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Research Article

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Fractional Choquard Equations with Confining Potential With or Without Subcritical Perturbations

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Abstract: In this paper, we consider fractional Choquard equations with confining potentials. First, we show that they admit a positive ground state and infinitely many bound states. Then we prove the existence of two signed solutions when a superlinear and subcritical perturbation is added; in this case, the main feature is that such a perturbation does not satisfy the usual Ambrosetti–Rabinowitz condition.

Keywords: Fractional Choquard Equation, Hardy–Littlewood–Sobolev Inequality, Ground State Solution, Subcritical Perturbation, Lack of Ambrosetti–Rabinowitz Condition

MSC 2010: 35A15, 47J30, 35S15, 47G10, 45G0

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1 Introduction

The starting point of this paper is a class of fractional Choquard equations of the form

$$(-\Delta)^s u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\ N \geq 1. \tag{P$_1$}$$

Here $p > 1$ varies in a suitable range, $s \in (0, 1)$, and $(-\Delta)^s$ is the fractional Laplacian defined as

$$(-\Delta)^s u = C(n, s)\,\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy$$

with the integral in the principal value sense, that is,

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

$$C(n, s) = \pi^{(-2s+N/2)} \frac{\Gamma(N/2 + s)}{\Gamma(-s)},$$

and Γ is Euler's Gamma function. Moreover, $V \in C(\mathbb{R}^N)$ is a potential such that $V(x) \geq V_0 > 0$ for every $x \in \mathbb{R}^N$.

Finally, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$, defined for every $x \in \mathbb{R}^N \setminus \{0\}$ as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}}.$$

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The reasons to consider problem (P_1) go back to physical motivations; indeed, the Choquard equation

$$-\Delta u + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

has appeared in the context of various physical models; for instance, see the models for polarons in a ionic lattice from Fröhlich in [9, 10]. The Choquard equation was actually introduced by Philippe Choquard in 1976 in the modelling of a one-component plasma; see [13]. More general versions of the Choquard equation have been introduced in recent years in the context of quantum mechanics; see [3, 5].

An interesting family of problems which extends (1.1) is given by the autonomous homogeneous Choquard equations

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \in \mathbb{N}$, $\alpha \in (0, N)$ and $p > 1$, studied in [15]. However, physical models in which particles are under the influence of an external electric field V lead to study Choquard equations in the form

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where generally V is a nonconstant electric potential in $L^1_{\text{loc}}(\mathbb{R}^N)$. Due to the presence of the potential V, the problem is not invariant under translation of the space, and the situation is more complicated; see [4, Chapter 1] and [25].

It is clear that problem (P_1) is the nonlocal counterpart of (1.2). In fact, recent research has shown that local interaction sometimes should be conveniently replaced by nonlocal ones (for instance, see [7, 11, 12, 17, 24, 27]), and indeed, our first set of results are related to those proved by Van Schaftingen and Xia in [25]; therein, they studied the problem in the case of a nonnegative potential V and proved the existence of a ground state solution, as well as a sequence of solutions whose energies are unbounded. In our case, due to the nonlocal nature of problem (P_1), we do not handle the case of a vanishing potential, so we assume $V\colon \mathbb{R}^N \to [V_0, +\infty)$ with $V_0 > 0$, and we first prove analogous results to those proved in [25]. In particular, the first two main results of this paper are Theorem 3.1, where the existence of infinitely many solutions is proved, and Theorem 3.3, where the existence of a ground state is given.

In the second part of the paper, we study a subcritical perturbation of problem (P_1), namely,

$$(-\Delta)^s u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u + f(x, u) \quad \text{in } \mathbb{R}^N. \tag{P_2}$$

Here f satisfies suitable conditions, but, in particular, it does not satisfy the Ambrosetti–Rabinowitz conditions. This means that the usual strategy to find critical points for the associated functional cannot be performed. For this reason, we assume a new condition on f (see Section 4), recently introduced in [18] and already exploited in other contexts (for instance, see [8]). This condition is quite general, but, on the other hand, it is enough to overcome the difficulties arising from the lack of the Ambrosetti–Rabinowitz condition and prove that the associated functional has critical points. In this way, we can prove the existence of two solutions, one being positive and the other being negative; see Theorem 4.2, the third main result of this paper.

The paper is organized as follows: in Section 2, we introduce the problem in detail, and we give the functional setting we shall use later on, in particular, proving some embedding and continuity results. In Section 3, we prove the existence of an unbounded sequence of solutions for problem (P_1) and that there exists a ground state solution. The former result is obtained by using the fountain theorem by Bartsch [2], while the latter is the consequence of a strategy which goes back to Rabinowitz [20].

Finally, in Section 4, we consider problem (P_2) and prove that there exist two nontrivial constant-sign solutions. In this case, due to the general behavior of the nonlinearity f, we are not able to apply the usual mountain pass theorem with the (PS) condition, but we need a version under the validity of the (C) condition. Indeed, the fact that f does not satisfy the Ambrosetti–Rabinowitz conditions makes the proof of the boundedness of (C) sequences very hard, but, following the approach of [18], we succeeded in proving it, gaining the desired result.

2 Functional Setting

We divide this section in two parts: in the first one, we study some properties of the convolution term and some embedding properties related to the functional space where the problems are set; in the second part, we introduce the variational structure we will use.

2.1 Embedding Results

The leading operator in the equation forces to consider the quantity

$$\|u\|_{H^s_V} := \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} Vu^2\,dx \Big)^{\frac{1}{2}}.$$

Now, we claim that this is a norm and define $H^s_V(\mathbb{R}^N)$ as the normed space obtained by completion of the set of smooth functions with compact support $C^\infty_c(\mathbb{R}^N)$ with respect to the norm $\|u\|_{H^s_V}$. Indeed, $\|\cdot\|_{H^s_V}$ is clearly a semi-norm, but if $\|u\|_{H^s_V} = 0$, from the first term, u is constant with $\int_{\mathbb{R}^N} V|u|^2 = 0$, which leads to $u = 0$, so $\|\cdot\|_{H^s_V}$ is a norm.

Moreover, $H^s_V(\mathbb{R}^N)$ is a Hilbert space, endowed with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} Vuv\,dx \quad \text{for every } u, v \in H^s_V(\mathbb{R}^N).$$

With an assumption on the potential V, we can say that $H^s_V(\mathbb{R}^N)$ is continuously embedded in the fractional Hilbert space $H^s(\mathbb{R}^N)$.

Remark 2.1. If $\inf V > 0$, then $H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$, that is, there exists $C > 0$ such that

$$\|u\|_{H^s(\mathbb{R}^N)} \le C\|u\|_{H^s_V(\mathbb{R}^N)} \quad \text{for every } u \in H^s_V(\mathbb{R}^N). \tag{2.1}$$

Indeed, we have

$$\begin{aligned}\|u\|^2_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} u^2\,dx \\ &\le \max\Big\{1, \frac{1}{\inf V}\Big\}\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} Vu^2\,dx\Big) \le C\|u\|_{H^s_V(\mathbb{R}^N)},\end{aligned}$$

so we have the continuous embedding.

For further references, we also define $H^s_V(\Omega)$ as the completion of smooth functions with compact support with respect to the norm

$$\Big(\int_{\Omega^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\Omega} Vu^2\,dx \Big)^{\frac{1}{2}}.$$

We first study when we have an embedding of $H^s_V(\mathbb{R}^N)$ into the weighted space

$$L^2(\mathbb{R}^N; |x|^\gamma\,dx) := \Big\{u\colon \mathbb{R}^N \to \mathbb{R} : u \text{ measurable}, \int_{\mathbb{R}^N} |x|^\gamma u^2\,dx < +\infty\Big\}.$$

We consider this space with the norm

$$\|u\|^2_{L^2(\mathbb{R}^N;|x|^\gamma\,dx)} = \int_{\mathbb{R}^N} |x|^\gamma u^2\,dx.$$

More precisely, we show that, under suitable assumptions, this embedding is continuous and compact.

Proposition 2.2. *Let $N > 2s$ and $\gamma \in [0, +\infty)$. If $V \in C(\mathbb{R}^N)$, $V(x) \geq V_0 > 0$ and*

$$\liminf_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma} > 0, \tag{2.2}$$

then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |x|^\gamma u^2 \, dx \leq C\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} Vu^2 \, dx \Big) \quad \text{for every } u \in H^s_V(\mathbb{R}^N), \tag{2.3}$$

that is, the embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N; |x|^\gamma \, dx)$ is continuous.

If, in addition,

$$\lim_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma} = +\infty, \tag{2.4}$$

then the corresponding embedding is compact. Moreover, when $\gamma = 0$, the embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for every $q \in (2, 2^*)$.

Here $2^* = \frac{2N}{N-2s}$ is the usual Sobolev fractional exponent; see [6]. A similar result was proved in [25] for the embedding of $H^1_V(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ with $\frac{1}{q} \in (\frac{1}{2} - \frac{1}{N}, \frac{1}{2})$, only assuming $V \geq 0$. In our result, we require a stronger assumption, that is, V is far from 0, but we give a simpler proof.

