DE GRUYTER Adv. Nonlinear Stud. 2019; 19(2): 317–332

**Research Article**

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# Existence and Multiplicity of Periodic Solutions to Indefinite Singular Equations Having a Non-monotone Term with Two Singularities

https://doi.org/10.1515/ans-2018-2018
Received November 14, 2017; revised April 18, 2018; accepted April 19, 2018

**Abstract:** Efficient conditions guaranteeing the existence and multiplicity of $T$-periodic solutions to the second order differential equation $u'' = h(t)g(u)$ are established. Here, $g\colon (A, B) \to (0, +\infty)$ is a positive function with two singularities, and $h \in L(\mathbb{R}/T\mathbb{Z})$ is a general sign-changing function. The obtained results have a form of relation between multiplicities of zeros of the weight function $h$ and orders of singularities of the nonlinear term. Our results have applications in a physical model, where from the equation $u'' = \frac{h(t)}{\sin^2 u}$ one can study the existence and multiplicity of periodic motions of a charged particle in an oscillating magnetic field on the sphere. The approach is based on the classical properties of the Leray–Schauder degree.

**Keywords:** Singular Differential Equation, Indefinite Weight, Periodic Solution, Degree Theory, Kepler Problem on $\mathbb{S}^2$

**MSC 2010:** 34C25, 34B16, 47H11, 70F05

**Communicated by:** Fabio Zanolin

## 1 Introduction and Main Results

In a recent series of papers, see [9, 10, 12–14, 19, 20], Zanolin and his collaborators have studied the existence and multiplicity of solutions to Neumann and periodic boundary value problems for second order differential equations of the form

$$u'' = \sigma h(t)g(u). \tag{1.1}$$

Here, $h \in L(\mathbb{R}/T\mathbb{Z})$, $\sigma > 0$ is a parameter, and $g\colon (A, B) \to (0, +\infty)$ is a continuous function, with $-\infty \le A < B \le +\infty$. An elementary observation is that any solution to (1.1) satisfying the above-mentioned boundary conditions satisfies

$$\int_0^T h(t)g(u(t))\,dt = 0 \tag{1.2}$$

and, consequently, the function $h$ (sometimes called the weight function) has to change its sign. These types of equations with sign-changing weight function are called *indefinite equations*. The terminology "indefinite" was probably introduced by Hess and Kato in [27] under the framework of linear eigenvalue problems,

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meaning that $h$ is not sign-constant. In addition to the above-mentioned papers, it is worth mentioning here that the qualitative study of the solutions to the indefinite equation (1.1) has also been treated by many other authors in both ODEs and PDEs settings, see, e.g., [4, 6–8, 11, 16–18, 23]. However, most of them considered equation (1.1) without singularities (i.e., $A = -\infty$, $B = +\infty$).

As far as singular nonlinearities are concerned (that is, either $-\infty < A$ or $B < +\infty$), both periodic and Neumann problems become slightly more tricky. Indeed, the indefinite character of such problems increases the difficulty of finding a suitable region where a desired solution can be localized even for the non-singular case. However, in spite of this fact, in that case the nonlinearity $g$ can be extended on the whole axis in such a way that a point of equilibrium exists. This procedure considerably simplifies the way of finding a region containing the above-mentioned solution, where the degree of a certain fixed point operator is not equal to zero. Moreover, in contrast to the singular case, the set of solutions can be included in an open bounded subset of continuous functions (some a priori bounds can be found). Thus, it seems to be more natural to use the adaptation of the newly developed method of the topological degree in the non-singular case than in the singular one. That is why, contrarily to the non-singular case, the existence and multiplicity of solutions to (1.1) with periodic or Neumann boundary conditions still remain investigated insufficiently, and only a few papers nowadays can be found that deal with the problem in question. We can recommend the reader some recently published papers concerning the indefinite equations with singularities for better understanding the actual state, see, e.g., [15, 22, 24, 25, 31] for the case with one singularity, and [14, 26], where the case of two singularities is investigated.

In this paper we investigate equation (1.1) with two singularities, i.e., we consider the case when $A, B \in \mathbb{R}$. The results established in the paper have a form of relation between multiplicities of zeros of the weight function $h$ and orders of singularities of the nonlinear term. Therefore, as a mathematical model to illustrate the conditions obtained, we introduce the generalized Emden–Fowler equation with two negative exponents, that is,

$$u'' = \frac{\sigma h(t)}{u^{\lambda}(\pi - u)^{\mu}}, \tag{1.3}$$

where $\lambda > 0$, $\mu > 0$, and $\sigma > 0$ is a parameter.

In addition to the latter model, equations with two singularities appear frequently in models describing the problems in celestial mechanics. For instance, the dynamics of a charged particle moving on a line between two fixed charges can be mentioned as one of the simplest models in physics. Supposing that the distance between the fixed charges is equal to one, this model can be included into the following family of equations:

$$u'' = \frac{h_1(t)}{u^{\lambda}} + \frac{h_2(t)}{(1 - u)^{\mu}},$$

where $\lambda > 0$, $\mu > 0$, and $h_1, h_2$ are related to the interaction forces between the charges with the free-particle. We point out that the latter equation can be viewed as an equation with two indefinite singularities. We refer to [1, 3, 5, 21, 30] for a review of works dealing with this type of models (most of them in larger dimensions).

An equation modeling the Kepler problem on the sphere is considered in the paper as a physical application. By the Kepler problem we understand a motion of a particle moving on a sphere of radius one subjected to the influence of an electric field created by a charge of a time-depending magnitude fixed in the north pole. This problem can be modeled by

$$u'' = \frac{c^2 \cos u}{\sin^3 u} + \frac{\sigma h(t)}{\sin^2 u},$$

where $c$ corresponds to the angular momentum of the free-particle which is moving on the sphere, $h$ is a kind of force associated with an electric field caused by charge distribution, $\sigma$ is a positive parameter, and $u$ is a real variable connected with the angular position of the particle (see, e.g., [2] and references therein). In the case when $c = 0$, we obtain the equation

$$u'' = \frac{\sigma h(t)}{\sin^2 u}, \tag{1.4}$$

which is a particular case of (1.1), corresponding to the choice $A = 0$, $B = \pi$, $g(x) = \sin^{-2} x$.

Up to our knowledge, the periodic problem for equation (1.4) has been considered only under very restrictive assumptions on the weight function ($h$ is a piecewise constant function with two pieces of different signs, see [26]). However, a general setting may cover more realistic situations (e.g., the case when $h$ is a continuous function).

It is worth mentioning here that indefinite equations with one singularity were considered with a monotone function $g$ in the above-mentioned papers. Obviously, this is not the case anymore for equations with two singularities. The appearing of another singularity plays an important role in the dynamics of solutions. Roughly speaking, we could name two aspects that substantially change the attitude to the problem:

- the nonlinear term is not monotone,
- the right-hand side possesses two singularities.

