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Research Article

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A Morse Lemma for Degenerate Critical Points of Solutions of Nonlinear Equations in $\mathbb{R}^2$

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Abstract: In this paper we prove a Morse Lemma for degenerate critical points of a function u which satisfies

$$-\Delta u = f(u) \quad \text{in } B_1,$$

where $u \in C^2(B_1)$, B_1 is the unit ball of $\mathbb{R}^2$ and f is a smooth nonlinearity. Other results on the nondegeneracy of the critical points and the shape of the level sets are proved.

Keywords: Morse Theory, Elliptic Equations, Level Sets

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1 Introduction and Statement of the Main Results

A famous result of Morse theory concerns the classification of *nondegenerate* critical points of a smooth function u. We recall it in the particular case of the plane,

Lemma 1.1 (Morse Lemma). *If $(0,0)$ is a nondegenerate critical point of u, then there exists a C^2 change of coordinates in a neighborhood of $(0,0)$ such that the function u expressed with respect to the new local coordinates (x,y) takes one of the following three standard forms*
- $u(x,y) = u(0,0) - x^2 - y^2$ *if $(0,0)$ is a maximum point for u,*
- $u(x,y) = u(0,0) + x^2 + y^2$ *if $(0,0)$ is a minimum point for u,*
- $u(x,y) = u(0,0) + x^2 - y^2$ *if $(0,0)$ is a saddle point for u.*

By saddle point we mean a critical point for u which is neither a maximum nor a minimum. We recall that $(0,0)$ is a *nondegenerate* critical point of a function $u \in C^2(B_1)$ if the Hessian matrix

$$H_u((0,0)) = \begin{pmatrix} u_{xx}(0,0) & u_{xy}(0,0) \\ u_{xy}(0,0) & u_{yy}(0,0) \end{pmatrix}$$

is invertible.

A lot of study has been devoted to the case where the critical point is degenerate, mainly addressed to study the topological properties of the sublevels of u (critical groups, Betti number, etc.). It is virtually impossible to provide a complete bibliography on this subject; we simply mention the classic books by Conley [6] and Milnor [14] and the paper [8] where some applications to (1.1) have been considered.

Anyway if the critical point is *degenerate*, a complete classification like in the Morse Lemma is impossible without some additional assumptions. Here we require that u is a solution to

$$-\Delta u = f(u) \quad \text{in } B_1, \tag{1.1}$$

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where $u \in C^\infty(B_1)$ and B_1 is the unit ball of $\mathbb{R}^2$. We assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity, say $f \in C^\infty$. In many cases this assumption can be relaxed, we do not care about the optimal regularity for f and u.

We would like to stress that a Morse Lemma for solutions to (1.1) provides a precise qualitative information on the shape of u in a neighborhood of the critical points. This *local* information, jointly with some *global* properties like Dirichlet boundary conditions allows to derive some important properties of u as the number of critical points or the shape of the level sets. Some examples will be given in Sections 4 and 5.

In this paper we want to extend the Morse Lemma to *degenerate* critical points of solutions of problem (1.1). We would like to stress that

- we do not assume any sign assumption on f,
- our result is local, so no boundary condition is needed,
- the critical point $(0, 0)$ does not need to be isolated.

In order to state our results, we need to fix a suitable setting. For a complex number $z = x + iy \in \mathbb{C}$, let us denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ the real and imaginary part of z. Moreover, if $(0, 0)$ is a degenerate critical point of u, it is not restrictive to assume that (up to a suitable rotation)

$$u_{xx}(0,0) = u_{xy}(0,0) = 0. \tag{1.2}$$

Finally, if $u_x \not\equiv 0$, let us consider the minimum integer $n \geq 3$ such that

$$\frac{\partial^n u}{\partial x^{n-k}\partial y^k}(0,0) \neq 0 \quad \text{for some } k = 0, \dots, n-1. \tag{1.3}$$

By classical results (see [3] or [5, p. 422]), if $u_x \not\equiv 0$, such an n always exists. Throughout the paper, we refer to n as the integer satisfying (1.3).

Our first result considers the case where $(0, 0)$ is a degenerate maximum point.

Theorem 1.2 (Morse Lemma for Degenerate Maximum Points). *Let u be a nonconstant solution to* (1.1). *If* $(0, 0)$ *is a degenerate maximum point of u satisfying* (1.2), *then*

$$u_{yy}(0,0) < 0. \tag{1.4}$$

Moreover, if $u_x \not\equiv 0$, the following expansions hold:

(i) *If n is even, we have that*

$$u(x,y) = u(0,0) + \frac{u_{yy}(0,0)}{2}y^2 + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!}\mathrm{Re}(z^n) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}}{n!}(0,0)\,\mathrm{Im}(z^n) + O(|z|^{n+1}), \tag{1.5}$$

with $\frac{\partial^n u}{\partial x^n}(0,0) < 0$.

(ii) *If n is odd, we have that* $\frac{\partial^n u}{\partial x^n}(0,0) = 0$, $\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0) \neq 0$ *and there exists an integer* $l \in [n+1, 2n-2]$ *such that*

$$\frac{\partial^l u}{\partial x^l}(0,0) < 0 \tag{1.6}$$

and

$$u(x,y) = u(0,0) + \frac{u_{yy}(0,0)}{2}y^2 + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{n!}\mathrm{Im}(z^n) + \frac{\frac{\partial^l u}{\partial x^l}(0,0)}{l!}\mathrm{Re}(z^l) + O(|z|^{n+1}). \tag{1.7}$$

Finally, if $l = 2n - 2$, *we have that*

$$\Big(\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\Big)^2 \leq \frac{2[(n-1)!]^2}{(2n-2)!}u_{yy}(0,0)\frac{\partial^{2n-2}u}{\partial x^{2n-2}}(0,0). \tag{1.8}$$

Before we give an idea of the proof, let us make some comments on Theorem 1.2.

Remark 1.3. The same result holds if $(0, 0)$ is a degenerate minimum point of u. In this case, $u_{yy}(0,0) > 0$, $\frac{\partial^n u}{\partial x^n}(0,0) > 0$, $\frac{\partial^l u}{\partial x^l}(0,0) > 0$ and (1.8) does not change.

Remark 1.4. Note that in Theorem 1.2 the assumption $u_x \not\equiv 0$ is not restrictive. Indeed, if $u_x \equiv 0$, then (1.1) becomes

$$-u'' = f(u), \quad u'(0) = 0,$$

and expansions of the solution u follows immediately. This case appears when $u(x, y) = \cos y$, which verifies $-\Delta u = u$ and admits $y = 0$ as maximum points.

In our opinion, (1.4) is the most relevant result of the theorems (see [2] for some properties of solutions satisfying (1.11)). In particular, we get that there is no u verifying (1.1) such that $u(x, y) \sim u(0, 0) - x^4 - y^4$ in a neighborhood of $(0, 0)$!

Some similar properties to (1.5) and (1.7) can be found in [5], where the authors study the properties of the zero-set of solutions u to

$$\Delta u = f(x, u, \nabla u),$$

with

$$|f(x, u, \nabla u)| \le A|u|^\alpha + B|u|^\beta, \quad A, B > 0,\ \alpha, \beta \ge 1.$$

Our result can be seen as an extension of [5, Theorem 1.2] in an appropriate setting.

An interesting particular case of Theorem 1.2 is when the maximum (minimum) point is non-isolated. Here we show that equality holds in (1.8).

Proposition 1.5. *Let u be a nonconstant solution to* (1.1) *and assume that* $(0, 0)$ *is not an* isolated *degenerate maximum point of u satisfying* (1.2). *Then n is odd, $l = 2n - 2$ and the equality holds in* (1.8).

A known case is that of *radial* functions. Here we have that $n = 3$ always occurs in Theorem 1.2.