Proof of Proposition 2.2. Since (2.2) holds, we can take $\lambda \in (0, +\infty)$ such that

$$\lambda < \liminf_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma}.$$

Then there exists $k > 0$ sufficiently large so that, if $x \in \mathbb{R}^N \setminus B(0, k)$, we have $V(x) \geq \lambda |x|^\gamma$. So, multiplying by u^2 and integrating in $\mathbb{R}^N \setminus B(0, k)$, we obtain

$$\lambda \int_{\mathbb{R}^N \setminus B(0,k)} |x|^\gamma u^2 \, dx \leq \int_{\mathbb{R}^N \setminus B(0,k)} Vu^2 \, dx.$$

Then, since $\gamma \geq 0$ and $V(x) \geq V_0 > 0$, we can write

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\gamma u^2 \, dx &\leq k^\gamma \int_{B(0,k)} u^2 \, dx + \int_{\mathbb{R}^N \setminus B(0,k)} |x|^\gamma u^2 \, dx \\ &\leq k^\gamma \int_{B(0,k)} \frac{V}{V_0} u^2 \, dx + \frac{1}{\lambda} \int_{\mathbb{R}^N \setminus B(0,k)} Vu^2 \, dx \\ &\leq \max\Big\{ \frac{k^\gamma}{V_0}, \frac{1}{\lambda} \Big\} \int_{\mathbb{R}^N} Vu^2 \, dx \leq C\|u\|^2_{H^s_V}, \end{aligned}$$

so (2.3) holds.

As for the compactness, if (2.4) holds, let $\mathcal{F} \subset H^s_V(\mathbb{R}^N)$ be a bounded set, and take a sequence $(v_n)_n \subset \mathcal{F}$. Up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^s_V(\mathbb{R}^N)$ as $n \to \infty$, so we want to prove that $v_n \to v$ in $L^2(\mathbb{R}^N; |x|^\gamma \, dx)$ as $n \to \infty$. Of course, we can assume that $v \equiv 0$. Since v_n is bounded in $H^s_V(\mathbb{R}^N)$; by assumption (2.4), for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\Big(\sup_{|x|>R} \frac{|x|^\gamma}{V(x)} \Big) \|v_n\|^2_{H^s_V} \leq \varepsilon. \tag{2.5}$$

Since $v_n \in H^s_V(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, we have $v_n \in H^s_V(B(0, R))$ and $v_n \rightharpoonup 0$ in $H^s_V(B(0, R))$ as $n \to \infty$. By the fractional Rellich–Kondrakov theorem, $H^s_V(B(0, R))$ is compactly embedded in $L^2(B(0, R))$, so it follows that $v_n \to 0$ in $L^2(B(0, R))$ as $n \to \infty$. Since $\gamma \geq 0$, the space $L^2(B(0, R))$ is naturally embedded in the weighted space $L^2(B(0, R); |x|^\gamma \, dx)$, so $v_n \to 0$ in $L^2(B(0, R); |x|^\gamma \, dx)$. Therefore, there exists $N_1 > 0$ such that, for every $n \geq N_1$, we have

$$\int_{B(0,R)} |x|^\gamma v_n^2 \, dx \leq \varepsilon. \tag{2.6}$$

Then, for $n \geq N_1$, by (2.5) and (2.6), we obtain

$$\int_{\mathbb{R}^N} |x|^\gamma v_n^2\,dx = \int_{B(0,R)} |x|^\gamma v_n^2\,dx + \int_{\mathbb{R}^N\setminus B(0,R)} |x|^\gamma v_n^2\,dx$$
$$\leq \varepsilon + \Big(\sup_{|x|>R} \frac{|x|^\gamma}{V(x)}\Big) \int_{\mathbb{R}^N\setminus B(0,R)} V v_n^2\,dx \leq \varepsilon + \Big(\sup_{|x|>R} \frac{|x|^\gamma}{V(x)}\Big)\|v_n\|^2_{H^s_V} \leq 2\varepsilon.$$

This proves that $\int_{\mathbb{R}^N} |x|^\gamma v_n^2\,dx \to 0$ as $n \to \infty$, that is, $v_n \to 0$ in $L^2(\mathbb{R}^N; |x|^\gamma\,dx)$, so the desired embedding is compact.

To conclude the proof, in the case $\gamma = 0$, we use some interpolation to show that the previous sequence $(v_n)_n$ in $H^s_V(\mathbb{R}^N)$ is compact in $L^q(\mathbb{R}^N)$ with $q \in (2, 2^*)$. To this purpose, take $\bar{q} = \frac{2N}{N-2s}$ if $N > 2s$; then there exists $\beta \in (0, 1)$ such that

$$\frac{1}{q} = \frac{\beta}{2} + \frac{1-\beta}{\bar{q}}.$$

Using the interpolation inequality, we have $\|v_n\|_{L^q} \leq C_1\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{L^{\bar{q}}}$. By the fractional Sobolev inequality (see [6]), we know that $\|v_n\|_{L^{\bar{q}}} \leq C\|v_n\|_{H^s}$, so

$$C_1\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{L^{\bar{q}}} \leq C_2\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{H^s}.$$

By (2.1), we have

$$C_2\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{H^s} \leq C_3\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{H^s_V}.$$

In the end, we have shown that

$$\|v_n\|_{L^q} \leq C\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{H^s_V}.$$

Since we have just proved that $v_n \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, $(v_n)_n$ being bounded in $H^s_V(\mathbb{R}^N)$, we get

$$\|v_n\|_{L^q} \leq C\|v_n\|^\beta_{L^2}\|v_n\|^{1-\beta}_{H^s_V} \to 0 \quad \text{as } n \to \infty.$$

This concludes the proof. □

Remark 2.3. Although we are not interested in the case $N \leq 2s$, we notice that the previous result holds true also in this situation with minor adaptations.

Before describing the link between the convolution term and the space H^s_V, we recall some basic results known as the Hardy–Littlewood–Sobolev inequality and the Stein–Weiss inequality (see [14, 22]).

Theorem 2.4 (Hardy–Littlewood–Sobolev Inequality). *Let $0 < \alpha < N$ and $1 < p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$. Then there exists $C = C(p, \alpha, N) > 0$ such that*

$$\Big\| \int_{\mathbb{R}^N} \frac{f(y)\,dy}{|x-y|^{N-\alpha}} \Big\|_{L^q(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)} \quad \textit{for every } f \in L^p(\mathbb{R}^N).$$

Remark 2.5. Since, in the previous result, we have $q = \frac{Np}{N-\alpha p}$, we can simply say that $I_\alpha * f \in L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * f|^{\frac{Np}{N-\alpha p}}\,dx \leq C\Big(\int_{\mathbb{R}^N} |f|^p\,dx\Big)^{\frac{N}{N-\alpha p}}.$$

Theorem 2.6 (Stein–Weiss Inequality). *Let*

$$T_\lambda f(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^\lambda}\,dy$$

with $0 < \lambda < N$, $1 < p < \infty$, $\alpha < N - \frac{N}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ and

$$\frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{N} - 1.$$

If $p \leq q < \infty$, then $\|T_\lambda f(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^N)} \leq A\|f(x)|x|^\alpha\|_{L^p(\mathbb{R}^N)}$, where $A = A(p, \alpha, \beta, \lambda)$.

In the next proposition, we will define two maps, and we will prove that they are continuous and of weak to strong type, that is, they map weakly convergent sequences into strongly convergent sequences.

Proposition 2.7. *Let $N > 2s$ and $\alpha \in (0, N)$. If $V \in C(\mathbb{R}^N)$, $V \geq V_0 > 0$, satisfies*

$$\liminf_{|x|\to+\infty} \frac{V(x)}{1+|x|^{\frac{N+\alpha}{p}-N}} > 0, \tag{2.7}$$

then there are two well-defined mappings

$$\varphi\colon H_V^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad u \in H_V^s(\mathbb{R}^N) \mapsto I_{\frac{\alpha}{2}} * |u|^p \in L^2(\mathbb{R}^N), \tag{2.8}$$

$$\psi\colon H_V^s(\mathbb{R}^N) \to (H_V^s(\mathbb{R}^N))', \quad u \in H_V^s(\mathbb{R}^N) \mapsto (I_\alpha * |u|^p)|u|^{p-2}u \in (H_V^s(\mathbb{R}^N))', \tag{2.9}$$

which are continuous for $p \in (1, \frac{N+\alpha}{N-2s})$. If, in addition,

$$\lim_{|x|\to+\infty} \frac{V(x)}{1+|x|^{\frac{N+\alpha}{p}-N}} = +\infty, \tag{2.10}$$

the mappings above are of weak to strong type.

Proof. The sign of the exponent in (2.7), that is, $\frac{N+\alpha}{p} - N$, gives us the asymptotic behavior of $V(x)$ when $|x| \to \infty$, so we study separately the cases $\frac{N+\alpha}{p} - N < 0$ and $\frac{N+\alpha}{p} - N \geq 0$.

(I) The case $p > \frac{N+\alpha}{N}$.

