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Research Article

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Nonlinear Scalar Field Equations with L^2 Constraint: Mountain Pass and Symmetric Mountain Pass Approaches

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Abstract: We study the existence of radially symmetric solutions of the following nonlinear scalar field equations in $\mathbb{R}^N$ ($N \geq 2$):

$$\begin{cases} -\Delta u = g(u) - \mu u \quad \text{in } \mathbb{R}^N, \\ \|u\|^2_{L^2(\mathbb{R}^N)} = m, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(*)_m$}$$

where $g(\xi) \in C(\mathbb{R}, \mathbb{R})$, $m > 0$ is a given constant and $\mu \in \mathbb{R}$ is a Lagrange multiplier. We introduce a new approach using a Lagrange formulation of problem $(*)_m$. We develop a new deformation argument under a new version of the Palais–Smale condition. For a general class of nonlinearities related to [H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I: Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983), no. 4, 313–345], [H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, Arch. Ration. Mech. Anal. 82 (1983), no. 4, 347–375], [J. Hirata, N. Ikoma and K. Tanaka, Nonlinear scalar field equations in $\mathbb{R}^N$: Mountain pass and symmetric mountain pass approaches, Topol. Methods Nonlinear Anal. 35 (2010), no. 2, 253–276], it enables us to apply minimax argument for L^2 constraint problems and we show the existence of infinitely many solutions as well as mountain pass characterization of a minimizing solution of the problem

$$\inf\left\{ \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - G(u)\,dx : \|u\|^2_{L^2(\mathbb{R}^N)} = m \right\}, \quad G(\xi) = \int_0^\xi g(\tau)\,d\tau.$$

Keywords: L^2-Constraint Problem, Normalized Solutions, Deformation Theory

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0 Introduction

In this paper, we study the existence of radially symmetric solutions of the following nonlinear scalar field equations in $\mathbb{R}^N$ $(N \ge 2)$:

$$\begin{cases} -\Delta u = g(u) - \mu u & \text{in } \mathbb{R}^N, \\ \|u\|^2_{L^2(\mathbb{R}^N)} = m, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(*)_m$}$$

where $g(\xi) \in C(\mathbb{R}, \mathbb{R})$, $m > 0$ is a given constant and $\mu \in \mathbb{R}$ is a Lagrange multiplier.

Solutions of $(*)_m$ can be characterized as critical points of the constraint problem

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) : S_m \to \mathbb{R},$$

where $S_m = \{u \in H^1_r(\mathbb{R}^N) : \|u\|^2_{L^2(\mathbb{R}^N)} = m\}$ and $G(\xi) = \int_0^\xi g(\tau)\, d\tau$.

When $g(\xi)$ has L^2-subcritical growth, Cazenave and Lions [7] and Shibata [16] successfully found a solution of $(*)_m$ via minimizing method:

$$\mathcal{I}_m = \inf_{u \in S_m} \mathcal{F}(u). \tag{0.1}$$

See also Ruppen [15] and Stuart [18] for earlier works. The paper [7] dealt with $g(\xi) = |\xi|^{q-1}\xi$ $(1 < q < 1 + \frac{4}{N})$ and [16] dealt with a class of more general nonlinearities, which satisfy the following conditions:
(g1) $g(\xi) \in C(\mathbb{R}, \mathbb{R})$.
(g2) $\lim_{\xi \to 0} \frac{g(\xi)}{\xi} = 0$.
(g3) For $p = 1 + \frac{4}{N}$, $\lim_{|\xi| \to \infty} \frac{|g(\xi)|}{|\xi|^p} = 0$.
(g4) There exists some $\xi_0 > 0$ such that $G(\xi_0) > 0$.
Shibata [16] showed:
(i) There exists $m_S \ge 0$ such that for $m > m_S$, $\mathcal{I}_m$ defined in (0.1) is achieved and $(*)_m$ has at least one solution for $m > m_S$.
(ii) It is important to check $m_S = 0$ or not. Shibata showed

$$m_S = 0 \quad \text{if } \liminf_{\xi \to 0} \frac{g(\xi)}{|\xi|^{\frac{4}{N}}\xi} = \infty, \tag{0.2}$$

$$m_S > 0 \quad \text{if } \limsup_{\xi \to 0} \frac{g(\xi)}{|\xi|^{\frac{4}{N}}\xi} < \infty. \tag{0.3}$$

We remark that the authors of [7, 16] also studied orbital stability of the minimizer. We also refer to Jeanjean [11] and Bartsch and de Valeriola [2] for the study of the L^2-supercritical case (e.g. $g(\xi) \sim |\xi|^{p-1}\xi$ with $p \in (1 + \frac{4}{N}, \frac{N+2}{N-2})$).

We note that conditions (g1)–(g4) are related to those in [4, 5] (see also [3, 9]) as *almost necessary and sufficient conditions* for the existence of solutions of nonlinear scalar field equations

$$\begin{cases} -\Delta u = g(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{0.4}$$

More precisely, replacing (g2) by $\limsup_{\xi \to 0} \frac{g(\xi)}{\xi} < 0$ and replacing p with $\frac{N+2}{N-2}$ in (g3), they showed the existence of a least energy solution and they also showed the existence of a unbounded sequence of possibly sign-changing solutions assuming oddness of $g(\xi)$ in addition:
(g5) $g(-\xi) = -g(\xi)$ for all $\xi \in \mathbb{R}$.
We remark that if $g(\xi)$ satisfies (g1)–(g4), then $\tilde{g}(\xi) = g(\xi) - \mu\xi$ satisfies the conditions of [4, 5] for $\mu \in (0, \infty)$ small.

In [7, 16], to show the achievement of $\mathcal{I}_m$ on S_m and orbital stability of solutions, the following *sub-additivity inequality* plays an important role:

$$\mathcal{I}_m < \mathcal{I}_s + \mathcal{I}_{m-s} \quad \text{for all } s \in (0, m),$$

which ensures compactness of minimizing sequences for $\mathcal{I}_m$. See also [17] for the sub-additivity inequality.

In this paper, we take another approach to $(*)_m$ and we try to apply minimax methods to a Lagrange formulation of problem $(*)_m$:

$$\mathcal{L}(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) + \frac{\mu}{2}(\|u\|^2_{L^2(\mathbb{R}^N)} - m) : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R}. \tag{0.5}$$

That is, we obtain solutions (μ, u) of $(*)_m$ as critical points of $\mathcal{L}(\mu, u)$. We give another proof to the existence result of [16]; we take an approach related to Hirata, Ikoma and Tanaka [9] and Jeanjean [11], which made use of the scaling properties of the problems to generate Palais–Smale sequences in augmented spaces with extra properties related to the Pohozaev identities. We remark that such approaches were successfully applied to other problems with suitable scaling properties. See Azzollini, d'Avenia and Pomponio [1], Byeon and Tanaka [6], Chen and Tanaka [8], Ikoma [10], and Moroz and Van Schaftingen [13]. In this paper we develop this idea further to establish a deformation argument, which enables us to apply minimax methods and genus theory in the space $\mathbb{R} \times H^1_r(\mathbb{R}^N)$. We also give a mountain pass characterization of the minimizing value $\mathcal{I}_m$ through the functional (0.5), which we expect to be useful in the study of singular perturbation problems. We remark that a mountain pass characterization of the least energy solutions for nonlinear scalar field equations (0.4) was given in [12].

Theorem 0.1. *Assume* (g1)–(g4). *Then the following statements hold:*

(i) *There exists $m_0 \in [0, \infty)$ such that for $m > m_0$, problem $(*)_m$ has at least one solution.*

(ii) *In addition to* (g1)–(g4), *assume*

$$\liminf_{\xi \to 0} \frac{g(\xi)}{|\xi|^{\frac{4}{N}}\xi} = \infty. \tag{0.6}$$

Then problem $(*)_m$ has at least one solution for all $m > 0$.

(iii) *In the setting of* (i)–(ii), *a solution is obtained through a mountain pass minimax method:*

$$b_{mp} = \inf_{\gamma \in \Gamma_{mp}} \max_{t \in [0,1]} I(\gamma(t)).$$

See (0.7) *below for the definition of I and see also Section 5 for a precise definition of the minimax class Γ_{mp}. We also have*

$$b_{mp} = \mathcal{I}_m,$$

where $\mathcal{I}_m$ is defined in (0.1).

We will give a presentation of m_0 using least energy levels of $-\Delta u + \mu u = g(u)$ in Section 5. We also show $m > m_0$ if and only if $\mathcal{I}_m < 0$.

We also deal with the existence of infinitely many solutions assuming oddness of $g(\xi)$. It seems that the existence of infinitely many solutions for the L^2-constraint problem is not well-studied. Our main result is the following:

Theorem 0.2. *Assume* (g1)–(g4) *and* (g5). *Then the following statements hold:*

(i) *For any $k \in \mathbb{N}$ there exists $m_k \geq 0$ such that for $m > m_k$, problem $(*)_m$ has at least k solutions.*

(ii) *Assume* (0.6) *in addition to* (g1)–(g5). *Then for any $m > 0$, problem $(*)_m$ has countably many solutions $(u_n)_{n=1}^\infty$, which satisfy*

$$\mathcal{F}(u_n) < 0 \quad \text{for all } n \in \mathbb{N},$$
$$\mathcal{F}(u_n) \to 0 \quad \text{as } n \to \infty.$$

To show Theorem 0.2, we develop a version of symmetric mountain pass methods, in which genus plays an important role.

In the following sections, we give proofs to our Theorems 0.1 and 0.2. Since the existence part of Theorem 0.1 is already known by [7, 16], we mainly deal with Theorem 0.2 in Sections 1–4.

In Section 1, first we give a variational formulation of problem $(*)_m$. For a technical reason, we write $\mu = e^\lambda$ ($\lambda \in \mathbb{R}$) and we try to find critical points of

$$I(\lambda, u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - G(u) + \frac{e^\lambda}{2}\left(\int_{\mathbb{R}^N} |u|^2 - m\right) \in C^1(\mathbb{R} \times H^1_r(\mathbb{R}^N), \mathbb{R}). \tag{0.7}$$

We also setup function spaces. Second for a fixed $\lambda \in \mathbb{R}$, we study the symmetric mountain pass value $a_k(\lambda)$ of

$$u \mapsto \widehat{I}(\lambda, u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{e^\lambda}{2}u^2 - G(u).$$

Behavior of $a_k(\lambda)$ is important in our study. In particular, m_k in Theorem 0.2 is given by

$$m_k = 2 \inf_{\lambda \in (-\infty, \lambda_0)} \frac{a_k(\lambda)}{e^\lambda}.$$

See (1.5) for the definition of λ_0.

In Sections 2–3, we find that $I(\lambda, u) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ has a kind of symmetric mountain pass geometry and we give a family of minimax sets for $I(\lambda, u)$, which involve the notion of genus under $\mathbb{Z}_2$-invariance: $I(\lambda, -u) = I(\lambda, u)$.

In Section 4, we develop a new deformation argument to justify the minimax methods in Section 2. Usually deformation theories are developed under the so-called Palais–Smale condition. However, under conditions (g1)–(g4), it is difficult to check the standard Palais–Smale condition for $I(\lambda, u)$. We introduce a new version (PSP) of Palais–Smale condition, which is inspired by our earlier work [9] and Jeanjean [11] (See Section 4.1 for the (PSP) condition. See also below for the (PSP) condition for scalar field equation (0.4).) Here we extend the ideas in [9, 11] and we establish a new deformation argument under condition (PSP). Our deformation flow is constructed in a special way; we use the scaling property of our functional $I(\lambda, u)$ effectively and our flow is obtained through an ODE in a higher-dimensional space $\mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$. More precisely, we construct our flow through a pseudo-gradient flow for

$$J(\theta, \lambda, u) = \frac{1}{2}e^{(N-2)\theta} \int_{\mathbb{R}^N} |\nabla u|^2 - e^{N\theta} \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}\left(e^{N\theta} \int_{\mathbb{R}^N} |u|^2 - m\right).$$

We note that $J(\theta, \lambda, u) \in C^1(\mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N), \mathbb{R})$ enjoys the following scaling property:

$$J(\theta, \lambda, u(x)) = I(\lambda, u(\tfrac{x}{e^\theta})).$$

In Section 5, we deal with Theorem 0.1 and we study the minimizing problem (0.1). Applying the mountain pass approach to $I(\lambda, u)$, we give another proof of the existence result as well as a mountain pass characterization of $\mathcal{I}_m$ in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ via our new deformation argument.

Our new deformation argument is applicable for problems with suitable scaling properties. In Section 6, we give a typical example and we deal with nonlinear scalar field equations (0.4). We show that the corresponding functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) \in C^1(H_r^1(\mathbb{R}^N), \mathbb{R})$$

satisfies our (PSP) condition and our deformation argument works also for $I(u)$ under condition (PSP). We give a simplified proof to the results of [9]. In our argument, a functional $P(u)$ given by

$$P(u) = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - N \int_{\mathbb{R}^N} G(u)$$

plays an important role and our (PSP) condition for $I(u)$ is given as follows:

Condition (PSP). If a sequence $(u_n)_{n=1}^\infty \subset H_r^1(\mathbb{R}^N)$ satisfies as $n \to \infty$,

$$\begin{aligned} I(u_n) &\to b, \\ \partial_u I(u_n) &\to 0 \quad \text{strongly in } (H_r^1(\mathbb{R}^N))^*, \\ P(u_n) &\to 0, \end{aligned}$$

then $(u_n)_{n=1}^\infty$ has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$.

We note that $P(u)$ corresponds to the Pohozaev identity for (0.4) and our (PSP) condition is weaker than the standard Palais–Smale condition.

We believe that our argument is based on an essential property of our problem – $I(u)$ satisfies our (PSP) condition – and it is of interest. We also believe that our argument is applicable to other problems.

1 Preliminaries

1.1 Functional Settings

In Sections 1–4, we deal with Theorem 0.2 and we assume (g1)–(g5). We denote by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions $u(x) = u(|x|)$ which satisfy $u(x), \nabla u(x) \in L^2(\mathbb{R}^N)$. We also use notation

$$\|u\|_r = \Big(\int_{\mathbb{R}^N} |u(x)|^r\Big)^{\frac{1}{r}} \qquad \text{for } r \in [1,\infty) \text{ and } u \in L^r(\mathbb{R}^N),$$
$$\|u\|_{H^1} = (\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \qquad \text{for } u \in H_r^1(\mathbb{R}^N).$$

We also write

$$(u,v)_2 = \int_{\mathbb{R}^N} uv \quad \text{for } u, v \in L^2(\mathbb{R}^N).$$

In what follows, we denote by p the L^2 critical exponent, i.e.,

$$p = 1 + \frac{4}{N}.$$

In particular, we have

$$\frac{p+1}{p-1} - \frac{N}{2} = 1, \tag{1.1}$$

which we will use repeatedly in this paper.

