Research Article

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Singularly Perturbed Fractional Schrödinger Equation Involving a General Critical Nonlinearity

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Abstract: In this paper, we are concerned with the existence and concentration phenomena of solutions for the following singularly perturbed fractional Schrödinger problem:

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $N > 2s$ and the nonlinearity f has critical growth. By using the variational approach, we construct a localized bound-state solution concentrating around an isolated component of the positive minimum point of V as $\varepsilon \to 0$. Our result improves the study made in [X. He and W. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, *Calc. Var. Partial Differential Equations* **55** (2016), no. 4, Article ID 91], in the sense that, in the present paper, the *Ambrosetti–Rabinowitz* condition and the *monotonicity* condition on $f(t)/t$ are not required.

Keywords: Fractional Laplacian, Nonlinear Schrödinger Equation, Standing Wave, Critical Growth, s-Harmonic Extension

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1 Introduction

1.1 Background

In this paper, we are concerned with the standing waves for the nonlinear fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

which is derived from the nonlinear fractional Schrödinger equation

$$i\hbar\varphi_t - \hbar^2(-\Delta)^s\varphi - V(x)\varphi + f(\varphi) = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

where $\hbar$ is the Plank constant, which is a very small physical quantity, i is the imaginary unit and $N > 2s$. The

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solutions of (1.2) of the form

$$\varphi(x,t) = e^{-iwt/\hbar}u(x), \quad w \in \mathbb{R}, \tag{1.3}$$

are called standing waves. Assuming that $f\colon \mathbb{C} \to \mathbb{C}$ is continuous such that $f(e^{i\theta}u) = e^{i\theta}f(u)$ $(u, \theta \in \mathbb{R})$, and inserting (1.3) to (1.2), we have

$$\hbar^2(-\Delta)^s u + (V(x) - w)u = f(u), \quad x \in \mathbb{R}^N.$$

Let $\hbar = \varepsilon^s$ and write $V - w$ as V. Then we get (1.1). In quantum mechanics, these standing waves are referred as semiclassical states, whose existence and concentration phenomena are particularly important as $\varepsilon \to 0$. Here $(-\Delta)^s$ $(0 < s < 1)$ is the fractional Laplacian operator, which can be seen as the infinitesimal generator of Lévy stable diffusion processes (see [2]). This operator arises in many areas such as physics, biology, chemistry and finance (see [2, 22]). The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Laskin [21, 22] as a result of extending the Feynman path integral from the Brownian-like quantum mechanical path to the Lévy-like one, where the Feynman path integral leads to the classical Schrödinger equation, and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. For further background in this field, we refer to [15] and the references therein.

1.2 Motivation

An interesting class of solutions to (1.1) is a family of solutions that develop a spike shape around some point in $\mathbb{R}^N$ as $\varepsilon \to 0$. When $s = 1$, equation (1.1) is reduced to a local elliptic equation, namely,

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

During the last three decades, many papers have been devoted to the singularly perturbed Schrödinger equation (1.4) involving subcritical growth or critical growth. Based on a Lyapunov–Schmidt reduction method, Floer and Weinstein [17] first studied the existence of single-peak solutions for $N = 1$ and $f(t) = t^3$. They constructed a single-peak solution which concentrates around any given nondegenerate critical point of V. More related results can be seen in [7, 14, 18, 24, 25] and the references therein. In these papers, the conditions such as the Ambrosetti–Rabinowitz condition, the monotonicity condition or the nondegenerate condition are needed. To remove or weaken these conditions, Byeon and Jeanjean [8] introduced a new penalization approach. Under the Berestycki–Lions conditions (see [4]), they proved that for small $\varepsilon > 0$, there exists a positive solution which clusters near a local minimum point of V if the following hold:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x)$,

(V2) there exists a bounded domain O such that

$$m \equiv \inf_{x\in O} V(x) < \min_{x\in\partial O} V(x).$$

For the critical nonlinearity f, similar results were obtained in [9, 29, 30].

