi in [17] is recovered:

$$K_{-Q+2} \mathcal{L} K_{-Q+2} u = d^4 \mathcal{L} u,$$

where K_{-Q+2} is the CR Kelvin transform (actually Korányi states the formula for the Heisenberg group; see [8] for the general formula). Just as the CR Kelvin transform sends harmonic functions to harmonic functions, the σ-Kelvin transform relates harmonic functions $v = K_\sigma u$ satisfying $\mathcal{L} v = 0$ to solutions of the more general sub-elliptic equation

$$d^4 \mathcal{L} u - 2(\sigma + Q - 2) d^3 \langle \nabla u, \nabla d\rangle + \sigma(\sigma + Q - 2)|x|^2 u = 0.$$

As a corollary to Proposition 1.3, K_σ is an isometry on more general weighted Sobolev spaces since it is easily confirmed that

$$\int_G |u(g)|^\alpha d(g)^{-\frac{1}{2}(2Q+\alpha\sigma)}\, dH(g) = \int_G |K_\sigma u(g')|^\alpha d(g')^{-\frac{1}{2}(2Q+\alpha\sigma)}\, dH(g').$$

Compare with [13] when $\sigma = -Q + 2$ and $\alpha = \frac{2Q}{Q-2}$.

It is worth mentioning that Ruzhansky and Suragan proved in [28] the following CKN-type inequalities for stratified groups:

Theorem 1.5. *Let G be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(G\backslash\{x' \neq 0\})$, and all $1 < p < \infty$ we have*

$$\frac{|N-\gamma|}{p} \|u|x'|^{-\frac{\gamma}{p}}\|^p_{L^p(\mathbb{G})} \le \||\nabla_H u||x'|^{-\alpha}\|_{L^p(\mathbb{G})} \|u|x'|^{-\frac{\beta}{p-1}}\|^{p-1}_{L^p(\mathbb{G})}$$

where $\gamma = \alpha + \beta + 1$, $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$, and x' are the first stratum coordinates of $x \in G$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

If in the proof of Theorem 1.1 the homogeneous norm is replaced by $|x'|$, then Ruzhansky and Suragan's inequality is recovered for some range of the parameters. Actually, this CKN-type inequality follows directly from the Euclidean case proof since $\nabla_H |x'|$ agrees with the usual $\mathbb{R}^m$ gradient acting on $|x'|$.

Concerning more general CKN-type inequalities, several generalized CKN-type inequalities with horizontal monomial and horizontal norm weights may be established for Carnot groups by using the proof techinque of Theorem 1.1. For Heisenberg-type groups, CKN-type inequalities may be established with monomial and horizontal norm weights, where the monomials depend on both horizontal and vertical coordinates.

To set up the theorem, let G be a Carnot group, and, if $g \in G$, let $(x_1, \dots, x_N)$ denote the first stratum coordinates of g. Set $|x|^2 = x_1^2 + \cdots + x_N^2$. Given $A = (A_1, \dots, A_N) \in \mathbb{R}^N$, define the horizontal monomials $x^A = |x_1|^{A_1} \cdots |x_N|^{A_N}$. Let $|A| = A_1 + \cdots + A_N$, noting that this number may have any sign.

Proposition 1.6. *Let G be a Carnot group, and let $A, B \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$, $j = 1, \dots, N$, and $0 \le q' < q < \infty$. Let $u \in C_0^\infty(G)$ be compactly supported where the weights are defined. Then:*

(i) *There holds*

$$\hat{C}\|x^{\frac{A+B}{q+q'+1}} u\|_{L^{q+q'+1}(G)} \le \|x^{\frac{B}{q}} |x|^{\frac{1}{q}} u\|^{\frac{q+q'}{q+q'+1}}_{L^{2q}(G)} \|x^{A-\frac{q'B}{q}} |x|^{-\frac{q'}{q}} \nabla u\|^{\frac{1}{q+q'+1}}_{L^{\frac{2q}{q-q'}}(G)},$$

where

$$\hat{C} = \Big(\frac{|A| + |B| + N}{q + q' + 1}\Big)^{\frac{1}{q+q'+1}}.$$

(ii) *There holds*

$$\hat{C}\|ux^{\frac{A+B}{q+q'+1}}|x_j|^{\frac{\alpha-1}{q+q'+1}}\|_{L^{q+q'+1}} \leq \|x^{A-\frac{q'B}{q}}|x_j|^{\frac{q'\alpha}{q}}\nabla u\|_{L^{\frac{2q}{q-q'}}}^{\frac{1}{q+q'+1}}\|ux^{\frac{B}{q}}|x_j|^{\alpha}\|_{L^{2q}}^{\frac{q+q'}{q+q'+1}},$$

where

$$\hat{C} = \left(\frac{\alpha + A_j + B_j}{q + q' + 1}\right)^{\frac{1}{q+q'+1}}.$$

Now suppose G is a Heisenberg-type group with the coordinates $g = (x, t)$ as above. We define, for a vector $A = (A_{11}, \dots, A_{m1}, A_{12}, \dots, A_{n2}) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$g^A = |x_1|^{A_{11}} \cdots |x_m|^{A_{m1}}|t_1|^{A_{12}} \cdots |t_n|^{A_{n2}},$$
$$|A|_1 = A_{11} + \cdots + A_{m1}$$

nothing that $|A|_1$ may have any sign. For sake of concreteness, only the L^2 case is reported. More general exponents may be obtained by Hölder's inequality as is done for the other CKN inequalities in this paper.