Corollary 1.6. *Let u be a nonconstant solution to* (1.1) *and suppose that* $(0, 0)$ *verifies* (1.2) *and it is a maximum point of a radial function $u = u(r)$, with $r^2 = x^2 + (y - P)^2$, $P \ne 0$. Then the following expansion holds:*

$$\begin{aligned} u(x, y) = u(0, 0) &- \Big(\frac{f(u(P))}{2} + o(1)\Big)y^2 + \Big(\frac{f(u(P))}{6P} + o(1)\Big)(3x^2 y - y^3) \\ &- \Big(\frac{f(u(P))}{8P^2} + o(1)\Big)(x^4 - 6x^2y^2 + y^4), \end{aligned} \tag{1.9}$$

and

$$(u_{xxy}(0, 0))^2 = \frac{f^2(u(P))}{P^2} = \frac{u_{yy}(0, 0)u_{xxxx}(0, 0)}{3}. \tag{1.10}$$

The next step is to get an analogous result of Theorem 1.2 when $(0, 0)$ is a saddle point of a solution u to (1.1). Here we cannot expect that (1.4) holds for any saddle point to u. Indeed, if we consider

$$u(x, y) = \mathrm{Re}(z^n), \quad \text{with } n \ge 3,$$

we get that u satisfies $\Delta u = 0$, $u_{yy}(0, 0) = 0$, and $(0, 0)$ is a degenerate saddle point. Note that in this case $i[u, (0, 0)] = 1 - n \le -2$, where $i[u, (0, 0)]$ denotes the *index* of ∇u at $(0, 0)$.

Recall that if $(0, 0)$ is an isolated critical point, then the *index* of ∇u at $(0, 0)$ is given by (denoting by $B(0, \epsilon)$ the ball centered at the origin with radius ϵ)

$$i[\nabla u, (0, 0)] = \lim_{\epsilon \to 0} \deg(\nabla u, B(0, \epsilon), (0, 0)).$$

The next result shows that if $(0, 0)$ is an *isolated* saddle point with index greater than -2, then the condition $u_{yy}(0, 0) \ne 0$ is again verified.

Theorem 1.7 (Morse Lemma for Degenerate Saddle Points). *Let u be a solution to* (1.1) *and assume* (1.2). *If* $(0, 0)$ *is an isolated degenerate saddle point of u verifying*

$$i[\nabla u, (0, 0)] \ge -1,$$

then we have that

$$u_{yy}(0, 0) \ne 0. \tag{1.11}$$

Moreover, if $i[u,(0,0)]=-1$, *the following expansions hold:*

(i) *If n is even, we have that*

$$u(x,y)=u(0,0)+\frac{u_{yy}(0,0)}{2}y^2+\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!}\operatorname{Re}(z^n)+\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{n!}\operatorname{Im}(z^n)+O(|z|^{n+1}), \tag{1.12}$$

and if $\frac{\partial^n u}{\partial x^n}(0,0)\neq 0$, *then*

$$\frac{\partial^n u}{\partial x^n}(0,0)u_{yy}(0,0)<0. \tag{1.13}$$

(ii) *If n is odd, we have that*

$$u(x,y)=u(0,0)+\frac{u_{yy}(0,0)}{2}y^2+\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{n!}\operatorname{Im}(z^n)+O(|z|^{n+1}). \tag{1.14}$$

Finally, if $i[u,(0,0)]=0$, *we can only say that*

$$u(x,y)=u(0,0)+\frac{u_{yy}(0,0)}{2}y^2+\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!}\operatorname{Re}(z^n)+\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{n!}\operatorname{Im}(z^n)+O(|z|^{n+1}).$$

We stress that (1.11) holds even if $i[u,(0,0)]=0$. In general this is the "worst" critical point to handle, since its existence does not imply a change of topology in the sub-levels of u. A typical example is given by $u(x,y)=y^2-x^3+3xy^2$. Despite this, (1.11) is still valid.

Now we describe the proof of Theorems 1.2 and 1.7. The basic idea is to mix some algebraic identities satisfied by the derivatives of u and topological properties of ∇u.

The first step (Proposition 3.2) is to prove that if $u_{yy}(0,0)=0$, then necessarily $(0,0)$ is *isolated* and furthermore we have that

$$i[\nabla u,(0,0)]\leq -2,$$

where $i[\nabla u,(0,0)]$ denotes the index of ∇u at $(0,0)$. In this way, recalling that a maximum (minimum) point has index 1 (see Theorem 2.4), we have the claim of (1.4) and (1.11). Next, assuming that (1.11) holds, we use again some algebraic identities satisfied by the derivatives of u in order to write the following expansion (Proposition 3.2):

$$u(x,y)=u(0,0)+\frac{u_{yy}(0,0)+o(1)}{2}y^2+\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!}\operatorname{Re}(z^n)+\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{n!}\operatorname{Im}(z^n)+R(x,y), \tag{1.15}$$

with $R(x,y)=O(|x|^n+|y|^n)$ and n given by (1.3). If $(0,0)$ is a maximum point from (1.15), we deduce (1.5)–(1.8). If u is a saddle point, of course is more difficult to deduce general properties of the coefficients of (1.2). However, if $i[\nabla u,(0,0)]=-1$, some topological arguments allow to deduce (1.13) and (1.14). We believe that Theorems 1.2 and 1.7 are very useful tools for studying qualitative properties of solutions to (1.1).

The last part of the paper (Section 4) is strongly influenced by the paper [4]. Here we assume that u has a *unique* critical point satisfying

$$\begin{cases}-\Delta u=f(u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.16}$$

and extend some results of [4] without requiring that u is a semi-stable solution to (1.1).

More precisely, we want to understand if the maximum point is degenerate or not jointly with the properties of the level set of u. First let us recall some known results where nondegeneracy is proved:

- The Gidas–Ni–Nirenberg theorem [11] in convex and symmetric domains.
- The Cabré–Chanillo Theorem [4] for semi-stable solutions in strictly convex domains.
- Solutions which concentrate at some point [12].

All the previous examples will be discussed with more details in Section 2.

Now we want to explore more closely the properties of the solution u according the degeneracy of its critical points. Our first result generalizes those considered in the previous examples because neither the symmetry of the domain nor any hypothesis on the solution is required.

Theorem 1.8. *Let us consider a solution u to (1.16) and assume that $\partial\Omega$ has strictly positive curvature, $f(0) \geq 0$, and $(0, 0)$ is the only critical point of u in Ω. Then $(0, 0)$ is nondegenerate.*

Note that $f(0) \geq 0$ is used to apply the Hopf lemma to the boundary of Ω but it can be relaxed to $\nabla u \neq (0, 0)$ on $\partial\Omega$ (see Remark 4.1).

The proof of Theorem 1.8 relies on the study of the zero-set of some partial derivatives of u. Similar techniques was used in [7] (see also [15]). An interesting consequence of the previous theorem is the following result.

Corollary 1.9. *Let us consider a solution u to (1.16) with $f(0) \geq 0$. Suppose that $(0, 0)$ is the only critical point of u. Then the following alternative holds: either*
(a) $(0, 0)$ *is a nondegenerate critical point for u, or*
(b) *all level sets of u have a point with nonpositive curvature.*

A curious consequence of the previous proposition is that the behavior of the level sets of a solution to (1.16) seems to be more difficult to predict if the maximum point of u is nondegenerate! In fact, in this case, we can have both convex and non-convex super-level sets (the latter case appears in [13]). An explicit example where (b) holds is given in Example 4.4.

Another consequence of Theorem 1.8 concerns solutions in the whole space.

Corollary 1.10. *Suppose that u verifies*

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^2,$$

with $f(0) \geq 0$. Assume that $(0, 0)$ is the only critical point of u and all level set of u are Jordan curves. Then the following alternative holds: either
(a) $(0, 0)$ *is a nondegenerate critical point for u, or*
(b) *all level sets of u have a points with nonpositive curvature.*

We end this introduction by pointing out that Theorems 1.2 and 1.7 suggest some nice explicit examples. In one of them we prove the existence of a star-shaped domain Ω, where a semi-stable solution for (1.16) admits exactly two critical points. This proves that the assumption on the positivity of the curvature of $\partial\Omega$ in Cabré–Chanillo's result cannot be relaxed.

Our project is to continue to study properties of the level sets of u and the degeneracy of its critical points when u admits two or more critical points.

The paper is organized as follows. In Section 2 we recall some useful preliminaries. In Section 3 we prove Theorems 1.2 and 1.7. In Section 4 we prove Theorem 1.8 and its consequences. Finally, in Section 5 we prove Proposition 1.5 and Corollary 1.6.

2 Known Results

The first result of this section is a direct consequence of the Gidas–Ni–Nirenberg theorem [11]. Since in [11] it is not explicitly stated, we give the proof.