For technical reasons, we set $\mu = e^\lambda$ in (0.5) and we set for a given $m > 0$,

$$I(\lambda,u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}(\|u\|_2^2 - m) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}.$$

It is easy to see that $I(\lambda,u) \in C^1(\mathbb{R} \times H_r^1(\mathbb{R}^N), \mathbb{R})$ and solutions of $(*)_m$ can be characterized as critical points of $I(\lambda,u)$, that is, (μ,u) with $\mu = e^\lambda > 0$ solves $(*)_m$ if and only if $\partial_\lambda I(\lambda,u) = 0$ and $\partial_u I(\lambda,u) = 0$. We also have

$$I(\lambda,-u) = I(\lambda,u) \quad \text{for all } (\lambda,u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N).$$

The following functionals will play important roles in our argument:

$$\widehat{I}(\lambda,u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{e^\lambda}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R},$$
$$P(\lambda,u) = \frac{N-2}{2}\|\nabla u\|_2^2 + N\bigg(\frac{e^\lambda}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u)\bigg) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}. \tag{1.2}$$

We note that:

(i) For a fixed $\lambda \in \mathbb{R}$, $u \mapsto \widehat{I}(\lambda,u)$ is corresponding to

$$\begin{cases} -\Delta u + e^\lambda u = g(u) & \text{in } \mathbb{R}^N, \\ u \in H_r^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

It is easy to see that

$$I(\lambda,u) = \widehat{I}(\lambda,u) - \frac{e^\lambda}{2}m \quad \text{for all } (\lambda,u). \tag{1.4}$$

(ii) $P(\lambda,u)$ is related to the Pohozaev identity for (1.3). It is well known that for $\lambda \in \mathbb{R}$, if $u(x) \in H_r^1(\mathbb{R}^N)$ solves (1.3), then $P(\lambda,u) = 0$.

1.2 Some Estimates for $\widehat{I}(\lambda, u)$

First we observe that for $\lambda \ll 0$, $u \mapsto \widehat{I}(\lambda, u)$ satisfies the assumptions of [3–5, 9] and possesses the symmetric mountain pass geometry. In what follows, we write

$$S^{k-1} = \{\xi \in \mathbb{R}^k : |\xi| = 1\}, \quad D_k = \{\xi \in \mathbb{R}^k : |\xi| \le 1\}$$

and set

$$\lambda_0 = \begin{cases} \log\Big(2 \sup_{\xi \ne 0} \frac{G(\xi)}{\xi^2}\Big) & \text{if } \sup_{\xi \ne 0} \frac{G(\xi)}{\xi^2} < \infty, \\ \infty & \text{if } \sup_{\xi \ne 0} \frac{G(\xi)}{\xi^2} = \infty. \end{cases} \tag{1.5}$$

Lemma 1.1. *The following statements hold:*

(i) *For $\lambda \in (-\infty, \lambda_0)$,*

$$G(\xi_0) - \frac{e^\lambda}{2}|\xi_0|^2 > 0 \quad \text{for some } \xi_0 > 0.$$

In particular, $\tilde{g}(\xi) = g(\xi) - e^\lambda \xi$ satisfies the assumptions of [4, 5, 9], *that is, $\tilde{g}(\xi)$ satisfies* (g1), (g3)–(g5) *and*

$$\lim_{\xi \to 0} \frac{\tilde{g}(\xi)}{\xi} < 0.$$

(ii) *For any $\lambda \in (-\infty, \lambda_0)$ and for any $k \in \mathbb{N}$, there exists a continuous odd map $\zeta : S^{k-1} \to H^1_r(\mathbb{R}^N)$ such that*

$$\widehat{I}(\lambda, \zeta(\xi)) < 0 \quad \text{for all } \xi \in S^{k-1}.$$

(iii) *When $\lambda_0 < \infty$, for $\lambda \ge \lambda_0$ we have*

$$G(\xi) - \frac{e^\lambda}{2}|\xi|^2 \le 0 \quad \text{for all } \xi \in \mathbb{R}.$$

In particular, $\widehat{I}(\lambda, u) \ge 0$ for all $u \in H^1_r(\mathbb{R}^N)$.

Proof. By (g1)–(g5) and definition (1.5) of λ_0, we can easily see (i) and (iii). By the arguments in [5, 9], we can observe that $u \mapsto \widehat{I}(\lambda, u)$ has property (ii). □

For $k \in \mathbb{N}$ and $\lambda \in (-\infty, \lambda_0)$, we set

$$\widehat{\Gamma}_k(\lambda) = \{\zeta \in C(D_k, H^1_r(\mathbb{R}^N)) : \zeta(-\xi) = -\zeta(\xi) \text{ for } \xi \in D_k,\ \widehat{I}(\lambda, \zeta(\xi)) < 0 \text{ for } \xi \in \partial D_k = S^{k-1}\}, \tag{1.6}$$

$$a_k(\lambda) = \inf_{\zeta \in \widehat{\Gamma}_k(\lambda)} \max_{\xi \in D_k} \widehat{I}(\lambda, \zeta(\xi)). \tag{1.7}$$

We note that $\widehat{\Gamma}_k(\lambda) \ne \emptyset$ by Lemma 1.1 (ii). Since $\widehat{I}(\lambda, u) = \frac{1}{2}(\|\nabla u\|_2^2 + e^\lambda \|u\|_2^2) + o(\|u\|_{H^1}^2)$ as $\|u\|_{H^1} \sim 0$, we have $a_k(\lambda) > 0$ for all $\lambda \in (-\infty, \lambda_0)$ and $k \in \mathbb{N}$. By the results of [9], we observe that $a_k(\lambda)$ is a critical value of $u \mapsto \widehat{I}(\lambda, u)$. See also Section 6.

We also have

$$\begin{aligned} &0 < a_1(\lambda) \le a_2(\lambda) \le \cdots \le a_k(\lambda) \le a_{k+1}(\lambda) \le \cdots && \text{for all } \lambda \in (-\infty, \lambda_0), \\ &a_k(\lambda) \le a_k(\lambda') && \text{for all } \lambda < \lambda' < \lambda_0 \text{ and } k \in \mathbb{N}. \end{aligned} \tag{1.8}$$

For the behavior of $a_k(\lambda)$ as $\lambda \to -\infty$, condition (0.6) is important. We have:

Lemma 1.2. *Assume* (g1)–(g5).

(i) *Assume* (0.6) *in addition. Then for any $k \in \mathbb{N}$,*

$$\lim_{\lambda \to -\infty} \frac{a_k(\lambda)}{e^\lambda} = 0.$$

(ii) *If*

$$\limsup_{\xi \to 0} \frac{g(\xi)}{|\xi|^{\frac{4}{N}} \xi} < \infty, \tag{1.9}$$

then for any $k \in \mathbb{N}$,

$$\liminf_{\lambda \to -\infty} \frac{a_k(\lambda)}{e^\lambda} > 0.$$

Proof. (i) Choose $r \in (1+\frac{4}{N}, \frac{N+2}{N-2})$ when $N \geq 3$ and choose $r \in (1+\frac{4}{N}, \infty)$ when $N = 2$. By (0.6) and (g3), for any $L > 0$ there exists $C_L > 0$ such that

$$\xi g(\xi) \geq L|\xi|^{p+1} - C_L|\xi|^{r+1} \quad \text{for all } \xi \in \mathbb{R},$$

from which we have

$$G(\xi) \geq \frac{L}{p+1}|\xi|^{p+1} - \frac{C_L}{r+1}|\xi|^{r+1} \qquad \text{for all } \xi \in \mathbb{R},$$
$$\hat{I}(\lambda, u) \leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{e^\lambda}{2}\|u\|_2^2 - \frac{L}{p+1}\|u\|_{p+1}^{p+1} + \frac{C_L}{r+1}\|u\|_{r+1}^{r+1} \quad \text{for all } u \in H_r^1(\mathbb{R}^N).$$

Setting $u(x) = e^{\frac{\lambda}{p-1}}v(e^{\frac{\lambda}{2}}x)$, $v(x) \in H_r^1(\mathbb{R}^N)$, we have from (1.1)

$$\hat{I}(\lambda, u) \leq e^\lambda\Big(\frac{1}{2}\|v\|_{H^1}^2 - \frac{L}{p+1}\|v\|_{p+1}^{p+1} + \frac{C_L}{r+1}e^{\frac{r-p}{p-1}\lambda}\|v\|_{r+1}^{r+1}\Big).$$

We note that

$$\bar{I}(v) = \frac{1}{2}\|v\|_{H^1}^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} : H_r^1(\mathbb{R}^N) \to \mathbb{R}$$

has the symmetric mountain pass geometry and thus there exists an odd continuous map $\bar{\zeta}(\xi) : D_k \to H_r^1(\mathbb{R}^N)$ such that $\bar{I}(\bar{\zeta}(\xi)) < 0$ for all $\xi \in \partial D_k$. By (1.4), $\zeta_\lambda(\xi) = e^{\frac{\lambda}{p-1}}\bar{\zeta}(\xi)(e^{\frac{\lambda}{2}}x)$ satisfies for $L \geq 1$,

$$\hat{I}(\lambda, \zeta_\lambda(\xi)) \leq e^\lambda\Big(\bar{I}(\bar{\zeta}(\xi)) - \frac{L-1}{p+1}\|\bar{\zeta}(\xi)\|_{p+1}^{p+1} + \frac{C_L}{r+1}e^{\frac{r-p}{p-1}\lambda}\|\bar{\zeta}(\xi)\|_{r+1}^{r+1}\Big).$$

Thus for $\lambda \ll 0$, we have $\zeta_\lambda(\xi) \in \hat{\Gamma}_k(\lambda)$ and we have

$$\limsup_{\lambda\to-\infty}\frac{a_k(\lambda)}{e^\lambda} \leq \max_{\xi\in D_k}\Big(\frac{1}{2}\|\bar{\zeta}(\xi)\|_{H^1}^2 - \frac{L}{p+1}\|\bar{\zeta}(\xi)\|_{p+1}^{p+1}\Big).$$

Since $L \geq 1$ is arbitrary, we have the conclusion.

(ii) By (1.9) and (g3), there exists $C > 0$ such that

$$G(\xi) \leq C|\xi|^{p+1} \quad \text{for all } \xi \in \mathbb{R}.$$

Thus we have

$$\hat{I}(\lambda, u) \geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{e^\lambda}{2}\|u\|_2^2 - \frac{C}{p+1}\|u\|_{p+1}^{p+1}.$$

As in (i),

$$\hat{I}(\lambda, e^{\frac{\lambda}{p-1}}u(e^{\frac{\lambda}{2}}x)) \geq e^\lambda\Big(\frac{1}{2}\|u\|_{H^1}^2 - \frac{C}{p+1}\|u\|_{p+1}^{p+1}\Big),$$

from which we deduce that $\frac{a_k(\lambda)}{e^\lambda}$ is estimated from below by the mountain pass minimax value for

$$u \mapsto \frac{1}{2}\|u(x)\|_{H^1}^2 - \frac{C}{p+1}\|u(x)\|_{p+1}^{p+1}.$$

Thus (ii) holds. □

We define for $k \in \mathbb{N}$,

$$m_k = 2\inf_{\lambda\in(-\infty,\lambda_0)}\frac{a_k(\lambda)}{e^\lambda} \geq 0. \tag{1.10}$$

By (1.8), we have

$$0 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} \leq \cdots. \tag{1.11}$$

In what follows, we fix $m > m_k$ arbitrary and try to show that $I(\lambda, u)$ has at least k pairs of critical points.

As a corollary to Lemma 1.2, we have the following result, which is analogous to (0.2)–(0.3).

Corollary 1.3. *The following statements hold:*

(i) *Under condition* (0.6),

$$m_k = 0 \quad \textit{for all } k \in \mathbb{N}.$$

(ii) *Under condition* (1.9),

$$m_k > 0 \quad \textit{for all } k \in \mathbb{N}.$$

1.3 An Estimate from Below

By (g2) and (g3), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\xi g(\xi) \le C_\delta |\xi|^2 + \delta |\xi|^{p+1} \quad \text{for all } \xi \in \mathbb{R}. \tag{1.12}$$

Then we also have

$$G(\xi) \le \frac{1}{2} C_\delta |\xi|^2 + \frac{\delta}{p+1} |\xi|^{p+1} \quad \text{for all } \xi \in \mathbb{R}.$$

Setting

$$\underline{I}(\lambda, u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(e^\lambda - C_\delta)\|u\|_2^2 - \frac{\delta}{p+1}\|u\|_{p+1}^{p+1},$$

we have

$$\widehat{I}(\lambda, u) \ge \underline{I}(\lambda, u) \quad \text{for all } (\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N). \tag{1.13}$$

The functional $u \mapsto \underline{I}(\lambda, u)$ has a typical mountain pass geometry if $e^\lambda > C_\delta + 1$, which enables us to give an estimate of $\widehat{I}(\lambda, u)$ from below.

In what follows, we denote by $E_0 > 0$ the least energy level for $-\Delta u + u = |u|^{p-1}u$ in $\mathbb{R}^N$, that is,

$$E_0 = \inf\Big\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} : u \ne 0,\ \|\nabla u\|_2^2 + \|u\|_2^2 = \|u\|_{p+1}^{p+1}\Big\}.$$

Lemma 1.4. *For $e^\lambda \ge C_\delta + 1$,*

$$\widehat{I}(\lambda, u) \ge \delta^{-\frac{2}{p-1}}(e^\lambda - C_\delta)E_0 \quad \textit{if } u \ne 0,\ \|\nabla u\|_2^2 + (e^\lambda - C_\delta)\|u\|_2^2 = \delta\|u\|_{p+1}^{p+1}, \tag{1.14}$$

$$\widehat{I}(\lambda, u) \ge 0 \quad \textit{if } \|\nabla u\|_2^2 + (e^\lambda - C_\delta)\|u\|_2^2 \ge \delta\|u\|_{p+1}^{p+1}. \tag{1.15}$$

Proof. Let $\omega(x)$ be the least energy solution of $-\Delta u + u = |u|^{p-1}u$. Then it is easy to see that

$$u_{\lambda,\delta}(x) = \Big(\frac{e^\lambda - C_\delta}{\delta}\Big)^{\frac{1}{p-1}} \omega((e^\lambda - C_\delta)^{\frac{1}{2}} x)$$

is a least energy solution of $-\Delta u + (e^\lambda - C_\delta)u = \delta|u|^{p-1}u$ in $\mathbb{R}^N$. Set

$$S_{\lambda,\delta} = \{u \in H_r^1(\mathbb{R}^N) \setminus \{0\} : \|\nabla u\|_2^2 + (e^\lambda - C_\delta)\|u\|_2^2 = \delta\|u\|_{p+1}^{p+1}\}.$$

By (1.1), it is easy to see that for $e^\lambda \ge C_\delta + 1$

$$\underline{I}(\lambda, u) \ge \delta^{-\frac{2}{p-1}}(e^\lambda - C_\delta)E_0 \quad \text{for } u \in S_{\lambda,\delta}.$$

Thus we get (1.14) from (1.13). Noting

$$\{u \in H_r^1(\mathbb{R}^N) : \|\nabla u\|_2^2 + (e^\lambda - C_\delta)\|u\|_2^2 \ge \delta\|u\|_{p+1}^{p+1}\} = \{tu : t \in [0,1],\ u \in S_{\lambda,\delta}\}$$

and that for $u \in S_{\lambda,\delta}$, $\underline{I}(\lambda, tu)$ is increasing for $t \in (0,1)$, we have (1.15). □

2 Minimax Methods for $I(\lambda, u)$

2.1 Symmetric Mountain Pass Methods

We fix $k \in \mathbb{N}$ and $m > 0$ such that

$$m > m_k, \tag{2.1}$$

where $m_k \ge 0$ is given in (1.10). We will show that $I(\lambda, u)$ has at least k pairs of critical points.