Proposition 1.7. *Let G be a Heisenberg-type group, and let $A, B \in \mathbb{R}^m \times \mathbb{R}^n$. Then, for $u \in C_0^\infty(G \backslash \{g^A, g^B \neq 0\})$, there holds*

$$\frac{||A|_1 + |B|_1 + m|}{2}\|g^{\frac{A+B}{2}}u\|_{L^2(G)}^2 \leq \|g^A \nabla u\|_{L^2(G)}\|g^B|x|u\|_{L^2(G)}.$$

The proof presented here relies decidedly on a characterizing property of Heisenberg-type groups, and so whether or not such inequalities hold of Carnot groups is currently unclear. Indeed, vector fields of Carnot groups acting on monomials depending on horizontal and vertical coordinates is considerably complicated.

Lastly, this paper is organized as follows. First, definitions and notions for Carnot groups, such as Heisenberg-type groups, Iwasawa groups, $\mathcal{L}$-gauges, etc., are recalled in Section 2. Then, in Section 3, the proofs of the results are given. At last, a remark is given to provide a heuristic as to why $A(u)s^2+B(u)s+C(u) \geq 0$ holding for all $s \in \mathbb{R}$, and all u in some class, does not necessarily imply $\frac{1}{4}B(u)^2 \leq A(u)C(u)$ is sharp.

2 Preliminaries

The following closely follows [1]. First definitions and notions are recalled for Heisenberg-type groups, then for Iwasawa groups, and then for general Carnot groups. Then properties of $\mathcal{L}$-gauges are recalled. Lastly, dH will always denote the standard Haar measure on a given Carnot group.

Let $\mathfrak{g}$ be a Heisenberg-type algebra; i.e., $\mathfrak{g}$ is a finite-dimensional real Lie algebra with inner product $\langle \cdot, \cdot \rangle$ satisfying

(i) $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, and $\mathfrak{z}^\perp$ its $\langle \cdot, \cdot \rangle$-orthogonal complement,

(ii) the map $J_z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_Z(U), w \rangle = \langle Z, [U, W] \rangle, \quad U, W \in \mathfrak{z}^\perp,$$

is an isometry for $\langle Z, Z \rangle = 1$, $Z \in \mathfrak{z}$.

Elements of $\mathfrak{v} := \mathfrak{z}^\perp$, and of $\mathfrak{z}$ are respectively called horizontal, and vertical directions. If G is a simply connected Lie group with Heisenberg-type algebra $\mathfrak{g}$, then G is called a Heisenberg-type group.

Let G be a Heisenberg-type group, and give G the usual stratified coordinates: identify $G \cong \mathbb{R}^m \times \mathbb{R}^n$ so that, if $g \in G$, then g may be written as $g = (x, t)$ where $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, and $(0, t)$ is in the center $\mathrm{Cent}(G) = \{(0, t) \in G\}$ of G. Correspondingly, let ∂_{x_j}, $j = 1, \dots, m$, and ∂_{t_s}, $s = 1, \dots, n$, denote the standard coordinate derivatives. The vector fields

$$X_j = \partial_{x_j} + 2\sum_{s=1}^{n}\sum_{i=1}^{m} U_{j,i}^{(s)} x_i \partial_{t_s}, \quad j = 1, \dots, m,$$

are left-invariant vector fields belonging to $\mathfrak{g}$ which agree with ∂_{x_j}, $j = 1, \dots, m$, respectively, at the origin. In fact, they form a Hörmander basis of step 2 for $\mathfrak{g}$; i.e., $X_1, \dots, X_m$ and their commutators of length 2 form a basis for $\mathfrak{g}$. Here, $U^{(s)} = (U_{i,j}^{(s)})$, $i, j = 1, \dots, m$, $s = 1 \dots, n$, are skew-symmetric orthogonal matrices satisfying anticommutativity: $U^{(s)}U^{(s')} + U^{(s')}U^{(s)} = 0$ for $s, s' = 1, \dots, n$ with $s \neq s'$. Note that these properties of $U^{(s)}$ characterize the Heisenberg-type groups. The horizontal gradient, divergence, and sub-Laplacian of G are then defined by

$$\nabla_H u = (X_1 u, \dots, X_m u),$$
$$\operatorname{div}_H(v_1, \dots, v_m) = X_1 v_1 + \dots + X_m v_m,$$
$$\mathcal{L} = \sum_{j=1}^{N} X_j^2.$$

A natural homogeneous norm on G is given by

$$|(x, t)| = (|x|^4 + |t|^2)^{\frac{1}{4}}, \tag{2.1}$$

where $|\cdot|$ denotes the respective Euclidean norms of $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$. Lastly, let $Q = m + 2n$ denote the homogeneous dimension of G.

An important subclass of Heisenberg-type groups is the class of Iwasawa N-groups, which is comprised of the nilpotent parts of the Iwasawa decompositions $G = KAN$ for rank one real semisimple Lie groups G; namely, the nilpotent parts for the groups $\mathrm{SO}(1, n)$, $\mathrm{SU}(1, n)$, $\mathrm{Sp}(1, n)$, and $\mathrm{F}_{4(-20)}$. In [8], Cowling, Dooley, Korányi, and Ricci characterized Iwasawa groups according to the so-called J^2-condition and the geometric properties equivalent to it. Here, G satisfies the J^2-condition if and only if, for any $X \in \mathfrak{v}$, and $Z, Z' \in \mathfrak{z}$ such that $\langle Z, Z' \rangle = 0$, there exists a $Z'' \in \mathfrak{z}$ satisfying $J_Z J_{Z'} X = J_{Z''} X$. Moreover, if $\bar{A}(x, t) = |x|^2 - J_t$, then the CR inversion

$$h(x, t) = (-\bar{A}(x, t) d(x, t)^{-4} x, -d(x, t)^{-4} t),$$

and the CR Kelvin transform $K_{-Q+2} u = d^{-Q+2} f \circ h$ are such that K_{-Q+2} preserves harmonicity if and only if G satisfies the J^2-condition, i.e., if and only if G is an Iwasawa group. Note that h preserves the gauge ball $\{d = 1\}$, that $A(x, t) = |x|^2 + J_t$ satisfies $A(x, t)\bar{A}(x, t) = d^4$, and that $d \circ h = d^{-1}$. Moreover, if $h(x, t) = (h_1(x), h_2(t))$, then

$$|h_1(x, t)| = \frac{|x|}{d(x, t)^2},$$
$$|h_2(x, t)| = \frac{|t|}{d(x, t)^4},$$

and, for Iwasawa groups, there holds

$$dH \circ h(g) = d(g)^{-2Q}\, dH(g), \tag{2.2}$$

but a formula for Heisenberg-type groups in general is not known.