Theorem 2.1. *Let Ω an arbitrary bounded domain in $\mathbb{R}^N$, which is convex in the x_i direction and symmetric with respect to the plane $x_i = 0$ for any $i = 1, \ldots, N$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution to*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in C^1(\Omega)$. Then the Hessian matrix at the origin is diagonal and strictly negative definite.

Proof. We give the proof only for $N = 2$; there is no difference in higher dimensions. By the Gidas–Ni–Nirenberg theorem, the function $v = \frac{\partial u}{\partial x_1}$ satisfies

$$\begin{cases} -\Delta v = f'(u)v & \text{in } \Omega \cap \{x_1 < 0\}, \\ v > 0 & \text{in } \Omega \cap \{x_1 < 0\}, \\ v = 0 & \text{on } \Omega \cap \{x_1 = 0\}. \end{cases}$$

Since $v(0, x_2) = 0$ in Ω, we get that $0 = \frac{\partial v}{\partial x_2}(0,0) = \frac{\partial^2 u}{\partial x_1 \partial x_2}(0,0)$, and by the Hopf lemma, it follows that $\frac{\partial^2 u}{\partial x_1^2}(0,0) < 0$. In the same way, we get that $\frac{\partial^2 u}{\partial x_2^2}(0,0) < 0$, which ends the proof. □

The second result, due to Cabré and Chanillo, removes the symmetry assumption requiring that u belongs to a suitable class of solutions.

Theorem 2.2. *Let Ω be a smooth bounded and convex domain in $\mathbb{R}^2$ whose boundary has positive curvature. Let u be a smooth nontrivial solution to*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with $f \in C^1(\Omega)$ and $f \geq 0$. Moreover, assume that u is a semi-stable solution to (2.1), i.e., the first eigenvalue of the linearized operator $\mathcal{L}$ to (2.1) at u defined as

$$\mathcal{L} = -\Delta - f'(u)I$$

is nonnegative. Then u has a unique critical point x_0 in Ω. Moreover, x_0 is the maximum of u and it is nondegenerate.

For the proof, see [4].

The last result concerns solutions which concentrate at some points. There are several results of this type (also in higher dimensions) for various nonlinearities. We just mention one of them.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let u_λ be a solution to*

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfying $\lambda \int_\Omega e^u \to 8\pi$ as $\lambda \to 0$. Let $x_\lambda \in \Omega$ be the point where $u_\lambda(x_\lambda) = \|u_\lambda\|_\infty$. Then x_λ is a nondegenerate critical point of u_λ for λ small enough.

Proof. In [12, Lemma 5] it was shown that

$$z_\lambda(x) = u_\lambda(\delta_\lambda x + x_\lambda) - u_\lambda(x_\lambda) \to \log \frac{1}{(1 + \frac{|x|^2}{8})^2} \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2),$$

with $\delta_\lambda^2 = \frac{1}{\lambda e^{u_\lambda(x_\lambda)}} \to 0$. Hence, for $i, j = 1, 2$, we get

$$\det\Big\{\frac{\partial^2 u}{\partial x_i \partial x_j}(x_\lambda)\Big\} = \det\Big\{\frac{\partial^2 z}{\partial x_i \partial x_j}(0,0)\Big\} \to \frac{1}{4},$$

and so the claim follows. □

The next result is classical and it will be used in the proof of Theorem 1.2.

Theorem 2.4. *Let $(0,0)$ be an isolated minimum (maximum) point of $f\colon B_1 \to \mathbb{R}$, where $B_1 \subset \mathbb{R}^2$. Then*

$$i[\nabla f, (0,0)] = 1,$$

where $i(\nabla f, (0,0))$ denotes the index of f at $(0,0)$.

For the proof, see [1].

In the proof of Theorem 1.7 we need to compute the index of some suitable vector field. The next lemma will be useful for it.

Lemma 2.5. *For $m > 1$ being an integer, and $B_1 \subset \mathbb{R}^2$ being an open neighborhood of $(0,0)$, assume that $f \in C(B_1, \mathbb{R}^2)$ can be written in the form*

$$f(x,y) = L(x,y) + H_m(x,y) + R(x,y),$$

where $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and not invertible, H_m is homogeneous of order m and $R(x,y)$ verifies

$$\lim_{(x,y)\to(0.0)} \frac{R(x,y)}{(x^2+y^2)^{\frac{m}{2}}} = 0.$$

If $H_m(x,y) \notin \mathrm{Im}(L)$ for any $x(x,y) \in \mathrm{Ker}(L) \setminus \{(0,0)\}$, then $i(f,(0,0))$ is well defined and

$$i(f,(0,0)) = i(L + JP,(0,0)) \cdot i(J^{-1}QH_m|_{\mathrm{Ker}(L)},(0,0)),$$

where Q is any projector such that $\mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $J\colon \mathrm{Ker}(L) = \mathrm{Im}(Q)$ any isomorphism.

For the proof, see [10, Corollary 6.5.1, p. 51].

3 Proof of Theorems 1.2 and 1.7

In this section we prove Theorems 1.2 and 1.7. We have the following.

Proposition 3.1. *Let us suppose that $(0,0)$ is a degenerate critical point of a nonconstant function u satisfying (1.1) and (1.2). If*

$$u_{yy}(0,0) = 0,$$

then we have that $i[\nabla u,(0,0)]$ is well defined and

$$i[\nabla u,(0,0)] \le -2. \tag{3.1}$$

Proof. Differentiating (1.1) with respect to x and y, we get that u_x and u_y are solutions to

$$-\Delta v = f'(u)v \quad \text{in } \Omega. \tag{3.2}$$

First let us consider the case where $u_x \equiv 0$, and consider $z(y) = u_y(0,y)$ which satisfies (since $u_{yy}(0,0) = 0$)

$$\begin{cases} -z'' = f'(u(0,y))z & \text{in } (-\delta,\delta), \\ z(0) = z'(0) = 0. \end{cases}$$

So, $z \equiv 0$ and $u(0,y)$ is constant. Since $u_x \equiv 0$, we get that u is constant in B_1, which contradicts our assumption.

Hence, $u_x \not\equiv 0$, and since u_x solves (3.2) as pointed out in the introduction, there is an integer $n \ge 3$ such that

$$\frac{\partial^n u}{\partial y^{n-k}\partial x^k}(0,0) \ne 0 \quad \text{for some } 1 \le k \le n. \tag{3.3}$$

Let us choose the minimum integer n such that (3.3) holds for some $1 \le k \le n$.

Next we prove that all derivatives of u with order less than n are zero, i.e.,

$$\frac{\partial^m u}{\partial y^{m-h}\partial x^h}(0,0) = 0 \quad \text{for any } 0 \le h \le m < n,\ m \ge 3. \tag{3.4}$$

This is obvious if $h > 0$ by the definition of n. Hence, consider $h = 0$ and by contradiction suppose that

$$\frac{\partial^m u}{\partial y^m}(0,0) \ne 0, \tag{3.5}$$

where we take the minimum integer m such that (3.5) holds. Differentiating (1.1) $m-2$ times, we get (using the minimality of m)

$$0 \neq \frac{\partial^m u}{\partial y^m}(0,0) = -\frac{\partial^m u}{\partial y^{m-2}\partial x^2}(0,0),$$

which is not possible again by the definition of n. Hence, (3.4) holds.