Now we describe our setting in more detail. Throughout the paper, we will assume that $-\infty < A < B < +\infty$, the function $g\colon (A, B) \to (0, +\infty)$ is continuously differentiable, and there exists $P \in (A, B)$ such that

$$g'(x) \le 0 \quad \text{for } x \in (A, P), \qquad g'(x) \ge 0 \quad \text{for } x \in (P, B), \tag{1.5}$$

$$\lim_{x\to A^+} \int_x^P g(s)\,ds = +\infty, \qquad \lim_{x\to B^-} \int_P^x g(s)\,ds = +\infty. \tag{1.6}$$

By $\overline{h}$ we understand the mean value of $h$, i.e., $\overline{h} = \frac{1}{T}\int_0^T h(s)ds$. Moreover, we set

$$H_+ = \int_0^T h^+(s)\,ds, \quad H_- = \int_0^T h^-(s)\,ds,$$

where $h^+(s) = \max\{0, h(s)\}$ and $h^-(s) = \max\{0, -h(s)\}$. Obviously, from (1.2) it follows that $H_+H_- \ne 0$ is a necessary condition for the solvability of a periodic problem for (1.1).

From (1.6) it follows, in particular, that $g$ has singularities at the points $x = A$ and $x = B$. We are interested in solutions that avoid these singularities. More precisely, we are looking for $T$-periodic functions $u\colon \mathbb{R} \to (A, B)$, which are continuous together with their first derivative on $[0, T]$ and satisfy (1.1) almost everywhere on $[0, T]$. The main result of this paper is the following.

**Theorem 1.1.** *Let $\overline{h} \ne 0$, $g$ satisfy (1.5), (1.6), and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ ($k = 1, \dots, n$, $i = 1, \dots, m$) such that*

$$\bigcup_{k=1}^{n}[a_k, b_k] \cup \bigcup_{i=1}^{m}[x_i, y_i] = [0, T], \tag{1.7}$$

$$h(t) \ge 0 \quad \textit{for a.e. } t \in \bigcup_{k=1}^{n}(a_k, b_k), \tag{1.8}$$

$$h(t) \le 0 \quad \textit{for a.e. } t \in \bigcup_{i=1}^{m}(x_i, y_i). \tag{1.9}$$

*Assume, moreover, that there exist $c_k \in (a_k, b_k)$ and $z_i \in (x_i, y_i)$ ($k = 1, \dots, n$, $i = 1, \dots, m$) such that*

$$\lim_{t\to t_0^+} \int_t^{t_0+\frac{P-A}{C_k}} h(s)g(A + C_k(s - t_0))\,ds = +\infty \quad \textit{for every } t_0 \in [a_k, c_k]\ (k = 1, \dots, n), \tag{1.10}$$

$$\lim_{t\to t_0^-} \int_{t_0-\frac{P-A}{D_k}}^{t} h(s)g(A + D_k(t_0 - s))\,ds = +\infty \quad \textit{for every } t_0 \in [c_k, b_k]\ (k = 1, \dots, n), \tag{1.11}$$

$$\lim_{t\to t_0^+} \int_t^{t_0+\frac{B-P}{K_i}} |h(s)|g(B - K_i(s - t_0))\,ds = +\infty \quad \textit{for every } t_0 \in [x_i, z_i]\ (i = 1, \dots, m), \tag{1.12}$$

$$\lim_{t\to t_0^-} \int_{t_0-\frac{B-P}{L_i}}^{t} |h(s)|g(B - L_i(t_0 - s))\,ds = +\infty \quad \textit{for every } t_0 \in [z_i, y_i]\ (i = 1, \dots, m), \tag{1.13}$$

*where*

$$C_k = \frac{B-A}{b_k - c_k}, \quad D_k = \frac{B-A}{c_k - a_k}, \quad K_i = \frac{B-A}{y_i - z_i}, \quad L_i = \frac{B-A}{z_i - x_i}. \tag{1.14}$$

*Then there exists $\sigma_* > 0$ such that equation* (1.1) *has at least two $T$-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one $T$-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that equation* (1.1) *has no $T$-periodic solution for every $\sigma > \sigma^*$.*

For the particular cases introduced above, Theorem 1.1 can be reformulated as follows.

**Corollary 1.2.** *Let $\overline{h} \neq 0$, $\lambda \geq 1$, $\mu \geq 1$ and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ ($k = 1, \dots, n$, $i = 1, \dots, m$) such that* (1.7)–(1.9) *hold. Assume, moreover, that there exist $c_k \in (a_k, b_k)$ and $z_i \in (x_i, y_i)$ ($k = 1, \dots, n$, $i = 1, \dots, m$) such that*

$$\lim_{x\to 0^+} \int_x^{\frac{\lambda(b_k-c_k)}{\lambda+\mu}} \frac{h(s+t_0)}{s^\lambda}\,ds = +\infty \quad \text{for every } t_0 \in [a_k, c_k]\ (k = 1, \dots, n),$$

$$\lim_{x\to 0^+} \int_x^{\frac{\lambda(c_k-a_k)}{\lambda+\mu}} \frac{h(t_0-s)}{s^\lambda}\,ds = +\infty \quad \text{for every } t_0 \in [c_k, b_k]\ (k = 1, \dots, n),$$

$$\lim_{x\to 0^+} \int_x^{\frac{\mu(y_i-z_i)}{\lambda+\mu}} \frac{|h(s+t_0)|}{s^\mu}\,ds = +\infty \quad \text{for every } t_0 \in [x_i, z_i]\ (i = 1, \dots, m),$$

$$\lim_{x\to 0^+} \int_x^{\frac{\mu(z_i-x_i)}{\lambda+\mu}} \frac{|h(t_0-s)|}{s^\mu}\,ds = +\infty \quad \text{for every } t_0 \in [z_i, y_i]\ (i = 1, \dots, m).$$

*Then there exists $\sigma_* > 0$ such that equation* (1.3) *has at least two $T$-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one $T$-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that equation* (1.3) *has no $T$-periodic solution for every $\sigma > \sigma^*$.*

**Corollary 1.3.** *Let $\overline{h} \neq 0$, $H_+H_- \neq 0$, and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$ and numbers $\delta_k \in \{-1, 1\}$ ($k = 1, \dots, n$) such that*

$$\bigcup_{k=1}^{n} [a_k, b_k] = [0, T], \tag{1.15}$$

$$\delta_k h(t) \geq 0 \quad \text{for a.e. } t \in [a_k, b_k]\ (k = 1, \dots, n). \tag{1.16}$$

*Assume, moreover, that there exist $c_k \in (a_k, b_k)$ ($k = 1, \dots, n$) such that*

$$\lim_{x\to 0^+} \int_x^{\frac{\pi}{2}} \frac{|h(s+t_0)|}{\sin^2 s}\,ds = +\infty \quad \text{for every } t_0 \in [a_k, c_k]\ (k = 1, \dots, n), \tag{1.17}$$

$$\lim_{x\to 0^+} \int_x^{\frac{\pi}{2}} \frac{|h(t_0-s)|}{\sin^2 s}\,ds = +\infty \quad \text{for every } t_0 \in [c_k, b_k]\ (k = 1, \dots, n). \tag{1.18}$$

*Then there exists $\sigma_* > 0$ such that equation* (1.4) *has at least two $T$-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one $T$-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that equation* (1.4) *has no $T$-periodic solution for every $\sigma > \sigma^*$.*

Explicit corollaries of the above-mentioned results are the following statements.

