DE GRUYTER Adv. Nonlinear Stud. 2018; 18(3): 453–468

**Research Article**

Sergio Cruz-Blázquez and David Ruiz*

# Prescribing Gaussian and Geodesic Curvature on Disks

https://doi.org/10.1515/ans-2018-2021
Received March 23, 2018; revised May 29, 2018; accepted May 31, 2018

**Abstract:** In this paper, we consider the problem of prescribing the Gaussian and geodesic curvature on a disk and its boundary, respectively, via a conformal change of the metric. This leads us to a Liouville-type equation with a non-linear Neumann boundary condition. We address the question of existence by setting the problem in a variational framework which seems to be completely new in the literature. We are able to find minimizers under symmetry assumptions.

**Keywords:** Prescribed Gaussian Curvature Problem, Variational Methods, Moser–Trudinger Inequality

**MSC 2010:** 35J20, 35R01, 53A30

**Communicated by:** Antonio Ambrosetti

## 1 Introduction

The problem of prescribing the Gaussian curvature on a compact surface $\Sigma$ under a conformal change of the metric is a classical one, and dates back to [4, 21, 22]. Let us denote by $g$ the original metric, by $\tilde{g}$ the new one and by $e^u$ the conformal factor (that is, $\tilde{g} = e^u g$). This problem reduces to solving the problem

$$-\Delta_g u + 2K_g = 2K_{\tilde{g}} e^u,$$

where $K_g$ and $K_{\tilde{g}}$ denote the curvatures with respect to $g$ and $\tilde{g}$, respectively. The solvability of this equation has been studied for a long time, and it is not possible to give here a comprehensive list of references.

If $\Sigma$ has a boundary, then boundary conditions are in order. Homogeneous Dirichlet and Neumann boundary conditions have already been considered in the literature. In this paper, our aim is to prescribe not only the Gaussian curvature in $\Sigma$, but also the geodesic curvature on $\partial\Sigma$. In this case, we are led with the boundary value problem

$$\begin{cases} -\Delta_g u + 2K_g = 2K_{\tilde{g}} e^u & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} + 2h_g = 2h_{\tilde{g}} e^{u/2} & \text{on } \partial\Sigma, \end{cases} \tag{1.1}$$

where $h_g$ and $h_{\tilde{g}}$ are the geodesic curvatures of $\partial\Sigma$ relative to $g$ and $\tilde{g}$, respectively.

Some versions of this problem have been studied in the literature. The case $h_{\tilde{g}} = 0$ has been treated by Chang and Yang in [7]. Moreover, the case $K_{\tilde{g}} = 0$ has been treated in [6, 23, 25]. There is also some progress in the blow-up analysis, see [3, 10], although a complete description of the phenomenon is still missing.

The case of constants $K_{\tilde{g}}$, $h_{\tilde{g}}$ has also been considered. For instance, Brendle [5] uses a parabolic flow to show that this problem admits always a solution for some constant curvatures. By using complex analysis techniques, explicit expressions for the solutions and the exact values of the constants are determined

**Sergio Cruz-Blázquez,** Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, e-mail: sergio.cruzblazquez@sns.it
***Corresponding author: David Ruiz,** Departamento de Análisis Matemático, Universidad de Granada, Campus Fuentenueva, 18071 Granada, Spain, e-mail: daruiz@ugr.es

if $\Sigma$ is a disk or an annulus; see [19, 20]. The case of the half-plane has also been studied; see [15, 24, 33]. However, the case in which both curvatures are not constant has not been much considered. In [9], some partial existence results are given, but they include a Lagrange multiplier which is out of control. Moreover, a Kazdan–Warner type of obstruction to existence has been found in [16]. In a forthcoming work, the case of $K < 0$ in domains different from the disk is treated, and also a blow-up analysis is performed; see [26]. At present, as far as we know, those are the only works considering non-constant curvatures.

The higher-dimensional analogue of this question (that is, prescribing scalar curvature of a manifold and mean curvature of the boundary) has been studied more. The case of zero scalar curvature and constant mean curvature is known as the Escobar problem, in strong analogy with the Yamabe problem. In this regard, see [1, 11–14, 17, 18, 28] and the references therein.

By integrating (1.1) and applying the Gauss–Bonnet theorem, one obtains

$$\int_{\Sigma} K_{\tilde g} e^{u} + \int_{\partial\Sigma} h_{\tilde g} e^{u/2} = 2\pi\chi(\Sigma). \tag{1.2}$$

In this paper, we shall consider the case in which $\chi(\Sigma) = 1$. By the Uniformization Theorem, we can pass via a conformal map to a disk, obtaining $K_{\tilde g} = 0$, $h_{\tilde g} = 1$. Taking this into account, we can consider the problem

$$\begin{cases} -\Delta u = 2Ke^{u} & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{1.3}$$

where now $K$, $h$ are the curvatures to be prescribed.

Generally speaking, the case of a disk is especially challenging because of the non-compact action of the group of conformal maps of the disk, as happens in the Nirenberg problem for $\Sigma = \mathbb{S}^2$. This issue has been only treated in [6] for $K = 0$ (see also [10]). A blow-up analysis in this case for non-constant $K$, $h$ is yet to be done, and will be the target of further research. In this paper, as a first step in the understanding of the problem, we shall impose symmetry conditions on $K$, $h$ in order to rule out this phenomenon. This idea goes back to Moser [30] for the Nirenberg problem.

Being more specific, we fix a symmetry group as follows:

(G) We denote by $G$ one of the following groups of symmetries of the disk:
- The dihedric group $\mathbb{D}_k$ with $k \geq 3$.
- The group of rotations with minimal angle $\frac{2\pi}{k}$, $k \geq 2$.
- The whole group of symmetries $O(2)$.

Notice that none of the groups above has fixed points on $\mathbb{S}^1$, that is, for each $x \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x) \neq x$. We say that a function $f$ is $G$-symmetric if $f(x) = f(g(x))$ for all $g \in G$ and for all $x$ in the domain of $f$.

Our main results are the following theorems.

**Theorem 1.1.** *Let $G$ be as in (G), and let $K : \mathbb{D}^2 \to \mathbb{R}$, $h : \mathbb{S}^1 \to \mathbb{R}$ be $G$-symmetric, Hölder continuous and nonnegative functions, not both of them identically equal to* 0. *Then problem* (1.3) *admits a solution.*

We can also deal with changing sign curvatures $K$, $h$ as long as their negative part is small.

**Theorem 1.2.** *Let $G$ be as in (G), and let $K_0 : \mathbb{D}^2 \to \mathbb{R}$, $h_0 : \mathbb{S}^1 \to \mathbb{R}$ be $G$-symmetric, Hölder continuous and nonnegative functions, none of them identically equal to* 0. *Then there exists $\varepsilon > 0$ such that problem* (1.3) *admits a solution for any Hölder continuous and $G$-symmetric functions $K$, $h$ with $\|K - K_0\|_{L^\infty} + \|h - h_0\|_{L^\infty} < \varepsilon$.*

One of the main goals of this paper is to find an original variational setting to this problem, which we think is natural and could be of use in future research on the topic. Let us be more specific. We define the parameter

$$\rho := \int_{\mathbb{D}^2} Ke^{u} = 2\pi - \int_{\mathbb{S}^1} he^{u/2}.$$

In order to fix the ideas, let us assume that both $K$, $h$ are nonnegative functions; by (1.2), $0 < \rho < 2\pi$.