Finally, we differentiate (1.1) $n-2$ times with respect to x and y. We get that

$$\frac{\partial^n u}{\partial y^{n-k}\partial x^k}(0,0) = -\frac{\partial^n u}{\partial y^{n+2-k}\partial x^{k-2}}(0,0) \quad \text{for any } k = 2,\dots,n,\ n>2, \tag{3.6}$$

An important consequence of (3.6) is that

$$\begin{cases} \dfrac{\partial^n u}{\partial x^n}(0,0) = -\dfrac{\partial^n u}{\partial x^{n-2}\partial y^2}(0,0) = \cdots \Rightarrow \dfrac{\partial^n u}{\partial x^{n-2h}\partial y^{2h}}(0,0) = (-1)^h\dfrac{\partial^n u}{\partial x^n}(0,0), \\ \dfrac{\partial^n u}{\partial x^{n-1}\partial y}(0,0) = -\dfrac{\partial^n u}{\partial x^{n-3}\partial y^3}(0,0) = \cdots \Rightarrow \dfrac{\partial^n u}{\partial x^{n-2h-1}\partial y^{2h+1}}(0,0) = (-1)^h\dfrac{\partial^n u}{\partial x^{n-1}\partial y}(0,0). \end{cases} \tag{3.7}$$

This gives, by the definition of n, that

$$\Big(\frac{\partial^n u}{\partial x^n}(0,0), \frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\Big) \neq (0,0). \tag{3.8}$$

The previous computations allow to write the following Taylor formula for u:

$$\begin{aligned} n!u(x,y) &= n!u(0,0) + \sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n u}{\partial x^{n-k}\partial y^k}(0,0)x^{n-k}y^k + R(x,y) \\ &= n!u(0,0) + \sum_{h=0}^{[\frac{n}{2}]}\binom{n}{2h}\frac{\partial^n u}{\partial x^{n-2h}\partial y^{2h}}(0,0)x^{n-2h}y^{2h} \\ &\quad + \sum_{h=0}^{[\frac{n-1}{2}]}\binom{n}{2h+1}\frac{\partial^n u}{\partial x^{n-2h-1}\partial y^{2h+1}}(0,0)x^{n-2h-1}y^{2h+1} + R(x,y) \\ &= n!u(0,0) + \frac{\partial^n u}{\partial x^n}(0,0)\sum_{h=0}^{[\frac{n}{2}]}\binom{n}{2h}(-1)^k x^{n-2h}y^{2h} + \frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0) \\ &\quad + \sum_{h=0}^{[\frac{n-1}{2}]}\binom{n}{2h+1}(-1)^h x^{n-2h-1}y^{2h+1} + R(x,y) \\ &= n!u(0,0) + \frac{\partial^n u}{\partial x^n}(0,0)\operatorname{Re}(z^n) + \frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\operatorname{Im}(z^n) + R(x,y), \end{aligned} \tag{3.9}$$

with $R(x,y) = O(|x|^{n+1} + |y|^{n+1})$. Note that in the last line of (3.9), we used that

$$z^n = (x+iy)^n = \underbrace{\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(-1)^k x^{n-2k}y^{2k}}_{\operatorname{Re}(z^n)} + i\underbrace{\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}(-1)^k x^{n-2k-1}y^{2k+1}}_{\operatorname{Im}(z^n)}.$$

Then differentiating (3.9) with respect to x and y, we get

$$\begin{cases} u_x(x,y) = \dfrac{\partial^n u}{\partial x^n}(0,0)\dfrac{1}{(n-1)!}\operatorname{Re}(z^{n-1}) + \dfrac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\dfrac{1}{(n-1)!}\operatorname{Im}(z^{n-1}) + \dfrac{\partial R(x,y)}{\partial x}, \\ u_y(x,y) = -\dfrac{\partial^n u}{\partial x^n}(0,0)\dfrac{1}{(n-1)!}\operatorname{Im}(z^{n-1}) + \dfrac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\dfrac{1}{(n-1)!}\operatorname{Re}(z^{n-1}) + \dfrac{\partial R(x,y)}{\partial y}, \end{cases} \tag{3.10}$$

with

$$\Big|\frac{\partial R(x,y)}{\partial x}\Big|, \Big|\frac{\partial R(x,y)}{\partial y}\Big| \leq C\sum_{k=0}^{n}|x|^{n-k}|y|^k \leq C(|x|^n + |y|^n) \leq C|z|^n.$$

Set

$$\alpha = \frac{\partial^n u}{\partial x^n}(0,0), \quad \beta = \frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0).$$

A first consequence of (3.10) is that $(0,0)$ is an *isolated* critical point. Indeed, if $(u_x(x,y), u_y(x,y)) = (0,0)$, by (3.10), we get

$$\begin{aligned}(\alpha^2+\beta^2)([\mathrm{Re}(z^{n-1})]^2 + [\mathrm{Im}(z^{n-1})]^2) &= [(n-1)!]^2\Big[\Big(\frac{\partial R(x,y)}{\partial x}\Big)^2 + \Big(\frac{\partial R(x,y)}{\partial y}\Big)^2\Big]\\ \Leftrightarrow (\alpha^2+\beta^2)|z|^{2n-2} &\le C|z|^{2n}\end{aligned} \tag{3.11}$$

and since $(\alpha,\beta) \neq (0,0)$, we get that any other critical point satisfies $|z| \ge C$. Hence, the index $i(\nabla u,(0,0))$ is well defined.

The final step of the proof is to show that $i(\nabla u,(0,0)) \le -2$. Let us introduce the vector field

$$\mathcal{L} = (\mathcal{L}_1,\mathcal{L}_2) = \Big(\frac{\alpha}{(n-1)!}\mathrm{Re}(z^{n-1}) + \frac{\beta}{(n-1)!}\mathrm{Im}(z^{n-1}), \ \frac{\beta}{(n-1)!}\mathrm{Re}(z^{n-1}) - \frac{\alpha}{(n-1)!}\mathrm{Im}(z^{n-1})\Big)$$

and the following homotopy for $t \in [0,1]$:

$$H(t,x,y) = t\nabla u + (1-t)\mathcal{L}.$$

Let us show that $H(t,x,y) \neq (0,0)$ for any $t \in [0,1]$ and $x^2+y^2 = \delta > 0$ small. By contradiction, suppose that $H(t,x,y) = (0,0)$ for some $t \in [0,1]$ and $x^2+y^2 = \delta$ small enough. Then arguing as in (3.11), we get a contradiction. This means that

$$i(\nabla u,(0,0)) = i(\mathcal{L},(0,0))$$

(in other words we can neglect the remainder term ∇R in (3.10)). Finally, let us compute $i(\mathcal{L},(0,0))$. Suppose that $\alpha \neq 0$ (if $\alpha = 0$, then, by (3.8), $\beta \neq 0$ and the proof is the same) and set

$$\mathcal{M} = \alpha(n-1)!(\mathrm{Re}(z^{n-1}), -\mathrm{Im}(z^{n-1})).$$

Using again the homotopy $H(t,x,y) = t\mathcal{L} + (1-t)\mathcal{M}$ for $t \in [0,1]$, we get that $H(t,x,y) = (0,0)$ implies

$$\alpha\,\mathrm{Re}(z^{n-1}) + t\beta\,\mathrm{Im}(z^{n-1}) = 0, \quad t\beta\,\mathrm{Re}(z^{n-1}) - t\alpha\,\mathrm{Im}(z^{n-1}) = 0,$$

and then $z = 0$. This implies that

$$i(\mathcal{L},(0,0)) = i(\mathcal{M},(0,0)),$$

and by known arguments (see, for example, [9, Theorem 3.1]),

$$i(\mathcal{M},(0,0)) = 1-n.$$

So, we have that

$$i(\nabla u,(0,0)) = 1-n \le -2,$$

since $n \ge 3$. Then (3.1) follows. □

Proposition 3.2. *Let us suppose that* $(0,0)$ *is a degenerate critical point of u satisfying* (1.1) *and* (1.2) *with*

$$i[\nabla u,(0,0)] > -2. \tag{3.12}$$

Then if $u_x \not\equiv 0$, *the following expansion holds:*

$$u(x,y) = u(0,0) + \frac{u_{yy}(0,0)+o(1)}{2}y^2 + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!}\mathrm{Re}(z^n) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{n!}\mathrm{Im}(z^n) + R(x,y), \tag{3.13}$$

with $R(x,y) = O(|x|^n + |y|^n)$ *and n is given by* (1.3). *Moreover,*

$$\Big(\frac{\partial^n u}{\partial x^n}(0,0), \frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)\Big) \neq (0,0). \tag{3.14}$$

Proof. By (3.12) and Proposition 3.1, we get that $u_{yy}(0,0) \neq 0$.