Now, we return our attention to problem (1.1). In contrast to the case $s = 1$, when $s \in (0, 1)$, $(-\Delta)^s$ is a nonlocal operator, and some difficulties arise. Even in the subcritical case, there are only few references on the existence and concentration phenomena for (1.1). In [13], Dávila, del Pino and Wei investigated (1.1) with $f(u) = u^p$ $(1 < p < 2_s^* - 1,\ 2_s^* = 2N/(N - 2s))$. By applying the Lyapunov–Schmidt reduction method, they proved the existence of positive solutions which exhibit multiple spikes near given topologically nontrivial critical points of V or cluster near a given local maximum point of V. More recently, Alves and Miyagaki [1] considered (1.1) with a general nonlinearity f. By the penalization method, due to del Pino and Felmer [14], they constructed a spike solution around the local minimum point of V. In particular, in [1], the nonlinearity f is subcritical and satisfies the Ambrosetti–Rabinowitz condition and the monotonicity condition. Inspired by [8], Seok [26] considered (1.1) just under Berestycki–Lions type conditions. By using the extension approach and the local deformation argument, Seok obtained positive solutions exhibiting multiple spikes near any given local minimum components of the potential V. For more information about the singularly perturbed fractional Schrödinger problems, we refer to [11, 16, 27] and the references therein.

In the works above, only the subcritical case was considered. To study the semiclassical states of (1.1), the limit problem

$$(-\Delta)^s u + mu = f(u), \quad u \in H^s(\mathbb{R}^N), \tag{1.5}$$

plays a crucial role. The existence of (ground state) solutions for fractional Schrödinger equations when the nonlinearity f satisfies the subcritical growth or critical growth has been studied in many papers (cf. [5, 28, 31]). For the critical case, the lack of compactness in the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ makes problem (1.1) tough. A first breakthrough was given by He and Zou [19]. Precisely, they obtained the existence and concentration results for the problem $\varepsilon^{2s}(-\Delta)^s u + V(x)u = g(u) + |u|^{2_s^*-2}u$. Here we should point out that in [19], g satisfies the Ambrosetti–Rabinowitz condition and the monotonicity condition. A natural open question is *whether in the critical case, a similar result as in* [19] *holds for a more general nonlinearity f, particularly, without the Ambrosetti–Rabinowitz condition and the monotonicity condition*. In this paper, we give an affirmative answer to this question.

1.3 Main Hypothesis

In the present paper, we assume that V satisfies (V1)–(V2) and the nonlinearity f satisfies the following:
(F1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\lim_{t\to 0} f(t)/t = 0$,
(F2) $\lim_{t\to\infty} f(t)/t^{2_s^*-1} = 1$,
(F3) there exist $\tilde{C} > 0$ and $p < 2_s^*$ such that $f(t) \geq t^{2_s^*-1} + \tilde{C}t^{p-1}$ for $t \geq 0$.

1.4 Main Result

Let

$$\mathcal{M} \equiv \{x \in O : V(x) = m\}.$$

Theorem 1.1. *Let $N > 2s$ and $s \in (0, 1)$. Suppose that $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies* (V1)–(V2) *and f satisfies* (F1)–(F3). *Then, for small $\varepsilon > 0$,* (1.1) *admits a positive solution v_ε if* $\max\{2_s^* - 2, 2\} < p < 2_s^*$. *Moreover, there exists a maximum point y_ε of v_ε such that* $\lim_{\varepsilon\to 0} \operatorname{dist}(y_\varepsilon, \mathcal{M}) = 0$, *and for any such y_ε, $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + y_\varepsilon)$ converges (up to a subsequence) uniformly to a least energy solution of* (1.5).

Remark 1.2. In the following, we give an example showing that the nonlinearity f satisfies (F1)–(F3), while the Ambrosetti–Rabinowitz condition and the monotonicity condition on $f(t)/t$ do not hold. For example,

$$f(t) = \begin{cases} t^{2_s^*-1} + \mu t^2 |\ln t| + \gamma t^{p-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\max\{2, 2_s^* - 2\} < p < 2_s^*$ and μ, γ are positive constants.