More generally, a (finite-dimensional and real) stratified Lie algebra $\mathfrak{g}$ of step r is one with subspaces $V_1, \dots, V_r$ satisfying

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_r, \qquad [V_1, V_i] = V_{i+1}, \quad i = 1, \dots, r-1, \qquad [V_1, V_r] = 0.$$

Let $N_j = \dim V_j$. If G is a simply connected Lie group with a stratified Lie algebra $\mathfrak{g}$, then G is called a Carnot group. By the exponential map, G may be identified with $\mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_r}$, and a point $g \in G$ may be identified with $(x^{(1)}, \dots, x^{(r)})$, where $x^{(j)} \in \mathbb{R}^{N_j}$. Dilations on G are then defined by $\delta_\lambda(g) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$, and a symmetric homogeneous norm is a function $d : G \to \mathbb{R}$ which satisfies

(i) $d(g) = 0$ if and only if $g = 0$,
(ii) $d(g^{-1}) = d(g)$, and
(iii) $d(\delta_\lambda g) = \lambda d(g)$.

For convenience, write $N = N_1$, and the components of $x^{(1)}$ will be denoted by $x_1, \dots, x_N$.

The left-invariant vector fields of the first stratum which agree with the coordinate derivatives $\partial_{x_1}, \dots, \partial_{x_N}$ at the origin, and which form a Hörmander basis for the Lie algebra $\mathfrak{g}$ of G are given by

$$X_j = \partial_{x_j} + \sum_{h=2}^{r} \sum_{k=1}^{N_h} a_{j,k}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \partial_{x_k^{(h)}}, \quad j = 1, \dots, N,$$

where $a_{j,k}^{(h)}$ are degree $h-1$ polynomials which are homogeneous with respect to the dilation δ_λ. The horizontal gradient, divergence, and sub-Laplacian of G are then respectively given by

$$\nabla_H u = (X_1 u, \dots, X_N u),$$
$$\mathrm{div}_H(v_1, \dots, v_N) = X_1 v_1 + \cdots + X_N v_N,$$
$$\mathcal{L} = \sum_{j=1}^{N} X_j^2.$$

For notational convenience, ∇_H will henceforth be denoted by ∇.

While all homogeneous norms are equivalent, the focus is on the so-called $\mathcal{L}$-gauges, which are symmetric homogeneous norms d smooth away from the origin, and satisfying

$$\mathcal{L}(d^{2-Q}) = 0$$

in $G\backslash\{0\}$. These homogeneous norms are unique up to positive multiplicative constant. Also, in case G is a Heisenberg-type group, d is given by (2.1). Henceforth, d will always denote an $\mathcal{L}$-gauge.

It will be necessary to consider $\mathcal{L}$-radial functions, which are functions $u : G\backslash\{0\} \to \mathbb{C}$ satisfying

$$u(x) = f(d(x))$$

for a suitable $f : (0, \infty) \to \mathbb{C}$. If u is a smooth $\mathcal{L}$-radial function, then (see [1])

$$\mathcal{L}u(x) = \mathcal{L}f(d) = |\nabla d|^2\Big(f''(d) + \frac{Q-1}{d} f'(d)\Big).$$

In particular,

$$\mathcal{L}d = (Q-1)\frac{|\nabla d|^2}{d} = (Q-1)d\psi, \tag{2.3}$$

where $\psi = \frac{|\nabla d|^2}{d^2}$ is the density function given in (1.5). In case G is a Heisenberg-type group, $|\nabla d|^2 = |x|^2 d^{-2}$, and so $\psi = |x|^2 d^{-4}$.

The following polar integration for Carnot groups will be used (see [10, Proposition 1.15]): for a homogeneous norm $|\cdot|$ on G, set $S = \{|x| = 1\}$. Then there is a unique Radon measure σ on S such that, for all $u \in L^1(G)$,

$$\int_G u(g)\, dH(g) = \int_0^\infty \int_S u(ry) r^{Q-1}\, d\sigma(y)\, dr. \tag{2.4}$$

Lastly, $A \lesssim B$ will mean there is an absolute constant C such that $A \le CB$, and $A \sim B$ will mean $A \lesssim B$ and $B \lesssim A$ hold.

3 Proofs of Main Results

Proofs of Propositions 1.3 and 1.4. By density, it is sufficient to consider $u \in C_0^\infty(G\backslash\{0\})$. Letting $u^* = K_\sigma u$, it is to be shown that

$$\int_G |\nabla u(g)|^2 d(g)^{-\alpha-Q+2}\, dH(g) = \int_G |\nabla u^*(g')|^2 d(g')^{-\alpha-Q+2}\, dH(g').$$