Next we prove (3.13). Since $u_x \not\equiv 0$, as in Proposition (3.1), let us consider the integer $n \geq 3$ such that (1.3) holds. Then we have that

$$\frac{\partial^m u}{\partial x^{m-k} \partial y^k}(0,0) = 0 \quad \text{for any } m < n \text{ and } k = 0, \dots, m-1. \tag{3.15}$$

Note that, unlike what happens in (3.4) of Proposition 3.1, since $u_{yy}(0,0) \neq 0$, (3.15) does not hold for $k = m$. Next let us differentiate (1.1) $n-2$ times with respect to x and y. We get that

$$\frac{\partial^n u}{\partial x^{n-k} \partial y^k}(0,0) = -\frac{\partial^n u}{\partial x^{n+2-k} \partial y^{k-2}}(0,0) + \frac{\partial^{n-2}(f(u))}{\partial x^{n-k} \partial y^{k-2}}(0,0) \quad \text{for any } k = 2, \dots, n. \tag{3.16}$$

Observe that in $\frac{\partial^{n-2}(f(u))}{\partial x^{n-k} \partial y^{k-2}}(0,0)$ appear the derivatives of u with order $1, \dots, n-2$. All these are zero by (3.15) except when $k = n$. So, we have

$$\frac{\partial^{n-2}(f(u))}{\partial x^{n-k} \partial y^{k-2}}(0,0) = 0 \quad \text{for any } k = 2, \dots, n-1. \tag{3.17}$$

As in the proof of Proposition 3.1, we get, from (3.16) and (3.17),

$$\begin{cases} \frac{\partial^n u}{\partial x^n}(0,0) = -\frac{\partial^n u}{\partial x^{n-2} y^2}(0,0) = \cdots \\ \qquad \Rightarrow \frac{\partial^n u}{\partial x^{n-2k} \partial y^{2k}}(0,0) = (-1)^k \frac{\partial^n u}{\partial x^n}(0,0), \qquad k = 1, \dots, \left[\frac{n-1}{2}\right], \\ \frac{\partial^n u}{\partial x^{n-1} \partial y}(0,0) = -\frac{\partial^n u}{\partial x^{n-3} \partial y^3}(0,0) = \cdots \\ \qquad \Rightarrow \frac{\partial^n u}{\partial x^{n-2k-1} \partial y^{2k+1}}(0,0) = (-1)^k \frac{\partial^n u}{\partial x^{n-1} \partial y}(0,0), \quad k = 1, \dots, \left[\frac{n-2}{2}\right], \end{cases}$$

By the previous equality and the definition of n, we get (3.14).

Now using the previous identities in Taylor's formula, we get

$$\begin{aligned} u(x,y) &= u(0,0) + \frac{u_{yy}(0,0)}{2} y^2 + \sum_{m=3}^{n} \frac{\partial^m u}{\partial y^m}(0,0) y^m + \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} \frac{\partial^n u}{\partial x^{n-k} \partial y^k}(0,0) x^{n-k} y^k + R(x,y) \\ &= u(0,0) + \frac{u_{yy}(0,0) + o(1)}{2} y^2 + \frac{1}{n!} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{\partial^n u}{\partial x^{n-2k} \partial y^{2k}}(0,0) x^{n-2k} y^{2k} \\ &\quad + \frac{1}{n!} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \frac{\partial^n u}{\partial x^{n-2k-1} \partial y^{2k+1}}(0,0) x^{n-2k-1} y^{2k+1} + R(x,y) \\ &= u(0,0) + \frac{u_{yy}(0,0) + o(1)}{2} y^2 + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (-1)^k y^{n-2k} x^{2k} \\ &\quad + \frac{\frac{\partial^n u}{\partial x^{n-1} \partial y}(0,0)}{n!} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1} + R(x,y) \\ &= u(0,0) + \frac{u_{yy}(0,0) + o(1)}{2} y^2 + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} \operatorname{Re}(z^n) + \frac{\frac{\partial^n u}{\partial x^{n-1} \partial y}(0,0)}{n!} \operatorname{Im}(z^n) + R(x,y), \end{aligned}$$

with $R(x,y) = O(|x|^n + |y|^n)$. This ends the proof. ☐

Now we are in position to give the proof of our theorems.

Proof of Theorem 1.2. Since $(0,0)$ is a maximum point, by Theorem 2.4, we get that $i[\nabla u, (0,0)] = 1$. Hence, Propositions 3.1 implies (1.4). Next to verify (1.5) and (1.7), we consider the cases n even and n odd separately.

The case n even. From (3.13), we have to show that $\frac{\partial^n u}{\partial x^n}(0,0) < 0$. If

$$\frac{\partial^n u}{\partial x^n}(0,0) \neq 0,$$

taking $y = 0$ in (3.13) and using that $(0,0)$ is a maximum point, we get

$$u(0,0) \geq u(x,0) = u(0,0) + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} x^n + O(|x|^{n+1}),$$

and because n is even the claim follows by choosing $|x|$ small enough. Hence, suppose that $\frac{\partial^n u}{\partial x^n}(0,0) = 0$. By (3.14), we have that $\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0) \neq 0$, and choosing $y = Bx^{n-1}$ in (3.13), we get

$$u(0,0) \geq u(x, Bx^{n-1}) = u(0,0) + \Big(B^2 \frac{u_{yy}(0,0) + o(1)}{2} + B \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{n!}\Big) x^{2n-2} + o(|x|^{2n-2}) > u(0,0)$$

if B satisfies

$$B^2 \frac{u_{yy}(0,0) + o(1)}{2} + B \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{n!} > 0.$$

This gives a contradiction.

The case n odd. First let us prove that $\frac{\partial^n u}{\partial x^n}(0,0) = 0$. As in the previous case let us test (3.13) for $y = 0$. We get

$$u(0,0) \leq u(x,y) = u(0,0) + \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} x^n + O(|x|^{n+1}),$$

and since n *is odd* we necessarily have that $\frac{\partial^n u}{\partial x^n}(0,0) = 0$ and by (3.14) we deduce that $\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0) \neq 0$.

Now we prove (1.7) and (1.8). We claim that

$$\text{there exists } l \leq 2n-2 \text{ such that } \frac{\partial^l u}{\partial x^l}(0,0) \neq 0. \tag{3.18}$$

By contradiction, let us suppose that

$$\frac{\partial^{n+1} u}{\partial x^{n+1}}(0,0) = \frac{\partial^{n+2} u}{\partial x^{n+2}}(0,0) = \cdots = \frac{\partial^{2n-2} u}{\partial x^{2n-2}}(0,0) = 0, \tag{3.19}$$

and complete the expansion in (3.13) adding the terms up to the order $2n-2$. We get

$$\begin{aligned} u(x,y) = u(0,0) &+ \frac{u_{yy}(0,0) + o(1)}{2} y^2 + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{n!} \operatorname{Im}(z^n) + \frac{1}{(n+1)!} \sum_{k=1}^{n+1} \binom{n+1}{k} \frac{\partial^{n+1} u}{\partial x^{n+1-k}\partial y^k}(0,0) x^{n+1-k} y^k \\ &+ \cdots + \frac{1}{(2n)!} \sum_{k=1}^{2n} \binom{2n}{k} \frac{\partial^{2n} u}{\partial x^{2n-k}\partial y^k}(0,0) x^{2n-k} y^k + R(x,y), \end{aligned} \tag{3.20}$$

with $R(x,y) = O(|x|^{2n+1} + |y|^{2n+1})$. Setting $y = Ax^{n-1}$ in (3.20) ($A > 0$ will be chosen later) and observing that $l - k + (n-1)k > 2n-2$ for any $k \geq 1$ and $l \in [n+1, 2n-2]$, we derive that

$$u(x, x^{n-1}) = u(0,0) + \Big(\frac{u_{yy}(0,0)}{2} A^2 + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!} A + o(1)\Big) x^{2n-2} + o(x^{2n-2}).$$

Choosing

$$\frac{u_{yy}(0,0)}{2} A^2 + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!} A > 0$$

and $|x|$ small enough, we have that $u(x, x^{n-1}) > u(0,0)$, which contradicts the fact that $(0,0)$ is a maximum point. Hence, (3.18) holds, and choosing the minimum integer l in (3.18), we get that (3.20) becomes

$$u(x,y) = u(0,0) + \frac{u_{yy}(0,0) + o(1)}{2} y^2 + \Big(\frac{\frac{\partial^n u}{\partial x^{n-1} y}(0,0)}{n!} + o(1)\Big) \operatorname{Im}(z^n) + \Big(\frac{1}{l!} + o(1)\Big) \frac{\partial^l u}{\partial x^l}(0,0) x^l,$$

which is equivalent to (1.7). Since $(0,0)$ is a maximum point, we get that $\frac{\partial^l u}{\partial x^l}(0,0) < 0$ for $l \in [n+1, 2n-2]$ and, finally, if $l = 2n-2$, setting $y = Ax^{n-1}$ in (1.7), we get

$$u(x,y) = u(0,0) + \Big(\frac{u_{yy}(0,0)}{2}A^2 + \frac{\frac{\partial^n u}{\partial x^{n-1} y}(0,0)}{(n-1)!}A + \frac{\frac{\partial^l u}{\partial x^l}(0,0)}{l!} + o(1)\Big)x^{2n-2},$$

which implies

$$\frac{u_{yy}(0,0)}{2}A^2 + \frac{\frac{\partial^n u}{\partial x^{n-1} y}(0,0)}{(n-1)!}A + \frac{\frac{\partial^l u}{\partial x^l}(0,0)}{l!} \le 0 \quad \text{for any } A \in \mathbb{R}.$$

Then (1.8) holds and this ends the proof. □

Proof of Theorem 1.7. Since $i[\nabla f, (0,0)] \ge -1$, Proposition 3.1 applies and so $u_{yy}(0,0) \ne 0$.