Continuity and weak to strong property for φ. In this case, $|x|^{\frac{N+\alpha}{p}-N} \to 0$ as $|x| \to \infty$, so (2.7) implies that we are in the case $\gamma = 0$ of Proposition 2.2, that is, $\liminf_{|x|\to+\infty} V(x) > 0$. Take $u \in H_V^s(\mathbb{R}^N)$; then, by (2.1), we have $H_V^s(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ and

$$\|u\|_{H^s} \leq C\|u\|_{H_V^s}. \tag{2.11}$$

By the fractional Sobolev embedding, we have $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ and

$$\|u\|_{L^{\frac{2Np}{N+\alpha}}} \leq C\|u\|_{H^s}, \tag{2.12}$$

provided that the exponent $\frac{2Np}{N+\alpha}$ satisfies the condition

$$2 \leq \frac{2Np}{N+\alpha} \leq \frac{2N}{N-2s}.$$

This implies

$$\frac{N-2s}{N+\alpha} \leq \frac{1}{p} \leq \frac{N}{N+\alpha}, \tag{2.13}$$

which is indeed the case we are considering. Thus, taking $u \in H_V^s(\mathbb{R}^N)$, we have $u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, and so $|u|^p \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, being

$$\|u\|_{L^{\frac{2Np}{N+\alpha}}} = \Big(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}}\Big)^{\frac{N+\alpha}{2Np}} = \Big(\int_{\mathbb{R}^N} |u|^{p\frac{2N}{N+\alpha}}\Big)^{\frac{N+\alpha}{2Np}} = \||u|^p\|_{L^{\frac{2N}{N+\alpha}}}^{\frac{1}{p}}.$$

By the Hardy–Littlewood–Sobolev inequality with $0 < \frac{\alpha}{2} < N$, $p = \frac{2N}{N+\alpha}$ and $f = |u|^p$, being

$$1 < \frac{2N}{N+\alpha} < \frac{2N}{\alpha},$$

since $N + \alpha > \alpha$ and $\alpha < N$, and

$$\frac{N\frac{2N}{N+\alpha}}{N - \frac{\alpha}{2}\frac{2N}{N+\alpha}} = \frac{2N^2}{N^2 + N\alpha - N\alpha} = 2, \tag{2.14}$$

we get

$$I_{\frac{\alpha}{2}} * |u|^p \in L^2(\mathbb{R}^N) \quad \text{and} \quad \|I_{\frac{\alpha}{2}} * |u|^p\|_{L^2(\mathbb{R}^N)} \leq C\||u|^p\|_{L^{\frac{2N}{N+\alpha}}}.$$

This means that the Riesz integral operator, which maps

$$|u|^p \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \mapsto I_{\frac{\alpha}{2}} * |u|^p \in L^2(\mathbb{R}^N), \tag{2.15}$$

is a linear and bounded operator. In the end, we get that the first map we are considering, that is, (2.8) is well defined.

As for the continuity, by (2.1), we have $H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$, and by the fractional Sobolev embedding, $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Moreover, the Nemytskii operator

$$u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \mapsto |u|^p \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \tag{2.16}$$

is continuous; see [19]. Then, as we said before, the Riesz integral operator (2.15) is linear and bounded, so it is continuous. It follows that the composition of these maps (2.8) is continuous, so we get the first part of the claim.

If (2.10) holds, the embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ is compact for $\frac{N-2s}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$ by Proposition 2.2. As a consequence, if we take $(u_n)_n$ in $H^s_V(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^s_V(\mathbb{R}^N)$, we have $u_n \to u$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. From the continuity of the previous maps, we get $I_{\frac{\alpha}{2}} * |u_n|^p \to I_{\frac{\alpha}{2}} * |u|^p$ in $L^2(\mathbb{R}^N)$. This proves that (2.8) is of weak to strong type, as we claimed.

Continuity and weak to strong property of ψ. For the second map, starting with $u \in H^s_V(\mathbb{R}^N)$, we have, as before,

$$H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \quad \text{and} \quad \|u\|_{L^{\frac{2Np}{N+\alpha}}} \le C_1\|u\|_{H^s} \le C_2\|u\|_{H^s_V}$$

for p such that (2.13) holds. Then, from $u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, it follows that $|u|^p \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Now, we can use again the Hardy–Littlewood–Sobolev inequality with the same p and f as before, but with α in place of $\frac{\alpha}{2}$, being

$$1 < \frac{2N}{N+\alpha} < \frac{N}{\alpha} \quad \text{and} \quad \frac{N\frac{2N}{N+\alpha}}{N-\alpha\frac{2N}{N+\alpha}} = \frac{2N^2}{N^2-N\alpha} = \frac{2N}{N-\alpha},$$

so we get

$$I_\alpha * |u|^p \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \quad \text{with} \quad \|I_\alpha * |u|^p\|_{L^{\frac{2N}{N-\alpha}}} \le C\||u|^p\|_{L^{\frac{2N}{N+\alpha}}}.$$

By $u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, we also have $|u|^{p-2}u \in L^{\frac{2N}{N+\alpha}\frac{p}{p-1}}(\mathbb{R}^N)$ since

$$\|u\|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}}\,dx\Big)^{\frac{N+\alpha}{2Np}} = \Big(\int_{\mathbb{R}^N} \big||u|^{p-1}\big|^{\frac{2N}{N+\alpha}\frac{p}{p-1}}\,dx\Big)^{\frac{N+\alpha}{2Np}\frac{p-1}{p-1}} = \||u|^{p-2}u\|^{\frac{1}{p-1}}_{L^{\frac{2N}{N+\alpha}\frac{p}{p-1}}}.$$

Now, we want to use the fact that $I_\alpha * |u|^p \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and $|u|^{p-2}u \in L^{\frac{2N}{N+\alpha}\frac{p}{p-1}}(\mathbb{R}^N)$ to prove that

$$(I_\alpha * |u|^p)|u|^{p-2}u \in L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N). \tag{2.17}$$

To do this, we will use the Hölder inequality. We have the exponents $\frac{2N}{N-\alpha}$ and $\frac{2N}{N+\alpha}\frac{p}{p-1}$, so, from

$$\frac{N-\alpha}{2N} + \frac{(N+\alpha)(p-1)}{2Np} = 1 - \frac{N+\alpha}{2Np},$$

we get

$$\frac{\frac{N-\alpha}{2N}}{1-\frac{N+\alpha}{2Np}} + \frac{\frac{(N+\alpha)(p-1)}{2Np}}{1-\frac{N+\alpha}{2Np}} = 1.$$

So, with these exponents, we can use the Hölder inequality to get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big|(I_\alpha * |u|^p)|u|^{p-2}u\big|^{\frac{1}{1-\frac{N+\alpha}{2Np}}}\,dx \\
&\quad\le \Big(\int_{\mathbb{R}^N} \big|I_\alpha * |u|^p\big|^{\frac{1}{1-\frac{N+\alpha}{2Np}}\frac{2N}{N-\alpha}(1-\frac{N+\alpha}{2Np})}\,dx\Big)^{\frac{1}{\frac{2N}{N-\alpha}(1-\frac{N+\alpha}{2Np})}} \Big(\int_{\mathbb{R}^N} \big||u|^{p-2}u\big|^{\frac{1}{1-\frac{N+\alpha}{2Np}}\frac{2Np}{(N+\alpha)(p-1)}(1-\frac{N+\alpha}{2Np})}\,dx\Big)^{\frac{1}{\frac{2Np}{(N+\alpha)(p-1)}(1-\frac{N+\alpha}{2Np})}} \\
&\quad\le \Big(\int_{\mathbb{R}^N} \big|I_\alpha * |u|^p\big|^{\frac{2N}{N-\alpha}}\,dx\Big)^{\frac{1}{\frac{2N}{N-\alpha}(1-\frac{N+\alpha}{2Np})}} \Big(\int_{\mathbb{R}^N} \big||u|^{p-2}u\big|^{\frac{2Np}{(N+\alpha)(p-1)}}\,dx\Big)^{\frac{1}{\frac{2Np}{(N+\alpha)(p-1)}(1-\frac{N+\alpha}{2Np})}},
\end{aligned}$$

which means that (2.17) holds, as we claimed. Since $\frac{2Np}{N+\alpha}$ is the Hölder conjugate of $\frac{1}{1-\frac{N+\alpha}{2Np}}$, we can identify $L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N)$ with the dual of $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, so we have

$$(I_\alpha * |u|^p)|u|^{p-2}u \in (L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N))' \cong L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N).$$

Now, since $H^s_V(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, by duality, $(L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N))' \hookrightarrow (H^s_V(\mathbb{R}^N))'$, so $(I_\alpha * |u|^p)|u|^{p-2}u \in (H^s_V(\mathbb{R}^N))'$. In the end, we obtain that the second map (2.9) is well defined.

As for the continuity again, we start with the continuous embeddings

$$H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N).$$

As above, the map (2.16) is continuous, and with the Hardy–Littlewood–Sobolev inequality, we showed that the linear map

$$|u|^p \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \mapsto I_\alpha * |u|^p \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$$

is bounded; hence it is continuous. With the same arguments, we also have that the map

$$u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \mapsto |u|^{p-2}u \in L^{\frac{2N}{N+\alpha}\frac{p}{p-1}}(\mathbb{R}^N)$$

is continuous, and by the Hölder inequality, the map

$$u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \mapsto (I_\alpha * |u|^p)|u|^{p-2}u \in L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N)$$

is continuous. Then, as before, we identify $L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N)$ with the dual of $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, that is,

$$L^{\frac{1}{1-\frac{N+\alpha}{2Np}}}(\mathbb{R}^N) \cong (L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N))'.$$

Then, from the continuous embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, considering the dual spaces, the embedding $(L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N))' \hookrightarrow (H^s_V(\mathbb{R}^N))'$ is continuous. So, composing the maps, we get (2.9) that is, a continuous map, as we claimed.

If, in addition, (2.10) holds, then the embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ is compact for $\frac{N-2s}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$ by Proposition 2.2. Hence, if we take a sequence $(u_n)_n$ in $H^s_V(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^s_V(\mathbb{R}^N)$, then $u_n \to u$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Again from the continuity of the maps, it follows that $(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n \to (I_\alpha * |u|^p)|u|^{p-2}u$ in $(H^s_V(\mathbb{R}^N))'$. So we proved that the map (2.9) is of weak to strong type, as we wanted.

(II) The case $p \le \frac{N+\alpha}{N} (< 2)$.