We choose $\delta_m > 0$ such that

$$\delta_m^{-\frac{2}{p-1}} E_0 > \frac{m}{2} \tag{2.2}$$

and take $C_{\delta_m} > 0$ so that (1.12) holds. For

$$\lambda_m = \log(C_{\delta_m} + 1),$$

we set

$$\begin{aligned}\Theta_\lambda &= \{u : \|\nabla u\|_2^2 + (e^\lambda - C_{\delta_m})\|u\|_2^2 > \delta_m \|u\|_{p+1}^{p+1}\} \cup \{0\} \quad \text{for } \lambda \geq \lambda_m,\\ \Omega_m &= \bigcup_{\lambda \in (\lambda_m, \infty)} (\{\lambda\} \times \Theta_\lambda).\end{aligned} \tag{2.3}$$

We note that Ω_m is a domain whose section $\Theta_\lambda \subset H_r^1(\mathbb{R}^N)$ is a set surrounded by the Nehari manifold

$$\{u \in H_r^1(\mathbb{R}^N) \setminus \{0\} : \|\nabla u\|_2^2 + (e^\lambda - C_{\delta_m})\|u\|_2^2 = \delta_m \|u\|_{p+1}^{p+1}\}.$$

In particular, $(\lambda_m, \infty) \times \{0\} \subset \Omega_m$.

Using Lemma 1.4, we have:

Lemma 2.1. *The following statements hold:*
(i) $B_m \equiv \inf_{(\lambda,u)\in\partial\Omega_m} I(\lambda, u) > -\infty$,
(ii) $\widehat{I}(\lambda, u) \geq 0$ *for* $(\lambda, u) \in \Omega_m$.

Proof. Note that $\partial\Omega_m = \mathcal{C}_0 \cup \mathcal{C}_1$, where

$$\begin{aligned}\mathcal{C}_0 &= \{(\lambda, u) \in \mathbb{R} \times (H_r^1(\mathbb{R}^N) \setminus \{0\}) : \lambda \geq \lambda_m,\ \|\nabla u\|_2^2 + (e^\lambda - C_{\delta_m})\|u\|_2^2 = \delta_m \|u\|_{p+1}^{p+1}\},\\ \mathcal{C}_1 &= \{(\lambda_m, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) : \|\nabla u\|_2^2 + (e^{\lambda_m} - C_{\delta_m})\|u\|_2^2 \geq \delta_m \|u\|_{p+1}^{p+1}\}.\end{aligned}$$

By Lemma 1.4,

$$\begin{aligned}I(\lambda, u) &= \widehat{I}(\lambda, u) - \frac{e^\lambda}{2} m \geq \delta_m^{-\frac{2}{p-1}} (e^\lambda - C_{\delta_m}) E_0 - \frac{e^\lambda}{2} m \quad &&\text{for } (\lambda, u) \in \mathcal{C}_0,\\ I(\lambda, u) &= \widehat{I}(\lambda, u) - \frac{e^\lambda}{2} m \geq -\frac{e^{\lambda_m}}{2} m &&\text{for } (\lambda, u) \in \mathcal{C}_1.\end{aligned}$$

By our choice (2.2) of δ_m, we have $\inf_{(\lambda,u)\in\partial\Omega_m} I(\lambda, u) > -\infty$ and (i) holds. Part (ii) is also clear. □

We introduce a family of minimax methods. For $j \in \mathbb{N}$ we set

$$\Gamma_j = \{\gamma(\xi) = (\varphi(\xi), \zeta(\xi)) \in C(D_j, \mathbb{R} \times H_r^1(\mathbb{R}^N)) : \gamma(\xi) \text{ satisfies conditions } (\gamma1)\text{–}(\gamma3) \text{ below}\},$$

where
(γ1) $\varphi(-\xi) = \varphi(\xi)$, $\zeta(-\xi) = -\zeta(\xi)$ for all $\xi \in D_j$.
(γ2) There exists $\lambda \in (-\infty, \lambda_0)$ such that

$$\varphi(\xi) = \lambda, \quad (\lambda, \zeta(\xi)) \in (\mathbb{R} \times H_r^1(\mathbb{R}^N)) \setminus \Omega_m, \quad I(\lambda, \zeta(\xi)) \leq B_m - 1 \quad \text{for } \xi \in \partial D_j.$$

(γ3) $\varphi(0) \in (\lambda_m, \infty)$ and $\zeta(0) = 0$. Moreover,

$$I(\varphi(0), \zeta(0)) = -\frac{e^{\varphi(0)}}{2} m \leq B_m - 1.$$

We note that:
(i) For $\lambda \in (-\infty, \lambda_0)$, $u \mapsto \widehat{I}(\lambda, u)$ has the symmetric mountain pass geometry.
(ii) $I(\lambda, 0) = -\frac{e^\lambda}{2} m \to -\infty$ as $\lambda \to \infty$.

From these facts, we have $\Gamma_j \neq \emptyset$ for all $j \in \mathbb{N}$. We remark that Γ_j is a family of j-dimensional symmetric mountain paths joining points in $[\lambda_m, \infty) \times \{0\} \subset \Omega_m$ and $(\mathbb{R} \times H_r^1(\mathbb{R}^N)) \setminus \Omega_m$.

We set

$$b_j = \inf_{\gamma\in\Gamma_j} \max_{\xi\in D_j} I(\gamma(\xi)) \quad \text{for } j \in \mathbb{N}.$$

Proposition 2.2. *The following statements hold:*
(1) $b_j \geq B_m$ *for all* $j \in \mathbb{N}$.
(2) $b_j < 0$ *for* $j = 1, 2, \ldots, k$.

To show Proposition 2.2, we need:

Lemma 2.3. *We have*

$$b_j \le a_j(\lambda) - \frac{e^\lambda}{2} m \quad \textit{for } \lambda \in (-\infty, \lambda_0). \tag{2.4}$$

Proof. First we note that by (ii) of Lemma 2.1 that

$$(\lambda, \zeta(\xi)) \in (\mathbb{R} \times H_r^1(\mathbb{R}^N)) \setminus \Omega_m \quad \text{for } \zeta \in \widehat{\Gamma}_j(\lambda) \text{ and } \xi \in \partial D_j.$$

Second we remark that we may assume for $\zeta(\xi) \in \widehat{\Gamma}_j(\lambda)$,

$$I(\lambda, \zeta(\xi)) \le B_m - 1 \quad \text{for } \xi \in \partial D_j. \tag{2.5}$$

In fact, for $u \in H_r^1(\mathbb{R}^N)$ and $\nu > 0$ we have

$$\widehat{I}(\lambda, u(\tfrac{x}{\nu})) = \frac{1}{2}\nu^{N-2}\|\nabla u\|_2^2 + \nu^N\bigg(\frac{e^\lambda}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u)\bigg),$$

from which we deduce that if $\widehat{I}(\lambda, u(x)) < 0$, then $\nu \mapsto \widehat{I}(\lambda, u(\frac{x}{\nu}))$, $[1, \infty) \to \mathbb{R}$ is decreasing and

$$\lim_{\nu \to \infty} \widehat{I}\Big(\lambda, u\Big(\frac{x}{\nu}\Big)\Big) = -\infty.$$

Thus, for a given $\zeta(\xi) \in \widehat{\Gamma}_j(\lambda)$, setting

$$\widetilde{\zeta}(\xi)(x) = \begin{cases} \zeta(2\xi) & \text{for } |\xi| \in [0, \frac{1}{2}], \\ \zeta(\frac{\xi}{|\xi|})(\frac{x}{L(2|\xi|-1)+1}) & \text{for } |\xi| \in (\frac{1}{2}, 1], \end{cases}$$

we find for $L \gg 1$, $\widetilde{\zeta}(\xi) \in \widehat{\Gamma}_j(\lambda)$ and

$$\max_{\xi \in D_j} \widehat{I}(\lambda, \widetilde{\zeta}(\xi)) = \max_{\xi \in D_j} \widehat{I}(\lambda, \zeta(\xi)),$$
$$I(\lambda, \widetilde{\zeta}(\xi)) \le B_m - 1 \quad \text{for } \xi \in \partial D_j.$$

Thus we may assume (2.5) for $\zeta(\xi) \in \widehat{\Gamma}_j(\lambda)$.

Next we show (2.4). For $\zeta(\xi) \in \widehat{\Gamma}_j(\lambda)$ with (2.5), we define $\check{\gamma}(\xi) = (\check{\varphi}(\xi), \check{\zeta}(\xi))$ by

$$\check{\varphi}(\xi) = \begin{cases} \lambda + R(1 - 2|\xi|) & \text{for } |\xi| \in [0, \frac{1}{2}], \\ \lambda & \text{for } |\xi| \in (\frac{1}{2}, 1], \end{cases}$$

$$\check{\zeta}(\xi) = \begin{cases} 0 & \text{for } |\xi| \in [0, \frac{1}{2}], \\ \zeta(\frac{\xi}{|\xi|}(2|\xi| - 1)) & \text{for } |\xi| \in (\frac{1}{2}, 1]. \end{cases}$$

Then for R large, we have $\check{\gamma}(\xi) \in \Gamma_j$ and

$$I(\check{\gamma}(\xi)) = I(\lambda + R(1 - 2|\xi|), 0) = -\frac{e^{\lambda + R(1-2|\xi|)}}{2} m \le -\frac{e^\lambda}{2} m \qquad \text{for } |\xi| \in \Big[0, \frac{1}{2}\Big],$$
$$I(\check{\gamma}(\xi)) = I(\lambda, \check{\zeta}(\xi)) = \widehat{I}(\lambda, \check{\zeta}(\xi)) - \frac{e^\lambda}{2} m \le \max_{\xi \in D_j} \widehat{I}(\lambda, \zeta(\xi)) - \frac{e^\lambda}{2} m \quad \text{for } |\xi| \in \Big(\frac{1}{2}, 1\Big].$$

Since $\zeta(\xi) \in \widehat{\Gamma}_j(\lambda)$ is arbitrary, we have (2.4). □

Now we give a proof to Proposition 2.2.

Proof of Proposition 2.2. (i) By $(\gamma 2)$ and $(\gamma 3)$, we have

$$\gamma(\partial D_j) \cap \Omega_m = \emptyset \quad \text{and} \quad \gamma(0) \in \Omega_m \quad \text{for all } \gamma \in \Gamma_j.$$

Thus $\gamma(D_j) \cap \partial\Omega_m \ne \emptyset$ for all $\gamma \in \Gamma_j$ and it follows from Lemma 2.1 (i) that

$$\max_{\xi \in D_j} I(\gamma(\xi)) \ge \inf_{(\lambda,u) \in \partial\Omega_m} I(\lambda, u) \equiv B_m.$$

Since $\gamma \in \Gamma_j$ is arbitrary, we have (i).

(ii) By Lemma 2.3, for any $\lambda \in (-\infty, \lambda_0)$,

$$\frac{b_j}{e^\lambda} \le \frac{a_j(\lambda)}{e^\lambda} - \frac{m}{2}.$$

Since

$$2 \inf_{\lambda \in (-\infty, \lambda_0)} \left(\frac{a_j(\lambda)}{e^\lambda} - \frac{m}{2} \right) = m_j - m,$$

conclusion (ii) follows from (1.11) and (2.1). □

In Section 3, we will see that $I(\lambda, u)$ satisfies a version of Palais–Smale-type condition $(\mathrm{PSP})_b$ for $b < 0$, which enables us to develop a deformation argument and to show b_j $(j = 1, 2, \dots, k)$ are critical values of $I(\lambda, u)$. However, to show multiplicity, i.e., to deal with the case $b_i = \dots = b_{i+\ell}$ $(1 \le i < i + \ell \le k)$, we need another family of minimax methods, which involve the notion of genus.

2.2 Symmetric Mountain Pass Methods Using Genus

In this subsection, we use an idea from Rabinowitz [14] to define another family of minimax methods. Here the notion of genus plays a role.

Definition. Let E be a Banach space. For a closed set $A \subset E \setminus \{0\}$, which is symmetric with respect to 0, i.e., $-A = A$, we define genus$(A) = n$ if and only if there exists an odd map $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property. When there is no odd map $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ with this property for any $n \in \mathbb{N}$, we define genus$(A) = \infty$. Finally, we set genus$(\emptyset) = 0$.

We refer to [14] for fundamental properties of the genus.

Our setting is different from [14]; our functional is invariant under the following $\mathbb{Z}_2$-action:

$$\mathbb{Z}_2 \times \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R} \times H^1_r(\mathbb{R}^N), \quad (\pm 1, \lambda, u) \mapsto (\lambda, \pm u), \tag{2.6}$$

that is, $I(\lambda, -u) = I(\lambda, u)$. Remarking that there is no critical points in the $\mathbb{Z}_2$-invariants $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$, we modify the arguments in [14].

We define our second family of minimax sets as follows:

$$\Lambda_j = \{\gamma(\overline{D_{j+\ell} \setminus Y}) : \ell \ge 0,\ \gamma \in \Gamma_{j+\ell},\ Y \subset D_{j+\ell} \setminus \{0\} \text{ is closed, symmetric with respect to 0 and genus}(Y) \le \ell\},$$
$$c_j = \inf_{A \in \Lambda_j} \max_{(\lambda, u) \in A} I(\lambda, u).$$

Here we summarize fundamental properties of Λ_j. Here we use a projection $P_2 : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to H^1_r(\mathbb{R}^N)$ defined by

$$P_2(\lambda, u) = u \quad \text{for } (\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N).$$

Lemma 2.4. *The following statements hold:*
(i) $\Lambda_j \ne \emptyset$ *for all* $j \in \mathbb{N}$.
(ii) $\Lambda_{j+1} \subset \Lambda_j$ *for all* $j \in \mathbb{N}$.
(iii) *Let* $\psi(\lambda, u) = (\psi_1(\lambda, u), \psi_2(\lambda, u)) : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R} \times H^1_r(\mathbb{R}^N)$ *be a continuous map with properties*

$$\psi_1(\lambda, -u) = \psi_1(\lambda, u), \quad \psi_2(\lambda, -u) = -\psi_2(\lambda, u) \quad \text{for all } (\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N), \tag{2.7}$$
$$\psi(\lambda, u) = (\lambda, u) \quad \text{if } I(\lambda, u) \le B_m - 1. \tag{2.8}$$

Then for $A \in \Lambda_j$, *we have* $\psi(A) \in \Lambda_j$.
(iv) *For* $A \in \Lambda_j$ *and a closed set* Z, *which is invariant under* $\mathbb{Z}_2$*-action* (2.6), *i.e.,* $(\lambda, -u) \in Z$ *for all* $(\lambda, u) \in Z$, *with* $0 \notin \overline{P_2(Z)}$,

$$\overline{A \setminus Z} \in \Lambda_{j-i}, \quad \text{where } i = \text{genus}(\overline{P_2(Z)}).$$

(v) $A \cap \partial\Omega_m \ne \emptyset$ *for any* $A \in \Lambda_j$. *Here* Ω_m *is defined in* (2.3).

Proof. Parts (i) and (ii) follow from the definition of Λ_j.

(iii) Suppose $\psi(\lambda, u) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ satisfies (2.7)–(2.8). Then it is easy to see $\psi \circ \gamma \in \Gamma_j$ for all $\gamma \in \Gamma_j$. Thus (iii) holds.