In order to prove (1.12)–(1.14), we first remark that $u_x \not\equiv 0$. Otherwise, we get that $u = u(y)$ verifies $-u'' = f(u)$, and since $u_{yy} \ne 0$, we deduce that $(0,0)$ is a maximum or minimum point for u. So, Proposition (3.2) applies, and as in the proof of Theorem 1.2 we use (3.13) to handle the cases n even and n odd. However, here the condition

$$i(\nabla u, (0,0)) = -1$$

plays a crucial role.

The case n even. From (3.13), we get that

$$\begin{cases} u_x = \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1}) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}) + \frac{\partial R(x,y)}{\partial x}, \\ u_y = (u_{yy}(0,0) + o(1))y - \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}) + n\frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1}) + \frac{\partial R(x,y)}{\partial y}. \end{cases} \tag{3.21}$$

Then let us consider the vector field $f(x,y) = (f_1(x,y), f_2(x,y))$ given by

$$\begin{cases} f_1(x,y) = \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1}) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}) + \frac{\partial R(x,y)}{\partial x}, \\ f_2(x,y) = u_{yy}(0,0)y - \frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1}) + \frac{\partial R(x,y)}{\partial y}. \end{cases} \tag{3.22}$$

If $\frac{\partial^n u}{\partial x^n}(0,0) \ne 0$, then f satisfied the assumptions of Lemma 2.5, with

$$m = n-1, \quad L(x,y) = (0, u_{yy}(0)y),$$
$$H_{n-1}(x,y) = \Big(\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1}) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}),\ -\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{(n-1)!}\operatorname{Im}(z^{n-1}) + \frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!}\operatorname{Re}(z^{n-1})\Big),$$
$$R(x,y) = \Big(\frac{\partial R(x,y)}{\partial x}, \frac{\partial R(x,y)}{\partial y}\Big).$$

Moreover,

$$N(L) = (x,0), \quad \text{with } x \in \mathbb{R}, \qquad R(L) = (0,y), \quad \text{with } y \in \mathbb{R},$$
$$Q(x,y) = (x,0) = P(x,y) \quad \text{and} \quad J = I.$$

We claim that the main assumption in Lemma 2.5 , $H_{n-1}|_{N(L)\setminus 0} \notin R(L)$, is satisfied. Indeed, if $(x,y) \in N(L)\setminus 0$, then $y = 0$ and $x \ne 0$. Hence,

$$H_{n-1}(x,0) = \frac{1}{(n-1)!}\Big(\frac{\partial^n u}{\partial x^n}(0,0)x^{n-1}, \frac{\partial^n u}{\partial x^{n-1} y}(0,0)x^{n-1}\Big),$$

and since $\frac{\partial^n u}{\partial x^n}(0,0) \ne 0$, this does not belong to $R(L) = (0,y)$. Then the claim of Lemma 2.5 yields

$$i[f,(0,0)] = i[L + JP, 0]\, i[J^{-1}QH_{n-1}|_{N(L)}, 0],$$

and since in our case,

$$(L+JP)(x,y) = L(x,y) + P(x,y) = (0, u_{yy}(0,0)y) + (x,0) = (x, u_{yy}(0,0)y),$$

we get that

$$i[L+JP,(0,0)] = \operatorname{sgn}(u_{yy}(0,0)).$$

On the other hand, we have that

$$(J^{-1}QH_{n-1})(x,0) = Q\frac{1}{(n-1)!}\Big(\frac{\partial^n u}{\partial x^n}(0,0)x^{n-1}, \frac{\partial^n u}{\partial x^{n-1}y}(0,0)x^{n-1}\Big) = \Big(\frac{1}{(n-1)!}\frac{\partial^n u}{\partial x^n}(0,0)x^{n-1}, 0\Big),$$

and since n is even,

$$i[J^{-1}QH_{n-1}|_{N(L)}, (0,0)] = \operatorname{sgn}\Big(\frac{\partial^n u}{\partial x^n}(0,0)\Big).$$

Hence, we get that

$$i[f,(0,0)] = \operatorname{sgn}(u_{yy}(0,0))\operatorname{sgn}\Big(\frac{\partial^n u}{\partial x^n}(0,0)\Big).$$

Finally, by (3.21) and (3.22), we get that ∇u is a small perturbation of the vector field f in a suitable small ball centered at $(0,0)$. Then $i[f,(0,0)] = i[\nabla u,(0,0)]$ and, since by assumption $i[\nabla u,(0,0)] = -1$, (1.13) follows.

The case n odd. Here we proceed as in the case where n is even. Let us suppose that $\frac{\partial^n u}{\partial x^n}(0,0) \neq 0$ and using the same notation, we get that $i[f,(0,0)]$ is well posed and again

$$i[f,(0,0)] = i[L+JP,0]\,i[J^{-1}QH_{n-1}|_{N(L)},0].$$

However, since in this case n is odd, we have that $(J^{-1}QH_{n-1})(x,0) = 0$, and so $i[f,(0,0)] = 0$. As in the previous case, this implies $i[\nabla u,(0,0)] = 0$, a contradiction. Then we have that

$$\frac{\partial^n u}{\partial x^n}(0,0) = 0. \tag{3.23}$$

Hence, (3.13) and (3.23) imply (1.14). □

4 Nondegeneracy of Solutions with One Critical Point

In this section we consider a solution u to (1.16) with $f(0,0) \geq 0$ and $\partial\Omega$ with positive curvature. Moreover, we assume that $(0,0) \in \Omega$ is the *unique* critical point of u (of course its maximum).

First let us recall some results proved in [4]. For $\theta \in [0,\pi)$, set

$$u_\theta = \cos\theta u_x + \sin\theta u_y,$$

and

$$\mathcal{N}_\theta = \{(x,y) \in \Omega : u_\theta(x,y) = 0\}.$$

We have the following result.

Lemma 4.1. *We have that $\mathcal{N}_\theta \cap \partial\Omega$ consists of exactly two points P_1 and P_2 and, around each P_i, $\mathcal{N}_\theta$ is a smooth curve that intersects $\partial\Omega$ transversally at its end-point P_i, $i = 1,2$.*

Proof. Let $P \in \mathcal{N}_\theta \cap \partial\Omega$ and let $\nu = (\nu_x, \nu_y)$ be the exterior unit normal to Ω. Then, since $u = 0$ on $\partial\Omega$, we have that $u_\theta(P) = \frac{\partial u}{\partial\nu}(P)(\cos\theta\nu_x(P) + \sin\theta\nu_y(P))$. Since $f(0) \geq 0$, by the Hopf lemma, we get that $\frac{\partial u}{\partial\nu}(P) < 0$, and so $u_\theta(P) = 0$ if and only if the vector $(\cos\theta, \sin\theta)$ is orthogonal to the normal ν, and since the curvature of $\partial\Omega$ is positive, it happens exactly at two points P_1 and P_2.