Continuity and weak to strong property of φ. Again, we start with $u \in H^s_V(\mathbb{R}^N)$. Since $\frac{N+\alpha}{p} - N \ge 0$, by Proposition 2.2 with $\gamma = \frac{N+\alpha}{p} - N$, we have a continuous embedding

$$H^s_V(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx), \tag{2.18}$$

and

$$\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N}|u(x)|^2\,dx \le C\|u\|^2_{H^s_V}. \tag{2.19}$$

Moreover, the operator

$$u \in L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx) \mapsto |u|^p \in L^{\frac{2}{p}}(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx) \tag{2.20}$$

is well defined since

$$\|u\|_{L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)} = \Big(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N}|u(x)|^2\,dx\Big)^{\frac{1}{2}} = \||u|^p\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)}. \tag{2.21}$$

Then, using the Stein–Weiss inequality with $\lambda = N - \frac{\alpha}{2}$, $\beta = 0$, $q = 2$, $\frac{N+\alpha-Np}{2}$ in place of α and $\frac{2}{p}$ in place of p, we claim that

$$\int_{\mathbb{R}^N} \big|I_{\frac{\alpha}{2}} * |u|^p\big|^2 \le C\Big(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N}|u(x)|^2\,dx\Big)^p,$$

that is,

$$\|I_{\frac{\alpha}{2}} * |u|^p\|_{L^2}^2 \le C\||u|^p\|^2_{L^{\frac{2}{p}}(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)}. \tag{2.22}$$

Now, we show that the conditions on the exponents hold. Indeed, we have

$$\frac{N+\alpha-Np}{2} < N - \frac{Np}{2}$$

since $N > \alpha$. Then, from $p \le \frac{N+\alpha}{N}$, we get

$$\frac{N+\alpha-Np}{2} \ge 0.$$

Finally, the equation

$$\frac{1}{2} = \frac{p}{2} + \frac{N-\frac{\alpha}{2}+\frac{N+\alpha-Np}{2}}{N} - 1$$

is verified, and we can apply the Stein–Weiss inequality, as claimed.

As a consequence, the linear operator

$$|u|^p \in L^{\frac{2}{p}}(|x|^{\frac{N+\alpha}{p}-N}\,dx;\mathbb{R}^N) \mapsto I_{\frac{\alpha}{2}} * |u|^p \in L^2(\mathbb{R}^N) \tag{2.23}$$

is bounded, and so it is continuous. In the end, the composition map

$$u \in H_V^s(\mathbb{R}^N) \mapsto I_{\frac{\alpha}{2}} * |u|^p \in L^2(\mathbb{R}^N) \tag{2.24}$$

is well defined.

As for the continuity, from the continuous embedding (2.18), we get that the map

$$u \in H_V^s(\mathbb{R}^N) \mapsto u \in L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)$$

is continuous. As before, the map (2.20) is continuous. Then, by (2.22), the map (2.23) is continuous. By composition, we obtain that (2.24) is continuous.

If we are in the case (2.10), then the weak to strong property follows from Proposition 2.2, which gives the compactness of the embedding (2.18). So, if we take a sequence $(u_n)_n$ in $H_V^s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H_V^s(\mathbb{R}^N)$, then $u_n \to u$ in $L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)$. From the continuity of the maps, the strong convergence still holds, so we have $I_{\frac{\alpha}{2}} * |u_n|^p \to I_{\frac{\alpha}{2}} * |u|^p$ in $L^2(\mathbb{R}^N)$. This means that the desired map (2.24) is of weak to strong type, so we get the first part of the claim.

Continuity and weak to strong property of ψ. For the second map, starting again with $u \in H_V^s(\mathbb{R}^N)$, by Proposition 2.2, we have the continuous embedding (2.18) with

$$\|u\|_{L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)} \le C\|u\|_{H_V^s},$$

so $u \in L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)$. Then, as before, the operator (2.20) is well defined, being

$$\|u\|_{L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)} = \Big(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N}|u(x)|^2\,dx\Big)^{\frac{1}{2}} = \||u|^p\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)}.$$

Similarly, the operator

$$u \in L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx) \mapsto |u|^{p-2}u \in L^{\frac{2}{p-1}}(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx) \tag{2.25}$$

is well defined; in fact,

$$\|u\|_{L^2(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)} = \Big(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N}|u(x)|^2\,dx\Big)^{\frac{1}{2}} = \||u|^{p-1}\|^{\frac{1}{p-1}}_{L^{\frac{2}{p-1}}(\mathbb{R}^N;|x|^{\frac{N+\alpha}{p}-N}\,dx)}.$$

By the Stein–Weiss inequality, we claim that

$$\|I_\alpha * |u|^p|x|^{-\frac{N+\alpha-pN}{2}}\|_{L}^{\frac{2}{2-p}} \le A\||u|^p|x|^{\frac{N+\alpha-pN}{2}}\|_{L}^{\frac{2}{p}},$$

that is,

$$\int_{\mathbb{R}^N} \big|I_\alpha * |u|^p\big|^{\frac{2}{2-p}} |x|^{-\frac{N+\alpha-pN}{2-p}}\,dx \le \bigg(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p}-N} |u|^2\,dx\bigg)^{\frac{p}{2-p}}.$$

This implies that the linear operator

$$|u|^p \in L^{\frac{2}{p}}(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx) \mapsto I_\alpha * |u|^p \in L^{\frac{2}{2-p}}(\mathbb{R}^N; |x|^{-\frac{N+\alpha-pN}{2-p}}\,dx) \tag{2.26}$$

is bounded, so it is continuous as well. To prove this, we use again the Stein–Weiss inequality with $\lambda = N - \alpha$, $f = |u|^p$, $q = \frac{2}{2-p}$, $\beta = \frac{N+\alpha-Np}{2}$, $\frac{2}{p}$ in place of p and $\frac{N+\alpha-pN}{2}$ in place of α.

Now, using $I_\alpha * |u|^p \in L^{\frac{2}{2-p}}(\mathbb{R}^N; |x|^{-\frac{N+\alpha-pN}{2-p}}\,dx)$ together with $|u|^{p-2}u \in L^{\frac{2}{p-1}}(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx)$, we want to prove that

$$(I_\alpha * |u|^p)|u|^{p-2}u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx).$$

Indeed, using the Hölder inequality with exponents $\frac{1}{p-1}$ and $\frac{1}{2-p}$, being $p-1+2-p = 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \big|(I_\alpha * |u|^p)|u|^{p-2}u\big|^2 |x|^{N-\frac{N+\alpha}{p}}\,dx &= \int_{\mathbb{R}^N} \big|(I_\alpha * |u|^p)\big|^2 |x|^{-(N+\alpha-pN)} \big||u|^{p-2}u\big|^2 |x|^{\frac{p-1}{p}(N+\alpha-pN)}\,dx \\ &\le \bigg(\int_{\mathbb{R}^N} \big|(I_\alpha * |u|^p)\big|^{\frac{2}{2-p}} |x|^{-\frac{N+\alpha-pN}{2-p}}\bigg)^{2-p} \bigg(\int_{\mathbb{R}^N} \big||u|^{p-2}u\big|^{\frac{2}{p-1}} |x|^{\frac{N+\alpha-pN}{p}}\bigg)^{p-1}.\end{aligned}$$

From this, we have that

$$u \in L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx) \mapsto (I_\alpha * |u|^p)|u|^{p-2}u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx) \tag{2.27}$$

is well defined. Now, we claim that

$$L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx) = (L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx))'. \tag{2.28}$$

As a matter of fact, starting with $u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx)$, we can consider $\bar{u} := |x|^{\frac{N}{2}-\frac{N+\alpha}{2p}} u \in L^2(\mathbb{R}^N)$ and define

$$T(v) := \int_{\mathbb{R}^N} |x|^{-\frac{N}{2}+\frac{N+\alpha}{2p}} \bar{u} v\,dx \quad \text{for every } v \in L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx)$$

so that $T \in (L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx))'$.

On the other hand, starting with $T \in (L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx))'$, by the Riesz representation theorem, there exists a unique $\bar{u} \in L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx)$ such that

$$T(v) = \int_{\mathbb{R}^N} \bar{u} v\,dx \quad \text{for every } v \in L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx).$$

Now, we can define $u := \bar{u}|x|^{\frac{N+\alpha}{p}-N}$ so that $u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx)$. So, for every $f \in (L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx))'$, we proved that there exists a unique $u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx)$, and thus we get (2.28), as claimed.

From this, we know that $(I_\alpha * |u|^p)|u|^{p-2}u \in (L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx))'$. Then, from Proposition 2.2, we have the continuous embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^2(|x|^{\frac{N+\alpha}{p}-N}\,dx; \mathbb{R}^N)$, so, reasoning with the dual spaces, we get that the embedding

$$(L^2(|x|^{\frac{N+\alpha}{p}-N}\,dx; \mathbb{R}^N))' \hookrightarrow (H^s_V(\mathbb{R}^N))' \tag{2.29}$$

is continuous. Thus, composing the maps, we get

$$u \in H^s_V(\mathbb{R}^N) \mapsto (I_\alpha * |u|^p)|u|^{p-2}u \in (H^s_V(\mathbb{R}^N))', \tag{2.30}$$

which is well defined.

For the continuity, we take $u \in H^s_V(\mathbb{R}^N)$, and from Proposition 2.2, we have the continuous map

$$u \in H^s_V(\mathbb{R}^N) \mapsto u \in L^2(|x|^{\frac{N+\alpha}{p}-N}\,dx; \mathbb{R}^N).$$

As above, the map (2.20) is continuous, as well as the map (2.25), doing the same calculations with $p-1$ instead of p. From the Stein–Weiss inequality, we obtained that the linear map (2.26) is bounded, so it is continuous. Then, combining the last two maps with the Hölder inequality, we obtain that the map (2.27) is continuous. Then we identify $L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx)$ with the dual of $L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx)$ as before. As a consequence of Proposition 2.2, we have the continuous embedding (2.29), so the map

$$(I_\alpha * |u|^p)|u|^{p-2}u \in L^2(\mathbb{R}^N; |x|^{N-\frac{N+\alpha}{p}}\,dx) \mapsto (I_\alpha * |u|^p)|u|^{p-2}u \in (H^s_V(\mathbb{R}^N))'$$

is continuous. Composing the maps, we obtain (2.30), which is a continuous map, as we stated.