Integration by parts yields

$$\begin{aligned}\int_G \langle \nabla u(g), d(g)^{-\sigma-Q+2}\nabla u(g)\rangle\, dH(g) &= -\int_G u(g)\langle \nabla d(g)^{-\sigma-Q+2}, \nabla u(g)\rangle\, dH(g) - \int_G u(g) d^{-\sigma-Q+2}\mathcal{L}u(g)\, dH(g)\\ &= \int_G u^2(g)\mathcal{L}(d(g)^{-\sigma-Q+2})\, dH(g) + \int_G u(g)\langle \nabla d(g)^{-\sigma-Q+2}, \nabla u(g)\rangle\, dH(g)\\ &\quad - \int_G u(g) d(g)^{-\sigma-Q+2}\mathcal{L}u(g)\, dH(g),\end{aligned}$$

and so, using

$$|\nabla d(g)|^2 = \frac{|x|^2}{d(g)^2} = \frac{|y|^2 d(g)^4}{d(g)^2} = \frac{|y|^2}{d(g')^2} = |\nabla d(g')|^2,$$
$$\mathcal{L} d^{-\sigma-Q+2} = \sigma(\sigma+Q-2)d^{-\sigma-Q}|\nabla d|^2,$$

where $g=(x,t)$, and $g'=(y,s)=h(g)$, there holds

$$\begin{aligned}\int_G u(g)\langle \nabla d(g)^{-\sigma-Q+2}, \nabla u(g)\rangle\, dH(g) &= -\frac{1}{2}\int_G u^2(g)\mathcal{L}(d(g)^{-\sigma-Q+2})\, dH(g)\\ &= -\frac{1}{2}\sigma(\sigma+Q-2)\int_G u^2(g)|\nabla d(g)|^2 d(g)^{-\sigma-Q}\, dH(g),\end{aligned}$$

and whence, using the Jacobian (2.2) of h,

$$\begin{aligned}\int_G u(g)\langle \nabla d(g)^{-\sigma-Q+2}, \nabla u(g)\rangle\, dH(g) &= -\frac{1}{2}\sigma(\sigma+Q-2)\int_G u^2(h(g'))|\nabla d(g')|^2 d(g')^{\sigma-Q}\, dH(g')\\ &= -\frac{1}{2}\sigma(\sigma+Q-2)\int_G (u^*(g'))^2|\nabla d(g')|^2 d(g')^{-\sigma-Q}\, dH(g')\\ &= \int_G u^*(g')\langle \nabla d(g')^{-\sigma-Q+2}, \nabla u^*(g')\rangle\, dH(g').\end{aligned}$$

It is left to be shown that

$$\int_G u(g)d(g)^{-\sigma-Q+2}\mathcal{L}u(g)\, dH(g) = \int_G u^*(g')d(g')^{-\sigma-Q+2}\mathcal{L}(u^*(g'))\, dH(g'),$$

which will follow from Proposition 1.4 and integration by parts. To see that Proposition 1.4 holds, observe that, since G is assumed to be an Iwasawa group, there holds (see [8, p. 27]) for $f \in C^2(G\backslash\{0\})$,

$$d^{-Q+2}\mathcal{L}(d^{-Q+2} f\circ h)\circ h = d^4 \mathcal{L} f,$$

and so, by taking $f = d^{-\sigma-Q+2}u$, one finds

$$\begin{aligned}d^{\sigma}\mathcal{L}(d^{\sigma}u\circ h)\circ h &= d^{\sigma+Q-2}d^{-Q+2}\mathcal{L}(d^{-Q+2}(d^{-\sigma-Q+2}u)\circ h)\circ h\\ &= d^{\sigma+Q+2}\mathcal{L}(d^{-\sigma-Q+2}u)\\ &= d^4\mathcal{L}u + ud^{\sigma+Q+2}\mathcal{L}(d^{-\sigma-Q+2}) + 2d^{\sigma+Q+2}\langle \nabla u, \nabla d^{-\sigma-Q+2}\rangle. \end{aligned} \tag{3.1}$$

Proposition 1.4 follows from (2.3) and $|\nabla d|^2 = |x|^2 d^{-2}$ since

$$\mathcal{L} d^{-\sigma-Q+2} = |x|^2\sigma(\sigma+Q-2)d^{-\sigma-Q-2}$$

and so

$$K_\sigma \mathcal{L} K_\sigma u = d^4\mathcal{L}u - 2(\sigma+Q-2)d^3\langle \nabla u, \nabla d\rangle + \sigma(\sigma+Q-2)|x|^2 u.$$

Consequently, by (3.1), it follows that

$$\begin{aligned}\int_G u(g)d(g)^{-\sigma-Q+2}\mathcal{L}u(g)\, dH(g) &= \int_G u(g)d(g)^{-\sigma-Q-2}d(g)^4\mathcal{L}u\, dH(g)\\ &= \int_G u(g)d(g)^{-Q-2}\mathcal{L}(d(g)^{\sigma}u\circ h)\circ h\, dH(g) - \int_G u^2(g)\mathcal{L}(d(g)^{-\sigma-Q+2})\, dH(g)\\ &\quad - 2\int_G \langle u(g)\nabla u(g), \nabla d(g)^{-\sigma-Q+2}\rangle\, dH(g).\end{aligned}$$

But

$$\int_G \langle u(g)\nabla u(g), \nabla d(g)^{-\sigma-Q+2}\rangle\, dH(g) = -\int_G u^2(g)\mathcal{L}(d(g)^{-\sigma-Q+2})\, dH(g) - \int_G \langle u(g)\nabla u(g), \nabla d(g)^{-\sigma-Q+2}\rangle\, dH(g),$$

whence

$$-2\int_G \langle u(g)\nabla u(g), \nabla d(g)^{-\sigma-Q+2}\rangle\, dH(g) = \int_G u^2(g)\mathcal{L}(d(g)^{-\sigma-Q+2})\, dH(g),$$

and

$$\begin{aligned}\int_G u(g)d(g)^{-\sigma-Q+2}\mathcal{L}u(g)\, dH(g) &= \int_G u(g)d(g)^{-Q-2}\mathcal{L}(d(g)^{\sigma}u\circ h)\circ h\, dH(g)\\ &= \int_G u\circ h(g')d(g')^{-Q+2}\mathcal{L}u^*\, dH(g')\\ &= \int_G u^*(g')d(g')^{-\sigma-Q+2}\mathcal{L}u^*(g')\, dH(g').\end{aligned}$$