(iv) Following and modifying the argument in [14, Sections 7–8], we can show (iv). For the sake of completeness, we give a proof in the Appendix.

(v) Suppose $A = \gamma(\overline{D_{j+\ell} \setminus Y}) \in \Lambda_j$, where $\gamma \in \Gamma_{j+\ell}$, $\operatorname{genus}(Y) \le \ell$, and let U be the connected component of $\mathcal{O} = \gamma^{-1}(\Omega_m)$ containing 0. It is easy to see

$$0 \in U, \quad U \subset \operatorname{int} D_{j+\ell},$$

from which we have $\operatorname{genus}(\partial U) = j + \ell$. Thus

$$\operatorname{genus}(\overline{\partial U \setminus Y}) \ge \operatorname{genus}(\partial U) - \operatorname{genus}(Y) \ge j.$$

In particular, $\overline{\partial U \setminus Y} \ne \emptyset$. Since $\gamma(\overline{\partial U \setminus Y}) \subset A \cap \partial\Omega_m$, we have $A \cap \partial\Omega_m \ne \emptyset$. □

As fundamental properties of c_n, we have:

Lemma 2.5. *The following statements hold:*
(i) $B_m \le c_1 \le c_2 \le \cdots \le c_j \le c_{j+1} \le \cdots$.
(ii) $c_j \le b_j$ *for all* $j \in \mathbb{N}$.

Proof. (i) By Lemma 2.4 (v), we have for any $A \in \Lambda_j$,

$$\max_{(\lambda,u)\in A} I(\lambda, u) \ge \inf_{(\lambda,u)\in\partial\Omega_m} I(\lambda, u) = B_m,$$

which implies $c_j \ge B_m$ for all $j \in \mathbb{N}$. Lemma 2.4 (ii) implies $c_j \le c_{j+1}$.

(ii) It is easy to see $\gamma(D_j) \in \Lambda_j$ for any $\gamma \in \Gamma_j$. Thus we have $c_j \le b_j$. □

In the following section, we use a special deformation lemma to show c_j ($j = 1, 2, \dots, k$) are attained by critical points.

3 Deformation Argument and Existence of Critical Points

In this section we introduce a deformation result for $I(\lambda, u)$ and we show that c_j ($j = 1, 2, \dots, k$) given in the previous section are achieved by critical points.

3.1 Deformation Result for $I(\lambda, u)$

For $b \in \mathbb{R}$ we set

$$K_b = \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) : I(\lambda, u) = b,\ \partial_\lambda I(\lambda, u) = 0,\ \partial_u I(\lambda, u) = 0,\ P(\lambda, u) = 0\}. \tag{3.1}$$

Here $P(\lambda, u)$ is introduced in (1.2). We note that $\partial_u I(\lambda, u) = 0$ implies $P(\lambda, u) = 0$. We also use the following notation:

$$[I \le c] = \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) : I(\lambda, u) \le c\} \quad \text{for } c \in \mathbb{R}.$$

We have the following deformation result.

Proposition 3.1. *Assume* (g1)–(g5) *and* $b < 0$. *Then:*
(i) K_b *is compact in* $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ *and* $K_b \cap (\mathbb{R} \times \{0\}) = \emptyset$.
(ii) *For any open neighborhood* $\mathcal{O}$ *of* K_b *and* $\overline{\varepsilon} > 0$ *there exist* $\varepsilon \in (0, \overline{\varepsilon})$ *and a continuous map* $\eta(t, \lambda, u) : [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ *such that:*
   (1) $\eta(0, \lambda, u) = (\lambda, u)$ *for all* $(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$.
   (2) $\eta(t, \lambda, u) = (\lambda, u)$ *if* $(\lambda, u) \in [I \le b - \overline{\varepsilon}]$.
   (3) $I(\eta(t, \lambda, u)) \le I(\lambda, u)$ *for all* $(t, \lambda, u) \in [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$.

(4) $\eta(1, [I \le b + \varepsilon] \setminus \mathcal{O}) \subset [I \le b - \varepsilon]$, $\eta(1, [I \le b + \varepsilon]) \subset [I \le b - \varepsilon] \cup \mathcal{O}$.
(5) *If* $K_b = \emptyset$, *then* $\eta(1, [I \le b + \varepsilon]) \subset [I \le b - \varepsilon]$.
(6) *Writing* $\eta(t, \lambda, u) = (\eta_1(t, \lambda, u), \eta_2(t, \lambda, u))$, *we have*

$$\eta_1(t, \lambda, -u) = \eta_1(t, \lambda, u), \quad \eta_2(t, \lambda, -u) = -\eta_2(t, \lambda, u)$$

for all $(t, \lambda, u) \in [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$.

Such a deformation result is usually obtained under the Palais–Smale compactness condition. However, it seems difficult to verify the standard Palais–Smale condition under (g1)–(g4). In Section 4, we introduce a new version (PSP) of Palais–Smale condition and we develop a new deformation argument to prove Proposition 3.1. We postpone a proof of Proposition 3.1 until Section 4 and in this section we show c_j $(j = 1, 2, \dots, k)$ are attained by critical points.

Remark 3.2. Our deformation flow $\eta(t, \lambda, u)$ stated in Proposition 3.1 is generated in a special way and it does not have the following properties in general:
(i) $\eta(s + t, \lambda, u) = \eta(t, \eta(s, \lambda, u))$ for $s, t \in [0, 1]$ with $s + t \in [0, 1]$ and $(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$.
(ii) For $t \in (0, 1]$, a map $(\lambda, u) \mapsto \eta(t, \lambda, u)$, $\mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ is a homeomorphism.

We note that the above properties (i)–(ii) hold for the standard deformation flow. See [14] and see also Remark 4.10 in Section 4. We also note that properties (1)–(6) in Proposition 3.1 enable us to apply minimax arguments and the genus arguments.

3.2 Existence of Critical Points

As an application of our Proposition 3.1 we show the following proposition.

Proposition 3.3. *The following statements hold:*
(i) *For* $j = 1, 2, \dots, k$, $c_j < 0$ *and* c_j *is a critical value of* $I(\lambda, u)$.
(ii) *If* $c_j = c_{j+1} = \cdots = c_{j+q} \equiv b < 0$ $(j + q \le k)$, *then*

$$\mathrm{genus}(P_2(K_b)) \ge q + 1.$$

In particular, $\#(K_b) = \infty$ *if* $q \ge 1$.

Proof. $c_j < 0$ $(j = 1, 2, \dots, k)$ follows from Proposition 2.2 and Lemma 2.5 (ii). The argument for the fact that $K_{c_j} \ne \emptyset$ is similar to the proof of (ii). So we omit it.

(ii) Suppose that $c_j = c_{j+1} = \cdots = c_{j+q} = b < 0$. Since K_b is compact and $K_b \cap (\mathbb{R} \times \{0\}) = \emptyset$, the projection $P_2(K_b)$ of K_b onto $H_r^1(\mathbb{R}^N)$ is compact, symmetric with respect to 0 and $0 \notin P_2(K_b)$. Thus by the fundamental property of the genus,
(a) $\mathrm{genus}(P_2(K_b)) < \infty$,
(b) there exists $\delta > 0$ small such that $\mathrm{genus}(\overline{P_2(N_\delta(K_b))}) = \mathrm{genus}(P_2(K_b))$.

Here we denote a δ-neighborhood of a set $A \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ by $N_\delta(A)$, that is,

$$N_\delta(A) = \{(\lambda, u) : \mathrm{dist}((\lambda, u), A) \le \delta\},$$

where $\mathrm{dist}(\cdot\,, \cdot)$ is the standard distance on $\mathbb{R} \times H_r^1(\mathbb{R}^N)$, i.e.,

$$\mathrm{dist}((\lambda, u), (\lambda', u')) = \sqrt{|\lambda - \lambda'|^2 + \|u - u'\|_{H^1}^2} \quad \text{for } (\lambda, u),\ (\lambda', u') \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

and

$$\mathrm{dist}((\lambda, u), A) = \inf_{(\lambda', u') \in A} \mathrm{dist}((\lambda, u), (\lambda', u')).$$

By Proposition 3.1, there exist $\varepsilon > 0$ small and $\eta : [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ such that

$$\eta(1, [I \le b + \varepsilon] \setminus N_\delta(K_b)) \subset [I \le b - \varepsilon],$$

$$\eta(t, \lambda, u) = (\lambda, u) \quad \text{if } I(\lambda, u) \le b - \frac{1}{2}.$$

We note that $B_m - 1 \le b - \frac{1}{2}$.

We take $A \in \Lambda_{j+q}$ such that $A \subset [I \le b + \varepsilon]$. Then

$$\eta(1, \overline{A \setminus N_\delta(K_b)}) \subset [I \le b - \varepsilon]. \tag{3.2}$$

If $\mathrm{genus}(P_2(K_b)) \le q$, we have $\mathrm{genus}(\overline{P_2(N_\delta(K_b))}) \le q$. By (iv) of Lemma 2.4,

$$\overline{A \setminus N_\delta(K_b)} \in \Lambda_j. \tag{3.3}$$

Equations (3.2) and (3.3) imply $c_j \le b - \varepsilon$, which is a contradiction. Thus $\mathrm{genus}(P_2(K_b)) \ge q + 1$. □

Now we can show:

Proof of Theorem 0.2 (i). Clearly (i) of Theorem 0.2 follows from Proposition 3.3. □

Proof of Theorem 0.2 (ii). Under condition (0.6), we have $m_k = 0$ for all $k \in \mathbb{N}$. Thus we have $c_j \le b_j < 0$ for all $j \in \mathbb{N}$ and c_j ($j \in \mathbb{N}$) are critical values of $I(\lambda, u)$. We need to show $c_j \to 0$ as $j \to \infty$.

Arguing indirectly, we assume $c_j \to \overline{c} < 0$ as $j \to \infty$. Then $K_{\overline{c}}$ is compact and $K_{\overline{c}} \cap (\mathbb{R} \times \{0\}) = \emptyset$. Set

$$q = \mathrm{genus}(P_2(K_{\overline{c}})) < \infty$$

and choose $\delta > 0$ small such that

$$\mathrm{genus}(\overline{P_2(N_\delta(K_{\overline{c}}))}) = \mathrm{genus}(P_2(K_{\overline{c}})) = q.$$

As in the proof of Proposition 3.3, there exist $\varepsilon > 0$ small and $\eta : [0, 1] \times \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R} \times H^1_r(\mathbb{R}^N)$ such that

$$\eta(1, [I \le \overline{c} + \varepsilon] \setminus N_\delta(K_{\overline{c}})) \subset [I \le \overline{c} - \varepsilon], \tag{3.4}$$
$$\eta(t, \lambda, u) = (\lambda, u) \quad \text{if } I(\lambda, u) \le B_m - 1.$$

We choose $j \gg 1$ so that $c_j > \overline{c} - \varepsilon$ and take $A \in \Lambda_{j+q}$ such that $A \subset [I \le \overline{c} + \varepsilon]$. Then we have

$$\overline{A \setminus N_\delta(K_{\overline{c}})} \in \Lambda_j. \tag{3.5}$$

Equations (3.4) and (3.5) imply $c_j \le \overline{c} - \varepsilon$. Since we can take j arbitrary large, we have $\lim_{j\to\infty} c_j \le \overline{c} - \varepsilon$. This is a contradiction. □

4 (PSP) Condition and Construction of a Flow

In this section we give a new type of deformation argument for our functional $I(\lambda, u)$. Our deformation argument is inspired by our previous work [9].

4.1 (PSP) Condition

Since it is difficult to verify the standard Palais–Smale condition for $I(\lambda, u)$ under conditions (g1)–(g5), we introduce a new type of Palais–Smale condition $(\mathrm{PSP})_b$, which is weaker than the standard Palais–Smale condition and which takes the scaling property of $I(\lambda, u)$ into consideration through the Pohozaev functional $P(\lambda, u)$.

Definition. For $b \in \mathbb{R}$, we say that $I(\lambda, u)$ satisfies $(\mathrm{PSP})_b$ condition if and only if the following holds:

Condition $(\mathrm{PSP})_b$. If a sequence $(\lambda_n, u_n)_{n=1}^\infty \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$ satisfies as $n \to \infty$,

$$I(\lambda_n, u_n) \to b, \tag{4.1}$$
$$\partial_\lambda I(\lambda_n, u_n) \to 0, \tag{4.2}$$
$$\partial_u I(\lambda_n, u_n) \to 0 \quad \text{strongly in } (H^1_r(\mathbb{R}^N))^*, \tag{4.3}$$
$$P(\lambda_n, u_n) \to 0, \tag{4.4}$$

then $(\lambda_n, u_n)_{n=1}^\infty$ has a strongly convergent subsequence in $\mathbb{R} \times H^1_r(\mathbb{R}^N)$.

First we observe that $(\mathrm{PSP})_b$ holds for $I(\lambda, u)$ for $b < 0$.

Proposition 4.1. *Assume* (g1)–(g4). *Then $I(\lambda, u)$ satisfies* $(\mathrm{PSP})_b$ *for $b < 0$.*

Proof. Let $b < 0$ and suppose that $(\lambda_n, u_n)_{n=1}^\infty$ satisfies (4.1)–(4.4). We will show that $(\lambda_n, u_n)_{n=1}^\infty$ has a strongly convergent subsequence. The proof consists of several steps.

Step 1: λ_n is bounded from below as $n \to \infty$. Since

$$P(\lambda_n, u_n) = N\Big(I(\lambda_n, u_n) + \frac{m}{2} e^{\lambda_n}\Big) - \|\nabla u_n\|_2^2,$$

we have from (4.1) and (4.4) that

$$\frac{m}{2} \liminf_{n\to\infty} e^{\lambda_n} \geq -b > 0.$$

Thus λ_n is bounded from below as $n \to \infty$.

Step 2: $\|u_n\|_2^2 \to m$ as $n \to \infty$. Since

$$\partial_\lambda I(\lambda_n, u_n) = \frac{e^{\lambda_n}}{2}(\|u_n\|_2^2 - m),$$

it follows from (4.2) and Step 1 that $\|u_n\|_2^2 \to m$.

Step 3: $\|\nabla u_n\|_2^2$ and λ_n are bounded as $n \to \infty$. We have

$$\partial_u I(\lambda_n, u_n) u_n = \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} g(u_n) u_n + e^{\lambda_n} \|u_n\|_2^2. \tag{4.5}$$

By (g2) and (g3), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|g(\xi)\xi| \leq C_\delta |\xi|^2 + \delta |\xi|^{p+1} \quad \text{for all } \xi \in \mathbb{R}.$$

Thus

$$\Big|\int_{\mathbb{R}^N} g(u) u\Big| \leq C_\delta \|u\|_2^2 + \delta \|u\|_{p+1}^{p+1} \quad \text{for all } u \in H_r^1(\mathbb{R}^N).$$

Since $p = 1 + \frac{4}{N}$, by the Gagliardo–Nirenberg inequality there exists $C_N > 0$ such that

$$\|u\|_{p+1}^{p+1} \leq C_N \|\nabla u\|_2^2 \|u\|_2^{p-1} \quad \text{for all } u \in H_r^1(\mathbb{R}^N).$$

Thus it follows from (4.5) that

$$\|\nabla u_n\|_2^2 - C_\delta \|u_n\|_2^2 - \delta C_N \|\nabla u_n\|_2^2 \|u_n\|_2^{p-1} + e^{\lambda_n} \|u_n\|_2^2 \leq \varepsilon_n \sqrt{\|\nabla u_n\|_2^2 + \|u_n\|_2^2},$$

where $\varepsilon_n = \|\partial_u I(\lambda_n, u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0$. By Step 2,

$$\Big(1 - \delta C_N (m + o(1))^{\frac{p-1}{2}}\Big) \|\nabla u_n\|_2^2 + (e^{\lambda_n} - C_\delta)(m + o(1)) \leq \varepsilon_n \sqrt{\|\nabla u_n\|_2^2 + m + o(1)}.$$

Choosing $\delta > 0$ small so that $\delta C_N m^{\frac{p-1}{2}} < \frac{1}{2}$, we observe that $\|\nabla u_n\|_2^2$ and e^{λ_n} are bounded as $n \to \infty$.