Proof. It is easy to check that $f(t)$ satisfies (F1)–(F3). Set $g(t) = f(t)/t$ and let $t \in (0, 1)$. Then we have

$$g'(t) = (2_s^* - 2)t^{2_s^*-3} + \gamma(p-2)t^{p-3} - \mu(\ln t + 1).$$

So, there exist $\delta > 0$ and $\mu_0 > 0$ large enough such that for any $t \in (1-\delta, 1)$ and $\mu > \mu_0$, $g'(t) < 0$, which implies that the monotonicity condition on $f(t)/t$ does not hold for $t \in (0, +\infty)$. On the other hand, for $t \in (0, 1)$, we have

$$tf(t) - 2F(t) = \Big(\frac{2_s^* - 2}{2_s^*}\Big)t^{2_s^*} + \gamma\Big(\frac{p-2}{p}\Big)t^p - \mu t^3\Big(\frac{1}{3}\ln t + \frac{2}{9}\Big).$$

Then there exists $\delta > 0$ and $\mu_0 > 0$ large enough such that for any $t \in (1-\delta, 1)$ and $\mu > \mu_0$, $tf(t) - 2F(t) < 0$, which means that the Ambrosetti–Rabinowitz condition does not hold. □

Remark 1.3. The condition $f \in C^1$ in (F1) is to guarantee that a solution u of (1.5) satisfies the fractional Pohožaev identity. Since we are concerned with the positive solution, we assume $f(t) \equiv 0$ for $t \leq 0$.

1.5 Main Difficulties and Ideas

The main difficulties are three-fold. Firstly, without the Ambrosetti–Rabinowitz condition, the boundedness of the (PS)-sequence is difficult to obtain. To overcome this difficulty, we seek the solutions in some neighborhood of the set of ground state solutions to the limit problem.

Secondly, with the presence of the critical exponent 2_s^*, the compactness of the (PS)-sequence does not hold in general. To recover the compactness, we apply a truncation argument. Precisely, by the Moser iteration argument, we get a priori L^∞-estimate of ground state solutions to the limit problem. Then we reduce the original problem to a subcritical problem and show the existence of spike solutions to the truncated problem. By the elliptic estimate, we show that the solution obtained is indeed a solution of the original problem.

Thirdly, in the truncation procedure, the uniform L^∞-estimate of ground states to the limit problem plays a crucial role. However, the method introduced in [3] cannot be used directly. In this present paper, we prove that up to a translation, the set of ground state solutions is compact. By virtue of the compactness, we show that the ground state solutions are uniformly bounded in $L^\infty(\mathbb{R}^N)$.

The paper is organized as follows. In Section 2, we introduce the variational setting and present some preliminary results. Section 3 is devoted to the study of ground state solutions of the limit problem (1.5). The property of ground state solutions of (1.5) such as uniform boundedness is obtained by using the Moser iteration technique. Section 4 is devoted to the proof of Theorem 1.1.

2 Preliminaries

By the scale change $x \to x/\varepsilon$ and setting $V_\varepsilon(x) = V(\varepsilon x)$, it follows that (1.1) is equivalent to

$$(-\Delta)^s u + V_\varepsilon(x)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Thus, to study (1.1), it suffices to consider (2.1). In the following, we present a quick survey of some preliminaries and properties about fractional Sobolev spaces.

2.1 Fractional Sobolev Spaces

The fractional Laplacian operator $(-\Delta)^s$, with $s \in (0,1)$, of a function $u\colon \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N, \tag{2.2}$$

where $\mathcal{F}$ is the Fourier transform. Consider the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\xi|^{2s}|\hat{u}|^2 + \hat{u}^2)\,d\xi < \infty\Big\},$$

where $\hat{u} := \mathcal{F}(u)$. The norm is defined by

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|\xi|^{2s}|\hat{u}|^2 + \hat{u}^2)\,d\xi\Big)^{\frac{1}{2}}.$$

By Plancherel's theorem, we have $\|u\|_{L^2(\mathbb{R}^N)} = \|\hat{u}\|_{L^2(\mathbb{R}^N)}$, and for any $u \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2\,dx = \int_{\mathbb{R}^N} |\widehat{(-\Delta)^{\frac{s}{2}} u}(\xi)|^2\,d\xi = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2\,d\xi.$$

It follows that

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u(x)|^2 + u^2)\,dx\Big)^{\frac{1}{2}}, \quad u \in H^s(\mathbb{R}^N).$$

The space $D^s(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norms

$$\|u\|^2_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2\,d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u(x)|^2\,dx.$$

Since we investigate problem (2.1), we need the fractional Sobolev space $H^s_{V_\varepsilon}(\mathbb{R}^N)$, which is a Hilbert space of $D^s(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s_{V_\varepsilon}(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u(x)|^2 + V_\varepsilon(x)u^2)\,dx\Big)^{\frac{1}{2}} < \infty.$$

For the reader's convenience, we recall the embedding results for fractional Sobolev spaces.