**Corollary 1.4.** *Let $\overline{h} \neq 0$, $\lambda \geq 1$, $\mu \geq 1$ and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ ($k = 1, \dots, n$, $i = 1, \dots, m$) such that* (1.7) *holds. Furthermore, let there exists $\alpha > 0$ such that*

$$h(t) \geq \alpha[(b_k - t)(t - a_k)]^{\lambda-1} \quad \text{for a.e. } t \in [a_k, b_k]\ (k = 1, \dots, n),$$

$$h(t) \leq -\alpha[(y_i - t)(t - x_i)]^{\mu-1} \quad \text{for a.e. } t \in [x_i, y_i]\ (i = 1, \dots, m).$$

*Then there exists* $\sigma_* > 0$ *such that equation* (1.3) *has at least two* $T$*-periodic solutions for every* $0 < \sigma < \sigma_*$ *and at least one* $T$*-periodic solution for* $\sigma = \sigma_*$*. Moreover, there exists* $\sigma^* \geq \sigma_*$ *such that equation* (1.3) *has no* $T$*-periodic solution for every* $\sigma > \sigma^*$.

**Corollary 1.5.** *Let* $\overline{h} \neq 0$, $H_+H_- \neq 0$, *and let there exist pairwise disjoint intervals* $(a_k, b_k) \subset [0, T]$ *and numbers* $\delta_k \in \{-1, 1\}$ $(k = 1, \dots, n)$, $\alpha > 0$ *such that* (1.15), (1.16) *and*

$$|h(t)| \geq \alpha(b_k - t)(t - a_k) \quad \text{for a.e. } t \in [a_k, b_k]\ (k = 1, \dots, n)$$

*hold. Then there exists* $\sigma_* > 0$ *such that equation* (1.4) *has at least two* $T$*-periodic solutions for every* $0 < \sigma < \sigma_*$ *and at least one* $T$*-periodic solution for* $\sigma = \sigma_*$*. Moreover, there exists* $\sigma^* \geq \sigma_*$ *such that equation* (1.4) *has no* $T$*-periodic solution for every* $\sigma > \sigma^*$.

The paper is structured as follows. In Section 2, by passing to an auxiliary equation, we introduce a functional analytical setting to apply the results established in [25] in the classical framework of the Leray–Schauder degree. Then we prove the existence of at least one $T$-periodic solution, provided that $\sigma$ is sufficiently small. In Section 3 we prove the nonexistence of a $T$-periodic solution to equation (1.1) when the parameter $\sigma$ is large enough. In Section 4 we consider a family of operators $M_\sigma$ whose fixed points correspond to $T$-periodic solutions to (1.1). Then some a priori estimates are established in order to prove that the above-mentioned homotopy is admissible on some adequate open set of continuous functions. Consequently, the degree of the operator $M_\sigma$ is equal to zero in the above-mentioned open set, according to the results of Section 3. Finally, we prove the multiplicity result using the excision property of the degree. The paper ends with Section 5, where we discuss the applicability of our results, we give an estimation of the parameter $\sigma_* > 0$, and in the end some open problems are introduced.

Before passing to the mathematical details of the paper, we observe that it is sufficient to assume that $\overline{h} < 0$ in the proofs of the main results. Indeed, if $\overline{h} > 0$, then the transformation

$$\begin{aligned} v(t) &= A + B - u(t) && \text{for } t \in [0, T], \\ \tilde{h}(t) &= -h(t) && \text{for a.e. } t \in [0, T], \\ \tilde{g}(x) &= g(A + B - x) && \text{for } x \in (A, B) \end{aligned}$$

leads to the equation

$$v'' = \sigma \tilde{h}(t)\tilde{g}(v),$$

where the mean value of the function $\tilde{h}$ is negative.

Furthermore, without loss of generality, we can assume that $g(P) < 1$. Indeed, if this is not the case, we can pass to the equation

$$u'' = \sigma \tilde{h}(t)\tilde{g}(u),$$

where $\tilde{h}(t) = (g(P) + 1)h(t)$ for a.e. $t \in [0, T]$ and $\tilde{g}(x) = g(x)/(g(P) + 1)$ for $x \in (A, B)$.

Finally, it is worth noting that the change of variable $v = u - A$ allows us to assume, without loss of generality, that $A = 0$ and $B > 0$. This will be assumed in the rest of the paper.

## 2 Existence of a Solution and Topological Degree

Let us momentarily consider the truncated nonlinearity $g_0 \colon (0, +\infty) \to \mathbb{R}$, defined by

$$g_0(x) = \begin{cases} g(x) & \text{if } x \in (0, P), \\ g(P) & \text{if } x \geq P, \end{cases}$$

and the modified equation

$$u'' = \sigma h(t) g_0(u). \tag{2.1}$$

We are about to rewrite the problem of finding $T$-periodic solutions to (2.1) as a fixed-point problem for an operator equation. For this purpose we need a suitable functional framework that is described briefly in what follows.

Let $C$ denote the Banach space of continuous functions $x\colon [0,T] \to \mathbb{R}$ endowed with a standard norm $\|x\|_\infty = \max\{|x(t)| : t \in [0,T]\}$. We consider the closed subspace

$$C_T = \{u \in C : u(0) = u(T)\}.$$

The open ball centered at zero and with radius $r > 0$ is denoted by $B(0,r)$. We denote by $P, Q\colon C \to C$ the continuous projectors

$$P[u](t) = u(0), \quad Q[u](t) = \frac{1}{T}\int_0^T u(s)\,ds \quad \text{for } t \in [0,T].$$

Furthermore, let $K\colon C \to C$ be a continuous linear operator, given by

$$K[u](t) = \int_0^t u(s)\,ds \quad \text{for } t \in [0,T].$$

All $T$-periodic solutions to (2.1) belong to the open set

$$\Lambda = \{u \in C_T : u(t) > 0 \text{ for } t \in [0,T]\}.$$

Thus, for each $\sigma > 0$, the operator $\widetilde{M}_\sigma\colon \Lambda \to C_T$ is defined by

$$\widetilde{M}_\sigma = P + Q\widetilde{N}_\sigma + K(I-Q)K(I-Q)\widetilde{N}_\sigma, \tag{2.2}$$

where $\widetilde{N}_\sigma$ is the Nemitsky operator associated with (2.1), i.e., $\widetilde{N}_\sigma[u](t) = \sigma h(t)g(u(t))$. The operator $\widetilde{M}_\sigma$ is continuous and maps bounded subsets of $\Lambda$, whose closure is contained in $\Lambda$, into relatively compact sets of $C_T$ (it is completely continuous on $\Lambda$). Now the periodic boundary value problem for (2.1) becomes equivalent to the fixed-point problem for the operator equation

$$u = \widetilde{M}_\sigma[u], \quad u \in \Lambda.$$

Our starting point will be the computation of the degree of the operator $I - \widetilde{M}_\sigma$ on a certain bounded open set (here $I$ denotes the identity map defined on the space $C_T$).