Step 4: Conclusion. By Steps 1–3, $(\lambda_n, u_n)_{n=1}^\infty$ is a bounded sequence in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$. After extracting a subsequence – still denoted by $(\lambda_n, u_n)_{n=1}^\infty$ –, we may assume that $\lambda_n \to \lambda_0$ and $u_n \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$ for some $(\lambda_0, u_0) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$. By (g2)–(g3), we have

$$\int_{\mathbb{R}^N} g(u_n) u_0 \to \int_{\mathbb{R}^N} g(u_0) u_0, \qquad \int_{\mathbb{R}^N} g(u_n) u_n \to \int_{\mathbb{R}^N} g(u_0) u_0.$$

Thus, we deduce from $\partial_u I(\lambda_n, u_n) u_n \to 0$ and $\partial_u I(\lambda_n, u_n) u_0 \to 0$ that

$$\|\nabla u_n\|_2^2 + e^{\lambda_0} \|u_n\|_2^2 \to \|\nabla u_0\|_2^2 + e^{\lambda_0} \|u_0\|_2^2,$$

which implies $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^N)$. □

Remark 4.2. For $b = 0$, condition $(\mathrm{PSP})_0$ does not hold for $I(\lambda, u)$. In fact, for a sequence $(\lambda_n, 0)_{n=1}^{\infty}$ with $\lambda_n \to -\infty$, we have

$$I(\lambda_n, 0) = -\frac{e^{\lambda_n}}{2}m \to 0, \quad \partial_\lambda I(\lambda_n, 0) = -\frac{e^{\lambda_n}}{2}m \to 0,$$
$$\partial_u I(\lambda_n, 0) = 0, \qquad P(\lambda_n, 0) = 0.$$

But $(\lambda_n, 0)_{n=1}^{\infty}$ has no convergent subsequences.

As a corollary to Proposition 4.1, we have:

Corollary 4.3. *For $b < 0$, K_b defined in* (3.1) *is compact in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ and satisfies $K_b \cap (\mathbb{R} \times \{0\}) = \emptyset$.*

Proof. The set K_b is compact since $I(\lambda, u)$ satisfies condition $(\mathrm{PSP})_b$. $K_b \cap (\mathbb{R} \times \{0\}) \neq \emptyset$ follows from the fact that $\partial_\lambda I(\lambda, 0) = -\frac{e^\lambda}{2}m \neq 0$. □

4.2 Functional $J(\theta, \lambda, u)$

To construct a deformation flow, we need an augmented functional $J(\theta, \lambda, u) : \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(\theta, \lambda, u) = \frac{1}{2}e^{(N-2)\theta}\|\nabla u\|_2^2 - e^{N\theta}\int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}(e^{N\theta}\|u\|_2^2 - m).$$

We introduce $J(\theta, \lambda, u)$ to make use of the scaling property of $I(\lambda, u)$. As a basic property of $J(\theta, \lambda, u)$ we have

$$I(\lambda, u(\tfrac{x}{e^\theta})) = J(\theta, \lambda, u) \quad \text{for all } (\theta, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N). \tag{4.6}$$

We will construct our deformation flow for $I(\lambda, u)$ through a deformation flow for $J(\theta, \lambda, u)$.

The functional $J(\theta, \lambda, u)$ satisfies the following properties.

Lemma 4.4. *For all $(\theta, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$, $h \in H_r^1(\mathbb{R}^N)$ and $\beta \in \mathbb{R}$,*

$$\partial_\theta J(\theta, \lambda, u(x)) = P(\lambda, u(\tfrac{x}{e^\theta})), \tag{4.7}$$
$$\partial_\lambda J(\theta, \lambda, u(x)) = \partial_\lambda I(\lambda, u(\tfrac{x}{e^\theta})), \tag{4.8}$$
$$\partial_u J(\theta, \lambda, u(x))h(x) = \partial_u I(\lambda, u(\tfrac{x}{e^\theta}))h(\tfrac{x}{e^\theta}), \tag{4.9}$$
$$J(\theta + \beta, \lambda, u(e^\beta x)) = J(\theta, \lambda, u(x)). \tag{4.10}$$

Proof. We compute that

$$\begin{aligned}
\partial_\theta J(\theta, \lambda, u(x)) &= \frac{N-2}{2}e^{(N-2)\theta}\|\nabla u\|_2^2 + Ne^{N\theta}\Big(\frac{e^\lambda}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u)\Big) \\
&= \frac{N-2}{2}\|\nabla(u(\tfrac{x}{e^\theta}x))\|_2^2 + N\Big(\frac{e^\lambda}{2}\|u(\tfrac{x}{e^\theta})\|_2^2 - \int_{\mathbb{R}^N} G(u(\tfrac{x}{e^\theta}))\Big) \\
&= P(\lambda, u(\tfrac{x}{e^\theta})), \\
\partial_\lambda J(\theta, \lambda, u(x)) &= \frac{e^\lambda}{2}(e^{N\theta}\|u\|_2^2 - m) = \frac{e^\lambda}{2}(\|u(\tfrac{x}{e^\theta})\|_2^2 - m) \\
&= \partial_\lambda I(\lambda, u(\tfrac{x}{e^\theta})), \\
\partial_u J(\theta, \lambda, u(x))h(x) &= e^{(N-2)\theta}(\nabla u, \nabla h)_2 + e^\lambda e^{N\theta}(u, h)_2 - e^{N\theta}\int_{\mathbb{R}^N} g(u(x))h(x) \\
&= (\nabla u(\tfrac{x}{e^\theta}), \nabla h(\tfrac{x}{e^\theta}))_2 + e^\lambda(u(\tfrac{x}{e^\theta}), h(\tfrac{x}{e^\theta}))_2 - \int_{\mathbb{R}^N} g(u(\tfrac{x}{e^\theta}))h(\tfrac{x}{e^\theta}) \\
&= \partial_u I(\lambda, u(\tfrac{x}{e^\theta}))h(\tfrac{x}{e^\theta}).
\end{aligned}$$

Thus we have (4.7)–(4.9). Equation (4.10) follows from (4.6). □

To analyze $J(\theta, \lambda, u)$, it is natural to regard $\mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$ as a Hilbert manifold with a metric related to (4.6). More precisely, we write $M = \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$. We note that

$$T_{(\theta,\lambda,u)}M = \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \quad \text{for } (\theta, \lambda, u) \in M$$

and we introduce a metric $\langle \cdot , \cdot \rangle_{(\theta,\lambda,u)}$ on $T_{(\theta,\lambda,u)}M$ by

$$\langle (\alpha, \nu, h), (\alpha', \nu', h') \rangle_{(\theta,\lambda,u)} = \alpha\alpha' + \nu\nu' + e^{(N-2)\theta}(\nabla h, \nabla h')_2 + e^{N\theta}(h, h')_2,$$

$$\|(\alpha, \nu, h)\|_{(\theta,\lambda,u)} = \sqrt{\langle (\alpha, \nu, h), (\alpha, \nu, h) \rangle_{(\theta,\lambda,u)}}$$

for $(\alpha, \nu, h), (\alpha', \nu', h') \in T_{(\theta,\lambda,u)}M$. We also denote the dual norm of $\| \cdot \|_{(\theta,\lambda,u)}$ by $\| \cdot \|_{(\theta,\lambda,u),*}$, that is,

$$\|f\|_{(\theta,\lambda,u),*} = \sup_{\|(\alpha,\nu,h)\|_{(\theta,\lambda,u)} \le 1} |f(\alpha, \nu, h)| \quad \text{for } f \in T^*_{(\theta,\lambda,u)}(M). \tag{4.11}$$

It is easily seen that $(M, \langle \cdot , \cdot \rangle)$ is a complete Hilbert manifold. We note that $\langle \cdot , \cdot \rangle_{(\theta,\lambda,u)}$ and $\| \cdot \|_{(\theta,\lambda,u)}$ depend only on θ. So sometimes we denote them by $\langle \cdot , \cdot \rangle_{(\theta,\cdot,\cdot)}$, $\| \cdot \|_{(\theta,\cdot,\cdot)}$. We have

$$\begin{aligned} \|(\alpha, \nu, h)\|^2_{(\theta,\cdot,\cdot)} &= \alpha^2 + \nu^2 + e^{(N-2)\theta}\|\nabla h\|_2^2 + e^{N\theta}\|h\|_2^2 \\ &= \alpha^2 + \nu^2 + \|h(\tfrac{x}{e^\theta})\|^2_{H^1} \\ &= \|(\alpha, \nu, h(\tfrac{x}{e^\theta}))\|^2_{(0,\cdot,\cdot)} \quad \text{for } (\alpha, \nu, h) \in T_{(\theta,\lambda,u)}M. \end{aligned} \tag{4.12}$$

We also have for all $(\alpha, \nu, h) \in T_{(\theta,\cdot,\cdot)}M$ and $\beta \in \mathbb{R}$,

$$\|(\alpha, \nu, h(e^\beta x))\|_{(\theta+\beta,\cdot,\cdot)} = \|(\alpha, \nu, h(x))\|_{(\theta,\cdot,\cdot)}. \tag{4.13}$$

We denote a natural distance induced by the metric $\langle \cdot , \cdot \rangle$ by

$$\begin{aligned} &\mathrm{dist}_M((\theta_0, \lambda_0, h_0), (\theta_1, \lambda_1, h_1)) \\ &\quad = \inf \left\{ \int_0^1 \|\dot\sigma(t)\|_{\sigma(t)}\, dt : \sigma(t) \in C^1([0, 1], M),\ \sigma(0) = (\theta_0, \lambda_0, h_0),\ \sigma(1) = (\theta_1, \lambda_1, h_1) \right\}. \end{aligned}$$

By property (4.13), we have for all $\beta \in \mathbb{R}$,

$$\mathrm{dist}_M((\theta_0 + \beta, \lambda_0, u_0(e^\beta x)), (\theta_1 + \beta, \lambda_1, u_1(e^\beta x))) = \mathrm{dist}_M((\theta_0, \lambda_0, u_0(x)), (\theta_1, \lambda_1, u_1(x))). \tag{4.14}$$

Using notation

$$\mathcal{D} = (\partial_\theta, \partial_\lambda, \partial_u),$$

we have:

Lemma 4.5. *For $(\theta, \lambda, u) \in M$, we have*

$$\|\mathcal{D}J(\theta, \lambda, u)\|_{(\theta,\lambda,u),*} = \Big(|P(\lambda, u(\tfrac{x}{e^\theta}))|^2 + |\partial_\lambda I(\lambda, u(\tfrac{x}{e^\theta}))|^2 + \|\partial_u I(\lambda, u(\tfrac{x}{e^\theta}))\|^2_{(H_r^1(\mathbb{R}^N))^*} \Big)^{\frac{1}{2}}.$$

Proof. By Lemma 4.4, we have

$$\mathcal{D}J(\theta, \lambda, u)(\alpha, \nu, h) = P(\lambda, u(\tfrac{x}{e^\theta}))\alpha + \partial_\lambda I(\lambda, u(\tfrac{x}{e^\theta}))\nu + \partial_u I(\lambda, u(\tfrac{x}{e^\theta}))h(\tfrac{x}{e^\theta}).$$

Noting (4.12), the conclusion of Lemma 4.5 follows from the definition (4.11). □

For $b \in \mathbb{R}$, we use notation

$$\widetilde{K}_b = \{(\theta, \lambda, u) \in M : J(\theta, \lambda, u) = b,\ \mathcal{D}J(\theta, \lambda, u) = (0, 0, 0)\}.$$

By (4.6)–(4.9), we observe that

$$\widetilde{K}_b = \{(\theta, \lambda, u(e^\theta x)) : \theta \in \mathbb{R},\ (\lambda, u) \in K_b\}.$$

We also use notation for $(\theta, \lambda, u) \in M$ and $\widetilde{A} \subset M$,

$$\mathrm{dist}_M((\theta, \lambda, u), \widetilde{A}) = \inf_{(\theta',\lambda',u') \in \widetilde{A}} \mathrm{dist}_M((\theta, \lambda, u), (\theta', \lambda', u')).$$

From condition $(\mathrm{PSP})_b$ for $I(\lambda, u)$, we deduce the following proposition.

Proposition 4.6. *For $b < 0$, $J(\theta, \lambda, u)$ satisfying the following property:*

Condition $(\widetilde{\mathrm{PSP}})_b$. For any sequence $(\theta_n, \lambda_n, u_n)_{n=1}^{\infty} \subset M$ with

$$J(\theta_n, \lambda_n, u_n) \to b, \tag{4.15}$$

$$\|\mathcal{D}J(\theta_n, \lambda_n, u_n)\|_{(\theta_n,\lambda_n,u_n),*} \to 0 \quad \text{as } n \to \infty, \tag{4.16}$$

we have

$$\mathrm{dist}_M((\theta_n, \lambda_n, u_n), \widetilde{K}_b) \to 0. \tag{4.17}$$

Proof. Suppose that $(\theta_n, \lambda_n, u_n)_{n=1}^{\infty}$ satisfies (4.15)–(4.16). It suffices to show that $(\theta_n, \lambda_n, u_n)_{n=1}^{\infty}$ has a subsequence with property (4.17). Setting $\hat{u}_n(x) = u_n(\frac{x}{e^{\theta_n}})$, we have by Lemma 4.5 that

$$\begin{aligned} &I(\lambda_n, \hat{u}_n) \to b < 0, \\ &P(\lambda_n, \hat{u}_n) \to 0, \quad \partial_\lambda I(\lambda_n, \hat{u}_n) \to 0, \quad \partial_u I(\lambda_n, \hat{u}_n) \to 0 \text{ strongly in } (H_r^1(\mathbb{R}^N))^*. \end{aligned}$$

Thus by Proposition 4.1, there exists a subsequence — still denoted by $(\lambda_n, \hat{u}_n)_{n=1}^{\infty}$ — and $(\lambda_0, \hat{u}_0) \in \mathbb{R}\times H_r^1(\mathbb{R}^N)$ such that

$$\lambda_n \to \lambda_0 \quad \text{and} \quad \hat{u}_n \to \hat{u}_0 \text{ strongly in } H_r^1(\mathbb{R}^N).$$

Note that $(\lambda_0, \hat{u}_0) \in K_b$ and thus $(\theta_n, \lambda_0, \hat{u}_0(e^{\theta_n}x)) \in \widetilde{K}_b$. By (4.14), we have

$$\begin{aligned} \mathrm{dist}_M((\theta_n, \lambda_n, u_n), \widetilde{K}_b) &\le \mathrm{dist}_M((\theta_n, \lambda_n, u_n), (\theta_n, \lambda_0, \hat{u}_0(e^{\theta_n}x))) \\ &= \mathrm{dist}_M((0, \lambda_n, \hat{u}_n), (0, \lambda_0, \hat{u}_0(x))) \\ &\le (|\lambda_n - \lambda_0|^2 + \|\hat{u}_n - \hat{u}_0\|_{H^1}^2)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $n \to \infty$. □

As a corollary to Proposition 4.6, we have the following uniform estimate of $\mathcal{D}J(\theta, \lambda, u)$ outside a ρ-neighborhood of $\widetilde{K}_b$.