Lemma 2.1 (see [23]). *Let $H^s_r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}$. For any $s \in (0,1)$, $N > 2s$, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_s]$ and compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [1, 2^*_s)$. Moreover, $H^s_r(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$.*

Lemma 2.2 (see [12, 15]). *For any $s \in (0,1)$, $D^s(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_s}(\mathbb{R}^N)$, i.e., there exists $S_s > 0$ such that $\|u\|_{L^{2^*_s}(\mathbb{R}^N)} \le S_s\|u\|_{D^s(\mathbb{R}^N)}$.*

2.2 The variational setting

Associated to (2.1), the energy functional $I\colon H^s_{V_\varepsilon(x)}(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u(x)|^2 + V_\varepsilon(x)u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx \quad \text{for all } u \in H^s_{V_\varepsilon}(\mathbb{R}^N),$$

where $F(t) = \int_0^t f(t)\,dt$. Conditions (F1)–(F3) imply that $I(u) \in C^1$.

Definition 2.3. We say that $u \in H^s_{V_\varepsilon}(\mathbb{R}^N)$ is a weak solution of (2.1) if

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi + V_\varepsilon(x)u\phi)\,dx = \int_{\mathbb{R}^N} f(u)\phi\,dx \quad \text{for all } \phi \in H^s_{V_\varepsilon}(\mathbb{R}^N).$$

Proposition 2.4 (see [31]). *Under the assumptions of Theorem 1.1, the following hold:*
(i) *The limit problem* (1.5) *admits a positive ground state solution, which is radially symmetric.*
(ii) *Let S_m be the set of positive radial ground state solutions of* (1.5) *whose maximum point is* 0. *Then S_m is compact in $H^s_r(\mathbb{R}^N)$.*
(iii) *Any solution $u \in S_m$ satisfies the Pohozǎev identity*

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 = N\int_{\mathbb{R}^N} G(u),$$

where $G(u) = \int_{\mathbb{R}^N} F(u) - \frac{m}{2}u^2$,
(iv) *Any solution $u \in S_m$ is also a mountain pass solution.*

2.3 Extended problems

The fractional Laplacian operator is defined in the whole space through the Fourier transform (see (2.2)). In [10], Caffarelli and Silvestre showed that the fractional Laplacian operator can also be realized in a local way by using one more variable and the so-called s-harmonic extension. They developed a local interpretation of the fractional Laplacian operator given in $\mathbb{R}^N$ by considering a Dirichlet to Neumann type operator in the domain $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0, +\infty)$. More precisely, for $u \in D^s(\mathbb{R}^N)$, the solution $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ of

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(x,0) = u & \text{in } \mathbb{R}^N \end{cases}$$

is called the s-harmonic extension of u and satisfies

$$-\lim_{t\to 0} t^{1-2s}\partial_t U(x,t) = N_s(-\Delta)^s u(x) \quad \text{in } \mathbb{R}^N,$$

where $N_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. Let $U := E_s(u)$ and $u = \operatorname{tr}(U) = U(x,0)$. From [10], the s-harmonic extension of u is defined by

$$U(x,t) = \int_{\mathbb{R}^N} P_s(x-\xi,t)u(\xi)\,d\xi, \tag{2.3}$$

where

$$P_s(x,t) = C_{N,s}\frac{t^{2s}}{(|x|^2+|t|^2)^{\frac{N+2s}{2}}},$$

with the constant $C_{N,s}$ satisfying $\int_{\mathbb{R}^N} P_s(x,1)\,dx = 1$.