If, in addition, (2.10) holds, as in the other cases, the weak to strong property again follows from the compactness of the embedding (2.18) given again by Proposition 2.2. So we take $(u_n)_n$ such that $u_n \rightharpoonup u$ in $H^s_V(\mathbb{R}^N)$, and then $u_n \to u$ in $L^2(\mathbb{R}^N; |x|^{\frac{N+\alpha}{p}-N}\,dx)$. From the continuity of the maps, we obtain $(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n \to (I_\alpha * |u|^p)|u|^{p-2}u$ in $(H^s_V(\mathbb{R}^N))'$. So the map (2.30) is of weak to strong type, and this concludes the proof. □

2.2 The Energy Functional

Now, we use the results of the previous section to prove that the functional $J_p : H^s_V \to \mathbb{R}$, defined as

$$J_p(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} Vu^2\,dx - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx,$$

is well defined and of class C^1 on $H^s_V(\mathbb{R}^N)$.

First of all, we prove an equality that will be useful in the proof of the next result. By the semi-group identity for the Riesz potential, that is, $I_\alpha = I_{\frac{\alpha}{2}} * I_{\frac{\alpha}{2}}$ (see [21, p. 118, equation 6]), we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx = \int_{\mathbb{R}^N} |I_{\frac{\alpha}{2}} * |u|^p|^2\,dx.$$

Proposition 2.8. *Let $N > 2s$, $\alpha > 0$, $p \in (1, \frac{N+\alpha}{N-2s})$. If* (2.10) *holds, then functional J_p is of class C^1 on $H^s_V(\mathbb{R}^N)$.*

Proof. We only need to consider the nonlinear term G_p of J_p, that is,

$$G_p(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = \int_{\mathbb{R}^N} \big|I_{\frac{\alpha}{2}} * |u|^p\big|^2.$$

By Proposition 2.7, the map $u \in H^s_V(\mathbb{R}^N) \mapsto (I_\alpha * |u|^p)|u|^p \in L^2(\mathbb{R}^N)$ is continuous, so G_p is continuous on $H^s_V(\mathbb{R}^N)$, and so J_p is continuous as well. Again by Proposition 2.7, which gives the continuity of the map (2.30), being

$$\langle G_p'(u), v\rangle = 2p\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uv \quad \text{for every } v \in H^s_V(\mathbb{R}^N),$$

we get the claim. □

3 Infinitely Many Solutions and Existence of a Ground State

3.1 Unbounded Sequence of Solutions

The first result is that equation (P_1) has infinitely many solutions.

Theorem 3.1. *Let $N > 2s$, $\alpha \in (0, N)$, $p \in (1, \frac{N+\alpha}{N-2s})$ and $V \in C(\mathbb{R}^N)$ with $V \geq V_0 > 0$. If* (2.10) *holds, then problem* (P_1) *has an infinite sequence of solutions whose critical values are unbounded.*

In order to prove Theorem 3.1, we start with the following proposition.

Proposition 3.2. *Under the assumptions of Theorem 3.1, the functional J_p satisfies the (PS) condition, that is, any sequence $(u_n)_n$ in $H^s_V(\mathbb{R}^N)$ with the property that $(J_p(u_n))_n$ is bounded and $J'_p(u_n) \to 0$ in $(H^s_V(\mathbb{R}^N))'$ as $n \to \infty$ has a convergent subsequence.*

Proof. Consider a Palais–Smale sequence $(u_n)_n$ for J_p, that is,

$$(J_p(u_n))_n \text{ is bounded} \quad \text{and} \quad J'_p(u_n) \to 0 \quad \text{in } (H^s_V(\mathbb{R}^N))' \quad \text{as } n \to \infty.$$

We want to show that $(u_n)_n$ is bounded in $H^s_V(\mathbb{R}^N)$. First, we observe that, by assumption, there exist $A, B > 0$ such that

$$J_p(u_n) \le A \quad \text{and} \quad \|J'_p(u_n)\|_{(H^s_V(\mathbb{R}^N))'} \le B,$$

so, by the Cauchy–Schwarz inequality, we have

$$J_p(u_n) - \frac{1}{2p}\langle J'_p(u_n), u_n\rangle \le A + \frac{B}{2p}\|u_n\|_{H^s_V}.$$

We also have

$$J_p(u_n) - \frac{1}{2p}\langle J'_p(u_n), u_n\rangle = \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2_{H^s_V}.$$

As a consequence,

$$\Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2_{H^s_V} \le A + \frac{B}{2p}\|u_n\|_{H^s_V},$$

which proves that the sequence $(u_n)_n$ is bounded in $H^s_V(\mathbb{R}^N)$. Up to a subsequence, we can assume that $(u_n)_n$ converges to some function u weakly in $H^s_V(\mathbb{R}^N)$ and strongly in $L^q(\mathbb{R}^N)$ with $\frac{1}{q} \in (\frac{1}{2} - \frac{s}{N}, \frac{1}{2})$. By Proposition 2.7 we have $G'_p(u_n) \to G'_p(u)$ as $n \to +\infty$ in $(H^s_V(\mathbb{R}^N))'$ because the map G'_p is of weak to strong type. Now, we can write

$$\begin{aligned}
\|u_n - u\|^2_{H^s_V} &= \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V|u_n - u|^2\,dx \\
&= \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u_n(x) - u(x) - u_n(y) + u(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(u_n(x) - u(x) - u_n(y) + u(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N} Vu(u_n - u)\,dx + \int_{\mathbb{R}^N} Vu_n(u_n - u)\,dx \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}(u_n - u)\,dx + \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}(u_n - u)\,dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}(u_n - u)\,dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}(u_n - u)\,dx \\
&= \langle J'_p(u_n), u_n - u\rangle - \langle J'_p(u), u_n - u\rangle + \frac{1}{2p}\langle G'_p(u_n) u_n - u\rangle - \frac{1}{2p}\langle G'_p(u), u_n - u\rangle \\
&= \langle J'_p(u_n) - J'_p(u), u_n - u\rangle + \frac{1}{2p}\langle G'_p(u_n) - G'_p(u), u_n - u\rangle.
\end{aligned}$$

Combining this with the fact that $J'_p(u_n) \to J'_p(u) = 0$ and $G'_p(u_n) \to G'_p(u)$ in $(H^s_V(\mathbb{R}^N))'$, $u_n \rightharpoonup u$ in $H^s_V(\mathbb{R}^N)$, we get

$$\|u_n - u\|^2_{H^s_V} = \langle J'_p(u_n) - J'_p(u), u_n - u\rangle - \frac{1}{2p}\langle G'_p(u_n) - G'_p(u), u_n - u\rangle \to 0 \quad \text{as } n \to +\infty,$$

so J_p satisfies the (PS) condition. □

We are now ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. From Proposition 2.8, we know that $J_p \in C^1(H^s_V(\mathbb{R}^N))$ and from Proposition 3.2 that it satisfies the (PS) condition, so we have to prove that the conditions (A_1) and (A_2) of the fountain theorem hold. Since $H^s_V(\mathbb{R}^N)$ is a subspace of $L^2(\mathbb{R}^N)$, which is a separable space, it is separable as well, so there exists an orthonormal basis $(e_j)_{j\geq 0}$ of $H^s_V(\mathbb{R}^N)$. Using this basis, we define $X_j := \mathbb{R}e_j$, $Y_k = \bigoplus_{j=0}^k X_j$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$.

Now, we find suitable r_k and ρ_k. First, let us denote by σ_k the positive minimum of G_p on the unit sphere of Y_k, and then, for any $u \in Y_k$ with $\|u\|_{H^s_V} = \rho_k$, compute

$$J_p(u) = \frac{1}{2}\|u\|^2_{H^s_V} - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p = \frac{1}{2}\|u\|^2_{H^s_V} - \frac{1}{2p}\|u\|^{2p}_{H^s_V} G_p\Big(\frac{u}{\|u\|_{H^s_V}}\Big) \leq \frac{1}{2}\rho_k^2 - \frac{\sigma_k}{2p}\rho_k^{2p}.$$

Since $p > 1$, we have

$$\lim_{\rho_k\to\infty}\Big(\frac{1}{2}\rho_k^2 - \frac{\sigma_k}{2p}\rho_k^{2p}\Big) = -\infty,$$

so, for sufficiently large ρ_k, we have $\sup_{u\in Y_k,\|u\|=\rho_k} J_p(u) \leq 0$, and so condition (A_1) holds.

Now, turn to (A_2). We define

$$\beta_k := \sup\{\|I_{\frac{\alpha}{2}} * |u|^p\|_{L^2} : u \in Z_k,\ \|u\|_{H^s_V} = 1\}$$

and show that $\beta_k \to 0$ as $k \to \infty$. Indeed, we observe that $0 < \beta_{k+1} \leq \beta_k$ since $Z_k \supset Z_{k+1}$, so $\beta_k \to \beta \geq 0$. By definition of β_k, we know that, for every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\|_{H^s_V} = 1$ and

$$\|I_{\frac{\alpha}{2}} * |u_k|^p\|_{L^2} > \frac{\beta_k}{2}. \tag{3.1}$$

By definition of Z_k, we have $u_k \rightharpoonup 0$ in $H^s_V(\mathbb{R}^N)$, and as a consequence of Proposition 2.7, we have

$$I_{\frac{\alpha}{2}} * |u_k|^p \to I_{\frac{\alpha}{2}} * |u|^p \quad \text{in } L^2(\mathbb{R}^N) \quad \text{with } u = 0.$$

Hence, by (3.1), we get $\beta = 0$.