We shall show that (1.3) is equivalent to

$$\begin{cases} -\Delta u = 2\rho \dfrac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 2(2\pi - \rho)\dfrac{he^{u/2}}{\int_{\mathbb{S}^1} he^{u/2}} & \text{on } \mathbb{S}^1, \\ \dfrac{(2\pi-\rho)^2}{\rho} = \dfrac{(\int_{\mathbb{S}^1} he^{u/2})^2}{\int_{\mathbb{D}^2} Ke^u} & \text{for } 0 < \rho < 2\pi. \end{cases} \tag{1.4}$$

Observe that problem (1.4) is now invariant under addition of constants to $u$, and $\rho$ is unknown here. This formulation may seem rather artificial but it has the advantage of being related to the critical points of the energy functional

$$I(u, \rho) = \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} Ke^u + 2\int_{\mathbb{S}^1} u - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} he^{u/2} + 4(2\pi-\rho)\log(2\pi - \rho) + 2\rho + 2\rho\log\rho. \tag{1.5}$$

We highlight the fact that the functional above depends on the couple $(u, \rho)$, where $u \in H^1(\mathbb{D}^2)$ and $\rho \in (0, 2\pi)$. The form of this energy functional seems to be completely new in the related literature.

If we freeze the variable $\rho$, the form of this functional is adequate for the use of Moser–Trudinger-type inequalities (or Onofri-type inequalities) which are already available also for boundary terms. Indeed, by interpolating these inequalities we will show that $I$ is bounded from below. We will gain coercivity in the $u$ variable by imposing symmetry, as done first by Moser in [30]. Finally, we will need to exclude the possibility of obtaining minima at the endpoints $\rho = 0$ or $\rho = 2\pi$. Those limit cases correspond to the problem in which $K = 0$ or $h = 0$, respectively, so some study of these cases is needed. By energy estimates we can assure that the minimum is attained at $\rho \in (0, 2\pi)$, concluding the proof.

If either $K$ or $h$ changes sign, the above approach fails. We shall also give in Theorem 3.4 a more general result; as a corollary, and making use of a compactness result for minima of the functional $I$, we will obtain the perturbation result stated in Theorem 1.2.

The rest of the paper is organized as follows: In Section 2, we set the notation and the variational formulation of the problem. After that, an analysis of the properties of the energy functional is performed by means of Moser–Trudinger-type inequalities. Section 3 is devoted to the proof of Theorem 1.1, for which we first need to address the limiting cases $\rho = 0$ and $\rho = 2\pi$. A more general version is also given. Finally, the proof of Theorem 1.2 is completed in Section 4.

# 2 Variational Setting

## 2.1 Notation

Let us first set some notation. Given a set $A \subset X$ in a metric space, we set

$$(A)^r = \{x \in X : \operatorname{dist}(x, A) < r\}.$$

Regarding the integrals, in this paper we shall consider only the Lebesgue measure and we drop the element of area or length, that is, we shall only write $\int_{\mathbb{D}^2} Ke^u$ or $\int_{\mathbb{S}^1} he^{u/2}$. We also use the symbol $⨍ f$ to denote the mean value of $f$, that is,

$$⨍_{\Sigma} f = \frac{1}{|\Sigma|}\int_{\Sigma} f.$$

In our estimates, we sometimes write $C$ to denote a positive constant, independent of the variables considered, that may change from line to line.

## 2.2 Variational Formulation

As commented in the introduction, we will consider the functional $I$ given by (1.5) and defined on the space

$$\mathbb{X} \times (0, 2\pi) = \Big\{ u \in H^1(\mathbb{D}^2) : \int_{\mathbb{D}^2} Ke^u > 0, \int_{\mathbb{S}^1} he^{u/2} > 0 \Big\} \times (0, 2\pi).$$

With the purpose of clarifying the notation, for a fixed $\rho \in (0, 2\pi)$ we denote by $I_\rho$ the functional $u \to I(u, \rho)$ defined for every $u \in \mathbb{X}$. We should notice that the functionals $I_\rho$ are invariant under the addition of constants.

**Lemma 2.1.** *The space $\mathbb{X}$ is nonempty if and only if $K$ and $h$ are positive somewhere.*

*Proof.* We reduce ourselves to prove that if $K$ and $h$ are positive somewhere, then $\mathbb{X}$ is nonempty as the reciprocal is immediate. As $K$ is continuous and there exists $x_0 \in \operatorname{Int}(\mathbb{D}^2)$ such that $K(x_0) > 0$, there exists $r > 0$ such that $(\{x_0\})^r \cap \mathbb{S}^1 = \emptyset$ and $K(x) > 0$ for all $x \in (\{x_0\})^r$.

Moreover, we know that there exists $x_1 \in \mathbb{S}^1$ satisfying $h(x_1) > 0$, and again by continuity we get $s > 0$ such that $h(x) > 0$ for all $x \in (\{x_1\})^s \cap \mathbb{S}^1$. It is not restrictive to assume $(\{x_0\})^r \cap (\{x_1\})^s = \emptyset$. We define $\Omega_0^r := (\{x_0\})^r$ and $\Omega_1^s := (\{x_1\})^s$ and consider a cutoff function $\varphi \in H^1(\mathbb{D}^2)$ satisfying

$$\varphi(x) = \begin{cases} a & \text{if } x \in \Omega_0^{r/2}, \\ b & \text{if } x \in \Omega_1^{s/2}, \\ 0 & \text{if } \mathbb{D}^2 \setminus (\Omega_0^r \cup \Omega_1^s), \end{cases}$$

where $a$ and $b$ are real constants to determine. We see that

$$\begin{aligned} \int_{\mathbb{S}^1} he^{\varphi/2} &= \int_{\Omega_1^{s/2} \cap \partial\mathbb{D}^2} he^{\varphi/2} + \int_{(\Omega_1^s \setminus \Omega_1^{s/2}) \cap \partial\mathbb{D}^2} he^{\varphi/2} + \int_{\partial\mathbb{D}^2 \setminus \Omega_1^s} he^{\varphi/2} \\ &\geq e^{b/2} \int_{\Omega_1^{s/2} \cap \partial\mathbb{D}^2} h + \int_{\partial\mathbb{D}^2 \setminus \Omega_1^{s/2}} h \\ &= C_1 e^{b/2} + C, \end{aligned}$$

where $C_1 > 0$ and $C \in \mathbb{R}$. We can choose $b$ large enough so that

$$\int_{\mathbb{S}^1} he^{\varphi/2} > 0.$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{D}^2} Ke^{\varphi} &= \int_{\Omega_0^{r/2}} Ke^{\varphi} + \int_{\Omega_1^{s/2}} Ke^{\varphi} + \int_{\mathbb{D}^2 \setminus (\Omega_0^r \cup \Omega_1^r)} Ke^{\varphi} + \int_{\Omega_1^s \setminus \Omega_1^{s/2}} Ke^{\varphi} + \int_{\Omega_0^r \setminus \Omega_0^{r/2}} Ke^{\varphi} \\ &\geq e^a \int_{\Omega_0^{r/2}} K - e^b C_2 \|K\|_\infty + C \\ &= C_1' e^a - C_2' e^b + C. \end{aligned}$$

So we can also set $a$ big enough so that

$$\int_{\mathbb{D}^2} Ke^{\varphi} > 0. \qquad \square$$

Let us point out that the Euler–Lagrange equation of $I$ is given by (1.4), which is a reformulation of (1.3) in view of the next lemma.

**Lemma 2.2.** *Problems* (1.3) *and* (1.4) *are equivalent.*

*Proof.* In order to check that every solution of (1.3) is a solution of (1.4) we just need to take

$$\rho = \int_{\mathbb{D}^2} Ke^u = 2\pi - \int_{\mathbb{S}^1} he^{u/2} > 0.$$

Reciprocally, if $u \in \mathbb{X}$ solves (1.4), applying the invariance under addition of constants of that problem, we have, for any $C \in \mathbb{R}$,

$$-\Delta(u + C) = 2\rho \frac{Ke^{u+C}}{e^C \int_{\mathbb{D}^2} Ke^u},$$
$$\frac{\partial(u + C)}{\partial\eta} + 2 = 2(2\pi - \rho)\frac{he^{u/2}}{e^{\frac{C}{2}} \int_{\mathbb{S}^1} he^{u/2}}.$$

If we want $u + C$ to solve (1.3), we need $C \in \mathbb{R}$ such that

$$e^C = \frac{\rho}{\int_{\mathbb{D}^2} Ke^u}, \qquad e^{\frac{C}{2}} = \frac{(2\pi - \rho)}{\int_{\mathbb{S}^1} he^{u/2}}.$$

The third equation of (1.4) tells us that both conditions are actually the same. Thus, it is enough to choose

$$C = \log\rho - \log \int_{\mathbb{D}^2} Ke^u. \qquad \square$$

## 2.3 Moser–Trudinger Inequalities

The Moser–Trudinger inequality (see [7, 29, 30, 32]) and their variations are useful tools to deal with the non-linear terms of exponential type which appear in our functional. In particular, we are interested in weaker versions of these inequalities, also called Onofri-type inequalities.