The space $D^1(t^{1-2s},\mathbb{R}_+^{N+1})$ denotes the completion of $C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$ with the norm

$$\|U\|^2_{D^1(t^{1-2s},\mathbb{R}_+^{N+1})} = \int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla U(x,t)|^2\,dx\,dt,$$

which satisfies (see [6])

$$\|U\|_{D^1(t^{1-2s},\mathbb{R}_+^{N+1})} = \sqrt{N_s}\|u\|_{D^s(\mathbb{R}^N)}.$$

It is known that (see [20]) for any $U \in D^1(t^{1-2s},\mathbb{R}_+^{N+1})$, the trace $U(x,0)$ belongs to $D^s(\mathbb{R}^N)$ and the trace map is continuous as follows:

$$\|U(x,0)\|_{D^s(\mathbb{R}^N)} \le C\|U\|_{D^1(t^{1-2s},\mathbb{R}_+^{N+1})}.$$

By the s-harmonic extension, we introduce the extended problems of (1.5) and (2.1), respectively,

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\dfrac{1}{N_s}\lim\limits_{t\to 0} t^{1-2s}\partial_t U(x,t) = -mU(x,0) + f(U(x,0)) & \text{in } \mathbb{R}^N \end{cases} \tag{2.4}$$

and

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\dfrac{1}{N_s}\lim\limits_{t\to 0} t^{1-2s}\partial_t U(x,t) = -V_\varepsilon(x)U(x,0) + f(U(x,0)) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.5}$$

Define the function spaces H_0 and H_{V_ε} by the sets of $U \in D^1(t^{1-2s},\mathbb{R}_+^{N+1})$ satisfying

$$\|U\|_0^2 = \int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla U(x,t)|^2\,dx\,dt + \int_{\mathbb{R}^N} U^2(x,0)\,dx < \infty,$$

$$\|U\|_{V_\varepsilon}^2 = \int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla U(x,t)|^2\,dx\,dt + \int_{\mathbb{R}^N} V_\varepsilon(x)U^2(x,0)\,dx < \infty.$$

We call $U \in H_0$ a weak solution of (2.4) if it satisfies

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s}\nabla U\nabla V\,dx\,dt + N_s\int_{\mathbb{R}^N}[mU(x,0) - f(U(x,0))]V(x,0)\,dx = 0$$

for any $V \in H_0$. It is well known that if $U \in H_0$ is a weak solution of (2.4), then $U(x,0) \in H^s(\mathbb{R}^N)$ is a weak solution of (1.5). Similarly, we can define the weak solution of (2.5) and get a similar relation between the solutions of (2.1) and (2.5).

Proposition 2.4 gives the main results about the existence and compactness of ground state solutions of (1.5). In the following, we investigate the existence and compactness of ground state solutions of (2.4).

Proposition 2.5. *Under the assumptions of Theorem 1.1, there exists a positive ground solution to* (2.4). *Denote by $\tilde{S}_m$ the set of ground state solutions of* (2.4) *such that, for any $U \in \tilde{S}_m$,* $\operatorname{tr}(U)$ *is radial and attains its maximum at* $0 \in \mathbb{R}^N$. *Then $\tilde{S}_m$ is compact.*

Proof. From Proposition 2.4, we know that under the assumptions of Theorem 1.1, there exists a least energy solution of (1.5). Let S_m be the set of positive radial and decreasing ground state solutions. Then S_m is compact in $H_r^s(\mathbb{R}^N)$. Define the energy functional associated to (1.5) and (2.4), respectively, by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx + \int_{\mathbb{R}^N}\Big[\frac{1}{2}mu^2 - F(u)\Big]\,dx,$$

$$J(U) = \frac{1}{2}\int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla U|^2\,dx\,dt + N_s\int_{\mathbb{R}^N}\Big[\frac{1}{2}mU(x,0)^2 - F(U(x,0))\Big]\,dx.$$

From the argument about extended problems, by a simple calculation, we can obtain that $J(U) = N_s I(u)$. So we can see that the solutions of (1.5) and (2.4) are one-to-one.