Propositions 1.3 and 1.4 are thus proven. □

Proof of Theorem 1.1. Note that, since $|u|$ is differentiable almost everywhere, and since $|\nabla|u|| = |\nabla u|$ almost everywhere, u may be assumed without loss of generality (and sake of notational convenience) to be nonnegative. Observe that, for all $s \in \mathbb{R}$,

$$\int_G \left|u^{q'}\frac{\nabla u}{d^b} + su^q\frac{\nabla d}{d^a}\right|^2 dH = s^2\int_G u^{2q}\frac{|\nabla d|^2}{d^{2a}}\, dH + 2s\int_G u^{q+q'}\frac{\nabla u\cdot\nabla d}{d^{a+b}}\, dH + \int_G u^{2q'}\frac{|\nabla u|^2}{d^{2b}}\, dH \geq 0.$$

Let

$$I = \int_G u^{q+q'}\frac{\nabla u\cdot\nabla d}{d^{a+b}}\, dH.$$

Then, by integration by parts, one has

$$I = -(q+q')I - \int_G u^{q+q'+1}\operatorname{div}_H\frac{\nabla d}{d^{a+b}}\, dH.$$

Now, by (2.3), compute

$$\begin{aligned}\operatorname{div}_H\frac{\nabla d}{d^{a+b}} &= d^{-a-b}\mathcal{L}d + \langle\nabla d, \nabla d^{-a-b}\rangle\\ &= (Q-1)d^{-a-b-1}|\nabla d|^2 + (-a-b)d^{-a-b-1}|\nabla d|^2\\ &= (Q-(a+b+1))d^{-a-b-1}|\nabla d|^2.\end{aligned}$$

Therefore

$$I = -\frac{[Q-(a+b+1)]}{q+q'+1}\int_G u^{q+q'+1}\frac{|\nabla d|^2}{d^{a+b+1}}\, dH.$$

Consequently, for all $s \in \mathbb{R}$, there holds

$$As^2 - Bs + C \geq 0$$

with

$$\begin{aligned}A &= \int_G u^{2q}\frac{|\nabla d|^2}{d^{2a}}\, dH,\\ B &= 2\frac{[Q-(a+b+1)]}{q+q'+1}\int_G u^{q+q'+1}\frac{|\nabla d|^2}{d^{a+b+1}}\, dH,\\ C &= \int_G u^{2q'}\frac{|\nabla u|^2}{d^{2b}}\, dH.\end{aligned}$$

Therefore, since $As^2 - Bs + C \geq 0$ holding for all s implies $\frac{1}{2}|B| \leq A^{\frac{1}{2}}C^{\frac{1}{2}}$, there holds

$$\frac{|Q-(a+b+1)|}{q+q'+1}\int_G u^{q+q'+1}\frac{|\nabla d|^2}{d^{a+b+1}}\, dH \leq \left(\int_G u^{2q}\frac{|\nabla d|^2}{d^{2a}}\, dH\right)^{\frac{1}{2}}\left(\int_G u^{2q'}\frac{|\nabla u|^2}{d^{2b}}\, dH\right)^{\frac{1}{2}}.$$

Using Hölder's inequality on the right most integral results in

$$\begin{aligned}\Big(\int_G u^{2q'}\frac{|\nabla u|^2}{d^{2b}}\,dH\Big)^{\frac12} &= \Big(\int_G u^{2q'}\Big[\frac{|\nabla d|^2}{d^{2a}}\Big]^{\frac{q'}{q}}|\nabla u|^2|\nabla d|^{-\frac{2q'}{q}}d^{-2b+\frac{2aq'}{q}}\,dH\Big)^{\frac12}\\ &\le \Big(\int_G u^{2q}\frac{|\nabla d|^2}{d^{2a}}\,dH\Big)^{\frac{q'}{2q}}\Big(\int_G |\nabla u|^{\frac{2q}{q-q'}}|\nabla d|^{-\frac{2q'}{q-q'}}d^{\frac{-2bq+2aq'}{q-q'}}\,dH\Big)^{\frac{q-q'}{2q}}.\end{aligned}$$

Therefore,

$$\frac12|B| \le \Big(\int_G u^{2q}\frac{|\nabla d|^2}{d^{2a}}\,dH\Big)^{\frac{q+q'}{2q}}\Big(\int_G |\nabla u|^{\frac{2q}{q-q'}}|\nabla d|^{-\frac{2q'}{q-q'}}d^{\frac{-2bq+2aq'}{q-q'}}\,dH\Big)^{\frac{q-q'}{2q}}.$$

Consequently,

$$\hat{C}\|u|\nabla d|^{\frac{2}{q+q'+1}}d^{-\frac{a+b+1}{q+q'+1}}\|_{L^{q+q'+1}(G)} \le \|u|\nabla d|^{\frac1q}d^{-\frac aq}\|^{\frac{q+q'}{q+q'+1}}_{L^{2q}(G)}\||\nabla u||\nabla d|^{-\frac{q'}{q}}d^{-b+\frac{aq'}{q}}\|^{\frac{1}{q+q'+1}}_{L^{\frac{2q}{q-q'}}(G)},$$

where

$$\hat{C} = \Big(\frac{|Q-(a+b+1)|}{q+q'+1}\Big)^{\frac{1}{q+q'+1}}.$$

Therefore the CKN inequalities are established, and all there is left to prove is that the functions given by (1.11), (1.12), (1.13), and (1.14) are extremizers for their respective parameter ranges. Thus the goal is to find a u so that $\frac12 B^2 = AC$. (Note that $\frac12 B^2 = AC$ implies that $As^2 - Bs + C = 0$ for $s = \frac{B}{2A}$.) It follows that, for such a u, a necessary condition is, for some s,