A result on the existence of a periodic solution to (2.1) was recently established in [25, Theorem 1.1] using the coincidence degree. More details on this theory can be found in [29] (see also [28] for an extension for $p$-Laplacian operators). Using the techniques applied in [25] one can obtain the following assertion, which will be useful in the proof of our main result.

**Proposition 2.1.** *Let $\overline{h} < 0$ and let there exist pairwise disjoint intervals $[a_k, b_k] \subset [0,T]$ ($k = 1, \dots, n$) such that*

$$h(t) \geq 0 \quad \text{for a.e. } t \in \bigcup_{k=1}^n [a_k, b_k],$$
$$h(t) \leq 0 \quad \text{for a.e. } t \in [0,T] \setminus \bigcup_{k=1}^n [a_k, b_k].$$

*Assume, moreover, that there exist $c_k \in (a_k, b_k)$ ($k = 1, \dots, n$) such that*

$$\lim_{t\to t_0^+} \int_t^{b_k} h(s) g_0(\widetilde{C}_k(s - t_0))\,ds = +\infty \quad \text{for every } t_0 \in [a_k, c_k]\ (k = 1, \dots, n),$$
$$\lim_{t\to t_0^-} \int_{a_k}^{t} h(s) g_0(\widetilde{D}_k(t_0 - s))\,ds = +\infty \quad \text{for every } t_0 \in [c_k, b_k]\ (k = 1, \dots, n),$$

*where*

$$\widetilde{C}_k = \frac{\Gamma}{b_k - c_k} + \frac{\sigma|\overline{h}|(b_k - c_k)}{4}, \quad \widetilde{D}_k = \frac{\Gamma}{c_k - a_k} + \frac{\sigma|\overline{h}|(c_k - a_k)}{4}, \quad \Gamma = g_0^{-1}(1) + \frac{\sigma T}{4}\|h\|_1.$$

*Then, for every $\sigma > 0$, there exists a bounded open set $\Omega_\sigma \subseteq C_T$ such that $\overline{\Omega}_\sigma \subseteq \Lambda$ and*

$$d_{\mathrm{LS}}[I - \widetilde{M}_\sigma, \Omega_\sigma, 0] = 1. \tag{2.3}$$

Now we establish two auxiliary lemmas dealing with a priori estimates of solutions to (2.1).

**Lemma 2.2.** *Let $u$ be a $T$-periodic solution to* (2.1). *Then*

$$\left|\frac{u'(t)}{g_0(u(t))}\right| \le \sigma\|h\|_1 \quad \textit{for } t \in [0, T]. \tag{2.4}$$

*Proof.* From (2.1) it follows that

$$\frac{u''(t)}{g_0(u(t))} = \sigma h(t) \quad \text{for a.e. } t \in [0, T]. \tag{2.5}$$

Thus, integrating from $t_0$ to $t$, where $t_0 \in [0, T]$ is such that $u'(t_0) = 0$, results in

$$\frac{u'(t)}{g_0(u(t))} + \int_{t_0}^{t} \frac{g_0'(u(s))u'^2(s)}{g_0^2(u(s))}\, ds = \sigma \int_{t_0}^{t} h(s)\, ds \quad \text{for } t \in [t_0, t_0 + T],$$

and so we get

$$\frac{u'(t)}{g_0(u(t))} \ge -\sigma\|h\|_1 \quad \text{for } t \in [t_0, t_0 + T].$$

Analogously, if we integrate (2.5) from $t$ to $t_0 + T$, we arrive at

$$\frac{u'(t)}{g_0(u(t))} \le \sigma\|h\|_1 \quad \text{for } t \in [t_0, t_0 + T].$$

Therefore, (2.4) holds. □

**Lemma 2.3.** *Let all the assumptions of Proposition 2.1 be fulfilled. Then there exists $\sigma_\star > 0$ such that, for every $0 < \sigma \le \sigma_\star$, an arbitrary $T$-periodic solution $u$ to equation* (2.1) *admits the estimate*

$$0 < u(t) < P \quad \textit{for } t \in [0, T]. \tag{2.6}$$

*Proof.* Let $u$ be an arbitrary $T$-periodic solution to (2.1). Set

$$m_u = \min\{u(t) : t \in [0, T]\}, \quad M_u = \max\{u(t) : t \in [0, T]\}.$$

At first, we note that necessarily $m_u < P$. Indeed, if $m_u \ge P$, then $g_0(u(t)) = g(P)$ for $t \in [0, T]$, and so a direct integration of both sides in (2.1) over the interval $[0, T]$ yields a contradiction to the fact that $\overline{h} < 0$. Moreover, we claim that there exists $y_0 \in (0, P)$ that does not depend on $\sigma$ such that

$$m_u \le y_0. \tag{2.7}$$

Indeed, the integration of (2.1) over $[0, T]$ results in

$$\int_0^T h^+(s)g_0(u)\, ds = \int_0^T h^-(s)g_0(u)\, ds.$$

Hence, according to (1.5), it follows that $g(m_u) \ge g(P)H_-/H_+$. Thus, since $m_u < P$, the continuity and monotonicity of $g$ on the interval $(0, P)$ imply that there exists a constant $y_0 \in (0, P)$ such that (2.7) holds; for instance, we can set

$$y_0 = \inf g^{-1}\Big(\frac{H_-}{H_+}g(P)\Big) \cap (0, P).$$

Now let $t_m \in [0, T]$ and $t_M \in (t_m, t_m + T)$ be such that $u(t_m) = m_u$, $u(t_M) = M_u$. According to Lemma 2.2, $u$ satisfies (2.4). Thus, the integration of (2.4) from $t_m$ to $t_M$ yields

$$\int_{m_u}^{M_u} \frac{ds}{g_0(s)} \leq \sigma T \|h\|_1.$$

If we assume that $M_u \geq P$, then (2.7) and the latter inequality imply that

$$0 < \int_{y_0}^{P} \frac{ds}{g(s)} \leq \sigma T \|h\|_1. \tag{2.8}$$

However, this is a contradiction, provided that $\sigma$ is sufficiently small. □

A key step towards the proof of the existence of a solution to (1.1) is the result formulated below. Under the assumption that our parameter $\sigma$ is small enough, we have proven (see Lemma 2.3) that any $T$-periodic solution to (2.1) lies in the unmodified region of the nonlinearity $g$ (i.e., (2.6) holds). Hence, in view of (2.3) and an excision property of the degree, the following assertion immediately follows from Proposition 2.1.

**Lemma 2.4.** *Let all the assumptions of Proposition 2.1 be fulfilled. Then there exists $\sigma_\star > 0$ such that, for every $0 < \sigma \leq \sigma_\star$, the operator $\widetilde{M}_\sigma$ does not have any fixed point in $\partial[B(0, P) \cap \Omega_\sigma]$, where $\Omega_\sigma$ is the open set appearing in Proposition 2.1. In addition,*

$$d_{\mathrm{LS}}[I - \widetilde{M}_\sigma, \Omega_\sigma \cap B(0, P), 0] = 1. \tag{2.9}$$

To prove that the assumptions of Theorem 1.1 guarantee the existence of a $T$-periodic solution to (1.1), it remains only to observe that the conditions of Theorem 1.1 imply, in particular, the conditions of Proposition 2.1. Then, by applying Lemma 2.4, the existence of such a solution becomes clear. More precisely, the following assertion holds.