Next let us show that around each P_i, $i = 1,2$, $\mathcal{N}_\theta$ is a smooth curve. We have that

$$\langle\nabla u_\theta, (\cos\theta, \sin\theta)\rangle = \cos^2\theta u_{xx} + 2\cos\theta\sin\theta u_{xy} + \sin^2\theta u_{yy}, \tag{4.1}$$

and using that $u = 0$ on $\partial\Omega$ and $(\cos\theta, \sin\theta)$ is orthogonal to ν, we get that

$$\cos\theta = \frac{u_y(P)}{\frac{\partial u}{\partial\nu}(P)} \quad \text{and} \quad \sin\theta = -\frac{u_x(P)}{\frac{\partial u}{\partial\nu}(P)}.$$

Hence, (4.1) becomes

$$\langle\nabla u_\theta, (\cos\theta, \sin\theta)\rangle = u_y^2(P)u_{xx} + 2u_x u_y(P)u_{xy} + u_x^2(P)u_{yy} \neq 0,$$

because the curvature of $\partial\Omega$ is positive. Then the implicit function theorem applies and the claim follows. □

Remark 4.2. The condition $f(0) \geq 0$ is used to apply the Hopf Lemma on $\partial\Omega$ and deduce that $\frac{\partial u}{\partial\nu} < 0$. What we need in the proof of Lemma 4.1 is that $\nabla u \neq (0, 0)$ on $\partial\Omega$.

We say that $P \in \Omega$ is a singular point for u_θ if

$$\nabla u_\theta(P) = (0, 0).$$

Lemma 4.3. *If $P \in \mathcal{N}_\theta$ is a singular point for u_θ, then the curve $\mathcal{N}_\theta$ "encloses" a subdomain $\omega \subset \Omega$. More precisely, there exists a subdomain $\omega \subset \Omega$ such that*
- *$\partial\omega$ is a Jordan curve,*
- *$\partial\omega \cap \partial\Omega = \emptyset$,*
- *$\partial\omega \subset \mathcal{N}_\theta$.*

For the proof, see [4].

Now we are in position to prove Theorem 1.8.

Proof of Theorem 1.8. By contradiction, let us suppose that $(0, 0)$ is degenerate. Up to a rotation, we can assume (1.2) and then

$$\nabla u_x(0, 0) = 0.$$

By Lemma 4.3 the curve $\mathcal{N}_0$ encloses a region $\omega \subset\subset \Omega$. Moreover, u_x verifies

$$\begin{cases} -\Delta u_x = f'(u)u_x & \text{in } \omega, \\ u_x = 0 & \text{on } \partial\omega, \end{cases} \tag{4.2}$$

and we can suppose that $u_x > 0$ in ω. Next let us take $Q \in \omega$ and set

$$w(x, y) = u_y(Q)u_x(x, y) - u_x(Q)u_y(x, y)$$

and

$$\mathcal{M} = \{(x, y) \in \Omega : w(x, y) = 0\}.$$

Of course $Q \in \mathcal{M}$, and since $\nabla u(Q) \neq (0, 0)$ (uniqueness of the critical point), the definition of w is well posed. Moreover, w satisfies

$$-\Delta w = f'(u)w \quad \text{in } \Omega,$$

and by [5], we have that around Q the set $\mathcal{M}$ consists of a finite number of curves intersecting transversally at Q (this is actually a smooth curve if $\nabla w(Q) \neq 0$).

We have the following alternative:

Case 1: $\mathcal{M} \cap \partial\omega$ is given by at most one point. In this case, we have that $\mathcal{M}$ encloses at least one region in ω and there exists $D \subset \omega$ such that

$$\begin{cases} -\Delta w = f'(u)w & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

This implies that the first eigenvalue $\lambda_1(-\Delta - f'(u)I) < 0$ in ω, and this contradicts (4.2).

Case 2: $\mathcal{M} \cap \partial\omega$ is given by at least two points. In this case we would have that the derivatives u_θ and w intersect at two different points and this contradicts the uniqueness of the critical point unless $w = Cu_x$. This implies that $u_\theta(Q) = Cw(Q) = 0$ and so $Q \in \partial\omega$. This is a contradiction, which ends the proof. □

Proof of Corollary 1.9. Let us suppose that $(0,0)$ is a degenerate critical point of u and consider a level set $\mathcal{C}=\{u=c\}$ with $0<c<\|u\|_\infty$. Since $(0,0)$ is the unique critical point, we have that $\mathcal{C}$ is a smooth closed curve. By contradiction, suppose that the curvature of $\mathcal{C}$ is positive everywhere. We have that u solves the following problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \{u>c\},\\ u=c & \text{on } \mathcal{C}.\end{cases}$$

The proof of Theorem 1.8 applies with $\partial\Omega$ replaced by $\mathcal{C}$. The only difference is that here the condition $f(0)\ge 0$ is not sufficient to apply the Hopf Lemma to $\mathcal{C}$ in Lemma 4.1 because $c\neq 0$. However, if $\frac{\partial u}{\partial \nu}(P)=0$ for some $P\in\mathcal{C}$, then $\nabla u(P)=0$ and this contradicts the uniqueness of the critical point. The claim of Theorem 1.8 gives that $(0,0)$ is nondegenerate, a contradiction. This means that any level set $\{u=c\}$ must contain a point with nonpositive curvature. This ends the proof. □

The proof of Corollary 1.10 is the same as that of Corollary 1.9.

We end this section with two interesting examples of solution to (1.16), both suggested by Theorems 1.2 and 1.7. The first one shows that (b) of Corollary 1.9 can occur.

Example 4.4. If we put $n=4$ in (1.5), we are lead to consider the set

$$D_\alpha=\Big\{(x,y)\in\mathbb{R}^2:\frac{1}{2}y^2+x^4-6x^2y^2+y^4<\alpha\Big\}.$$

It is not difficult to show that there exists $\alpha_0>0$ such that for any $0<\alpha<\alpha_0$, the set ∂D_α is a closed curve. Note that ∂D_α shrinks to the origin as $\alpha\to 0$. Fix $\bar\alpha\in(0,\alpha_0)$ and let us consider the function

$$u(x,y)=\bar\alpha-\Big(\frac{1}{2}y^2+x^4-6x^2y^2+y^4\Big).$$

We have that u verifies

$$\begin{cases} -\Delta u = 1 & \text{in } D_{\bar\alpha},\\ u>0 & \text{in } D_{\bar\alpha},\\ u=0 & \text{on } \partial D_{\bar\alpha},\end{cases}\tag{4.3}$$

and $(0,0)$ is the unique critical (maximum) point to u in $D_{\bar\alpha}$, provided that we choose $\bar\alpha$ suitably smaller. Moreover, u is a semi-stable solution to (4.3). A straightforward computation shows that the curvature at a point $(0,y)\in\{u=c\}$ with $0<c<\bar\alpha$ is given by

$$k(0,y)=-\frac{u_{xx}u_y^2-2u_{xy}u_xu_y+u_{yy}u_x^2}{(u_x^2+u_y^2)^{\frac{3}{2}}}=-\frac{12|y|}{1+4y^3}<0$$

for $y<0$. This means that there is always at least a point in $\{u=c\}$ with negative curvature, which gives the claim.

Next example shows that semi-stable solutions to (1.16) can have two critical points in star-shaped domains.

Example 4.5. Theorem 1.7 with $n=3$ suggested the construction of this example.

For c small enough, let us consider

$$D_c=\Big\{(x,y)\in\mathbb{R}^2:\frac{1}{2}y^2+x^3-3xy^2+A(x^4-6x^2y^2+y^4)<c\Big\}.$$

Some tedious computations show that for $A>\frac{35}{2}$ and c small enough, D_c is a closed curve.

For the same parameters A and c let us consider the function

$$u(x,y)=c-\Big(\frac{1}{2}y^2+x^3-3xy^2+A(x^4-6x^2y^2+y^4)\Big).$$

We have that u verifies

$$\begin{cases} -\Delta u = 1 & \text{in } D_c,\\ u>0 & \text{in } D_c,\\ u=0 & \text{on } \partial D_c.\end{cases}$$

A straightforward computation shows that u admits exactly two critical points, $P_0 = (0,0)$ (saddle point) and $P_1 = (-\frac{3}{4A}, 0)$ (maximum point). As remarked in the introduction, D_c is a star-shaped domain with respect to the origin. Since u is a semi-stable solution to (1.16), this example shows that in Cabré–Chanillo's theorem, the assumption of positive curvature cannot be relaxed to star-shapedness with respect to some point.

5 Non-isolated Maximum Points

In this section we show some consequences of Theorem 1.2 when the maximum (minimum) points of u is not isolated.