**Theorem 2.3.** *Let $\Sigma$ be a compact surface with $C^1$ boundary. Then there exists a constant $C \in \mathbb{R}$, depending only on $\Sigma$, such that*

$$\log \int_\Sigma e^u \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 + C \quad \textit{for all } u \in H_0^1(\Sigma), \tag{2.1}$$

*and*

$$\log \int_\Sigma e^u \leq \frac{1}{8\pi} \int_\Sigma |\nabla u|^2 + \fint_\Sigma u + C \quad \textit{for all } u \in H^1(\Sigma). \tag{2.2}$$

The first inequality is classical, whereas the second is given in [7, Proposition 2.3 and the subsequent corollary]. In both cases the constant is optimal.

In order to address the non-linear boundary terms of the functional $I$, we will use an analogous version of Theorem 2.3 for the boundary of a compact surface that can be found for instance in [23].

**Proposition 2.4.** *Let $\Sigma$ be a compact surface with $C^1$ boundary. Then there exists a constant $C > 0$, depending only on $\Sigma$, such that*

$$\log \int_{\partial\Sigma} e^u \leq \frac{1}{4\pi} \int_\Sigma |\nabla u|^2 + \fint_{\partial\Sigma} u + C \quad \textit{for all } u \in H^1(\Sigma).$$

In the case of the disk, the above inequality is the so-called Lebedev–Milin inequality (with $C = 0$, see for instance [31, (4')]).

By interpolating the previous inequalities, we will obtain a lower bound for the functional $I$. First, we notice that inequality (2.2) can be manipulated so that the mean value of $u$ in $\partial\Sigma$ replaces the mean in $\Sigma$.

**Corollary 2.5.** *Let $\Sigma$ be a compact surface with $C^1$ boundary. There exists a constant $C \in \mathbb{R}$, depending only on $\Sigma$, such that*

$$\log \int_\Sigma e^v \leq \frac{1}{8\pi} \int_\Sigma |\nabla v|^2 + \fint_{\partial\Sigma} v + C \quad \textit{for all } v \in H^1(\Sigma).$$

*Proof.* We consider the problem

$$\begin{cases} -\Delta w = \dfrac{-4\pi}{|\Sigma|} & \text{in } \Sigma, \\ \dfrac{\partial w}{\partial \eta} = \dfrac{4\pi}{|\partial\Sigma|} & \text{on } \partial\Sigma. \end{cases} \tag{2.3}$$

Let us point out that (2.3) is solvable in $H^1(\Sigma)$ because

$$\int_{\partial\Sigma} \frac{4\pi}{|\partial\Sigma|} = -\int_{\Sigma} \frac{4\pi}{|\Sigma|} = 4\pi.$$

We fix a solution $w$ of (2.3) and apply (2.2) to $v + w$, obtaining

$$\log \int_{\Sigma} e^v \le \frac{1}{8\pi} \int_{\Sigma} |\nabla v|^2 + \frac{1}{4\pi} \int_{\partial\Sigma} \frac{\partial w}{\partial \eta} v - \frac{1}{4\pi} \int_{\Sigma} (\Delta w) v + ⨍_{\Sigma} v + C.$$

Finally, we use that $w$ solves (2.3):

$$\log \int_{\Sigma} e^v \le \frac{1}{8\pi} \int_{\Sigma} |\nabla v|^2 + ⨍_{\partial\Sigma} v - ⨍_{\Sigma} v + ⨍_{\Sigma} v + C = \frac{1}{8\pi} \int_{\Sigma} |\nabla v|^2 + ⨍_{\partial\Sigma} v + C. \qquad \square$$

In a similar way, one can obtain a modified version of Proposition 2.4 in which the mean value of $u$ on $\Sigma$ substitutes the mean on $\partial\Sigma$.

**Corollary 2.6.** *Let $\Sigma$ be a compact surface with $C^1$ boundary. There exists $C \in \mathbb{R}$, depending only on $\Sigma$, such that*

$$\log \int_{\partial\Sigma} e^u \le \frac{1}{4\pi} \int_{\Sigma} |\nabla u|^2 + ⨍_{\Sigma} u + C \quad \textit{for all } u \in H^1(\Sigma).$$

The combined use of inequality (2.3) and Corollary 2.5 allows us to prove that $I$ is bounded from below in $H^1(\mathbb{D}^2)$.

**Proposition 2.7.** *There exists a constant $C \in \mathbb{R}$ such that $I_\rho(u) \ge C$ for every $u \in \mathbb{X}$ and every $\rho \in [0, 2\pi]$.*

*Proof.* Let us define $f : (0, 2\pi) \to \mathbb{R}$ as the correction term in (1.5), that is,

$$f(\rho) = 4(2\pi - \rho)\log(2\pi - \rho) + 2\rho + 2\rho \log \rho.$$

It is clear that

$$\lim_{\rho \to 0} f(\rho) = 8\pi \log(2\pi), \qquad \lim_{\rho \to 2\pi} f(\rho) = 4\pi + 4\pi \log(2\pi).$$

Then $f$ can be continuously extended to the compact interval $[0, 2\pi]$. Thus, there exists a constant $M > 0$ such that $|f(\rho)| \le M$ for all $\rho \in [0, 2\pi]$. Moreover, since $K$ and $h$ are continuous, there exist $M_1, M_2 \in \mathbb{R}$ such that

$$\log \int_{\mathbb{D}^2} K e^u \le \log \int_{\mathbb{D}^2} e^u + C, \quad \log \int_{\mathbb{S}^1} h e^{u/2} \le \int_{\mathbb{S}^1} e^{u/2} + C.$$

Then for every $a, b \in \mathbb{R}$,

$$\begin{aligned} I_\rho(u) &\ge \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} e^u - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} e^{u/2} + 2 \int_{\mathbb{S}^1} u + C \\ &= \frac{8\pi - 2a - b}{16\pi} \int_{\mathbb{D}^2} |\nabla u|^2 + \frac{a}{8\pi} \int_{\mathbb{D}^2} |\nabla u|^2 + \frac{b}{16\pi} \int_{\mathbb{D}^2} |\nabla u|^2 - 2\rho \log \int_{\mathbb{D}^2} e^u - 4(2\pi - \rho) \log \int_{\mathbb{S}^1} e^{u/2} + 2 \int_{\mathbb{S}^1} u + C. \end{aligned}$$

As the functional $I$ is invariant under the addition of constants, we can assume that $\int_\Sigma u = 0$ and apply Corollary 2.5 and Proposition 2.4, taking $a = 2\rho$ and $b = 4(2\pi - \rho)$, and thus obtaining:

$$I_\rho(u) \ge -2\rho ⨍_{\mathbb{S}^1} u - 2(2\pi - \rho) ⨍_{\mathbb{S}^1} u + 2 \int_{\mathbb{S}^1} u + C = C.$$

We highlight that the constant $C$ does not depend on $\rho$. $\square$

Proposition 2.7 states that the functional $I$ is bounded from above, but we do not have coercivity. The reason for this is the non-compact action of the conformal group of the disk. This effect appears, for instance, also in the Nirenberg problem in the sphere and makes the problem rather difficult.