Moreover, for every $u \in Z_k$, we have

$$\begin{aligned} J_p(u) &= \frac{1}{2}\|u\|^2_{H^s_V} - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p \\ &= \frac{1}{2}\|u\|^2_{H^s_V} - \frac{1}{2p}\|u\|^{2p}_{H^s_V}\int_{\mathbb{R}^N}\Big[\Big(I_\alpha * \Big|\frac{u}{\|u\|_{H^s_V}}\Big|^p\Big)\Big]^2 \\ &\geq \frac{1}{2}\|u\|^2_{H^s_V} - \frac{\beta_k^2}{2p}\|u\|^{2p}_{H^s_V}. \end{aligned}$$

Now, setting $r_k := \frac{1}{\beta_k^{1/(p-1)}}$, for every $u \in Z_k$ with $\|u\|_{H^s_V} = r_k$, we obtain

$$J_p(u) \geq \frac{1}{2}\frac{1}{\beta_k^{\frac{2}{p-1}}} - \frac{1}{2p}\frac{1}{\beta_k^{\frac{2p}{p-1}-2}} = \Big(\frac{1}{2} - \frac{1}{2p}\Big)\frac{1}{\beta_k^{\frac{2}{p-1}}}.$$

This means that

$$\inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_p(u) \geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)\frac{1}{\beta_k^{\frac{2}{p-1}}}.$$

Taking the limit, we obtain

$$\lim_{k\to\infty}\inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_p(u) \geq \lim_{k\to\infty}\Big(\frac{1}{2} - \frac{1}{2p}\Big)\frac{1}{\beta_k^{\frac{2}{p-1}}} = +\infty,$$

so condition (A_2) holds. Of course, fix r_k first as above, choose ρ_k such that $\rho_k > r_k > 0$, and apply the fountain theorem. □

3.2 Ground State

Now, we will prove that equation (P_1) admits a ground state solution, and this will be done using the mountain pass theorem, following the lines of the celebrated paper by Rabinowitz [20].

Theorem 3.3. *Let $N > 2s$, $\alpha \in (0, N)$, $p \in (1, \frac{N+\alpha}{N-2s})$ and $V \in C(\mathbb{R}^N)$ with $V \geq V_0 > 0$. If (2.10) holds, then problem (P_1) has a positive ground state solution.*

Proof. We divide the proof in several steps.

Existence of a mountain pass solution. To prove the existence of a solution, we apply the mountain pass theorem to J_p. First, by Proposition 2.8, J_p is of class C^1 on H_V^s, and by Proposition 3.2, it satisfies the (PS) condition with $J_p(0) = 0$. Now, we observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \leq C\|u\|_{H_V^s}^{2p}. \tag{3.2}$$

Indeed, if $p > \frac{N+\alpha}{N}$, by (2.11), (2.12) and (2.14), inequality (3.2) holds at once. On the other hand, if $1 < p \leq \frac{N+\alpha}{N}$, the estimate follows by using (2.19), (2.21) and (2.22).

As a consequence of (3.2),

$$J_p(u) \geq \frac{1}{2}\|u\|_{H_V^s}^2 - \frac{C}{2p}\|u\|_{H_V^s}^{2p} = \frac{1}{2}\|u\|_{H_V^s}^2\Big(1 - \frac{C}{p}\|u\|_{H_V^s}^{2p-2}\Big).$$

So, for $\|u\|_{H_V^s} = \rho$ small enough, we have $\inf_{\|u\|=\rho} J_p(u) > 0$, so the functional has a strict local minimum at 0.

Finally, take $u \in H_V^s(\mathbb{R}^N) \setminus \{0\}$, and notice that

$$J_p(tu) = \frac{1}{2}t^2\|u\|_{H_V^s}^2 - \frac{1}{2p}t^{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \to -\infty \quad \text{as } t \to +\infty.$$

This means that J_p enjoys the geometric structure of the mountain pass, so

$$\beta = \inf_{g\in\Gamma}\max_{\theta\in[0,1]} f(g(\theta)) > 0 \tag{3.3}$$

is a critical value for the functional J_p, where

$$\Gamma = \{g \in C^0([0, 1], H_V^s(\mathbb{R}^N));\ g(0) = 0,\ g(1) < 0\}.$$

So we have found a nontrivial solution u for problem (P_1). Now, we want to show that such a point u is the desired ground state.

To do that, we introduce the usual Nehari manifold

$$\mathcal{N} := \{u \in H_V^s(\mathbb{R}^N) \setminus \{0\} : J_p'(u)u = 0\}.$$

The mountain pass solution is a ground state. As usual, we start defining a radial homeomorphism between $\mathcal{N}$ and the unit ball in $H_V^s(\mathbb{R}^N)$. To do that, for every $u \in H_V^s(\mathbb{R}^N)) \setminus \{0\}$, we define $\psi\colon (0, +\infty) \to \mathbb{R}$ as

$$\psi(t) := J_p(tu).$$

From the behavior of J_p, as we discussed above, $\psi(t) > 0$ for t small, and $\psi(t) < 0$ for t large enough. As a consequence, there exists $\max_{t\geq 0}\psi(t)$, and it is achieved at a certain $t := \varphi(u) > 0$. Since $\varphi(u)$ is a maximum point for ψ, we have $\psi'(\varphi(u)) = 0$. On the other hand, we have

$$0 = \varphi(u)\psi'(\varphi(u)) = \langle J_p'(\varphi(u)u), \varphi(u)u\rangle,$$

that is, $\varphi(u)u \in \mathcal{N}$.

Now, we claim that $\varphi(u)$ is the only value of $t > 0$ such that $tu \in \mathcal{N}$. Indeed, since

$$\psi'(t) = t\Big(\|u\|_{H_V^s}^2 - t^{2p-2}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\Big),$$

we have $\psi'(t) = 0$ if and only if

$$\bar{t} = \left(\frac{\|u\|^2_{H^s_V}}{\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p} \right)^{\frac{1}{2p-2}}$$

with $\psi'(t) > 0$ for $0 < t < \bar{t}$ and $\psi'(t) < 0$ for $t > \bar{t}$. This means that the equation

$$\|u\|^2_{H^s_V} = t^{2p-2} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

is solvable if and only if $t = \bar{t} = \varphi(u)$. As a consequence, there is a well-defined map φ,

$$u \in H^s_V(\mathbb{R}^N) \mapsto \varphi(u) \in (0, +\infty).$$

In particular, if $u \in B(0, 1)$, there exists a unique $\varphi(u) > 0$ such that $\varphi(u)u \in \mathcal{N}$. Moreover, we show that the map φ is continuous. Indeed,let $(u_n)_n$ be such that $u_n \to u$ in $H^s_V(\mathbb{R}^N) \setminus \{0\}$. From Proposition 2.7,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \to \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \text{ as } n \to \infty.$$

Since, for every $n > 0$, we have $\varphi(u_n)u_n \in \mathcal{N}$, then

$$\varphi(u_n)^2 \|u_n\|^2_{H^s_V} = \varphi(u_n)^{2p} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p. \tag{3.4}$$

From this, we have

$$\varphi(u_n)^{2p-2} = \frac{\|u_n\|^2_{H^s_V}}{\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p} \to \frac{\|u\|^2_{H^s_V}}{\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p} \quad \text{as } n \to +\infty.$$

So $\varphi(u_n)$ converges to a certain $\bar{\varphi}$, and since $u \neq 0$, we have $\bar{\varphi} \neq 0$. Taking the limit in (3.4), we get

$$\bar{\varphi}^2 \|u_n\|^2_{H^s_V} = \bar{\varphi}^{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p,$$

so $\bar{\varphi}u \in \mathcal{N}$. Then, by the uniqueness of φ, we get $\bar{\varphi} = \varphi(u)$, so $\varphi(u_n) \to \varphi(u)$. In conclusion, $\mathcal{N}$ is homeomorphic to the unit ball in $H^s_V(\mathbb{R}^N)$.

Now, define

$$\beta^* = \inf_{H^s_V(\mathbb{R}^N)\setminus\{0\}} \max_{\theta \geq 0} J_p(\theta u).$$

We claim that

$$\beta^* = \beta = \inf_{u \in \mathcal{N}} J_p(u),$$

where β is defined in (3.3). In fact, from the definition of φ, for every $u \in H^s_V(\mathbb{R}^N) \setminus \{0\}$, we have

$$\max_{\theta \geq 0} J_p(\theta u) = J_p(\varphi(u)u),$$

so

$$\inf_{u \in H^s_V(\mathbb{R}^N)\setminus\{0\}} \max_{\theta \geq 0} J_p(\theta u) = \inf_{u \in H^s_V(\mathbb{R}^N)\setminus\{0\}} J_p(\varphi(u)u) = \inf_{u \in \mathcal{N}} J_p(u)$$

so that $\beta^* = \inf_{u \in \mathcal{N}} J_p(u)$. Moreover,

$$\max_{t \in [0,1]} J_p(g(t)) \geq J_p(g(t')) \geq \inf_{u \in \mathcal{N}} J_p(u) = \beta^*$$

so that $\beta \geq \beta^*$.