**Proposition 2.5.** *Let $\overline{h} < 0$, $g$ satisfy (1.5), (1.6), and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$ ($k = 1, \dots, n$) such that (1.8) and*

$$h(t) \leq 0 \quad \text{for a.e. } t \in [0, T] \setminus \bigcup_{k=1}^{n} (a_k, b_k)$$

*hold. Assume, moreover, that there exist $c_k \in (a_k, b_k)$ ($k = 1, \dots, n$) such that conditions (1.10) and (1.11) are fulfilled, where $C_k$ and $D_k$ are given by (1.14) (with $A = 0$). Then there exists $\sigma_\star > 0$ with the following property: for every $0 < \sigma \leq \sigma_\star$, there exists a bounded open set $\Omega_\sigma \subseteq C_T$ satisfying $\overline{\Omega}_\sigma \subseteq \Lambda$, and (2.9) holds. In particular, there exists at least one $T$-periodic solution $u$ to equation (1.1) satisfying inequalities (2.6).*

*Proof.* It is sufficient to note that for $\sigma > 0$ sufficiently small, we have $\Gamma \leq P$, $\widetilde{C}_k \leq C_k$ and $\widetilde{D}_k \leq D_k$, where $\Gamma$, $\widetilde{C}_k$ and $\widetilde{D}_k$ are the constants defined in Proposition 2.1. Consequently, the conditions of Proposition 2.1 follows from (1.10) and (1.11) due to the monotonicity of the function $g$ in $(0, P)$. □

# 3 Nonexistence of Solutions for a Large Parameter

Now we turn our attention to equation (1.1) in order to investigate whether or not there are $T$-periodic solutions, provided that $\sigma$ is large enough. In particular, we will prove that there is no $T$-periodic solution, and hence the degree of the corresponding operator is equal to zero. To be more precise, we will introduce a suitable abstract framework in the same way as it was done in the previous section. The first elementary observation is that all solutions to (1.1) are located in the open set

$$\Delta = \{u \in C_T : 0 < u(t) < B \text{ for } t \in [0, T]\}.$$

So, for every $\sigma > 0$, the operator $M_\sigma \colon \Delta \to C_T$ defined by

$$M_\sigma = P + QN_\sigma + K(I - Q)K(I - Q)N_\sigma, \tag{3.1}$$

where $N_\sigma$ is the Nemitsky operator associated with equation (1.1) ($P$, $Q$ and $K$ were defined in Section 2), is completely continuous in the sense that it is continuous and maps subsets of $\Delta$ whose closure is contained in $\Delta$ into relatively compact sets in $C_T$. Now the periodic problem associated with (1.1) may be seen as a fixed-point problem for the operator equation

$$u = M_\sigma[u], \quad u \in \Delta, \tag{3.2}$$

for every $\sigma > 0$. In order to compute the degree of the operator $I - M_\sigma$ for $\sigma$ sufficiently large, we will prove a stronger result.

**Lemma 3.1.** *Let there exist an interval $[a, b] \subset [0, T]$ such that*

$$h(t) \geq 0 \quad \text{for a.e. } t \in [a, b], \qquad \int_a^b h(s)\,ds \neq 0. \tag{3.3}$$

*Then there exists $\sigma^* > 0$ such that for every $\sigma \geq \sigma^*$, there is no $T$-periodic solution to* (1.1).

*Proof.* Assume that (3.3) holds, i.e., the function $h$ is non-negative in $[a, b]$. Assume also that there exists a $T$-periodic solution $u$ to (1.1). Then

$$u''(t) \geq \sigma h(t) g(P) \quad \text{for a.e. } t \in [a, b],$$

and thus, for $t \in [a, b]$, we have

$$u(t) \leq \frac{u(b)(t-a) + u(a)(b-t)}{b-a} - \frac{\sigma g(P)}{b-a}\Bigg[(b-t)\int_a^t (s-a)h(s)\,ds + (t-a)\int_t^b (b-s)h(s)\,ds\Bigg].$$

Consequently, from the latter inequality, for $t = (a+b)/2$, we obtain

$$\frac{\sigma g(P)}{2}\Bigg[\int_a^{\frac{a+b}{2}} (s-a)h(s)\,ds + \int_{\frac{a+b}{2}}^b (b-s)h(s)\,ds\Bigg] \leq \frac{u(b)+u(a)}{2} - u\Big(\frac{a+b}{2}\Big) < B.$$

Therefore, the assertion holds. □

The assertion below is an elementary consequence of the degree theory and Lemma 3.1.

**Proposition 3.2.** *Let all the assumptions of Theorem 1.1 be fulfilled. Then there exists $\sigma^* > 0$ such that, for every $\sigma \geq \sigma^*$, there is no $T$-periodic solution to* (1.1). *In particular,*

$$d_{\mathrm{LS}}[I - M_{\sigma^*}, U, 0] = 0 \tag{3.4}$$

*for each open set $U \subset C_T$ such that $\overline{U} \subset \Delta$.*

Actually, Proposition 3.2 can be proven assuming only that condition (3.3) holds (that can be derived from the assumptions of Theorem 1.1). It is worth noting here that the appearance of two singularities (i.e., the boundedness of the interval $(0, B)$) plays an important role.

## 4 The Homotopy. Proof of the Main Result

In this section we complete the proof of Theorem 1.1. We will begin our study considering the "homotopy" operator $M_\sigma$ defined in (3.1). The degree of the operator $I - M_\sigma$ for $\sigma > 0$ large enough was derived in Proposition 3.2. Then, by virtue of the generalized homotopy invariance of the degree, we can calculate the degree of

$I - M_\sigma$ for every $\sigma > 0$, provided that we are able to find a suitable open set whose boundary does not contain solutions to equation (3.2) (that is, $T$-periodic solutions to (1.1)). In other words, before passing to the proof of Theorem 1.1, we take advantage of the structure of our nonlinear term in order to find a priori bounds for all possible $T$-periodic solutions to (1.1) (note that this is possible due to the fact that $g$ possesses two singularities; otherwise, [31, Corollary 3.4] shows that it cannot be done in general).

## 4.1 Some A Priori Bounds

Here the relation between the order of the singularities of $g$ and the order of the zeros of $h$ will play a crucial role.

Lemma 4.1, formulated below, proves that $T$-periodic solutions to (1.1) cannot arbitrarily approach to the singularity of $g$ at zero if the multiplicity of every zero of $h$ (in the intervals where it is non-negative) is sufficiently small in comparison with the order of the singularity.