Let $\{u_n\} \in S_m$. Then, up to a subsequence, $u_n \to u_0$ strongly in S_m, since S_m is compact. Denote by $U_n = E_s(u_n)$ and $U_0 = E_s(u_0)$ the solutions of (2.4) corresponding to u_n and u_0, respectively. Then $U_n \in \tilde{S}_m$ and $U_0 \in \tilde{S}_m$. To prove that $\tilde{S}_m$ is compact, it suffices to prove that $\|U_n - U_0\|_0 \to 0$. Since U_n and U_0 are (weak) solutions of (2.4), for any $\phi \in H_0$, we have

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s}(\nabla U_n \nabla\phi)\,dx\,dt = N_s\int_{\mathbb{R}^N}[f(U_n(x,0) - mU_n(x,0)]\phi(x,0)\,dx \tag{2.6}$$

and

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s}(\nabla U_0 \nabla\phi)\,dx\,dt = N_s\int_{\mathbb{R}^N}[f(U_0(x,0) - mU_0(x,0)]\phi(x,0)\,dx. \tag{2.7}$$

From (2.6)–(2.7), we get

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s}(\nabla U_n - \nabla U_0)\nabla\phi\,dx\,dt = N_s\int_{\mathbb{R}^N}[f(U_n(x,0)) - f(U_0(x,0)) - m(U_n(x,0) - U_0(x,0))]\phi(x,0)\,dx.$$

Since $u_n \to u_0$ in $H^s(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}_+^{N+1}} t^{1-2s}(\nabla U_n - \nabla U_0)\nabla\phi = 0.$$

Let $\phi = U_n - U_0$. Then

$$\lim_{n\to\infty}\int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla U_n - \nabla U_0|^2 = 0,$$

which implies $\lim_{n\to\infty}\|U_n - U_0\|^2_{D^1(t^{1-2s},\mathbb{R}_+^{N+1})} = 0$. Letting $n \to \infty$, we can obtain

$$\|U_n - U_0\|_0^2 = \|U_n - U_0\|^2_{D^1(t^{1-2s},\mathbb{R}_+^{N+1})} + \|U_n(x,0) - U_0(x,0)\|^2_{L^2(\mathbb{R}^N)} \to 0.$$

The proof is completed. □

In order to study the existence and concentration phenomena by harmonic extension, we need some elliptic estimates for extended problems.

2.4 Elliptic Estimates

Let $\Omega_r := B_r^N(0) \times (0, r)$. Consider the following nonlinear Neumann boundary value problem:

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \Omega_1, \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t U(x,t) = a(x)U(x,0) + g(x) & \text{in } B_1^N(0). \end{cases} \tag{2.8}$$

Let $H^1(t^{1-2s}, \Omega_r)$ be the weighted Sobolev space with the norm

$$\|U(x,t)\|^2_{H^1(t^{1-2s},\Omega_r)} = \int_{\Omega_r} |t|^{1-2s}(U^2 + |\nabla U|^2) = \|U(x,t)\|^2_{L^2(t^{1-2s},\Omega_r)} + \|U(x,t)\|^2_{D^1(t^{1-2s},\Omega_r)}.$$

Proposition 2.6 (De Giorgi–Nash–Moser type estimate, see [20, 26]). *Suppose that $a, g \in L^p(B_1^N(0))$ for some $p > \frac{N}{2s}$.*

(i) *Let $U \in H^1(t^{1-2s}, \Omega_1)$ be a weak solution of (2.8). Then $U \in L^\infty(\Omega_{1/2})$ and there exist a constant $C > 0$, depending only on N, s, p and $\|a\|_{L^p(B_1^N(0))}$, such that*

$$\sup_{\Omega_{1/2}} U \le C(\|U\|_{L^2(t^{1-2s},\Omega_1)} + \|g\|_{L^p(B_1^N(0))}).$$

(ii) *Let $U \in H^1(t^{1-2s}, \Omega_1)$ be a weak solution of (2.8). Then there exists $\alpha \in (0, 1)$, depending only on N, s, p, such that $U \in C^\alpha(\overline{\Omega_{1/2}})$, and there exist a constant $C > 0$, depending only on N, s, p and $\|a^+\|_{L^p(B_1^N(0))}$, such that*

$$\|U\|_{C^\alpha(\overline{\Omega_{1/2}})} \le C(\|U^+\|_{L^\infty(\Omega_1)} + \|g\|_{L^p(B_1^N(0))}).$$

3 A Priori Estimate of Ground State Solutions for the Limit Problems (1.5) and (2.4)

In this section, we study an a priori L^∞-estimate of ground state solutions to the limit problems (1.5) and (2.4). However, the method introduced in [3] is only used to get the L^∞-estimate for any fixed solution. With the help of the compactness of S_m, we modify the argument in [3] to get the uniform L^∞-boundedness of ground state solutions to (1.5) and (2.4).