We will show now that we can gain coercivity by restricting ourselves to spaces of symmetric functions. In order to do that, we introduce local versions of the inequalities above. This idea dates back to [2], and these inequalities are known as Chen–Li-type inequalities (see [8] for more details).

**Proposition 2.8.** *Assume $\Sigma$ to be a compact surface with $C^1$ boundary, and let $\Sigma_1 \subset \Sigma$ and $\delta > 0$ be such that $(\Sigma_1)^\delta \cap \partial\Sigma = \emptyset$. Then for every $\varepsilon > 0$ there exists a constant $C \in \mathbb{R}$ depending on $\varepsilon$ and $\delta$ such that*

$$16\pi \log \int_{\Sigma_1} e^u \leq \int_{(\Sigma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C \quad \textit{for all } u \in H^1(\Sigma) \textit{ with } \int_\Sigma u = 0.$$

The details of the proof of this precise statement can be found, for instance, in [27, Proposition 2.2], but the idea dates back to [8]. Roughly speaking, one applies (2.1) to the function $u$ multiplied by a cut-off function in $\Sigma_1$.

If the function $u$ has mass in several separated regions satisfying the hypothesis of the propositions above, the obtained bounds improve by a factor of the number of such regions. This information is collected in the following corollary (see for instance [27, Lemma 2.4] for the case $l = 2$; the case of general $l$ is analogous).

**Corollary 2.9.** *Let $\Sigma$ be a compact surface with $C^1$ boundary, let $l \in \mathbb{N}$ and let $\Sigma_1, \dots, \Sigma_l \subset \Sigma$ for which there exists a $\delta > 0$ such that $(\Sigma_i)^\delta \cap (\Sigma_j)^\delta = \emptyset$ if $i \neq j$. Assume that there exists $\gamma \in (0, \frac{1}{l})$ such that*

$$\frac{\int_{\Sigma_i} e^u}{\int_\Sigma e^u} \geq \gamma \quad \textit{for all } i = 1, \dots, l.$$

*Then for every $\varepsilon > 0$ there exists a constant $C \in \mathbb{R}$ depending on $\varepsilon$, $\delta$ and $\gamma$ such that*

$$8l\pi \log \int_\Sigma e^u \leq \int_\Sigma |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C \quad \textit{for all } u \in H^1(\Sigma) \textit{ with } \int_\Sigma u = 0.$$

Using the same techniques, we can give a localized version of Proposition 2.4.

**Proposition 2.10.** *Let $\Sigma$ be a compact surface with $C^1$ boundary, and let $\Gamma_1 \subset \partial\Sigma$. Then for every $\varepsilon, \delta > 0$ there exists a constant $C \in \mathbb{R}$ depending on $\varepsilon$ and $\delta$ such that*

$$4\pi \log \int_{\Gamma_1} e^u \leq \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon \int_\Sigma |\nabla u|^2 + C \quad \textit{for all } u \in H^1(\Sigma) \textit{ with } \int_\Sigma u = 0$$

*Proof.* Following [8], we consider a cutoff function $g_\delta : \Sigma \to [0, 1]$ satisfying

$$g_\delta = \begin{cases} 1 & \text{if } x \in \Gamma_1, \\ 0 & \text{if } x \in \Sigma \setminus (\Gamma_1)^{\delta/2}. \end{cases}$$

We have $g_\delta u \in H^1(\Sigma)$, hence we can apply Corollary 2.6:

$$4\pi \log \int_{\Gamma_1} e^u = 4\pi \log \int_{\Gamma_1} e^{g_\delta u} \leq 4\pi \log \int_{\partial\Sigma} e^{g_\delta u} \leq \int_\Sigma |\nabla(g_\delta u)|^2 + 4\pi \fint_\Sigma g_\delta u + C. \tag{2.4}$$

Then

$$\begin{aligned} \int_\Sigma |\nabla(g_\delta u)|^2 &= \int_\Sigma u^2 |\nabla g_\delta|^2 + 2 \int_\Sigma g_\delta u \langle \nabla u, \nabla g_\delta \rangle + \int_\Sigma (g_\delta)^2 |\nabla u|^2 \\ &\leq C_\delta \int_\Sigma u^2 + 2 \int_\Sigma g_\delta u |\nabla u|\, |\nabla g_\delta| + \int_{(\Gamma_1)^\delta} |\nabla u|^2. \end{aligned} \tag{2.5}$$

The central term can be bounded using Cauchy's inequality, and thus obtaining

$$\int_\Sigma g_\delta u|\nabla u|\,|\nabla g_\delta| \le C_\delta \int_\Sigma u|\nabla u| \le C_{\varepsilon,\delta}\int_\Sigma u^2 + \varepsilon\int_\Sigma |\nabla u|^2. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\int_\Sigma |\nabla(g_\delta u)|^2 \le \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon\int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta}\int_\Sigma u^2. \tag{2.7}$$

Also, we have the following bound for the mean value of $g_\delta u$ on $\partial\Sigma$:

$$\fint_\Sigma g_\delta u \le \fint_\Sigma \frac{1}{2}((g_\delta)^2 + u^2) \le \frac{1}{2}\fint_\Sigma (g_\delta)^2 + \frac{1}{2|\Sigma|}\int_\Sigma u^2 \le C_\delta + C\int_\Sigma u^2. \tag{2.8}$$

Now, apply both inequalities (2.7) and (2.8) to (2.4) to get

$$4\pi\log\int_{\Gamma_1} e^u \le \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon\int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta}\int_\Sigma u^2 + C. \tag{2.9}$$

Finally, we address the term $\int_\Sigma u^2$.

Let $a \in \mathbb{R}$, $\eta = |\{x \in \Sigma : u(x) \ge a\}|$ and $(u-a)^+ = \max\{0, u-a\}$. Clearly, $u \le (u-a)^+ + a$. We now apply formula (2.9) to the function $(u-a)^+$:

$$\begin{aligned} 4\pi\log\int_{\Gamma_1} e^u &\le 4\pi\log\Big(e^a\int_{\Gamma_1} e^{(u-a)^+}\Big) \\ &\le 4\pi a + \log\int_{\Gamma_1} e^{(u-a)^+} \\ &\le 4\pi a + \int_{(\Gamma_1)^\delta} |\nabla(u-a)^+|^2 + \varepsilon\int_\Sigma |\nabla(u-a)^+|^2 + C_{\varepsilon,\delta}\int_\Sigma ((u-a)^+)^2 \\ &\le 4\pi a + \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon\int_\Sigma |\nabla u|^2 + C_{\varepsilon,\delta}\int_\Sigma ((u-a)^+)^2. \end{aligned} \tag{2.10}$$

By means of the Sobolev, Hölder and Poincaré–Wirtinger inequalities,

$$\begin{aligned} \int_\Sigma ((u-a)^+)^2 &= \int_{\{x\in\Sigma:\, u(x)\ge a\}} ((u-a)^+)^2 \\ &\le \eta^{1/2}\Big(\int_\Sigma ((u-a)^+)^4\Big)^{1/2} \\ &\le \eta^{1/2}\|(u-a)^+\|^2_{H^1(\Sigma)} \\ &\le C\eta^{1/2}\int_\Sigma |\nabla u|^2. \end{aligned} \tag{2.11}$$

Again by the Poincaré–Wirtinger inequality,

$$a\eta \le \int_{\{x\in\Sigma:\, u(x)\ge a\}} u \le \int_\Sigma |u| \le C\Big(\int_\Sigma |u|^2\Big)^{1/2} \le C\Big(\int_\Sigma |\nabla u|^2\Big)^{1/2}. \tag{2.12}$$

From (2.12), using Cauchy's inequality, we obtain

$$a \le \theta\int_\Sigma |\nabla u|^2 + \frac{C^2}{\eta^2\theta} \quad \text{for all } \theta > 0. \tag{2.13}$$