Corollary 4.7. *Assume $b < 0$. Then the following statements hold:*

(i) *When $\widetilde{K}_b \ne \emptyset$, i.e., $K_b \ne \emptyset$, for any $\rho > 0$ there exists $\delta_\rho > 0$ such that for $(\theta, \lambda, u) \in M$,*

$$|J(\theta, \lambda, u) - b| < \delta_\rho \text{ and } \mathrm{dist}_M((\theta, \lambda, u), \widetilde{K}_b) \ge \rho \implies \|\mathcal{D}J(\theta, \lambda, u)\|_{(\theta,\lambda,u),*} \ge \delta_\rho.$$

(ii) *When $\widetilde{K}_b = \emptyset$, i.e., $K_b = \emptyset$, there exists $\delta_0 > 0$ such that*

$$|J(\theta, \lambda, u) - b| < \delta_0 \implies \|\mathcal{D}J(\theta, \lambda, u)\|_{(\theta,\lambda,u),*} \ge \delta_0.$$

We note that $\widetilde{K}_b$ is not compact in M but Corollary 4.7 gives us a uniform lower bound of $\|\mathcal{D}J(\theta, \lambda, u)\|_{(\theta,\lambda,u),*}$ outside a ρ-neighborhood of $\widetilde{K}_b$, which enables us to construct a deformation flow for $J(\theta, \lambda, u)$.

4.3 Deformation Flow for $J(\theta, \lambda, u)$

In this subsection we give a deformation result for $J(\theta, \lambda, u)$. We need the following notation:

$$\begin{aligned} [J \le c]_M &= \{(\theta, \lambda, u) \in M : J(\theta, \lambda, u) \le c\} && \text{for } c \in \mathbb{R}, \\ \widetilde{N}_\rho(\widetilde{A}) &= \{(\theta, \lambda, u) \in M : \mathrm{dist}_M((\theta, \lambda, u), \widetilde{A}) \le \rho\} && \text{for } \widetilde{A} \subset M \text{ and } \rho > 0. \end{aligned}$$

We have the following deformation result.

Proposition 4.8. *Assume $b < 0$. Then for any $\bar{\varepsilon} > 0$ and $\rho > 0$ there exist $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\tilde{\eta}(t, \theta, \lambda, u) : [0, 1] \times M \to M$ such that:*

(1) *$\tilde{\eta}(0, \theta, \lambda, u) = (\theta, \lambda, u)$ for all $(\theta, \lambda, u) \in M$.*

(2) *$\tilde{\eta}(t, \theta, \lambda, u) = (\theta, \lambda, u)$ if $(\theta, \lambda, u) \in [J \le b - \bar{\varepsilon}]_M$.*

(3) *$J(\tilde{\eta}(t, \theta, \lambda, u)) \le J(\theta, \lambda, u)$ for all $(t, \theta, \lambda, u) \in [0, 1] \times M$.*

(4) *$\tilde{\eta}(1, [J \le b + \varepsilon]_M \setminus \widetilde{N}_\rho(\widetilde{K}_b)) \subset [J \le b - \varepsilon]_M$, $\tilde{\eta}(1, [J \le b + \varepsilon]_M) \subset [J \le b - \varepsilon]_M \cup \widetilde{N}_\rho(\widetilde{K}_b)$.*

(5) *If* $K_b = \emptyset$*, then* $\tilde\eta(1, [J \le b+\varepsilon]_M) \subset [J \le b-\varepsilon]_M$.
(6) *We write* $\tilde\eta(t,\theta,\lambda,u) = (\tilde\eta_0(t,\theta,\lambda,u), \tilde\eta_1(t,\theta,\lambda,u), \tilde\eta_2(t,\theta,\lambda,u))$*. Then* $\tilde\eta_0(t,\theta,\lambda,u)$ *and* $\tilde\eta_1(t,\theta,\lambda,u)$ *are even in* u *and* $\tilde\eta_2(t,\theta,\lambda,u)$ *is odd in* u*. That is, for all* $(t,\theta,\lambda,u) \in [0,1]\times M$,

$$\tilde\eta_0(t,\theta,\lambda,-u) = \tilde\eta_0(t,\theta,\lambda,u),$$
$$\tilde\eta_1(t,\theta,\lambda,-u) = \tilde\eta_1(t,\theta,\lambda,u),$$
$$\tilde\eta_2(t,\theta,\lambda,-u) = -\tilde\eta_2(t,\theta,\lambda,u).$$

Proof. Let $M' = \{(\theta,\lambda,u) \in M : \mathcal{D}J(\theta,\lambda,u) \neq (0,0,0)\}$. It is well known that there exists a pseudo-gradient vector field $\mathcal{V} : M' \to TM$ such that for $(\theta,\lambda,u) \in M'$:
(1) $\|\mathcal{V}(\theta,\lambda,u)\|_{(\theta,\lambda,u)} \le 2\|\mathcal{D}J(\theta,\lambda,u)\|_{(\theta,\lambda,u),*}$,
(2) $\mathcal{D}J(\theta,\lambda,u)\mathcal{V}(\theta,\lambda,u) \ge \|\mathcal{D}J(\theta,\lambda,u)\|^2_{(\theta,\lambda,u),*}$,
(3) $\mathcal{V} : M' \to \mathbb{R}\times\mathbb{R}\times H^1_r(\mathbb{R}^N)$ is locally Lipschitz continuous.
We can also have the following:
(4) $\mathcal{V}(\theta,\lambda,u) = (\mathcal{V}_0(\theta,\lambda,u), \mathcal{V}_1(\theta,\lambda,u), \mathcal{V}_2(\theta,\lambda,u))$ satisfies

$$\mathcal{V}_0(\theta,\lambda,-u) = \mathcal{V}_0(\theta,\lambda,u),$$
$$\mathcal{V}_1(\theta,\lambda,-u) = \mathcal{V}_1(\theta,\lambda,u),$$
$$\mathcal{V}_2(\theta,\lambda,-u) = -\mathcal{V}_2(\theta,\lambda,u).$$

For a given $\rho > 0$ we choose $\delta_\rho > 0$ by Corollary 4.7 so that

$$|J(\theta,\lambda,u) - b| < \delta_\rho \text{ and } (\theta,\lambda,u) \notin \widetilde{N}_{\rho/3}(\widetilde{K}_b) \implies \|\mathcal{D}J(\theta,\lambda,u)\|_{(\theta,\lambda,u),*} \ge \delta_\rho. \tag{4.18}$$

We choose a locally Lipschitz continuous function $\varphi : M \to [0,1]$ such that

$$\begin{aligned} \varphi(\theta,\lambda,u) &= 1 && \text{for } (\theta,\lambda,u) \in M \setminus \widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b),\\ \varphi(\theta,\lambda,u) &= 0 && \text{for } (\theta,\lambda,u) \in \widetilde{N}_{\frac{1}{3}\rho}(\widetilde{K}_b),\\ \varphi(\theta,\lambda,-u) &= \varphi(\theta,\lambda,u) && \text{for all } (\theta,\lambda,u) \in M. \end{aligned}$$

We note that $\widetilde{K}_b$ is symmetric in the following sense:

$$(\theta,\lambda,u) \in \widetilde{K}_b \implies (\theta,\lambda,-u) \in \widetilde{K}_b.$$

For $\bar\varepsilon > 0$ we may assume $\bar\varepsilon \in (0,\delta_\rho)$ and we choose a locally Lipschitz continuous function $\psi : \mathbb{R} \to [0,1]$ such that

$$\psi(s) = \begin{cases} 1 & \text{for } s \in [b - \frac{\bar\varepsilon}{2}, b + \frac{\bar\varepsilon}{2}],\\ 0 & \text{for } s \in \mathbb{R} \setminus [b-\bar\varepsilon, b+\bar\varepsilon]. \end{cases}$$

We consider the following ODE in M:

$$\begin{cases} \dfrac{d\tilde\eta}{dt} = -\varphi(\tilde\eta)\psi(J(\tilde\eta))\dfrac{\mathcal{V}(\tilde\eta)}{\|\mathcal{V}(\tilde\eta)\|_{\tilde\eta}},\\ \tilde\eta(0,\theta,\lambda,u) = (\theta,\lambda,u). \end{cases}$$

For $\varepsilon \in (0,\bar\varepsilon)$ small, $\tilde\eta(t,\theta,\lambda,u)$ has the desired properties (1)–(6). We show just the first part of (4):

$$\tilde\eta(1, [J \le b+\varepsilon]_M \setminus \widetilde{N}_\rho(\widetilde{K}_b)) \subset [J \le b-\varepsilon]_M. \tag{4.19}$$

We can check properties (1)–(3) easily and we use them in what follows. We also note that

$$\left\|\frac{d\tilde\eta}{dt}(t)\right\|_{\tilde\eta(t)} \le 1 \quad \text{for all } t. \tag{4.20}$$

For $\varepsilon \in (0, \frac{\bar\varepsilon}{2})$, which we choose later, we assume $\tilde\eta(t) = \tilde\eta(t,\theta,\lambda,u)$ satisfies

$$\tilde\eta(0) \in [J \le b+\varepsilon]_M \setminus \widetilde{N}_\rho(\widetilde{K}_b).$$

If $\tilde\eta(1) \notin [J \le b-\varepsilon]_M$, we have $J(\tilde\eta(t)) \in [b-\varepsilon, b+\varepsilon]$ for all $t \in [0,1]$. We consider two cases:
- Case 1: $\tilde\eta(t) \notin \widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b)$ for all $t \in [0,1]$,
- Case 2: $\tilde\eta(t_0) \in \widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b)$ for some $t_0 \in [0,1]$.

First we consider Case 1. By (4.18) we have

$$\|\mathcal{D}J(\tilde{\eta}(t))\|_{\tilde{\eta}(t),*} \geq \delta_\rho \quad \text{for all } t \in [0,1].$$

By our choice of φ and ψ, we have

$$\frac{d}{dt}J(\tilde{\eta}(t)) = \mathcal{D}J(\tilde{\eta}(t))\frac{d\tilde{\eta}}{dt}(t) = -\mathcal{D}J(\tilde{\eta}(t))\frac{\mathcal{V}(\tilde{\eta}(t))}{\|\mathcal{V}(\tilde{\eta}(t))\|_{\tilde{\eta}(t)}} \leq -\frac{1}{2}\|\mathcal{D}J(\tilde{\eta}(t))\|_{\tilde{\eta}(t),*} \leq -\frac{1}{2}\delta_\rho.$$

Thus we have

$$J(\tilde{\eta}(1)) = J(\tilde{\eta}(0)) + \int_0^1 \frac{d}{dt}J(\tilde{\eta}(t))\,dt \leq J(\tilde{\eta}(0)) - \frac{\delta_\rho}{2} \leq b + \varepsilon - \frac{\delta_\rho}{2}.$$

If Case 2 takes a place, we can find an interval $[\alpha,\beta] \subset [0,1]$ such that

$$\tilde{\eta}(\alpha) \in \partial\widetilde{N}_\rho(\widetilde{K}_b), \quad \tilde{\eta}(\beta) \in \partial\widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b),$$
$$\tilde{\eta}(t) \in \widetilde{N}_\rho(\widetilde{K}_b) \setminus \widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b) \quad \text{for all } t \in [\alpha,\beta).$$

By (4.20),

$$\beta - \alpha \geq \int_\alpha^\beta \left\|\frac{d\tilde{\eta}}{dt}(t)\right\|_{\tilde{\eta}(t)} dt \geq \operatorname{dist}_M(\tilde{\eta}(\alpha),\tilde{\eta}(\beta)) \geq \frac{1}{3}\rho.$$

Thus,

$$\begin{aligned}
J(\tilde{\eta}(1)) \leq J(\tilde{\eta}(\beta)) &= J(\tilde{\eta}(\alpha)) + \int_\alpha^\beta \frac{d}{dt}J(\tilde{\eta}(t))\,dt \\
&\leq J(\tilde{\eta}(0)) + \int_\alpha^\beta \frac{d}{dt}J(\tilde{\eta}(t))\,dt \leq J(\tilde{\eta}(0)) + \int_\alpha^\beta -\frac{\delta_\rho}{2}\,dt \\
&\leq J(\tilde{\eta}(0)) - \frac{\delta_\rho}{2}(\beta-\alpha) \leq b + \varepsilon - \frac{\delta_\rho \rho}{6}.
\end{aligned}$$

Choosing $\varepsilon < \min\{\frac{\bar{\varepsilon}}{2}, \frac{\delta_\rho}{4}, \frac{1}{12}\delta_\rho\rho\}$, we have $J(\tilde{\eta}(1)) \leq b - \varepsilon$ in both cases. This is a contradiction and we have (4.19). □

In the following section, we can construct a deformation flow for $I(\lambda,u)$ using $\tilde{\eta}(t,\theta,\lambda,u)$.

4.4 Deformation Flow for $I(\lambda, u)$

In this subsection, we construct a deformation flow for $I(\lambda,u)$ and give a proof to our Proposition 3.1.

We use the following maps:

$$\pi : M \to \mathbb{R} \times H_r^1(\mathbb{R}^N), \quad (\theta,\lambda,u(x)) \mapsto (\lambda, u(\tfrac{x}{e^\theta})),$$
$$\iota : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to M, \quad (\lambda,u(x)) \mapsto (0,\lambda,u(x))$$

and we construct a deformation flow

$$\eta(t,\lambda,u) : [0,1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

as a composition $\pi \circ \tilde{\eta}(t,\cdot) \circ \iota$;

$$\eta(t,\lambda,u) = \pi(\tilde{\eta}(t,\iota(\lambda,u))) = \pi(\tilde{\eta}(t,0,\lambda,u)). \tag{4.21}$$

As fundamental properties of π and ι, we have

$$\begin{aligned}
\pi(\iota(\lambda,u)) &= (\lambda,u) && \text{for all } (\lambda,u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N),\\
\iota(\pi(\theta,\lambda,u)) &= (0,\lambda,u(\tfrac{x}{e^\theta})) && \text{for all } (\theta,\lambda,u) \in M,\\
J(\theta,\lambda,u) &= I(\pi(\theta,\lambda,u)) && \text{for all } (\theta,\lambda,u) \in M.
\end{aligned}$$

Clearly $\pi(\widetilde{K}_b) = K_b$. The following lemma gives us a relation between $\pi(\widetilde{N}_\rho(\widetilde{K}_b))$ and $N_\rho(K_b)$.