**Lemma 4.1.** *Let $\sigma_0 > 0$, $g$ satisfy (1.5), (1.6), and let there exist $[a, b] \subset [0, T]$ such that*

$$h(t) \geq 0 \quad \text{for a.e. } t \in [a, b].$$

*Assume, moreover, that there exist $c \in (a, b)$ such that*

$$\lim_{x \to 0^+} \int_{t_0}^{t_0+\frac{P}{C}} \int_{t_0}^{s} h(\xi) g(x + C(\xi - t_0))\, d\xi\, ds > \frac{P}{\sigma_0} \quad \text{for every } t_0 \in [a, c], \tag{4.1}$$

$$\lim_{x \to 0^+} \int_{t_0-\frac{P}{D}}^{t_0} \int_{s}^{t_0} h(\xi) g(x + D(t_0 - \xi))\, ds > \frac{P}{\sigma_0} \quad \text{for every } t_0 \in [c, b], \tag{4.2}$$

*where*

$$C = \frac{B}{b - c}, \quad D = \frac{B}{c - a}. \tag{4.3}$$

*Then there exists $m(a, b; \sigma_0) \in (0, P)$ such that, for every $\sigma \geq \sigma_0$, an arbitrary $T$-periodic solution $u$ to equation (1.1) admits the estimate*

$$m(a, b; \sigma_0) \leq u(t) \quad \text{for } t \in [a, b].$$

**Remark 4.2.** Conditions (4.1), respectively (4.2), are fulfilled, e.g., if

$$\lim_{t \to t_0^+} \int_{t}^{t_0+\frac{P}{C}} h(s) g(C(s - t_0))\, ds = +\infty \quad \text{for every } t_0 \in [a, c],$$

respectively,

$$\lim_{t \to t_0^-} \int_{t_0-\frac{P}{D}}^{t} h(s) g(D(t_0 - s))\, ds = +\infty \quad \text{for every } t_0 \in [c, b].$$

*Proof of Lemma 4.1.* Assume on the contrary that for every $n \in \mathbb{N}$, there exist $\sigma_n \geq \sigma_0$, a solution $u_n$ to (1.1), with $\sigma = \sigma_n$, and $t_n \in [a, b]$ such that $u_n(t_n) = \min\{u_n(t) : t \in [a, b]\} < P$ and

$$\lim_{n \to +\infty} u_n(t_n) = 0. \tag{4.4}$$

Obviously, $u_n'(t_n) = 0$ if $t_n \in (a, b)$, $u_n'(t_n) \geq 0$ if $t_n = a$, and $u_n'(t_n) \leq 0$ if $t_n = b$. Moreover, without loss of generality (passing to a subsequence if necessary), we have either

$$t_n \in [a, c] \quad \text{for every } n \in \mathbb{N} \tag{4.5}$$

or

$$t_n \in [c, b] \quad \text{for every } n \in \mathbb{N}. \tag{4.6}$$

We will assume that (4.5) holds; in the case when (4.6) holds one can argue analogously.

It is clear that

$$u_n'(t_n) \geq 0 \quad \text{for every } n \in \mathbb{N}. \tag{4.7}$$

Moreover, without loss of generality we can assume that there exists $t_0 \in [a, c]$ such that

$$\lim_{n \to +\infty} t_n = t_0. \tag{4.8}$$

Put

$$s_n = t_n + \frac{P - u_n(t_n)}{C} \quad \text{for } n \in \mathbb{N}. \tag{4.9}$$

On the other hand, since $u_n''(t) \geq 0$ for a.e. $t \in [a, b]$, we have

$$u_n(t) \leq \frac{u_n(t_n)(b-t) + u_n(b)(t-t_n)}{b - t_n} \leq u_n(t_n) + C(t - t_n) \quad \text{for } t \in [t_n, b],\ n \in \mathbb{N}, \tag{4.10}$$

and on account of (4.9), we have

$$u_n(t_n) + C(t - t_n) \leq P \quad \text{for } t \in [t_n, s_n]. \tag{4.11}$$

Using (4.10) in (1.1) (with $\sigma = \sigma_n$), with respect to (1.5) and (4.11), we obtain

$$u_n''(t) \geq \sigma_0 h(t) g(u_n(t_n) + C(t - t_n)) \quad \text{for a.e. } t \in [t_n, s_n],\ n \in \mathbb{N}. \tag{4.12}$$

Moreover, in view of (4.1) and (4.4), there exist $x_0 \in (0, P)$ and $n_0 \in \mathbb{N}$ such that

$$\int_{t_0}^{t_0+\frac{P-x_0}{C}} \int_{t_0}^{s} h(\xi) g(x_0 + C(\xi - t_0))\, d\xi\, ds > \frac{P}{\sigma_0}, \tag{4.13}$$

$$u_n(t_n) \leq x_0 \quad \text{for } n \geq n_0. \tag{4.14}$$

Furthermore, with respect to (4.7) and (4.9)–(4.14), for $n \geq n_0$, we have

$$\begin{aligned}
P > u_n(s_n) - u_n(t_n) &= \int_{t_n}^{s_n} u_n'(s)\, ds \geq \int_{t_n}^{s_n} (u_n'(s) - u_n'(t_n))\, ds \\
&= \int_{t_n}^{s_n} \int_{t_n}^{s} u_n''(\xi)\, d\xi\, ds \geq \sigma_0 \int_{t_n}^{s_n} \int_{t_n}^{s} h(\xi) g(u_n(t_n) + C(\xi - t_n))\, d\xi\, ds \\
&\geq \sigma_0 \int_{t_n}^{t_n+\frac{P-x_0}{C}} \int_{t_n}^{s} h(\xi) g(x_0 + C(\xi - t_n))\, d\xi\, ds.
\end{aligned}$$

Now passing to the limit as $n$ tends to $+\infty$, on account of (4.8), from the latter inequalities it follows that

$$P \geq \sigma_0 \int_{t_0}^{t_0+\frac{P-x_0}{C}} \int_{t_0}^{s} h(\xi) g(x_0 + C(\xi - t_0))\, d\xi\, ds. \tag{4.15}$$

However, (4.15) contradicts (4.13). □

Analogously, one can prove the following assertion just with the hypothesis that the multiplicity of every zero of $h$ (in the intervals where it is nonpositive) is sufficiently small in comparison with the order of the singularity at $B$.

**Lemma 4.3.** *Let $\sigma_0 > 0$, $g$ satisfy* (1.5), (1.6), *and let there exist* $[a, b] \subset [0, T]$ *such that*

$$h(t) \le 0 \quad \text{for a.e. } t \in [a, b].$$

*Assume, moreover, that there exist $c \in (a, b)$ such that*

$$\lim_{x \to B^-} \int_{t_0}^{t_0+\frac{B-P}{C}} \int_{t_0}^{s} |h(\xi)| g(x - C(\xi - t_0))\, d\xi\, ds > \frac{B-P}{\sigma_0} \quad \text{for every } t_0 \in [a, c], \tag{4.16}$$

$$\lim_{x \to B^-} \int_{t_0-\frac{B-P}{D}}^{t_0} \int_{s}^{t_0} |h(\xi)| g(x - D(t_0 - \xi))\, d\xi\, ds > \frac{B-P}{\sigma_0} \quad \text{for every } t_0 \in [c, b], \tag{4.17}$$

*where $C$ and $D$ are given by* (4.3). *Then there exists $M(a, b; \sigma_0) \in (P, B)$ such that, for every $\sigma \ge \sigma_0$, an arbitrary $T$-periodic solution $u$ to equation* (1.1) *admits the estimate*

$$M(a, b; \sigma_0) \ge u(t) \quad \text{for } t \in [a, b].$$

**Remark 4.4.** Conditions (4.16), respectively, (4.17), are fulfilled, e.g., if

$$\lim_{t \to t_0^+} \int_{t}^{t_0+\frac{B-P}{C}} |h(s)| g(B - C(s - t_0))\, ds = +\infty \quad \text{for every } t_0 \in [a, c],$$

respectively,

$$\lim_{t \to t_0^-} \int_{t_0-\frac{B-P}{D}}^{t} |h(s)| g(B - D(t_0 - s))\, ds = +\infty \quad \text{for every } t_0 \in [c, b].$$

Combining Lemma 4.1 and Lemma 4.3 one can easily prove the following assertion.