Mixing (2.10), (2.11) and (2.13), we obtain

$$4\pi \log \int_{\Gamma_1} e^u \leq 4\pi\theta \int_{\Sigma} |\nabla u|^2 + \int_{(\Gamma_1)^\delta} |\nabla u|^2 + \varepsilon \int_{\Sigma} |\nabla u|^2 + C_{\varepsilon,\delta}\eta^{1/2} \int_{\Sigma} |\nabla u|^2 + C,$$

and it is enough to take $\theta = \frac{1}{4\pi}$ and $\eta^{1/2} \leq \frac{\varepsilon}{C_{\varepsilon,\delta}}$ to conclude the proof. □

**Corollary 2.11.** *Let $\Sigma$ be a compact surface with $C^1$ boundary, let $l \in \mathbb{N}$ and let $\Gamma_1, \dots, \Gamma_l \subset \partial\Sigma$ for which there exists a $\delta > 0$ such that $(\Gamma_i)^\delta \cap (\Gamma_j)^\delta = \emptyset$ if $i \neq j$. Moreover, assume that there exists $\gamma \in (0, \frac{1}{l})$ such that*

$$\frac{\int_{\Gamma_i} e^u}{\int_{\partial\Sigma} e^u} \geq \gamma \quad \textit{for all } i = 1, \dots, l. \tag{2.14}$$

*Then for every $\varepsilon > 0$ there exists a constant $C \in \mathbb{R}$ depending on $\varepsilon$, $\delta$ and $\gamma$ such that*

$$4l\pi \log \int_{\partial\Sigma} e^u \leq \int_{\Sigma} |\nabla u|^2 + \varepsilon \int_{\Sigma} |\nabla u|^2 + C \quad \textit{for all } u \in H^1(\Sigma) \textit{ with } \int_{\Sigma} u = 0.$$

*Proof.* First, we apply to each $\Gamma_i$ the previous result, obtaining

$$4\pi \log \int_{\Gamma_i} e^u \leq \int_{(\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon \int_{\Sigma} |\nabla u|^2 + C.$$

Using (2.14), we get

$$4\pi \log \int_{\Gamma_i} e^u \geq 4\pi \log \int_{\partial\Sigma} e^u + C.$$

Then

$$4\pi \log \int_{\partial\Sigma} e^u \leq \int_{(\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon \int_{\Sigma} |\nabla u|^2 + C.$$

Finally, summing over $i \in \{1, \dots, l\}$, we obtain

$$4l\pi \log \int_{\partial\Sigma} e^u \leq \int_{\bigsqcup_i (\Gamma_i)^\delta} |\nabla u|^2 + \varepsilon l \int_{\Sigma} |\nabla u|^2 + C \leq \int_{\Sigma} |\nabla u|^2 + \varepsilon l \int_{\Sigma} |\nabla u|^2 + C.$$ □

We have just seen that the more regions the mass of a function is separated in, the better bounds we obtain using the local versions of the Moser–Trudinger inequalities. If an $H^1(\mathbb{D}^2)$ function is concentrated in an interior point of the disk, Proposition 2.8 gives us a lower bound which is sufficient to achieve coercivity, but that is not the case when a function concentrates around a boundary point. To avoid this we will restrict ourselves to consider functions satisfying a symmetry condition guaranteeing that a function cannot concentrate around a single point of the boundary. Hence we will obtain coercivity by interpolating Corollaries 2.9 and 2.11 with $l = 2$.

We let $G$ be a subgroup of the orthogonal transformation group of $\mathbb{D}^2$ such that the set of fixed points on $\mathbb{S}^1$ under the action of $G$ is empty; in other words,

$$\{x \in \mathbb{S}^1 : g(x) = x \text{ for all } g \in G\} = \emptyset.$$

For instance, we can take $G$ as the group of rotations generated by $g(z) = e^{(2\pi i)/k} z$ as well as the dihedral groups $\mathbb{D}_k$ ($k \in \mathbb{N}$, $k > 1$).

In the sequel, $K$ and $h$ will be assumed to be $G$-symmetric functions, and we set

$$H^1_G(\mathbb{D}^2) = \{u \in H^1(\mathbb{D}^2) : u \circ g = u \text{ for all } u \in G\}$$

and

$$\mathbb{X}_G = \{u \in \mathbb{X} : u \circ g = u \text{ for all } g \in G\}.$$

As in Lemma 2.1, we observe that if $K$ and $h$ are G-symmetric functions somewhere positive, then $\mathbb{X}_G$ is not empty.

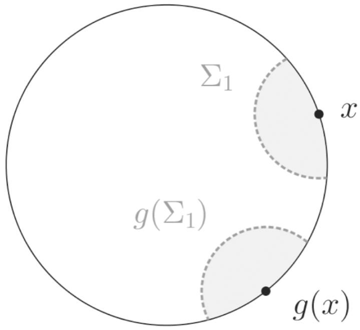


**Figure 1:** If a symmetric function concentrates around $x \in \mathbb{S}^1$, then it also concentrates around $g(x)$ for all $g \in G$.

**Proposition 2.12.** *Given $\rho \in [0, 2\pi]$, the functional $I_\rho$ is coercive on $\mathbb{X}_G$, that is,*

$$I_\rho(u) \to +\infty \quad (\|u\|_{H^1(\mathbb{D}^2)} \to +\infty,\ u \in \mathbb{X}_G).$$

*Proof.* Take a sequence $(u_n)$ in $\mathbb{X}_G$. We know that $I_\rho$ is invariant under the addition of constants, so we can assume that $\int_{\mathbb{D}^2} u_n = 0$ for every $n \in \mathbb{N}$. We have

$$I_\rho(u_n) \geq \frac{1}{2}\int_{\mathbb{D}^2} |\nabla u_n|^2 - 2\rho \log \int_{\mathbb{D}^2} e^{u_n} - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} e^{u_n/2} + 2\int_{\mathbb{S}^1} u_n + C.$$

Then for any $a, b \in \mathbb{R}$ one has

$$\begin{aligned} I_\rho(u_n) \geq{}& \frac{16\pi - 2a - b}{32\pi}\int_{\mathbb{D}^2} |\nabla u_n|^2 + \frac{a}{16\pi}\int_{\mathbb{D}^2} |\nabla u_n|^2 + \frac{b}{32\pi}\int_{\mathbb{D}^2} |\nabla u_n|^2 + 2\int_{\mathbb{S}^1} u_n \\ & - 2\rho \log \int_{\mathbb{D}^2} e^{u_n} - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} e^{u_n/2}. \end{aligned}$$

We can now apply Corollaries 2.9 and 2.11 with $l = 2$ (see Figure 1):

$$\begin{aligned} I_\rho(u_n) \geq{}& \frac{16\pi - 2a - b}{32\pi}\int_{\mathbb{D}^2} |\nabla u_n|^2 + a\log \int_{\mathbb{D}^2} e^{u_n} - a\varepsilon \int_{\mathbb{D}^2} |\nabla u_n|^2 + b\log \int_{\mathbb{S}^1} e^{u_n/2} \\ & - b\varepsilon \int_{\mathbb{D}^2} |\nabla u_n|^2 - 2\rho \log \int_{\mathbb{D}^2} e^{u_n} - 4(2\pi - \rho)\log \int_{\mathbb{S}^1} e^{u_n/2} + 2\int_{\mathbb{S}^1} u_n + C. \end{aligned}$$

Choosing $a = 2\rho$ and $b = 4(2\pi - \rho)$ and applying the trace inequality, we obtain

$$I_\rho(u_n) \geq \Big(\frac{1}{4} - \varepsilon\Big)\int_{\mathbb{D}^2} |\nabla u_n|^2 - 2C_2\|u_n\|_{H^1(\mathbb{D}^2)} + C, \quad C_2 > 0.$$

Finally, taking $\varepsilon$ small enough and using the Poincaré–Wirtinger inequality, we obtain

$$I_\rho(u_n) \geq C_1\|u_n\|^2_{H^1(\mathbb{D}^2)} - C_2\|u_n\|_{H^1(\mathbb{D}^2)} + C, \quad C_1, C_2 > 0.$$

Again, we remark that the constant $C_1$ is independent of $\rho$. □

# 3 Proof of Theorem 1.1 and Its Generalization

We begin this section by considering the limiting cases $\rho = 0$ and $\rho = 2\pi$. These cases have their own interest, as will be shown, but their study will be useful also for the proofs of Theorems 1.1, 1.2 and 3.4.