Lemma 4.9. *For any $\rho > 0$ there exists $R(\rho) > 0$ such that*

$$\pi(\widetilde{N}_\rho(\widetilde{K}_b)) \subset N_{R(\rho)}(K_b), \tag{4.22}$$

$$\iota((\mathbb{R} \times H^1_r(\mathbb{R}^N)) \setminus N_{R(\rho)}(K_b)) \subset M \setminus \widetilde{N}_\rho(\widetilde{K}_b). \tag{4.23}$$

Moreover,

$$R(\rho) \to 0 \quad \text{as } \rho \to 0. \tag{4.24}$$

Proof. For $\rho > 0$, suppose that $(\lambda_0, u_0) \in \mathbb{R} \times H^1_r(\mathbb{R}^N)$ satisfies $\mathrm{dist}_M((0, \lambda_0, u_0), \widetilde{K}_b) \le \rho$. First we show

$$\mathrm{dist}((\lambda_0, u_0), K_b) \le e^{\frac{N\rho}{2}}\rho + \sup\{\|\omega(e^\alpha x) - \omega(x)\|_{H^1} : |\alpha| \le \rho,\ \omega \in P_2(K_b)\}. \tag{4.25}$$

In fact, for any $\varepsilon > 0$ there exists $\sigma(t) = (\theta(t), \lambda(t), u(t)) \in C^1([0,1], M)$ such that $\sigma(0) = (0, \lambda_0, u_0)$, $\sigma(1) \in \widetilde{K}_b$ and

$$\int_0^1 \|\dot\sigma(t)\|_{\sigma(t)}\,dt \le \rho + \varepsilon.$$

In particular, since $\theta(0) = 0$, for any $t \in [0,1]$

$$|\theta(t)| \le \int_0^1 |\dot\theta(t)|\,dt \le \int_0^1 \|\dot\sigma(t)\|_{\sigma(t)}\,dt \le \rho + \varepsilon.$$

Thus

$$\begin{aligned}
\|(\lambda(0), u(0)) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^1_r(\mathbb{R}^N)} &\le \int_0^1 (|\dot\lambda(t)|^2 + \|\dot u(t)\|^2_{H^1})^{\frac12}\,dt \\
&\le e^{\frac{N(\rho+\varepsilon)}{2}} \int_0^1 (|\dot\theta(t)|^2 + |\dot\lambda(t)|^2 + e^{(N-2)\theta(t)}\|\nabla \dot u(t)\|_2^2 + e^{N\theta(t)}\|\dot u(t)\|_2^2)^{\frac12}\,dt \\
&= e^{\frac{N(\rho+\varepsilon)}{2}} \int_0^1 \|(\dot\theta(t), \dot\lambda(t), \dot u(t))\|_{\sigma(t)}\,dt \\
&\le e^{\frac{N(\rho+\varepsilon)}{2}}(\rho + \varepsilon).
\end{aligned}$$

On the other hand, since $(\theta(1), \lambda(1), u(1)) \in \widetilde{K}_b$, we have $(\lambda(1), u(1)(\frac{x}{e^{\theta(1)}})) \in K_b$, i.e., $u(1)(\frac{x}{e^{\theta(1)}}) \in P_2(K_b)$. Thus

$$\begin{aligned}
\mathrm{dist}((\lambda_0, u_0), K_b) &\le \|(\lambda(0), u(0)) - (\lambda(1), u(1)(\tfrac{x}{e^{\theta(1)}}))\|_{\mathbb{R}\times H^1_r(\mathbb{R}^N)} \\
&\le \|(\lambda(0), u(0)) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^1_r(\mathbb{R}^N)} + \|(\lambda(1), u(1)(x)) - (\lambda(1), u(1)(\tfrac{x}{e^{\theta(1)}}))\|_{\mathbb{R}\times H^1_r(\mathbb{R}^N)} \\
&\le \|(\lambda(0), u(0)) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^1_r(\mathbb{R}^N)} + \|u(1)(x) - u(1)(\tfrac{x}{e^{\theta(1)}})\|_{H^1} \\
&\le e^{\frac{N(\rho+\varepsilon)}{2}}(\rho+\varepsilon) + \sup\{\|\omega(e^\alpha x) - \omega(x)\|_{H^1} : |\alpha| \le \rho + \varepsilon,\ \omega \in P_2(K_b)\}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have (4.25).

We set

$$R(\rho) = e^{\frac{N\rho}{2}}\rho + \sup\{\|\omega(e^\alpha x) - \omega(x)\|_{H^1} : |\alpha| \le \rho,\ \omega \in P_2(K_b)\}.$$

Then

$$\mathrm{dist}_M((0, \lambda_0, u_0), \widetilde{K}_b) \le \rho \implies \mathrm{dist}((\lambda_0, u_0), K_b) \le R(\rho). \tag{4.26}$$

Since $P_2(K_b)$ is compact in $H^1_r(\mathbb{R}^N)$, we have

$$\sup\{\|\omega(e^\alpha x) - \omega(x)\|_{H^1} : |\alpha| \le \rho,\ \omega \in P_2(K_b)\} \to 0 \quad \text{as } \rho \to 0,$$

which implies (4.24). Noting $\mathrm{dist}_M((\theta, \lambda, u), \widetilde{K}_b) = \mathrm{dist}_M((0, \lambda, u(\frac{x}{e^\theta})), \widetilde{K}_b)$, we obtain that (4.26) implies (4.22) and (4.23). □

Now we can give a proof of Proposition 3.1 (ii).

Proof of Proposition 3.1 (ii). Let $\mathcal{O}$ be a given neighborhood of K_b and let $\bar{\varepsilon} > 0$ be a given positive number. We take small $\rho > 0$ such that $N_{R(\rho)}(K_b) \subset \mathcal{O}$. By Proposition 4.8, there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\widetilde{\eta} : [0,1] \times M \to M$ such that (1)–(6) in Proposition 4.8 hold. We define $\eta(t, \lambda, u) : [0,1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ by (4.21). We can check that $\eta(t, \lambda, u)$ satisfies properties (1)–(6) of Proposition 3.1. Here we just prove

$$\eta(1, [I \le b + \varepsilon] \setminus \mathcal{O}) \subset [I \le b - \varepsilon]. \tag{4.27}$$

Since $[I \le b + \varepsilon] \setminus \mathcal{O} \subset [I \le b + \varepsilon] \setminus N_{R(\rho)}(K_b)$, we have from (4.23) that

$$\iota([I \le b + \varepsilon] \setminus \mathcal{O}) \subset [J \le b + \varepsilon]_M \setminus \widetilde{N}_\rho(\widetilde{K}_b). \tag{4.28}$$

By (4) of Proposition 4.8,

$$\widetilde{\eta}(1, [J \le b + \varepsilon]_M \setminus \widetilde{N}_\rho(\widetilde{K}_b)) \subset [J \le b - \varepsilon]_M. \tag{4.29}$$

By the definition of π and (4.6),

$$\pi([J \le b - \varepsilon]_M) \subset [I \le b - \varepsilon]. \tag{4.30}$$

Combining (4.28)–(4.30), we have (4.27). □

Remark 4.10. By our construction,

$$t \mapsto \widetilde{\eta}(t, \theta, \lambda, u), \quad [0,1] \to \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

is of class C^1. However,

$$t \mapsto u(\tfrac{x}{e^t}), \quad \mathbb{R} \to H_r^1(\mathbb{R}^N)$$

is continuous but not of class C^1 for $u \in H_r^1(\mathbb{R}^N) \setminus H^2(\mathbb{R}^N)$ and thus

$$t \mapsto \eta(t, \lambda, u) = \pi(\widetilde{\eta}(t, 0, \lambda, u)), \quad [0,1] \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

is continuous but not of class C^1.

5 Minimizing Problem

In this section we assume (g1)–(g4) (without (g5)) and we deal with Theorem 0.1. Under the condition $\mathcal{I}_m < 0$, the existence of a solution is shown by Shibata [16], that is, he showed that $\mathcal{I}_m$ is achieved by a solution of problem $(*)_m$. First we give an approach using our functional $I(\lambda, u)$.

5.1 Mountain Pass Approach

Under conditions (g1)–(g4), as in Sections 1–2, we define $\lambda_0 \in (-\infty, \infty]$ by (1.5).

(i) For $\lambda < \lambda_0$, $u \mapsto \widehat{I}(\lambda, u)$ has the mountain pass geometry.

(ii) When $\lambda_0 < \infty$, $\widehat{I}(\lambda, u) \ge 0$ for all $\lambda \ge \lambda_0$ and $u \in H_r^1(\mathbb{R}^N)$.

We set for $\lambda < \lambda_0$,

$$\begin{aligned} \widehat{\Gamma}_{mp}(\lambda) &= \{\zeta(\tau) \in C([0,1], H_r^1(\mathbb{R}^N)) : \zeta(0) = 0,\ \widehat{I}(\lambda, \zeta(1)) < 0\}, \\ a_{mp}(\lambda) &= \inf_{\zeta \in \widehat{\Gamma}_{mp}(\lambda)} \max_{\tau \in [0,1]} \widehat{I}(\lambda, \zeta(\tau)). \end{aligned} \tag{5.1}$$

We note that if (g5) holds, $a_{mp}(\lambda)$ coincides with $a_1(\lambda)$ defined in (1.6)–(1.7). By the result of [9], we see that $a_{mp}(\lambda)$ is attained by a critical point of $u \mapsto \widehat{I}(\lambda, u)$. This fact can also be shown via our new deformation argument. See Section 6.

We set

$$m_0 = 2 \inf_{\lambda \in (-\infty, \lambda_0)} \frac{a_{mp}(\lambda)}{e^\lambda}. \tag{5.2}$$

As in Sections 1–4, we can show the following theorem.

Theorem 5.1. *Assume* (g1)–(g4). *Suppose $m > m_0$. Then $(*)_m$ has at least one solution $(\lambda_\sharp, u_\sharp)$, which is characterized by the following minimax method;*

$$I(\lambda_\sharp, u_\sharp) = b_{mp} < 0,$$

where

$$b_{mp} = \inf_{\gamma\in\Gamma_{mp}} \max_{\tau\in[0,1]} I(\gamma(\tau)),$$
$$\Gamma_{mp} = \{\gamma(\tau) \in C([0,1], \mathbb{R}\times H_r^1(\mathbb{R}^N)) : \gamma(0) \in (\lambda_m, \infty)\times\{0\},\ I(\gamma(0)) \le B_m - 1,$$
$$\gamma(1) \in (\mathbb{R}\times H_r^1(\mathbb{R}^N)) \setminus \Omega_m,\ I(\gamma(1)) \le B_m - 1\}.$$

Here $\lambda_m \in \mathbb{R}$, $\Omega_m \subset [\lambda_m, \infty)\times H_r^1(\mathbb{R}^N)$ and $B_m = \inf_{(\lambda,u)\in\partial\Omega_m} I(\lambda, u) > -\infty$ are chosen as in Section 2.

As a corollary, we have:

Corollary 5.2. *Assume* (g1)–(g4) *and suppose $m > m_0$. Then*

$$\mathcal{I}_m < 0.$$

Proof. The critical point $(\lambda_\sharp, u_\sharp)$ obtained in Theorem 5.1 satisfies

$$\|u_\sharp\|_2^2 = m \quad \text{and} \quad \mathcal{F}(u_\sharp) = I(\lambda_\sharp, u_\sharp) = b_{mp} < 0.$$

Thus $\mathcal{I}_m = \inf_{\|u\|_2^2=m} \mathcal{F}(u) \le \mathcal{F}(u_\sharp) < 0$. □

We also have:

Theorem 5.3. *Under the assumption of Theorem 5.1, there exists $\gamma_0 \in \Gamma_{mp}$ such that*

$$b_{mp} = \max_{\tau\in[0,1]} I(\gamma_0(\tau)).$$

Proof. Let $(\lambda_\sharp, u_\sharp)$ be the critical point corresponding to b_{mp}. In [12], we find a path $\zeta_0(\tau) \in \widehat{\Gamma}_{mp}(\lambda_\sharp)$ such that

$$u_\sharp \in \zeta_0([0,1]) \quad \text{and} \quad a_{mp}(\lambda_\#) = \widehat{I}(\lambda_\sharp, u_\sharp) = \max_{\tau\in[0,1]} \widehat{I}(\lambda_\sharp, \gamma_0(\tau)).$$

As in the proof of Lemma 2.3, we may assume $\widehat{I}(\lambda_\sharp, \zeta_0(1)) \le B_m - 1$. Joining paths

$$[0,1] \to \mathbb{R}\times H_r^1(\mathbb{R}^N), \quad \tau \mapsto (\lambda_\sharp\tau + L(1-\tau), 0)$$

and

$$[0,1] \to \mathbb{R}\times H_r^1(\mathbb{R}^N), \quad \tau \mapsto (\lambda_\sharp, \zeta_0(\tau)),$$

we find the desired path $\gamma_0 \in \Gamma_{mp}$. □

5.2 Mountain Pass Characterization of $\mathcal{I}_m$

Next we consider problem $(*)_m$ under conditions (g1)–(g4) and $\mathcal{I}_m < 0$.

Shibata [16] showed the following:

Theorem 5.4 ([16]). *There exists $m_S \in [0, \infty)$ such that:*
(i) $\mathcal{I}_m = 0$ *for* $m \in (0, m_S]$, $\mathcal{I}_m < 0$ *for* $m \in (m_S, \infty)$.
(ii) *If $\mathcal{I}_m < 0$, then $\mathcal{I}_m$ is attained and the minimizer is a solution of $(*)_m$.*

In what follows, we will show that m_0 given in (5.2) coincides with m_S and $\mathcal{I}_m = b_{mp}$. Precisely:
(i) $m > m_0$ if and only if $\mathcal{I}_m < 0$.
(ii) For $m > m_0$, $\mathcal{I}_m = b_{mp}$.

First we show the minimizer of $\mathcal{I}_m$ satisfies the following properties.

Lemma 5.5. *Suppose $\mathcal{I}_m < 0$ and let (μ_*, u_*) be the corresponding minimizer of $\mathcal{I}_m$, i.e., $\mathcal{F}(u_*) = \mathcal{I}_m$, $\|u_*\|_2^2 = m$. Then:*
(i) $\mu_* > 0$.
(ii) $\frac{N-2}{2}\|\nabla u_*\|_2^2 + N(\frac{\mu_*}{2}\|u_*\|_2^2 - \int_{\mathbb{R}^N} G(u_*)) = 0$.

Proof. First we deal with (ii). We can verify that the Pohozaev identity (ii) holds for u_* after the standard regularity argument. Here we give another proof of (ii).

We set $u_{*\theta}(x) = \theta^{\frac{N}{2}} u_*(\theta x)$ for $\theta > 0$. Since u_* is a minimizer of $\mathcal{F}(u)$ under the constraint $\|u\|_2^2 = m$ and $\|u_{*\theta}\|_2^2 = m$ for all $\theta > 0$, we have

$$\frac{d}{d\theta}\Big|_{\theta=1} \mathcal{F}(u_{*\theta}) = 0,$$

that is,

$$\|\nabla u_*\|_2^2 + N\int_{\mathbb{R}^N} G(u_*) - \frac{N}{2}\int_{\mathbb{R}^N} g(u_*)u_* = 0. \tag{5.3}$$

Since (μ_*, u_*) solves $(*)_m$, we also have

$$\|\nabla u_*\|_2^2 + \mu_*\|u_*\|_2^2 = \int_{\mathbb{R}^N} g(u_*)u_*. \tag{5.4}$$

Thus (ii) follows from (5.3) and (5.4).