Proof of Proposition 1.5. Since $(0,0)$ is not isolated we have that the sets

$$S_\epsilon = \{x^2 + y^2 = \epsilon\} \quad \text{and} \quad \mathcal{C} = \{(x,y) \in B_1 : u(x,y) = u(0,0)\}$$

verify

$$S_\epsilon \cap \mathcal{C} \neq \emptyset \tag{5.1}$$

for $\epsilon > 0$ small enough. Let us consider $(x_\epsilon, y_e) \in S_\epsilon \cap \mathcal{C}$. If, by contradiction, n is even in Theorem 1.2, using that for $(x,y) \in S_\epsilon$,

$$\operatorname{Re}(z^n) = x^n + o(y^2) \quad \text{and} \quad \operatorname{Im}(z^n) = nx^{n-1}y + o(y^2),$$

we get, by (1.5) and (5.1) and the Young inequality,

$$\begin{aligned} 0 &= \Big(\frac{u_{yy}(0,0)}{2} + o(1)\Big)y_\epsilon^2 + \Big(\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} + o(1)\Big)(\epsilon^2 - y_\epsilon^2)^{\frac{n}{2}} + \Big(\frac{\frac{\partial^n u}{\partial x^{n-1}\partial y}(0,0)}{(n-1)!} + o(1)\Big)x_\epsilon^{n-1}y_\epsilon \\ &= \Big(\frac{u_{yy}(0,0)}{2} + o(1)\Big)y_\epsilon^2 + \Big(\frac{\frac{\partial^n u}{\partial x^n}(0,0)}{n!} + o(1)\Big)\epsilon^n, \end{aligned}$$

which implies $y_\epsilon = \epsilon = 0$, contradicting (5.1).

So, n is odd and repeating the previous computation for (1.7), we obtain

$$\begin{aligned} 0 &= \Big(\frac{u_{yy}(0,0)}{2} + o(1)\Big)y_\epsilon^2 + \Big(\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{(n-1)!} + o(1)\Big)y_\epsilon(\epsilon^2 - y_\epsilon^2)^{\frac{n-1}{2}} + \Big(\frac{\frac{\partial^l u}{\partial x^l}(0,0)}{l!} + o(1)\Big)(\epsilon^2 - y_\epsilon^2)^{\frac{l}{2}} \\ &= \Big(\frac{u_{yy}(0,0)}{2} + o(1)\Big)y_\epsilon^2 + \Big(\frac{\frac{\partial^n u}{\partial x^{n-1}y}(0,0)}{(n-1)!} + o(1)\Big)\epsilon^{n-1}y_\epsilon + \Big(\frac{\frac{\partial^l u}{\partial x^l}(0,0)}{l!} + o(1)\Big)\epsilon^l. \end{aligned} \tag{5.2}$$

From (5.2), we deduce that

$$\frac{2}{l!}u_{yy}(0,0)\frac{\partial^l u}{\partial x^l}(0,0)\epsilon^l - \frac{1}{(n-1)!^2}\Big(\frac{\partial^n u}{\partial x^{n-1}y}(0,0)\Big)^2\epsilon^{2n-2} \le 0, \tag{5.3}$$

and if $l < 2n-2$, we get $u_{yy}(0,0)\frac{\partial^l u}{\partial x^l}(0,0) \le 0$, contradicting Theorem 1.2. So, $l = 2n-2$ in (5.3) and using (1.8), we deduce that

$$\frac{2}{l!}u_{yy}(0,0)\frac{\partial^l u}{\partial x^l}(0,0) = \frac{1}{(n-1)!^2}\Big(\frac{\partial^n u}{\partial x^{n-1}y}(0,0)\Big)^2,$$

which ends the proof. □

Proof of Corollary 1.6. Assume that $u = u(r)$ is a radial function, with $r^2 = x^2 + (y-P)^2$, $P \neq 0$, and u solves

$$-\Delta u = f(u).$$

Suppose that $u'(P) = 0$ and $u''(P) < 0$, and so $(0,0)$ belongs to a circle of maximum points. The derivatives at $(0,0)$ are given by

$$\begin{cases} u_{yy}(0,0) = u''(P) = -f(u(P)), \\ u_{xxx}(0,0) = 0, \\ u_{xxy}(0,0) = -\frac{1}{P}u''(P) = \frac{1}{P}f(u(P)), \\ u_{xxxx}(0,0) = \frac{3}{P^2}u''(P) = -\frac{3}{P^2}f(u(P)). \end{cases} \tag{5.4}$$

So, by (1.7), we get (1.9), and by (5.4), we have

$$u_{xxy}^2(0,0) = \frac{f^2(u(P))}{P^2} = \frac{u_{yy}(0,0)u_{xxxx}(0,0)}{3}.$$

So, the equality is achieved in (1.8) (equivalently in (1.10)). □

Remark 5.1. We do know examples of solutions with a curve of maximum point for $l > 3$ in Proposition 1.5. It should be interesting to construct it (if there exist!).

We end this section with some additional properties of solutions with curves of maximum points.

Proposition 5.2. *Assume the same assumptions of Theorem 1.2 and suppose that γ is a smooth curve of maxima for u. Then, denoting by $k(0,0)$ the curvature of γ at $(0,0)$ the following alternative holds: either*

$$k(0,0) \neq 0$$

or

$$u_x \text{ consists of 4 curves intersecting transversally at } (0,0). \tag{5.5}$$

Proof. Let us suppose that $k(0,0) = 0$ and show that (5.5) holds. Let us parametrize γ as follows:

$$\gamma(t) = \begin{cases} x = x(t), \\ y = y(t), \quad t \in (-\epsilon, \epsilon), \end{cases}$$

with $\gamma(0) = (0,0)$. Then since $u(\gamma(t)) = u(0,0)$ for any $t \in (-\epsilon, \epsilon)$, a straightforward computation gives

$$\begin{cases} 0 = \dfrac{d^2}{dt^2} u(\gamma(t))\Big|_{t=0} = u_{yy}(0,0)\,(\dot{y}(0))^2 \Rightarrow \dot{y}(0) = 0, \\ 0 = \dfrac{d^4}{dt^4} u(\gamma(t))\Big|_{t=0} = u_{xxxx}(0,0)\dot{x}(0))^4 + 5u_{xxy}(0,0)(\dot{x}(0))^2\ddot{y}(0) + 3u_{yy}(0,0)(\ddot{y}(0))^2. \end{cases} \tag{5.6}$$

Since $k(0,0) = 0$, from $\dot{y}(0) = 0$, we get that $\ddot{y}(0) = 0$ and by (5.6), we deduce that

$$u_{xxxx}(0,0) = 0. \tag{5.7}$$

Since by Proposition 1.5 we have that n is odd, (5.7) and (1.8) imply that $n \geq 5$ in Theorem 1.2. Finally, from (1.7), we deduce that in a neighborhood of $(0,0)$, it holds

$$u_x(x,y) = (A + o(1))\,\mathrm{Re}(z^{n-1}) + (B + o(1))\,\mathrm{Im}(z^{n-1}),$$

with $n - 1 \geq 4$ and $(A, B) \neq (0,0)$. This gives (5.5) and the claim follows. □

Let us recall that the Morse index $m(u)$ of a solution u to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

is given by the number of negative eigenvalue of the operator

$$\mathcal{L} = -\Delta - f'(u)I.$$

Our final result is a consequence of Proposition 5.2.

Corollary 5.3. *Assume the same assumptions as in Theorem 1.2 and suppose that γ is a smooth curve of maxima for u. Then if u is a solution to (5.8) with $m(u) < 3$, then*

$$k(0,0) \neq 0.$$

Proof. By Proposition 5.2 it is enough to prove that (5.5) cannot occur. Indeed, if it happens, we deduce that

$$\mathcal{N} = \{(x,y) \in \Omega : u_x(x,y) = 0\}$$

encloses at least three nodal regions $\omega_1, \omega_2, \omega_3 \subset \Omega$ (straightforward extension of Lemma 4.3). So, the functions

$$v_i(x,y) = \begin{cases} u_x & \text{if } (x,y) \in \omega_i, \\ 0 & \text{if } (x,y) \in \Omega \setminus \omega_i,\ i = 1,2,3, \end{cases}$$

verify that $v_i \in H_0^1(\Omega)$, $\int_\Omega \nabla v_i \nabla v_j = 0$ for $i \neq j$ and $\int_\Omega |\nabla v_i|^2 - f'(u) v_i^2 = 0$ for $i = 1, 2, 3$. Then, by the definition of Morse index, we derive that $m(u) \geq 3$ and this gives a contradiction. □

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