Observe that

$$I(u,0)=\frac{1}{2}\int_{\mathbb{D}^2}|\nabla u|^2+2\int_{\mathbb{S}^1}u-8\pi\log\int_{\mathbb{S}^1}he^{u/2}+8\pi\log(2\pi),$$

and, as $K$ does not play any role, it can be defined on the bigger space

$$\mathbb{X}^1=\Big\{u\in H^1(\mathbb{D}^2):\int_{\mathbb{S}^1}he^{u/2}>0\Big\}\supset\mathbb{X}.$$

The critical points of $I_0$ on $\mathbb{X}^1$ are weak solutions of the problem

$$\begin{cases}-\Delta u=0 & \text{in } \mathbb{D}^2,\\ \dfrac{\partial u}{\partial\eta}+2=4\pi\dfrac{he^{u/2}}{\int_{\mathbb{S}^1}he^{u/2}} & \text{on } \mathbb{S}^1,\end{cases}$$

which is clearly equivalent to the problem of prescribing Gaussian curvature $K=0$ and geodesic curvature $h$, that is,

$$\begin{cases}-\Delta u=0 & \text{in } \mathbb{D}^2,\\ \dfrac{\partial u}{\partial\eta}+2=2he^{u/2} & \text{on } \mathbb{S}^1.\end{cases}\tag{3.1}$$

Under the hypothesis that $h$ is $G$-symmetric, we can seek a minimizer of $I_0$ on the space of symmetric functions

$$\mathbb{X}^1_G=\{u\in\mathbb{X}^1:u\circ g=u \text{ for all } g\in G\}.$$

**Theorem 3.1.** *Let $G$ be as in (G), and let $h:\mathbb{S}^1\to\mathbb{R}$ be a $G$-symmetric, Hölder continuous and somewhere positive function. Then problem* (3.1) *admits a solution as a minimum of $I_0$ on $\mathbb{X}^1_G$.*

*Proof.* The functional is bounded from below as seen in Proposition 2.7, so there exists

$$\alpha=\inf_{u\in\mathbb{X}^1_G}I_0(u).$$

Let $(u_n)$ be a minimizing sequence in $\mathbb{X}^1_G$, that is, $I_0(u_n)\to\alpha$. By Proposition 2.12, we know that $I_0$ is coercive, so $u_n$ is bounded in the $H^1(\mathbb{D}^2)$ norm and we can assume that there exists $u_0$ in $H^1(\mathbb{D}^2)$ such that, up to a subsequence, $u_n\rightharpoonup u_0$. Then we also have

$$\int_{\mathbb{S}^1}u_n\to\int_{\mathbb{S}^1}u_0,\qquad\int_{\mathbb{S}^1}he^{u_n/2}\to\int_{\mathbb{S}^1}he^{u_0/2}.$$

Combining this information with the fact that the function $u\to\int_{\mathbb{D}^2}|\nabla u|^2$ is weakly lower semicontinuous, we have $I_0(u_0)\le\alpha$. It is easy to check that

$$\int_{\mathbb{S}^1}he^{\frac{u_0}{2}}>0$$

because if we had

$$\int_{\mathbb{S}^1}he^{u_n/2}\to0,$$

then $I_0(u_n)\to+\infty$, which contradicts that $u_n$ is minimizing. Also, notice that weak convergence respects symmetry, so $u_0$ is a $G$-symmetric function. □

Analogously, we can consider the functional related to the limiting case $\rho=2\pi$:

$$I(u,2\pi)=\frac{1}{2}\int_{\Sigma}|\nabla u|^2+2\int_{\mathbb{S}^1}u-4\pi\log\int_{\mathbb{D}^2}Ke^u+4\pi+4\pi\log(2\pi)$$

defined on the space

$$\mathbb{X}^2_G=\Big\{u\in H^1_G(\mathbb{D}^2):\int_{\mathbb{D}^2}Ke^u>0\Big\}\supset\mathbb{X}_G.$$

One can check that its variation with respect to $u$ produces weak solutions of the problem

$$\begin{cases} -\Delta u = 4\pi \dfrac{Ke^u}{\int_{\mathbb{D}^2} Ke^u} & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 0 & \text{on } \mathbb{S}^1, \end{cases}$$

which is equivalent to the problem of prescribing geodesic curvature $h = 0$ and Gaussian curvature $K$:

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 0 & \text{on } \mathbb{S}^1. \end{cases}$$

A trivial adaptation of the proof of Theorem 3.1 gives the following result.

**Theorem 3.2.** *Let $G$ be as in (G), and let $K : \mathbb{D}^2 \to \mathbb{R}$ be a $G$-symmetric, Hölder continuous and somewhere positive function. Then problem* (3.1) *admits a solution as a minimum of $I_{2\pi}$ on $\mathbb{X}_G^2$.*

**Remark 3.3.** The existence result of Theorem 3.1 is known; see for instance [25]. We have not found an explicit statement of the existence result of Theorem 3.2, but we guess that it must be also known. However, in this section we have reinterpreted those solutions as minimizers of $I_0$ and $I_{2\pi}$, respectively. This will be of use in what follows.

Let us now conclude the proof of Theorem 1.1.

*Proof of Theorem 1.1.* If $K = 0$ or $h = 0$, then we are under the assumptions of Theorems 3.1 or 3.2. Then we can assume that both $K$ and $h$ are positive in some point and nonnegative. In this case,

$$\mathbb{X}_G = \mathbb{X}_G^1 = \mathbb{X}_G^2 = H_G^1(\mathbb{D}^2).$$

By Proposition 2.12, there exists a minimizer $(\hat{u}, \hat{\rho}) \in H_G^1(\mathbb{D}^2) \times [0, 2\pi]$ for $I$. We conclude the proof if we exclude the possibilities $\hat{\rho} = 0$ or $\hat{\rho} = 2\pi$.

Assume that $\hat{\rho} = 0$. Observe that in this case $\hat{u}$ is a minimizer for $I(\cdot, 0)$. Then

$$\begin{aligned} I(\hat{u}, 0) &\le I(\hat{u}, \rho) \\ &= I(\hat{u}, 0) - 2\rho \log\Big(\int_{\mathbb{D}^2} Ke^{\hat{u}}\Big) + 4\rho \log\Big(\int_{\mathbb{S}^1} he^{\hat{u}/2}\Big) \\ &\quad + 8\pi \log\Big(\frac{2\pi - \rho}{2\pi}\Big) - 4\rho \log(2\pi - \rho) + 2\rho + 2\rho \log \rho. \end{aligned}$$

But observe that, as $\rho \to 0$, the main term above is $2\rho \log \rho$, which is negative. This gives a contradiction that excludes the case $\hat{\rho} = 0$. One can exclude the case $\hat{\rho} = 2\pi$ in an analogous way. □

The proof of Theorem 1.1 can be adapted to a more general setting as follows.

**Theorem 3.4.** *Let $G$ be as in (G), and let $K$, $h$ be $G$-symmetric, Hölder continuous functions that are positive somewhere. We define*

$$S_0 = \{u \in \mathbb{X}_G^1 : I_0(u) = \min_{\mathbb{X}_G^1} I_0\}, \quad S_{2\pi} = \{u \in \mathbb{X}_G^2 : I_{2\pi}(u) = \min_{\mathbb{X}_G^2} I_{2\pi}\}.$$

*If $S_0 \cap \mathbb{X}_G$ and $S_{2\pi} \cap \mathbb{X}_G$ are nonempty, then* (1.3) *admits a solution.*

Clearly, Theorem 1.1 is an immediate consequence of Theorem 3.4. Notice also that the sets $S_0$ and $S_{2\pi}$ of the hypotheses are nonempty because of Theorems 3.1 and 3.2.