$$\int_G \Big|u^{q'}\frac{\nabla u}{d^b} + su^q\frac{\nabla d}{d^a}\Big|^2\,dH = 0$$

and therefore

$$u^{q'}\frac{\nabla u}{d^b} + su^q\frac{\nabla d}{d^a} = 0$$

should a.e. hold. Now observe that, if $v:\mathbb{R}\to\mathbb{R}$ satisfies $v' = -v^{q-q'}$, then

$$v(X) = \begin{cases}(1+q'-q)^{\frac{1}{1+q'-q}}(c-X)^{\frac{1}{1+q'-q}}, & 1+q'-q>0,\\ (q-q'-1)^{\frac{1}{1+q'-q}}(X-c)^{\frac{1}{1+q'-q}}, & 1+q'-q<0,\\ c\exp(-X), & 1+q'-q=0,\end{cases}$$

for suitable values of $c, X\in\mathbb{R}$. It follows that, for each fixed $s\in\mathbb{R}$, then $u(g) = v(\frac{s}{\beta}d(g)^\beta)$, $\beta = b-a+1$, satisfies

$$\nabla u = \nabla v\Big(\frac{s}{\beta}d^\beta\Big) = (sd^{\beta-1}\nabla d)v'\Big(\frac{s}{\beta}d^\beta\Big) = -(sd^{\beta-1}\nabla d)v^{q-q'}\Big(\frac{s}{\beta}d^\beta\Big) = -su^{q-q'}d^{\beta-1}\nabla d,$$

where g is such that $v(\frac{s}{\beta}d(g)^\beta)$ is well-defined. The extremizers are thus motivated, and it will now be shown that (1.11), (1.12), (1.13), and (1.14) satisfy

$$\|u\|_{H^{1,q,q'}_{a,b}} = \Big(\int_G u^{2q}|\nabla d|^2 d^{-2a}\,dH\Big)^{\frac{1}{2q}} + \Big(\int_G |\nabla u|^{\frac{2q}{q-q'}}|\nabla d|^{-\frac{2q'}{q-q'}}d^{\frac{2(-bq+aq')}{q-q'}}\,dH\Big)^{\frac{q-q'}{2q}} < \infty.$$

In preparation, observe that, by (2.3), there holds

$$|\nabla d|^2 = \frac{d}{Q-1}\mathcal{L}d,$$

and so, since d is homogeneous of degree 1, and $\mathcal{L}d$ is homogeneous of degree -1, it follows that $|\nabla d|^2$ is homogeneous of degree 0. Hence $|\nabla d|^2$ is bounded on $G\backslash\{0\}$; indeed, $|\nabla d|^2$ depends only on the values $\frac{g}{d(g)}$ which comprise a compact set on which $|\nabla d|^2$ is continuous. Next, observe that, for $(a,b)\in\mathscr{A}$, we have $\operatorname{sgn}(s) = \operatorname{sgn}(\beta)$, and so $\frac{s}{\beta}\ge 0$ for $\beta\ne 0$.

First, suppose that $(a,b) \in \mathscr{A}_1$, i.e., $a < b+1 < \frac{Q}{2}$ and so $\beta > 0$, and suppose that $1+q'-q > 0$. Let u be given by (1.11). By the polar integration (2.4), there holds

$$\int_G u(g)^{2q} |\nabla d(g)|^2 d(g)^{-2a}\, dH(g) \lesssim \int_G \left| c_1 - \frac{s}{\beta} d(g)^\beta \right|^{\frac{2q}{1+q'-q}} d(g)^{-2a}\, dH(g) \sim \int_0^1 r^{-2a+Q-1}\, dr,$$

which is convergent since $-2a+Q-1 > -1$.

Since $\nabla u(g) = -s d(g)^{\beta-1} u^{q-q'} \nabla d$, there holds

$$|\nabla d|^{-\frac{2q'}{q-q'}} |\nabla u|^{\frac{2q}{q-q'}} \sim r^{\frac{2q(b-a)}{q-q'}} |c_1 - r^{b-a+1}|^{\frac{2q}{1+q'-q}} \sim r^{\frac{2q(b-a)}{q-q'}}$$

as $r \to 0$. Consequently,

$$\int_G |\nabla u|^{\frac{2q}{q-q'}} |\nabla d|^{-\frac{2q'}{q-q'}} d^{\frac{2(-bq+aq')}{q-q'}}\, dH \lesssim \int_0^1 r^{\frac{2q(b-a)}{q-q'}} r^{\frac{2(-bq+aq')}{q-q'}} r^{Q-1}\, dr < \infty,$$

by calculating

$$2q(b-a) + 2(-bq+aq') = -2a(q-q')$$

and using again $-2a+Q-1 > -1$. Therefore $u \in H^{1,q,q'}_{a,b}$.

Now suppose instead that $1+q'-q < 0$ with (a,b) as above, and let u be given by (1.13). Since $\beta > 0$, assume further that $(q,q') \in \mathscr{Q}^+_{a,b}$. Then

$$\begin{aligned}\int_G u(g)^{2q} |\nabla d|^2 d^{-2a}\, dH &\lesssim \int_G \left| c_3 + \frac{s}{\beta} d(g)^\beta \right|^{\frac{2q}{1+q'-q}} d(g)^{-2a}\, dH \\ &\lesssim \int_0^1 r^{-2a+Q-1} dr + \int_1^\infty r^{\frac{2q\beta}{1+q'-q}} r^{-2a+Q-1}\, dr,\end{aligned}$$

where the first integral converges by $-2a+Q-1 > -1$, and the second integral converges since $(q,q') \in \mathscr{Q}^+_{a,b}$, i.e.,

$$\frac{2q}{1+q'-q} < \frac{2a-Q}{1+b-a}.$$

A similar computation shows that the gradient integral also converges for this range of (q,q'), and so $u \in H^{1,q,q'}_{a,b}$. Observe that these integrals may diverge for $(q,q') \notin \mathscr{Q}^+_{a,b}$.