Next we show (i). By (ii), we have

$$\frac{\mu_* N}{2} m = \frac{\mu_* N}{2}\|u_*\|_2^2 = -N\mathcal{F}(u_*) + \|\nabla u_*\|_2^2 \ge -N\mathcal{I}_m > 0.$$

Thus we have $\mu_* > 0$. □

By Lemma 5.5, setting $\lambda_* = \log \mu_*$, (λ_*, u_*) is a critical point of $I(\lambda, u)$ with $I(\lambda_*, u_*) = \mathcal{I}_m$ and $P(\lambda_*, u_*) = 0$. Next we show:

Proposition 5.6. *Suppose $\mathcal{I}_m < 0$ and let (λ_*, u_*) be a critical point corresponding to $\mathcal{I}_m$. Then we have:*
(i) *$u \mapsto \widehat{I}(\lambda_*, u)$ has the mountain pass geometry, that is, $\lambda_* < \lambda_0$.*
(ii) *$\widehat{I}(\lambda_*, u_*) \ge a_{mp}(\lambda_*)$.*
(iii) *$m > m_0$, where m_0 is given in (5.2).*

Proof. (i) It suffices to show $\widehat{I}(\lambda_*, u) < 0$ for some $u \in H_r^1(\mathbb{R}^N)$. We set

$$\widehat{G}(\xi) = G(\xi) - \frac{e^{\lambda_*}}{2}\xi^2 \quad \text{for } \xi \in \mathbb{R}.$$

Then we have for some $v \in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \widehat{G}(v) > 0. \tag{5.5}$$

In fact, when $N \ge 3$, it follows from $P(\lambda_*, u_*) = 0$ that (5.5) holds with $v = u_*$. When $N = 2$, we have by $P(\lambda_*, u_*) = 0$ that

$$\int_{\mathbb{R}^N} \widehat{G}(u_*) = 0.$$

We also have from $(*)_m$

$$\frac{d}{ds}\Big|_{s=1} \int_{\mathbb{R}^N} \widehat{G}(su_*) = \int_{\mathbb{R}^N} g(u_*)u_* - e^{\lambda_*}\|u_*\|_2^2 = \|\nabla u_*\|_2^2 > 0.$$

Thus (5.5) holds with $v = su_*$ for $s > 1$ closed to 1. Since

$$\widehat{I}(\lambda_*, v(\tfrac{x}{\theta})) = \frac{1}{2}\theta^{N-2}\|\nabla v\|_2^2 - \theta^N \int_{\mathbb{R}^N} \widehat{G}(v) < 0 \quad \text{for large } \theta \gg 1,$$

(i) holds.

(ii) By the result of [12], the mountain pass minimax value $a_{mp}(\lambda_*)$ gives the least energy level for $\widehat{I}(\lambda_*, u)$. Thus $\widehat{I}(\lambda_*, u_*) \geq a_{mp}(\lambda_*)$.

(iii) Note that (ii) implies

$$\frac{e^{\lambda_*}}{2} m = \frac{e^{\lambda_*}}{2} \|u_*\|_2^2 = \widehat{I}(\lambda_*, u_*) - \mathcal{F}(u_*) \geq a_{mp}(\lambda_*) - \mathcal{I}_m > a_{mp}(\lambda_*).$$

Thus

$$m > 2\frac{a_{mp}(\lambda_*)}{e^{\lambda_*}} \geq m_0. \qquad \square$$

Proposition 5.7. *Suppose* $\mathcal{I}_m < 0$. *Then* $\mathcal{I}_m = b_{mp}$.

Proof. As in Lemma 2.3, we can show

$$b_{mp} \leq a_{mp}(\lambda) - \frac{e^{\lambda}}{2} m \quad \text{for all } \lambda \in (-\infty, \lambda_0).$$

Thus, by (ii) of Proposition 5.6, for a critical point (λ_*, u_*) corresponding to $\mathcal{I}_m$,

$$\mathcal{I}_m = I(\lambda_*, u_*) = \widehat{I}(\lambda_*, u_*) - \frac{e^{\lambda_*}}{2} m \geq a_{mp}(\lambda_*) - \frac{e^{\lambda_*}}{2} m \geq b_{mp}.$$

On the other hand, it follows from (iii) of Proposition 5.6 that $m > m_0$ and b_{mp} is attained by a critical point $(\lambda_\sharp, u_\sharp) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$. Thus

$$\|u_\sharp\|_2^2 = m, \quad \mathcal{F}(u_\sharp) = I(\lambda_\sharp, u_\sharp) = b_{mp},$$

and so

$$\mathcal{I}_m = \inf_{\|u\|_2^2 = m} \mathcal{F}(u) \leq \mathcal{F}(u_\sharp) = b_{mp}.$$

Therefore we have $\mathcal{I}_m = b_{mp}$. $\square$

Corollary 5.8. *We have* $\mathcal{I}_m < 0$ *if and only if* $m > m_0$.

Proof. The “if” part follows from Theorem 5.1 and the “only if” part follows from Proposition 5.6. $\square$

End of the proof of Theorem 0.1. Theorem 0.1 follows from Theorem 5.1, Proposition 5.6, Proposition 5.7 and Corollary 5.8. $\square$

6 Deformation Lemma for Scalar Field Equations

In this section we study the following nonlinear scalar field equations:

$$\begin{cases} -\Delta u + \mu u = g(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{6.1}$$

where $N \geq 2$, $\mu > 0$ and $g(\xi) \in C(\mathbb{R}, \mathbb{R})$ satisfies (g1)–(g3) with $p = \frac{N+2}{N-2}$ $(N \geq 3)$, $p \in (1, \infty)$ $(N = 2)$. Solutions of (6.1) are characterized as critical points of the following functional:

$$I(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{\mu}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u) \in C^1(H_r^1(\mathbb{R}^N), \mathbb{R}).$$

Here we use notation different from previous sections. We also write

$$P(u) = \frac{N-2}{2}\|\nabla u\|_2^2 + N\left(\frac{\mu}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u)\right).$$

In this section we give a new deformation result for (6.1) using ideas in Sections 3–4.

A key of our argument is the following proposition.

Proposition 6.1. *For any $b \in \mathbb{R}$, $I(u)$ satisfies the following property:*

Condition (PSP′)$_b$. If a sequence $(u_n)_{n=1}^{\infty} \subset H_r^1(\mathbb{R}^N)$ satisfies as $n \to \infty$,

$$I(u_n) \to b, \tag{6.2}$$
$$\partial_u I(u_n) \to 0 \quad \text{strongly in } (H_r^1(\mathbb{R}^N))^*, \tag{6.3}$$
$$P(u_n) \to 0, \tag{6.4}$$

then $(u_n)_{n=1}^{\infty}$ has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$.

Proof. First we note by (g2)–(g3) with $p = \frac{N+2}{N-2}$ $(N \geq 3)$, $p \in (0, \infty)$ $(N = 2)$ that $u_m \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$ implies for any $\varphi \in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} g(u_n)\varphi \to \int_{\mathbb{R}^N} g(u_0)\varphi, \quad \int_{\mathbb{R}^N} g(u_n)u_n \to \int_{\mathbb{R}^N} g(u_0)u_0. \tag{6.5}$$

The proof consists of several steps. Here we follow essentially the argument in [9, Propositions 5.1 and 5.3].

Step 1: $\|\nabla u_n\|_2$ is bounded as $n \to \infty$. Since $\|\nabla u_n\|_2^2 = NI(u_n) - P(u_n)$, Step 1 follows from (6.2) and (6.4).

From now on we prove that $\|u_n\|_2$ is bounded as $n \to \infty$. We argue indirectly and we assume

$$t_n = \|u_n\|_2^{-\frac{2}{N}} \to 0 \quad \text{as } n \to \infty.$$

We set $v_n(x) = u_n(\frac{x}{t_n})$. Since

$$\|v_n\|_2^2 = 1 \quad \text{and} \quad \|\nabla v_n\|_2^2 = t_n^{N-2}\|\nabla u_n\|_2^2, \tag{6.6}$$

$(v_n)_{n=1}^{\infty}$ is bounded in $H_r^1(\mathbb{R}^N)$. Thus we may assume after extracting a subsequence that

$$v_n \rightharpoonup v_0 \quad \text{weakly in } H_r^1(\mathbb{R}^N).$$

Step 2: $v_0 = 0$. Denoting $\varepsilon_n \equiv \|\partial_u I(u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0$, we have

$$\left|(\nabla u_n, \nabla\zeta)_2 + \mu(u_n, \zeta)_2 - \int_{\mathbb{R}^N} g(u_n)\zeta\right| \leq \varepsilon_n\|\zeta\|_{H^1} \quad \text{for any } \zeta \in H_r^1(\mathbb{R}^N).$$

Setting $u_n(x) = v_n(t_n x)$, $\zeta(x) = \varphi(t_n x)$, where $\varphi \in H_r^1(\mathbb{R}^N)$,

$$\left|t_n^{-(N-2)}(\nabla v_n, \nabla\varphi)_2 + \mu t_n^{-N}(v_n, \varphi)_2 - t_n^{-N}\int_{\mathbb{R}^N} g(v_n)\varphi\right| \leq \varepsilon_n\big(t_n^{-(N-2)}\|\nabla\varphi\|_2^2 + t_n^{-N}\|\varphi\|_2^2\big)^{\frac{1}{2}}.$$

Thus

$$\left|t_n^2(\nabla v_n, \nabla\varphi)_2 + \mu(v_n, \varphi)_2 - \int_{\mathbb{R}^N} g(v_n)\varphi\right| \leq \varepsilon_n t_n^{N/2}\big(t_n^2\|\nabla\varphi\|_2^2 + \|\varphi\|_2^2\big)^{\frac{1}{2}}, \tag{6.7}$$

from which we have

$$\int_{\mathbb{R}^N} (\mu v_0 - g(v_0))\varphi = 0 \quad \text{for any } \varphi \in H_r^1(\mathbb{R}^N).$$

Thus $\mu v_0 - g(v_0) = 0$. Since $\xi = 0$ is an isolated solution of $\mu\xi - g(\xi) = 0$ by (g2), we have $v_0(x) \equiv 0$.

Step 3: $\|u_n\|_2$ is bounded as $n \to \infty$. Setting $\varphi = v_n$ in (6.7),

$$\left|t_n^2\|\nabla v_n\|_2^2 + \mu\|v_n\|_2^2 - \int_{\mathbb{R}^N} g(v_n)v_n\right| \leq \varepsilon_n t_n^{N/2}\big(t_n^2\|\nabla v_n\|_2^2 + \|v_n\|_2^2\big)^{\frac{1}{2}}.$$

Thus, by (6.5), $\|v_n\|_2 \to 0$ as $n \to \infty$, which contradicts with (6.6). Thus $(u_n)_{n=1}^{\infty}$ is bounded in $H_r^1(\mathbb{R}^N)$.

Step 4: Conclusion. By Step 1 and Step 3, $(u_n)_{n=1}^{\infty}$ is bounded in $H_r^1(\mathbb{R}^N)$. After extracting a subsequence, we may assume that $u_n \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$ for some u_0. Since $\partial_u I(u_n)u_n \to 0$, $\partial_u I(u_n)u_0 \to 0$, we deduce from (6.5) that

$$\lim_{n\to\infty}(\|\nabla u_n\|_2^2 + \mu\|u_n\|_2^2) = \|\nabla u_0\|_2^2 + \mu\|u_0\|_2^2.$$

Thus $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^N)$. □

Arguing as in Sections 3–4, we obtain the following proposition.

Proposition 6.2. *Under the assumption of Proposition 6.1, for any $b \in \mathbb{R}$ we have:*

(i) $K_b = \{u \in H_r^1(\mathbb{R}^N) : I(u) = b,\ \partial_u I(u) = 0,\ P(u) = 0\}$ *is compact in* $H_r^1(\mathbb{R}^N)$.

(ii) *For any open neighborhood* $\mathcal{O}$ *of* K_b *and* $\overline{\varepsilon} > 0$ *there exist* $\varepsilon \in (0, \overline{\varepsilon})$ *and a continuous map* $\eta(t, u) : [0, 1] \times H_r^1(\mathbb{R}^N) \to H_r^1(\mathbb{R}^N)$ *such that*

  (1) $\eta(0, u) = u$ *for all* $u \in H_r^1(\mathbb{R}^N)$.

  (2) $\eta(t, u) = u$ *if* $u \in [I \le b - \overline{\varepsilon}]$.

  (3) $I(\eta(t, u)) \le I(u)$ *for all* $(t, u) \in [0, 1] \times H_r^1(\mathbb{R}^N)$.

  (4) $\eta(1, [I \le b + \varepsilon] \setminus \mathcal{O}) \subset [I \le b - \varepsilon]$, $\eta(1, [I \le b + \varepsilon]) \subset [I \le b - \varepsilon] \cup \mathcal{O}$.

  (5) *If* $K_b = \emptyset$, *then* $\eta(1, [I \le b + \varepsilon]) \subset [I \le b - \varepsilon]$.

  Here we use notation

$$[I \le c] = \{u \in H_r^1(\mathbb{R}^N) : I(u) \le c\}$$

  for $c \in \mathbb{R}$.

Using Proposition 6.2, we can show that $a_{mp}(\lambda)$ given in (5.1) is a critical value of $u \mapsto \widehat{I}(\lambda, u)$.

A Proof of Lemma 2.4 (iv)

Suppose that a closed set Z is invariant under $\mathbb{Z}_2$-action (2.6) and satisfies $0 \notin \overline{P_2(Z)}$. Then $\overline{P_2(Z)} \subset H_r^1(\mathbb{R}^N)$ is symmetric with respect to 0 and $\mathrm{genus}(\overline{P_2(Z)})$ is well defined.

For $A = \gamma(\overline{D_{j+\ell} \setminus Y})$, $\gamma \in \Gamma_{j+\ell}$, $\mathrm{genus}(Y) \le \ell$, we have

$$\overline{A \setminus Z} = \gamma(\overline{D_{j+\ell} \setminus (Y \cup \gamma^{-1}(Z))}). \tag{A.1}$$

In fact,

$$\gamma(D_{j+\ell} \setminus (Y \cup \gamma^{-1}(Z))) = \gamma(D_{j+\ell} \setminus Y) \setminus Z \subset \gamma(\overline{D_{j+\ell} \setminus Y}) \setminus Z = A \setminus Z. \tag{A.2}$$

Conversely, since $\overline{B} \setminus C \subset \overline{B \setminus C}$ for a set B and a closed set C, we have

$$A \setminus Z = \gamma(\overline{D_{j+\ell} \setminus Y}) \setminus Z = \gamma(\overline{D_{j+\ell} \setminus Y} \setminus \gamma^{-1}(Z)) \subset \gamma(\overline{D_{j+\ell} \setminus (Y \cup \gamma^{-1}(Z))}). \tag{A.3}$$

Thus (A.1) follows from (A.2) and (A.3). Since $P_2 \circ \gamma : \gamma^{-1}(Z) \to \overline{P_2(Z)}$ is an odd map,

$$\mathrm{genus}(\gamma^{-1}(Z)) \le \mathrm{genus}(\overline{P_2(Z)}) = i.$$

Thus,

$$\mathrm{genus}(Y \cup \gamma^{-1}(Z)) \le \mathrm{genus}(Y) + \mathrm{genus}(\gamma^{-1}(Z)) \le \mathrm{genus}(Y) + \mathrm{genus}(\overline{P_2(Y)}) \le \ell + i.$$

Therefore, by (A.1) we have $\overline{A \setminus Z} \in \Lambda_{j-i}$.

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