*Proof.* The proof follows from the same energy comparison argument as above, but a couple of details are worth writing down. First, the existence of a minimizer is not clear a priori. Let $(u_n, \rho_n) \in \mathbb{X}_G \times (0, 2\pi)$ be a minimizing sequence, that is, $I(u_n, \rho_n) \to \inf I$. Clearly, $u_n$ is bounded in $H_G^1(\mathbb{D}^2)$ by Proposition 2.12, but its weak limit $\hat{u}$ could fall outside $\mathbb{X}_G$.

If $\rho_n \to \hat{\rho} \in (0, 2\pi)$, from the fact that $I(u_n, \rho_n)$ is bounded we obtain

$$0 < \varepsilon < \int_{\mathbb{D}^2} Ke^{u_n} < C, \quad 0 < \varepsilon < \int_{\mathbb{S}^1} he^{u_n/2} < C$$

for some $\varepsilon > 0$ and $C > 0$. As a consequence, $u_n \rightharpoonup \hat{u} \in \mathbb{X}_G$ and we are done.

Assume now that $\rho_n \to 0$. If $n$ is sufficiently large, we have the estimate

$$I(u_n, \rho_n) \geq -2\rho_n \log\Big(\int_{\mathbb{D}^2} Ke^{u_n}\Big) - 4(2\pi - \rho_n)\log\Big(\int_{\mathbb{S}^1} he^{u_n/2}\Big) + C.$$

Notice that

$$\liminf_{n\to\infty} -2\rho_n \log\Big(\int_{\mathbb{D}^2} Ke^{u_n}\Big) \geq 0.$$

Thus, $-\log(\int_{\mathbb{S}^1} he^{u_n/2})$ must be bounded from above, which means that

$$0 < \varepsilon < \int_{\mathbb{S}^1} he^{u_n/2}.$$

Now, we write

$$\begin{aligned} I(u_n, \rho_n) = I(u_n, 0) - 2\rho_n \log\Big(\int_{\mathbb{D}^2} Ke^{u_n}\Big) + 4\rho_n \log\Big(\int_{\mathbb{S}^1} he^{u_n/2}\Big) \\ + 8\pi \log\Big(\frac{2\pi - \rho_n}{2\pi}\Big) - 4\rho_n \log(2\pi - \rho_n) + 2\rho_n + 2\rho_n \log \rho_n. \end{aligned}$$

From this we deduce that

$$\inf I = \lim_{n\to\infty} I(u_n, \rho_n) \geq \liminf_{n\to\infty} I(u_n, 0) \geq I(u_0, 0),$$

where $u_0 \in S_0 \cap \mathbb{X}_G$. But, as in the proof of Theorem 1.1,

$$I(u_0, 0) > I(u_0, \rho)$$

for small values of $\rho$. This contradiction shows that $\rho_n$ cannot converge to 0. In an analogous way, we can exclude its convergence to $2\pi$. □

# 4 A Perturbation Result

In this section, it is necessary to specify the dependence of $I$ on the curvature functions $K$ and $h$, so we are writing $I(u, \rho) = I[K, h](u, \rho)$. We begin with a compactness result:

**Lemma 4.1.** *Let $(K_n)$ and $(h_n)$ be sequences of Hölder continuous G-symmetric functions, defined on $\mathbb{D}^2$ and $\mathbb{S}^1$, respectively, such that*

$$\begin{aligned} K_n \to K \quad &\text{uniformly in } \mathbb{D}^2 \text{ and } K \in C^{0,\alpha}(\mathbb{D}^2), \\ h_n \to h \quad &\text{uniformly on } \mathbb{S}^1 \text{ and } h \in C^{0,\alpha}(\mathbb{S}^1). \end{aligned}$$

*Let us consider a sequence $(u_n)$, where each $u_n$ is a solution of the problem*

$$\begin{cases} -\Delta u = 2K_n e^u & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial n} + 2 = 2h_n e^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{4.1}$$

*satisfying*

$$\rho_n = \int_{\mathbb{D}^2} K_n e^{u_n} > 0, \quad \int_{\mathbb{S}^1} h_n e^{u_n/2} > 0 \qquad \textit{for all } n \in \mathbb{N}. \tag{4.2}$$

*Assume that* $I[K_n, h_n](u_n, \rho_n)$ *is uniformly bounded from above. Then* $u_n \rightharpoonup u_\infty$ *on* $H^1(\mathbb{D}^2)$*, with* $u_\infty$ *being a solution of the problem*

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial n} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1. \end{cases} \tag{4.3}$$

*Proof.* First, we notice that $\|K_n - K\|_\infty \to 0$ and $\|h_n - h\|_\infty \to 0$ imply that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\|K_n\|_\infty < \|K\|_\infty + \varepsilon, \quad \|h_n\|_\infty < \|h\|_\infty + \varepsilon.$$

Hypothesis (4.2) gives us $0 < \rho_n < 2\pi$ for all $n \in \mathbb{N}$. Then for $n \geq n_0$ we have the following bound:

$$I[K_n, h_n](u_n, \rho_n) \geq I(\|K\|_\infty + \varepsilon, \|h\|_\infty + \varepsilon)(u_n, \rho_n).$$

And then, by Proposition 2.12, there exist constants $C_1, C_2 > 0$ independent of $n$ such that

$$I[K_n, h_n](u_n, \rho_n) \geq C_1 \|u_n\|_{H^1}^2 - C_2 \|u_n\|_{H^1} + C_\varepsilon.$$

Taking into account the hypothesis that $I[K_n, h_n](u_n, \rho_n)$ is uniformly bounded from above, we have immediately that $u_n$ is bounded in the $H^1(\mathbb{D}^2)$ norm. Hence, up to a subsequence we can assume that there exists $u_\infty \in H^1(\mathbb{D}^2)$ such that $u_n \rightharpoonup u_\infty$.

Then it is known that

$$2K_n e^{u_n} \to 2Ke^{u_\infty} \quad \text{and} \quad 2h_n e^{u_n/2} \to 2he^{\frac{u_\infty}{2}}$$

on $L^p$ for $1 \leq p < +\infty$, and that $\langle \nabla u_n, w \rangle \to \langle \nabla u_\infty, w \rangle$ for all $w \in H^1(\mathbb{D}^2)$. In particular,

$$u_n|_{\mathbb{S}^1} \to u_\infty|_{\mathbb{S}^1} \quad \text{in } L^2(\mathbb{S}^1).$$

We now pass to the limit in the weak formulation of (4.1):

$$\int_{\mathbb{D}^2} \langle \nabla u_n, \nabla v \rangle - 2 \int_{\mathbb{D}^2} K_n e^{u_n} v + 2 \int_{\mathbb{S}^1} v - \int_{\mathbb{S}^1} h_n e^{u_n/2} v = 0$$

for all $v \in H^1(\mathbb{D}^2)$. As a consequence, $u_\infty$ is a weak solution of (4.3). By standard regularity estimates, $u_\infty$ is indeed a classical solution. □

The next step is to check that, when considering a sequence of minimum-type solutions, the hypotheses of Lemma 4.1 are automatically satisfied. Note that under our hypotheses we have $I[K_n, h_n](\cdot, \cdot) \to I[K, h](\cdot, \cdot)$ pointwise in $\mathbb{X} \times (0, 2\pi)$.