On the other hand, if we fix $u \in H^s_V(\mathbb{R}^N) \setminus \{0\}$, we have $J_p(\theta u) < 0$ for $\theta = \theta_u$ large enough. As a consequence, we can associate to each ray $\{\theta u : \theta \geq 0\}$ a function $g_u \in \Gamma$, defined as $g_u(t) = t\theta_u u$. From this, we have

$$\beta^* = \inf_{H^s_V(\mathbb{R}^N)\setminus\{0\}} \max_{\theta \geq 0} J_p(\theta u) = \inf_{H^s_V(\mathbb{R}^N)\setminus\{0\}} \max_{t \in [0,1]} J_p(g_u(t)).$$

Then, since $\{g_u : u \in H^s_V(\mathbb{R}^N) \setminus \{0\}\} \subset \Gamma$, we obtain

$$\beta^* = \inf_{H^s_V(\mathbb{R}^N)\setminus\{0\}} \max_{t\in[0,1]} J_p(g_u(t)) \geq \inf_{g\in\Gamma} \max_{t\in[0,1]} J_p(g(t)) = \beta.$$

Summing up, it follows that the mountain pass solution is also a minimizer on the Nehari manifold $\mathcal{N}$, so it is a ground state. By replacing u with $|u|$, we get, as usual, that u is positive. □

4 Perturbed Subcritical Problems

In this section, we study problem (P_2) with $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that $f(x, 0) = 0$ for almost every $x \in \mathbb{R}^N$. In addition, we assume the following hypotheses on f:

(f_1) there exists $a \in L^q(\mathbb{R}^N)$, $a \geq 0$, with $q \in ((2^*)', 2)$, $c > 0$ and $r \in (2, 2^*)$ such that $|f(x, u)| \leq a(x) + c|u|^{r-1}$ for almost every $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$;

(f_2) denoting $F(x, u) = \int_0^u f(x, t)\,dt$, we have $\lim_{u\to\pm\infty} \frac{F(x,u)}{u^2} = +\infty$ uniformly for almost every $x \in \mathbb{R}^N$;

(f_3) if $\sigma(x, u) = f(x, u)u - 2F(x, u)$, then there exists $\beta^* \in L^1(\mathbb{R}^N)$, $\beta^* \geq 0$, such that $\sigma(x, u_1) \leq \sigma(x, u_2) + \beta^*(x)$ for almost every $x \in \mathbb{R}^N$, all $0 \leq u_1 \leq u_2$ or $u_2 \leq u_1 \leq 0$;

(f_4) $\lim_{u\to 0} \frac{f(x,u)}{u} = 0$ uniformly for almost every $x \in \mathbb{R}^N$.

Remark 4.1. Condition (f_4) implies that $\lim_{u\to 0} \frac{F(x,u)}{u^2} = 0$ uniformly for almost every $x \in \mathbb{R}^N$.

Condition (f_3) was introduced in [18] to replace the frequently used Ambrosetti–Rabinowitz condition, which is not assumed here.

Now, we can prove that problem (P_2) admits solutions, and this will be done applying a version of the mountain pass theorem to some suitably truncated functionals of I_p.

Our main result is the following theorem.

Theorem 4.2. *Let $N > 2s$, $\alpha \in (0, N)$, $p \in (1, \frac{N+\alpha}{N-2s})$, $q \in (2, 2^*)$ and $V \in C(\mathbb{R}^N)$ with $V \geq V_0 > 0$. If hypotheses (f_1), (f_2), (f_3), (f_4) and (2.10) hold, then problem (P_2) admits two nontrivial constant-sign solutions.*

First, denoting by u^+ and u^- the positive part and the negative part of u, respectively, we introduce the functionals

$$I_\pm(u) := \frac{1}{2}\|u\|^2_{H^s_V} - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u^\pm|^p)|u^\pm|^p\,dx - \int_{\mathbb{R}^N} F(x, u^\pm)\,dx.$$

We start proving that both $I_\pm$ satisfy the Cerami, (C) for short, condition – a generalization of the (PS) condition –, which states that any sequence $(u_n)_n$ in $H^s_V(\mathbb{R}^N)$ such that $(I_\pm(u_n))_n$ is bounded and $(1 + \|u_n\|)I'_\pm(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence.

Proposition 4.3. *Under the assumptions of Theorem 4.2, the functionals $I_\pm$ satisfy the (C) condition.*

Proof. We do the proof for I_+, the proof for I_- being analogous.

Let $(u_n)_n$ in $H^s_V(\mathbb{R}^N)$ be such that

$$|I_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \geq 1, \tag{4.1}$$

and

$$(1 + \|u\|_{H^s_V})I'_+(u_n) \to 0 \quad \text{in } (H^s_V(\mathbb{R}^N))' \quad \text{as } n \to \infty. \tag{4.2}$$

From (4.2), we have $|(1 + \|u\|_{H^s_V})\langle I'_+(u_n), h\rangle| \leq \varepsilon_n\|h\|_{H^s_V}$ for every $h \in H^s_V(\mathbb{R}^N)$ and $\varepsilon_n \to 0$ as $n \to \infty$, that is,

$$\left|\langle u_n, h\rangle - \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^{p-2}u_n^+ h\,dx - \int_{\mathbb{R}^N} f(x, u_n^+)h\,dx\right| \leq \frac{\varepsilon_n\|h\|_{H^s_V}}{1 + \|u_n\|_{H^s_V}}. \tag{4.3}$$

In (4.3), if we take $h = -u_n^- \in H^s_V(\mathbb{R}^N)$, we obtain $|\langle u_n, u_n^-\rangle| \leq \varepsilon_n$ for all $n \geq 1$, that is,

$$\langle u_n^+, u_n^-\rangle - \|u_n^-\|^2_{H^s_V} \to 0 \tag{4.4}$$

with

$$\langle u^+, u^- \rangle = \int_{\mathbb{R}^{2N}} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \le -\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy \le 0,$$

so it follows that

$$u_n^- \to 0 \quad \text{in } H_V^s(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{4.5}$$

Now, we take $h = u_n^+ \in H_V^s(\mathbb{R}^N)$ in (4.3) and obtain

$$-\langle u_n, u_n^+ \rangle + \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N} f(x, u_n^+)u_n^+\,dx \le \varepsilon_n. \tag{4.6}$$

From (4.1) and (4.5), we get

$$\langle u_n, u_n^+ \rangle - \frac{1}{p}\int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx - 2\int_{\mathbb{R}^N} F(x, u_n^+)\,dx \le M_2 \tag{4.7}$$

for some $M_2 > 0$ and all $n \ge 1$. Adding (4.6) and (4.7), we get

$$\Big(1 - \frac{1}{p}\Big)\int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N} f(x, u_n^+)u_n^+\,dx - 2\int_{\mathbb{R}^N} F(x, u_n^+)\,dx \le M_3$$

for some $M_3 > 0$ and all $n \ge 1$, that is,

$$\Big(1 - \frac{1}{p}\Big)\int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N} \sigma(x, u_n^+)\,dx \le M_3. \tag{4.8}$$

Now, we claim that $(u_n^+)_n$ is bounded in $H_V^s(\mathbb{R}^N)$. To prove this, we argue by contradiction, and passing to a subsequence if necessary, we assume that $\|u_n^+\|_{H_V^s} \to \infty$. We set $y_n = u_n^+/\|u_n^+\|_{H_V^s}$, $n \ge 1$, so we can assume that

$$y_n \rightharpoonup y \quad \text{in } H_V^s(\mathbb{R}^N) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^P(\mathbb{R}^N) \tag{4.9}$$

for every $P \in (2, 2^*)$, with $y \ge 0$.

First, we assume $y \ne 0$. Then, defining $Z(y) := \{x \in \mathbb{R}^N : y(x) = 0\}$, we have $\operatorname{meas}(\mathbb{R}^N \setminus Z(y)) > 0$ and $u_n^+(x) \to \infty$ for almost every $x \in \mathbb{R}^N \setminus Z(y)$ as $n \to \infty$. By hypothesis (f_2), we have

$$\frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2} = \frac{F(x, u_n^+(x))}{u_n^+(x)} y_n(x)^2 \to \infty \quad \text{for almost every } x \in \mathbb{R}^N \setminus Z(y).$$

By Fatou's lemma, we have

$$\int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2}\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2}\,dx,$$

so

$$\int_{\mathbb{R}^N} \frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2}\,dx \to \infty \quad \text{as } n \to \infty. \tag{4.10}$$

Again from (4.1) and (4.5), we have

$$-\frac{1}{2}\|u_n^+\|_{H_V^s}^2 + \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N} F(x, u_n^+)\,dx \le M_4$$

for some $M_4 > 0$ and $n \ge 1$, so it follows that

$$-\frac{1}{2} + \int_{\mathbb{R}^N} \frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2}\,dx \le \frac{M_4}{\|u_n^+\|_{H_V^s}^2}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, u_n^+(x))}{\|u_n^+\|_{H_V^s}^2}\, dx \le M_5$$

for some $M_5 > 0$, which is a contradiction with (4.10), and this concludes the case $y \neq 0$.