**Lemma 4.5.** *Let $\sigma_0 > 0$, $g$ satisfy* (1.5), (1.6), *and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ $(k = 1, \dots, n,\ i = 1, \dots, m)$ such that* (1.7)–(1.9) *hold. Let, moreover, there exist $c_k \in (a_k, b_k)$ and $z_i \in (x_i, y_i)$ $(k = 1, \dots, n,\ i = 1, \dots, m)$ such that*

$$\lim_{x \to 0^+} \int_{t_0}^{t_0+\frac{P}{C_k}} \int_{t_0}^{s} h(\xi) g(x + C_k(\xi - t_0))\, d\xi\, ds > \frac{P}{\sigma_0} \quad \text{for every } t_0 \in [a_k, c_k]\ (k = 1, \dots, n), \tag{4.18}$$

$$\lim_{x \to 0^+} \int_{t_0-\frac{P}{D_k}}^{t_0} \int_{s}^{t_0} h(\xi) g(x + D_k(t_0 - \xi))\, d\xi\, ds > \frac{P}{\sigma_0} \quad \text{for every } t_0 \in [c_k, b_k] \quad (k = 1, \dots, n), \tag{4.19}$$

$$\lim_{x \to B^-} \int_{t_0}^{t_0+\frac{B-P}{K_i}} \int_{t_0}^{s} |h(\xi)| g(x - K_i(\xi - t_0))\, d\xi\, ds > \frac{B-P}{\sigma_0} \quad \text{for every } t_0 \in [x_i, z_i]\ (i = 1, \dots, m), \tag{4.20}$$

$$\lim_{x \to B^-} \int_{t_0-\frac{B-P}{L_i}}^{t_0} \int_{s}^{t_0} |h(\xi)| g(x - L_i(t_0 - \xi))\, d\xi\, ds > \frac{B-P}{\sigma_0} \quad \text{for every } t_0 \in [z_i, y_i]\ (i = 1, \dots, m), \tag{4.21}$$

*where*

$$C_k = \frac{B}{b_k - c_k}, \quad D_k = \frac{B}{c_k - a_k}, \quad K_i = \frac{B}{y_i - z_i}, \quad L_i = \frac{B}{z_i - x_i}.$$

*Then there exist $m(\sigma_0) \in (0, P)$ and $M(\sigma_0) \in (P, B)$ such that, for every $\sigma \ge \sigma_0$, an arbitrary $T$-periodic solution $u_\sigma$ to equation* (1.1) *admits the estimate*

$$m(\sigma_0) \le u_\sigma(t) \le M(\sigma_0) \quad \text{for } t \in [0, T].$$

**Remark 4.6.** Conditions (4.18)–(4.21) are fulfilled, e.g., if (1.10)–(1.13) hold.

## 4.2 Proof of Theorem 1.1

In this part we will use all the tools just presented in the previous sections to prove the main theorem of this manuscript.

*Proof of Theorem 1.1.* First of all, set

$$\Sigma = \{\tau > 0 : \text{there exist at least two } T\text{-periodic solutions to (1.1) for every } 0 < \sigma \le \tau\},$$

and let $\sigma_\star > 0$ be the number appearing in Proposition 2.5. We claim that $\sigma_\star \in \Sigma$.

Indeed, let $\sigma_0 \in (0, \sigma_\star]$ be arbitrary but fixed. Then, according to Proposition 2.5, there exists an open set $\Omega_{\sigma_0} \subseteq C_T$ such that $\overline{\Omega}_{\sigma_0} \subseteq \Delta$ and (2.9) holds with $\sigma = \sigma_0$. Since $\widetilde{M}_{\sigma_0} \equiv M_{\sigma_0}$ on $\Omega_{\sigma_0} \cap B(0, P)$ (see the definition of $\widetilde{M}_\sigma$ in (2.2) and $M_\sigma$ in (3.1)), we have

$$d_{\mathrm{LS}}[I - M_{\sigma_0}, \Omega_{\sigma_0} \cap B(0, P), 0] = 1. \tag{4.22}$$

This implies that there exists a $T$-periodic solution $u_1$ to (1.1) (with $\sigma = \sigma_0$) such that

$$u_1 \in \Omega_{\sigma_0} \cap B(0, P). \tag{4.23}$$

On the other hand, we define the open set

$$V = \{u \in C_T : m(\sigma_0) < u(t) < M(\sigma_0) \text{ for } t \in [0, T]\},$$

where $m(\sigma_0) \in (0, P)$ and $M(\sigma_0) \in (P, B)$ are the constants appearing in Lemma 4.5 (Remark 4.6 implies that the hypotheses of Theorem 1.1 guarantee the assumptions of Lemma 4.5). Since $\overline{V} \cup \overline{\Omega}_{\sigma_0} \subset \Delta$, there exists an open set $U \subset C_T$ such that

$$\overline{V} \cup \overline{\Omega}_{\sigma_0} \subset U \subset \overline{U} \subset \Delta. \tag{4.24}$$

By Proposition 3.2, there exists $\sigma^* > 0$ such that (3.4) holds. Thus, since $M_\sigma$ does not possess fixed points in $\partial U$ for every $\sigma_0 \le \sigma \le \sigma^*$ (this is a direct consequence of Lemma 4.5), the generalized invariance property to the degree assures that

$$d_{\mathrm{LS}}[I - M_\sigma, U, 0] = 0 \quad \text{for } \sigma \in [\sigma_0, \sigma^*]. \tag{4.25}$$

Now, taking into account (4.24), from (4.22), (4.25) and by the excision property of the degree, we get

$$d_{\mathrm{LS}}[I - M_{\sigma_0}, U \setminus \overline{\Omega_{\sigma_0} \cap B(0, P)}, 0] = -1.$$

Consequently, there exists another $T$-periodic solution $u_2$ to (1.1) in $U \setminus \overline{\Omega_{\sigma_0} \cap B(0, P)}$. Moreover, (4.23) ensures that $u_1 \neq u_2$. Thus, the set $\Sigma$ is non-empty.