If $(u_n, \rho_n)$ is a sequence of minimum-type solutions of (4.1), then

$$\limsup_{n \to +\infty} I[K_n, h_n](u_n, \rho_n) = \limsup_{n \to +\infty} \min_{\mathbb{X} \times (0, 2\pi)} I[K_n, h_n](\cdot, \cdot) \leq \min_{\mathbb{X} \times (0, 2\pi)} I.$$

The previous inequality is due to the fact that $(f_n)$ converging pointwise to $f$ implies

$$\lim_{n \to +\infty} \inf f_n(y) \leq \inf f(y).$$

*Proof of Theorem 1.2.* We apply Theorem 3.4 to the problem

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases}$$

for which we need that the limiting problems

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial n} + 2 = 0 & \text{on } \mathbb{S}^1, \end{cases} \tag{$P_K^1$}$$

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u}{\partial \eta} + 2 = 2he^{u/2} & \text{on } \mathbb{S}^1, \end{cases} \tag{$P_h^2$}$$

admit minimum-type solutions $u_1$ and $u_2$, respectively, verifying

$$\int_{\mathbb{D}^2} Ke^{u_2} > 0, \quad \int_{\mathbb{S}^1} he^{u_1/2} > 0.$$

By contradiction, take Hölder continuous functions $K_n$ and $h_n$ converging uniformly to $K_0$ and $h_0$. We can assume that $n$ is large enough so that $K_n$ and $h_n$ are somewhere positive, so that solutions for the limiting problems in the form of minimizers can be found via Theorems 3.1 and 3.2. Now, take a sequence of minimum-type solutions $(\tilde{u}_n)$ of problems $(P_{K_n}^1)$ and a sequence of minimum-type solutions $(\hat{u}_n)$ of problems $(P_{h_n}^2)$ such that

$$\text{either} \quad \int_{\mathbb{D}^2} K_n e^{\hat{u}_n} \leq 0 \quad \text{or} \quad \int_{\mathbb{S}^1} h_n e^{\tilde{u}_n/2} \leq 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

By Lemma 4.1, we know that $\hat{u}_n \rightharpoonup \hat{u}$ and $\tilde{u}_n \rightharpoonup \tilde{u}$ are solutions for the limiting problems $(P_{K_0}^1)$ and $(P_{h_0}^2)$, respectively. Taking the limit when $n \to +\infty$ in (4.4), we obtain

$$\text{either} \quad \int_{\mathbb{D}^2} K_0 e^{\tilde{u}} \leq 0 \quad \text{or} \quad \int_{\mathbb{S}^1} h_0 e^{\hat{u}/2} \leq 0,$$

which is a contradiction since both $K_0$ and $h_0$ are nonnegative functions somewhere positive. □

**Funding:** D. Ruiz has been supported by the Feder–Mineco Grant MTM2015-68210-P and by J. Andalucia (FQM116). S. Cruz-Blázquez is supported by the Istituto Nazionale di Alta Matematica "F. Severi", and by Marie Skłodowska-Curie Actions through grant number 713485.

## References

[1] A. Ambrosetti, Y. Li and A. Malchiodi, On the Yamabe problem and the scalar curvature problems under boundary conditions, *Math. Ann.* **322** (2002), no. 4, 667–699.

[2] T. Aubin, Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire, *J. Funct. Anal.* **32** (1979), no. 2, 148–174.

[3] J. Bao, L. Wang and C. Zhou, Blow-up analysis for solutions to Neumann boundary value problem, *J. Math. Anal. Appl.* **418** (2014), no. 1, 142–162.

[4] M. S. Berger, Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds, *J. Differential Geom.* **5** (1971), 325–332.

[5] S. Brendle, A family of curvature flows on surfaces with boundary, *Math. Z.* **241** (2002), no. 4, 829–869.

[6] K. C. Chang and J. Q. Liu, A prescribing geodesic curvature problem, *Math. Z.* **223** (1996), no. 2, 343–365.

[7] S.-Y. A. Chang and P. C. Yang, Conformal deformation of metrics on $S^2$, *J. Differential Geom.* **27** (1988), no. 2, 259–296.

[8] W. X. Chen and C. Li, Prescribing Gaussian curvatures on surfaces with conical singularities, *J. Geom. Anal.* **1** (1991), no. 4, 359–372.

[9] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés riemanniennes, *J. Funct. Anal.* **57** (1984), no. 2, 154–206.

[10] F. Da Lio, L. Martinazzi and T. Rivière, Blow-up analysis of a nonlocal Liouville-type equation, *Anal. PDE* **8** (2015), no. 7, 1757–1805.

[11] Z. Djadli, A. Malchiodi and M. Ould Ahmedou, Prescribing scalar and boundary mean curvature on the three dimensional half sphere, *J. Geom. Anal.* **13** (2003), no. 2, 255–289.

[12] J. F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, *Ann. of Math. (2)* **136** (1992), 1–50.
[13] J. F. Escobar, Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary, *Indiana Univ. Math. J.* **45** (1996), 917–943.
[14] V. Felli and M. Ould Ahmedou, Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries, *Math. Z.* **244** (2003), no. 1, 175–210.
[15] J. A. Gálvez and P. Mira, The Liouville equation in a half-plane, *J. Differential Equations* **246** (2009), no. 11, 4173–4187.
[16] H. Hamza, Sur les transformations conformes des variétés riemanniennes à bord, *J. Funct. Anal.* **92** (1990), no. 2, 403–447.
[17] Z.-C. Han and Y. Li, The Yamabe problem on manifolds with boundary: Existence and compactness results, *Duke Math. J.* **99** (1999), no. 3, 489–542.
[18] Z.-C. Han and Y. Li, The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature, *Comm. Anal. Geom.* **8** (2000), no. 4, 809–869.
[19] F. Hang and X. Wang, A new approach to some nonlinear geometric equations in dimension two, *Calc. Var. Partial Differential Equations* **26** (2006), no. 1, 119–135.
[20] A. Jiménez, The Liouville equation in an annulus, *Nonlinear Anal.* **75** (2012), no. 4, 2090–2097.
[21] J. L. Kazdan and F. W. Warner, Curvature functions for compact 2-manifolds, *Ann. of Math. (2)* **99** (1974), 14–47.
[22] J. L. Kazdan and F. W. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures, *Ann. of Math. (2)* **101** (1975), 317–331.
[23] Y. Li and P. Liu, A Moser–Trudinger inequality on the boundary of a compact Riemann surface, *Math. Z.* **250** (2005), no. 2, 363–386.
[24] Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* **80** (1995), no. 2, 383–417.
[25] P. Liu and W. Huang, On prescribing geodesic curvature on $D^2$, *Nonlinear Anal.* **60** (2005), no. 3, 465–473.
[26] R. López-Soriano, A. Malchiodi and D. Ruiz, Conformal metrics with pescribed Gaussian and geodesic curvatures, preprint (2018), https://arxiv.org/abs/1806.11533.
[27] R. López-Soriano and D. Ruiz, Prescribing the Gaussian curvature in a subdomain of $\mathbb{S}^2$ with Neumann boundary condition, *J. Geom. Anal.* **26** (2016), no. 1, 630–644.
[28] F. C. Marques, Existence results for the Yamabe problem on manifolds with boundary, *Indiana Univ. Math. J.* **54** (2005), no. 6, 1599–1620.
[29] J. Moser, A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.* **20** (1970/71), 1077–1092.
[30] J. Moser, On a non-linear problem in differential geometry, in: *Dynamical Systems*, Academic Press, New York (1973), 273–280.
[31] B. Osgood, R. Phillips and P. Sarnak, Extremals of determinants of Laplacians, *J. Funct. Anal.* **80** (1988), no. 1, 148–211.
[32] N. S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Sc. Norm. Super. Pisa (3)* **22** (1968), 265–274.
[33] L. Zhang, Classification of conformal metrics on $\mathbf{R}^2_+$ with constant Gauss curvature and geodesic curvature on the boundary under various integral finiteness assumptions, *Calc. Var. Partial Differential Equations* **16** (2003), no. 4, 405–430.