For the remaining cases of $(a,b) \in \mathscr{A}_2$, of $1+q'-q = 0$, and of $\beta < 0$, the computations are similar and thus omitted, and so sharpness is achieved for the given parameter ranges. □

Proof of Theorem 1.2. As already mentioned, the first part of this theorem follows from Theorem 1.1 by using $|\nabla d|^2 = |x|^2 d^{-2}$ for Heisenberg-type groups. Moreover, part (i) was already proved in Theorem 1.1. Thus, for G an Iwasawa group, it is left to prove cases (ii) and (iii). The proofs will be in the spirit of [5]. Lastly, let $C(Q,a,b)$ denote the sharp constant for (1.15).

Now suppose $(a,b) \in \mathscr{B}$. Applying Proposition 1.3 with $\sigma = 2b - Q + 2$, inequality (1.15) becomes

$$C(Q,a,b) \int_G |u|^2 \frac{|x|^2}{d^{3b-a+5}}\, dH \le \left(\int_G \frac{|\nabla u|^2}{d^{2b}}\, dH \right)^{\frac{1}{2}} \left(\int_G |u|^2 \frac{|x|^2}{d^{4b-2a+6}}\, dH \right).$$

By defining the transformation $(a,b) \mapsto (a',b')$ by $a' = 2b - a + 2$, $b' = b$, one sees that this inequality may be written as

$$C(Q,a,b) \int_G |u|^2 \frac{|x|^2}{d^{a'+b'+3}}\, dH \le \left(\int_G \frac{|\nabla u|^2}{d^{2b'}}\, dH \right)^{\frac{1}{2}} \left(\int_G |u|^2 \frac{|x|^2}{d^{2a'+2}}\, dH \right), \tag{3.2}$$

which is just the CKN inequality with parameters $(a',b') = (2b-a+2, b)$. Consequently, if $C(Q,a',b')$ denotes the best constant for inequality (3.2), then it may concluded that $C(Q,a,b) = C(Q,a',b')$. But

$\mathscr{B}_i \mapsto \mathscr{A}_i$, $i = 1, 2$, under the transformation $(a, b) \mapsto (a', b') = (2b - a + 2, b)$, and so, if $(a, b) \in \mathscr{B}$, then $(a', b') \in \mathscr{A}$. Consequently, as a result of case (i),

$$C(Q, a', b') = \frac{|Q - (a' + b + 1)|}{2} = \frac{|Q - (3b - a + 3)|}{2},$$

and it is achieved by the functions

$$u(g) = Dd^{2b-Q+2} \exp\Big(\frac{s}{b - a + 1} d(g)^{b-a+1}\Big),$$

noting that $-(b - a' + 1) = b - a + 1$. It is easy to see that $u \in H^1_{a,b}(G)$ since

$$\operatorname{sgn}(s) = \operatorname{sgn}(b - a' + 1) = -\operatorname{sgn}(b - a + 1).$$

This concludes the proof. □

Proof of Proposition 1.6. To establish the CKN-type inequality with monomial weights, the proof follows that of Theorem 1.1; for example, for the second part, one may consider

$$\int_G |g^A \nabla u + sug^B x_j^\alpha e_j|^2 \, dH(g),$$

where e_j is the horizontal vector satisfying $\langle \nabla u, e_j\rangle = X_j u$. The proof is thus omitted. □

Proof of Proposition 1.7. Compute

$$\begin{aligned}\int_G |g^A \nabla u + sug^B (x_1, \ldots, x_m)^T|^2 \, dH(g) &= \int_G g^{2A} |\nabla u|^2 \, dH(g) + s^2 \int u^2 g^{2B} |x|^2 \, dH(g) \\ &\quad + 2s \sum_{j=1}^m \int_G g^{A+B} u x_j X_j u \, dH(g) \\ &\geq 0.\end{aligned}$$

But

$$\begin{aligned}\sum_{j=1}^m \int_G g^{A+B} u x_j X_j u \, dH(g) &= -\sum_{j=1}^m \int_G g^{A+B} u x_j X_j u \, dH(g) - \sum_{j=1}^m \int_G u^2 \Big(\partial_{x_j} + 2 \sum_{s=1}^n \sum_{i=1}^m U_{j,i}^{(s)} x_i \partial_{t_s}\Big)(g^{A+B} x_j) \, dH(g) \\ &= -\sum_{j=1}^m \int_G g^{A+B} u x_j X_j u \, dH(g) - \sum_{j=1}^m (A_{j1} + B_{j1} + 1) \int_G u^2 g^{A+B} \, dH(g) \\ &\quad - \sum_{s=1}^n \sum_{i,j=1}^m 2(A_{s2} + B_{s2}) \int_G u^2 U_{j,i}^{(s)} x_i x_j g^{A+B} \, dH(g) \\ &= -\sum_{j=1}^m \int_G g^{A+B} u x_j X_j u \, dH(g) - \sum_{j=1}^m (A_{j1} + B_{j1} + 1) \int_G u^2 g^{A+B} \, dH(g),\end{aligned}$$

where the skew symmetry of the $U^{(s)}$ were used to conclude

$$\sum_{i,j=1}^m U_{j,i}^{(s)} x_i x_j = \langle U^{(s)} x, x\rangle = 0.$$

The rest of the proof follows as the proofs for the previous CKN inequalities. □

Remark. To clarify why the argument in [7] does not necessarily produce sharp inequalities, a general heuristic is given here. The following argument arose from an enlightening conversation with Hyun-Chul Jang.