In addition, according to Proposition 3.2, we have that the set $\Sigma$ is bounded. Therefore, we set

$$\sigma_* = \sup \Sigma.$$

Obviously, $\sigma_* \le \sigma^*$ and the estimates of solutions obtained in Lemma 4.5 together with the Arzelà–Ascoli theorem imply the existence of a $T$-periodic solution to (1.1) for $\sigma = \sigma_*$. □

# 5 Conclusions and Final Remarks

Now we turn our attention to (1.4) and (1.3), with $\mu \ge 1$ and $\lambda \ge 1$, to explain more about the applicability of our results. In particular, for equation (1.3), Corollary 1.2 can be applied when the number of sign changes of the weight function $h$ is finite. What is important in such a case is to assume that the multiplicity $\nu$ of every zero of the weight function $h$ on the intervals where it is positive has to be sufficiently small in comparison with the order of the singularity $\lambda > 0$ in the vicinity of zero ($\nu \le \lambda - 1$). But that is not all, since in order to deal with the second singularity ($x = \pi$), we need to assume that the same relation holds for the parameter

$\mu > 0$ and the multiplicity $\eta$ of every zero of the weight function $h$ where it is negative ($\eta \leq \mu - 1$). Roughly speaking, two relations between the multiplicities of the zeros of the weight function $h$ and the orders of the singularities of the nonlinear term, i.e.,

$$\nu \leq \lambda - 1, \quad \eta \leq \mu - 1,$$

are imposed in order to guarantee the existence of two solutions associated with the periodic problem to (1.3) (see Corollary 1.4). Here $\nu$ and $\eta$ are the above-mentioned multiplicities.

An interesting class of weight functions where we can apply our results easily is a family of so-called *piecewise-constant functions*. A $T$-periodic function $h\colon \mathbb{R} \to \mathbb{R}$ is said to be piecewise-constant if there is a partition $0 = t_0 < t_1 < \cdots < t_p = T$ of $[0, T]$ such that $h_{|(t_{i-1},t_i)} \equiv h_i$ is a constant for every $i = 1, \dots, p$. In particular, when we deal with these functions, Corollaries 1.2 and 1.3 are as follows.

**Corollary 5.1.** *Let $\overline{h} \neq 0$, $\lambda \geq 1$ and $\mu \geq 1$, and let us assume that h is a piecewise constant function with both positive and negative values. Then there exists $\sigma_* > 0$ such that* (1.3) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that equation* (1.3) *has no T-periodic solution for $\sigma > \sigma^*$.*

**Corollary 5.2.** *Let $\overline{h} \neq 0$, and let us assume that h is a piecewise constant function with both positive and negative values. Then there exists $\sigma_* > 0$ such that* (1.4) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that equation* (1.4) *has no T-periodic solution for $\sigma > \sigma^*$.*

Now we turn our attention to equation (1.4) in order to continue with the discussion of the main results. We consider this equation having in mind the physical model presented in Section 1, but the same discussion can be done for equation (1.3). A number $\sigma_* > 0$ satisfying certain properties with respect to the existence of a $T$-periodic solution to (1.4) was defined in the previous section. However, for applications it would be better to obtain some explicit estimation of $\sigma_*$. In this direction, the method of proof of Theorem 1.1 can be used. As for equation (1.4), the constants appearing in Proposition 2.1 play no role whatsoever, hence to prove Proposition 2.5, it is only required to check that any $T$-periodic solution to equation (2.1) (corresponding to the choice $g(x) = \sin^{-2} x$, $A = 0$, $B = \pi$, $P = \pi/2$) lies in the unmodified region of the nonlinearity $\sin^{-2} x$. Therefore, one way to estimate $\sigma_*$ is described in the proof of Lemma 2.3. Thus, in the case when $\overline{h} < 0$, using relation (2.8), for equation (1.4), we obtain

$$\sigma_* \geq \frac{\sqrt{\frac{H_+(H_- - H_+)}{H_-^2}} + \arccos\sqrt{\frac{H_+}{H_-}}}{2T\|h\|_1}.$$

Combining Corollary 1.2 and the above-mentioned estimate for $\sigma_* > 0$, we can study the existence of $T$-periodic motions in the Kepler problem on the sphere, i.e., we can study the existence of a $T$-periodic solution to the equation

$$u'' = \frac{h(t)}{\sin^2 u}. \tag{5.1}$$

As an illustration, we consider a weight function $h$ belonging to the class of piecewise-constant functions with both positive and negative values. Then we can guarantee the existence of at least two $T$-periodic solutions to (5.1), provided that $\overline{h} < 0$ and

$$2T\|h\|_1 \leq \sqrt{\frac{H_+(H_- - H_+)}{H_-^2}} + \arccos\sqrt{\frac{H_+}{H_-}}.$$

To complete this section, we would like to point out some open problems that still remain unresolved. In particular, the existence of a solution to (1.4) in the case when $\overline{h} = 0$ cannot be resolved using the result established in [25]. The reason is that the relation $\overline{h} < 0$ is not only essential in the proof of the mentioned result, but even necessary (see [25, Remark 1.2]). Moreover, also the approximation of $h$ by some functions $h_n$, with $\overline{h}_n < 0$ and $\overline{h}_n \to 0$, does not allow to apply the result of [25]. Indeed, it can be easily shown that in the case when $\overline{h} = 0$, every $T$-periodic solution $u$ to (1.4) has to cross the point $P = \pi/2$, and thus it is impossible to approximate $u$ by $T$-periodic solutions $u_n$ (corresponding to the weights $h_n$) whose values are located in the unchanged region, i.e., in $(0, \pi/2)$.

That is why one has to suggest another approach in order to establish results guaranteeing the existence of a $T$-periodic solution to (1.4) in the case when $\overline{h} = 0$. One of the suitable direction seems to work with a homotopy, where not only the equation but also the boundary conditions are modified. Nevertheless, also in this case, we expect to obtain the existence and multiplicity of solutions, provided that $\sigma$ is sufficiently small. This conclusion is based on our previous research published in [26], where the sign of the mean value of $h$ does not play any role. More precisely, we expect to obtain a result of the following type.

**Conjecture 5.3.** Let $H_+H_- \neq 0$ and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$ and numbers $\delta_k \in \{-1, 1\}$ $(k = 1, \dots, n)$ such that (1.15) and (1.16) are fulfilled. Assume, moreover, that there exist $c_k \in (a_k, b_k)$ $(k = 1, \dots, n)$ such that (1.17) and (1.18) hold. Then there exists a *critical value* $\sigma_* > 0$ such that

- there is no $T$-periodic solution to (1.4), provided that $\sigma > \sigma_*$,
- there exists at least one $T$-periodic solution to (1.4), provided that $\sigma = \sigma_*$,
- there exist at least two $T$-periodic solutions to (1.4), provided that $\sigma < \sigma_*$.

At this moment we would like to emphasize that the above-mentioned conjecture induces another open problem. More precisely, the validity of the relation

$$\sigma_* = \sigma^*.$$

However, the latter issue seems to be a very difficult problem when one deals with equations whose nonlinear term have indefinite singularities. For example, according to our experience, the classical lower and upper function approach cannot be applied. One of the suitable directions seems to analyze in details the topological properties of the set of solutions.

**Funding:** The authors gratefully acknowledge support from RVO: 67985840 (R. Hakl), and FONDECYT, project no. 11140203 (R. Hakl, M. Zamora)

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