Now, assume $y \equiv 0$. We consider the continuous functions $\gamma_n : [0,1] \to \mathbb{R}$ defined by $\gamma_n(t) := I_+(tu_n^+)$ for $t \in [0,1]$ and $n \ge 1$, and define t_n such that

$$\gamma_n(t_n) = \max_{t\in[0,1]} \gamma_n(t). \tag{4.11}$$

Now, for $\lambda > 0$, we define $v_n := (2\lambda)^{\frac{1}{2}} y_n \in H_V^s(\mathbb{R}^N)$. Then, by (4.9), we have $v_n \to 0$ in $L^P(\mathbb{R}^N)$ for all $P \in (2, 2^*)$. From (f_1), performing some integration, we obtain

$$\int_{\mathbb{R}^N} F(x, u_n^+(x)) \le \int_{\mathbb{R}^N} a(x)|v_n(x)|\, dx + C \int_{\mathbb{R}^N} |v_n(x)|^r\, dx,$$

so we have

$$\int_{\mathbb{R}^N} F(x, u_n^+(x)) \to 0 \quad \text{as } n \to \infty. \tag{4.12}$$

Since $\|u_n^+\|_{H_V^s} \to \infty$ as $n \to \infty$, there exists $n_0 \ge 1$ such that $(2\lambda)^{\frac{1}{2}}/\|u_n^+\|_{H_V^s} \in (0,1)$ for all $n \ge n_0$. Then, by (4.11), we have

$$\gamma_n(t_n) \ge \gamma_n\Big(\frac{(2\lambda)^{\frac{1}{2}}}{\|u_n^+\|_{H_V^s}}\Big) \quad \text{for all } n \ge n_0.$$

Hence,

$$I_+(t_n u_n^+) \ge I_+((2\lambda)^{\frac{1}{2}} y_n) = I_+(v_n) = \lambda \|y_n\|_{H_V^s}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p\, dx - \int_{\mathbb{R}^N} F(x, v_n)\, dx.$$

From (4.12) and Proposition 2.7, we have

$$I_+(t_n u_n^+) \ge \lambda + o(1) \quad \text{for all } n \ge n_1 \ge n_0,$$

and since $\lambda > 0$ is arbitrary, we have

$$I_+(t_n u_n^+) \to \infty \quad \text{as } n \to \infty. \tag{4.13}$$

Since $0 \le t_n u_n^+ \le u_n^+$ for all $n \ge 1$, by (f_3), we get

$$\int_{\mathbb{R}^N} \sigma(x, t_n u_n^+)\, dx \le \int_{\mathbb{R}^N} \sigma(x, u_n^+)\, dx + \|\beta^*\|_1 \quad \text{for all } n \ge 1. \tag{4.14}$$

In addition, we have $I_+(0) = 0$, and from (4.1), (4.4) and (4.5), there exists $M_6 > 0$ such that $I_+(u_n^+) \le M_6$ for all $n \ge 1$. This, together with (4.13), implies that $t_n \in (0,1)$ for all $n \ge n_2 \ge n_1$. Then, since t_n achieves a maximum, we have

$$0 = t_n \gamma_n'(t) = t_n \langle I_+'(t_n u_n^+), u_n^+\rangle = \langle J_p'(t_n u_n^+), t_n u_n^+\rangle - \int_{\mathbb{R}^N} f(x, t_n u_n^+) t_n u_n^+\, dx,$$

that is,

$$\|t_n u_n^+\|_{H_V^s}^2 - \int_{\mathbb{R}^N} (I_\alpha * |t_n u_n^+|^p)|t_n u_n^+|^p\, dx - \int_{\mathbb{R}^N} f(x, t_n u_n^+) t_n u_n^+\, dx = 0 \quad \text{for all } n \ge 1. \tag{4.15}$$

Now, adding (4.15) to (4.14), we obtain

$$\|t_n u_n^+\|_{H_V^s}^2 - \int_{\mathbb{R}^N} (I_\alpha * |t_n u_n^+|^p)|t_n u_n^+|^p\, dx - 2\int_{\mathbb{R}^N} F(x, t_n u_n^+\, dx) \le \int_{\mathbb{R}^N} \sigma(x, u_n^+)\, dx + \|\beta^*\|_1 \quad \text{for all } n \ge n_2,$$

and this implies that

$$\begin{aligned}2I_+(t_n u_n^+) &\le \Big(1-\frac{1}{p}\Big)\int_{\mathbb{R}^N}(I_\alpha * |t_n u_n^+|^p)|t_n u_n^+|^p\,dx + \int_{\mathbb{R}^N}\sigma(x,u_n^+)\,dx + \|\beta^*\|_1\\ &\le \Big(1-\frac{1}{p}\Big)\int_{\mathbb{R}^N}(I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N}\sigma(x,u_n^+)\,dx + \|\beta^*\|_1 \quad \text{for all } n\ge 2\end{aligned}$$

since $t_n \in (0,1)$. Thus, by (4.13), we get

$$\Big(1-\frac{1}{p}\Big)\int_{\mathbb{R}^N}(I_\alpha * |u_n^+|^p)|u_n^+|^p\,dx + \int_{\mathbb{R}^N}\sigma(x,u_n^+)\,dx \to \infty \quad \text{as } n\to\infty. \tag{4.16}$$

Now, if we combine (4.8) and (4.16), we reach a contradiction, so the claim follows.

So $(u_n^+)_n$ is bounded in $H_V^s(\mathbb{R}^N)$, and together with (4.5), this implies that $(u_n)_n$ is bounded in $H_V^s(\mathbb{R}^N)$. So we can assume that

$$u_n \rightharpoonup u \quad \text{in } H_V^s(\mathbb{R}^N) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^P(\mathbb{R}^N) \quad \text{for all } P\in(2,2^*).$$

Now, we choose $h = u_n - u$ in (4.3) and obtain

$$\langle u_n, u_n-u\rangle - \int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n^+|^{p-2}u_n^+(u_n-u)\,dx - \int_{\mathbb{R}^N} f(x,u_n^+)(u_n-u)\,dx = o(1). \tag{4.17}$$

But by (f_1), we have

$$\begin{aligned}\int_{\mathbb{R}^N}|f(x,u_n^+)(u_n-u)| &\le \int_{\mathbb{R}^N} a(x)|u_n-u|\,dx + \int_{\mathbb{R}^N}|u_n|^{r-1}|u_n-u|\,dx\\ &\le \|a\|_q\|u_n-u\|_{q'} + C\|u_n-u\|_r \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Hence, passing to the limit in (4.17), we obtain

$$\langle u_n, u_n-u\rangle - \int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n^+|^{p-2}u_n^+(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty.$$

By Proposition 2.7, we also have

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n^+|^{p-2}u_n^+(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty,$$

so we get

$$\langle u_n, u_n-u\rangle = \|u_n\|_{H_V^s}^2 - \langle u_n, u\rangle \to 0.$$

This implies that $\|u_n\|_{H_V^s}^2 \to \|u\|_{H_V^s}^2$ as $n\to\infty$, so $u_n \to u$ in $H_V^s(\mathbb{R}^N)$, and then I_+ satisfies the (C) condition, as desired. □

We are now ready to give the proof of Theorem 4.2.

Proof of Theorem 4.2. From Proposition 4.3, we know that I_+ satisfies the (C) condition, so we only have to verify the geometric conditions of the mountain pass theorem, and indeed, $I_+(0) = 0$.

By (f_1) and (f_4), there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$F(x,u) \le \frac{\varepsilon}{2}u_n^2 + C_\varepsilon|u|^r \quad \text{for almost every } x\in\mathbb{R}^N \text{ and all } u\in\mathbb{R}.$$

Then, by Proposition 2.7 and the Sobolev embedding theorem, there exist $C, S, C_1 > 0$ such that, for every $u \in H_V^s(\mathbb{R}^N)$, we have

$$\begin{aligned}I_+(u) &= \frac{1}{2}\|u\|_{H_V^s}^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u^+|^p)|u^+|^p\,dx - \int_{\mathbb{R}^N}F(x,u^+)\,dx\\ &\ge \frac{1}{2}\|u\|_{H_V^s}^2 - C\|u^+\|_{H_V^s}^{2p} - \frac{\varepsilon}{2}\|u^+\|_2^2 - C_\varepsilon\|u^+\|_r^r\\ &\ge \frac{1-\varepsilon S}{2}\|u\|_{H_V^s}^2 - C\|u^+\|_{H_V^s}^{2p} - C_1\|u^+\|_{H_V^s}^r.\end{aligned}$$

So, if $\|u\|_{H_V^s} = \rho$ small enough, we have $\inf_{\|u\|=\rho} I_+(u) > 0$.

Now, if we take $u \in H^s_V(\mathbb{R}^N) \setminus \{0\}$ with $u > 0$ and $t > 0$, then

$$\begin{aligned} I_+(tu) &= \frac{t^2}{2}\|u\|^2_{H^s_V} - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)u^p\,dx - \int_{\mathbb{R}^N} F(x,tu)\,dx \\ &= \frac{t^2}{2}\|u\|^2_{H^s_V} - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)u^p\,dx - t^2\int_{\mathbb{R}^N} \frac{F(x,tu)}{(tu)^2}u^2\,dx. \end{aligned}$$

By Fatou's lemma, we have

$$\int_{\mathbb{R}^N} \liminf_{t\to\infty} \frac{F(x,tu)}{(tu)^2}u^2\,dx \le \liminf_{t\to\infty}\int_{\mathbb{R}^N} \frac{F(x,tu)}{(tu)^2}u^2\,dx,$$

so, by (f_2), we have

$$\int_{\mathbb{R}^N} \frac{F(x,tu)}{(tu)^2}u^2\,dx \to \infty \quad \text{as } t \to \infty.$$

As a consequence, $I_+(tu) \to -\infty$ as $t \to \infty$, so there exists $e \in H^s_V(\mathbb{R}^N)$ such that $\|e\|_{H^s_V} \ge \rho$ and $I_+(e) < 0$.

Now, we can apply the mountain pass theorem to I_+ and obtain a nontrivial critical point u of I_+. In particular, we have

$$0 = \langle I'_+(u), u^-\rangle = \langle u, u^-\rangle - \int_{\mathbb{R}^N}(I_\alpha * |u^+|^p)|u^+|^{p-2}u^+u^-\,dx - \int_{\mathbb{R}^N} f(x,u^+)u^-\,dx = \langle u^+, u^-\rangle - \|u^-\|^2_{H^s_V}.$$

From this, we have

$$\begin{aligned} \|u^-\|^2_{H^s_V} = \langle u^+, u^-\rangle &= \int_{\mathbb{R}^{2N}} \frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &\le -\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)+u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy \le 0, \end{aligned}$$

so $u^- \equiv 0$. As a consequence, since $I_+(u) = I_p(u)$, u is a positive solution of (P_2).

In the same way, arguing with I_-, we can find a negative solution for problem (P_2). □

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