So let $F, G, H : X \to \mathbb{R}$ be nonnegative functionals on a real vector space X, and suppose they are homogeneous of degree 1 with respect to positive reals; e.g., $F(r\xi) = |r| F(\xi)$ for $r \in \mathbb{R}$. Suppose $F(\xi)s^2 + G(\xi)s + H(\xi) \geq 0$ holds for all $s \in \mathbb{R}$, $\xi \in X$, and hence

$$\frac{1}{4} G(\xi)^2 \leq F(\xi) H(\xi) \tag{3.3}$$

for all $\xi \in X$. Let

$$A = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{1}{4}x^2 \leq yz\right\},$$

and let

$$B = \{(F(\xi), G(\xi), H(\xi)) \in \mathbb{R}^3 : \xi \in X\}$$

be the image of $\xi \mapsto (F(\xi), G(\xi), H(\xi))$. Since A and B are both cones, it is easy to see that (3.3) is sharp if and only if $A \cap \overline{B}$ contains at least two elements (and hence a line emanating from the origin). Therefore, if B is a cone whose closure intersects A only at 0 (e.g., if B has a smaller aperture than A), it follows that (3.3) is not sharp.

References

[1] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monogr. Math., Springer, Berlin, 2007.

[2] X. Cabré and X. Ros-Oton, Sobolev and isoperimetric inequalities with monomial weights, *J. Differential Equations* **255** (2013), no. 11, 4312–4336.

[3] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compos. Math.* **53** (1984), no. 3, 259–275.

[4] H. Castro, Hardy–Sobolev-type inequalities with monomial weights, *Ann. Mat. Pura Appl. (4)* **196** (2017), no. 2, 579–598.

[5] F. Catrina and D. G. Costa, Sharp weighted-norm inequalities for functions with compact support in $\mathbb{R}^N_0$, *J. Differential Equations* **246** (2009), no. 1, 164–182.

[6] F. Catrina and Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, *Comm. Pure Appl. Math.* **54** (2001), no. 2, 229–258.

[7] D. G. Costa, Some new and short proofs for a class of Caffarelli–Kohn–Nirenberg type inequalities, *J. Math. Anal. Appl.* **337** (2008), no. 1, 311–317.

[8] M. Cowling, A. H. Dooley, A. Korányi and F. Ricci, *H*-type groups and Iwasawa decompositions, *Adv. Math.* **87** (1991), no. 1, 1–41.

[9] M. Dong, Existence of extremal functions for higher-order Caffarelli–Kohn–Nirenberg inequalities, *Adv. Nonlinear Stud.* **18** (2018), no. 3, 543–553.

[10] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton University, Princeton, 1982.

[11] B. Franchi, G. Lu and R. L. Wheeden, Representation formulas and weighted Poincaré inequalities for Hörmander vector fields, *Ann. Inst. Fourier (Grenoble)* **45** (1995), no. 2, 577–604.

[12] N. Garofalo and E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier (Grenoble)* **40** (1990), no. 2, 313–356.

[13] N. Garofalo and D. Vassilev, Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups, *Math. Ann.* **318** (2000), no. 3, 453–516.

[14] J. A. Goldstein and I. Kombe, The Hardy inequality and nonlinear parabolic equations on Carnot groups, *Nonlinear Anal.* **69** (2008), no. 12, 4643–4653.

[15] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, *Trans. Amer. Math. Soc.* **258** (1980), no. 1, 147–153.

[16] I. Kombe, Sharp hardy type inequalities on the carnot group, preprint (2005), https://arxiv.org/abs/math/0501522v1.

[17] A. Korányi, Kelvin transforms and harmonic polynomials on the Heisenberg group, *J. Funct. Anal.* **49** (1982), no. 2, 177–185.

[18] N. Lam, Sharp Trudinger–Moser inequalities with monomial weights, *NoDEA Nonlinear Differential Equations Appl.* **24** (2017), no. 4, Article ID 39.

[19] N. Lam, General sharp weighted Caffarelli–Kohn–Nirenberg inequalities, *Proc. Roy. Soc. Edinburgh Sect. A* **149** (2019), no. 3, 691–718.

[20] N. Lam and G. Lu, Sharp constants and optimizers for a class of Caffarelli–Kohn–Nirenberg inequalities, *Adv. Nonlinear Stud.* **17** (2017), no. 3, 457–480.

[21] C. S. Lin, Interpolation inequalities with weights, *Comm. Partial Differential Equations* **11** (1986), no. 14, 1515–1538.

[22] G. Lu, Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications, *Rev. Mat. Iberoam.* **8** (1992), no. 3, 367–439.

[23] G. Lu, The sharp Poincaré inequality for free vector fields: An endpoint result, *Rev. Mat. Iberoam.* **10** (1994), no. 2, 453–466.

[24] G. Lu, Local and global interpolation inequalities on the Folland–Stein Sobolev spaces and polynomials on stratified groups, *Math. Res. Lett.* **4** (1997), no. 6, 777–790.

[25] G. Lu, Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland–Stein spaces on stratified groups, *Acta Math. Sin. (Engl. Ser.)* **16** (2000), no. 3, 405–444.

[26] H.-M. Nguyen and M. Squassina, Fractional Caffarelli–Kohn–Nirenberg inequalities, *J. Funct. Anal.* **274** (2018), no. 9, 2661–2672.

[27] P. Niu, H. Zhang and Y. Wang, Hardy type and Rellich type inequalities on the Heisenberg group, *Proc. Amer. Math. Soc.* **129** (2001), no. 12, 3623–3630.

[28] M. Ruzhansky and D. Suragan, On horizontal Hardy, Rellich, Caffarelli–Kohn–Nirenberg and p-sub-Laplacian inequalities on stratified groups, *J. Differential Equations* **262** (2017), no. 3, 1799–1821.

[29] S. Zhang, Y. Han and J. Dou, A class of Caffarelli–Kohn–Nirenberg type inequalities on the H-type group, *Rend. Semin. Mat. Univ. Padova* **132** (2014), 249–266.