Proposition 3.1. *Under the assumptions of Theorem 1.1, for any $u \in S_m$, we have $u \in L^\infty(\mathbb{R}^N)$. Moreover, $\sup\{\|u\|_\infty : u \in S_m\} < \infty$.*

Proof. It suffices to prove that for any $\{u_n\} \subset S_m$, with $u_n \to u_0$ strongly in $H^s(\mathbb{R}^N)$, we have $u_n \in L^\infty(\mathbb{R}^N)$ and $\sup_n \|u_n\|_\infty < \infty$. Let $\gamma \ge 1$ and $T > 0$. We define

$$\varphi(t) = \varphi_{\gamma,T}(t) = \begin{cases} 0, & t \le 0, \\ t^\gamma, & 0 < t < T, \\ \gamma T^{\gamma-1}(t-T) + T^\gamma, & t \ge T. \end{cases}$$

It is easy to verify that, for any $t \in \mathbb{R}$, $\varphi'(t) \le \gamma T^{\gamma-1}$ and $t\varphi'(t) \le \gamma\varphi(t)$. Since $\varphi(t)$ is convex, we have $(-\Delta)^s \varphi(u_n) \le \varphi'(u_n)(-\Delta)^s u_n$ and consequently $\|\varphi(u_n)\|_{D^s(\mathbb{R}^N)} \le \gamma T^{\gamma-1}\|u_n\|_{D^s(\mathbb{R}^N)}$. On the other hand, from Lemma 2.2, we obtain

$$S_s \|\varphi(u_n)\|_{D^s(\mathbb{R}^N)} \ge \|\varphi(u_n)\|_{L^{2_s^*}(\mathbb{R}^N)}. \tag{3.1}$$

It follows from (F1) and (F2) that there exists a constant C such that $f(t) \le \frac{mt}{2} + Ct^{2_s^*-1}$ for $t > 0$. Noting that $u_n \ge 0$, we have $(-\Delta)^s u_n \le Cu_n^{2_s^*-1}$ in $\mathbb{R}^N$ and then

$$\|\varphi(u_n)\|^2_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \varphi(u_n)\varphi'(u_n)(-\Delta)^s u_n \le C \int_{\mathbb{R}^N} \varphi(u_n)\varphi'(u_n)u_n^{2_s^*-1}.$$

Together with (3.1) and $u_n\varphi'(u_n) \le \gamma\varphi(u_n)$, we deduce that

$$\|\varphi(u_n)\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le CS_s^2 \int_{\mathbb{R}^N} \varphi'(u_n)u_n\varphi(u_n)u_n^{2_s^*-2} \le C_\gamma \int_{\mathbb{R}^N} \varphi^2(u_n)u_n^{2_s^*-2}, \tag{3.2}$$

where $C_\gamma = CS_s^2\gamma$. By Hölder's inequality, we infer that

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi^2(u_n)u_n^{2_s^*-2} &= \int_{\{u_n \le R\}} \varphi^2(u_n)u_n^{2_s^*-2} + \int_{\{u_n > R\}} \varphi^2(u_n)u_n^{2_s^*-2} \\ &\le \int_{\{u_n \le R\}} \varphi^2(u_n)R^{2_s^*-2} + \|\varphi(u_n)\|^2_{L^{2_s^*}(\mathbb{R}^N)}\Big(\int_{\{u_n > R\}} u_n^{2_s^*}\Big)^{\frac{2_s^*-2}{2_s^*}}. \end{aligned} \tag{3.3}$$

Since $u_n \to u_0$ strongly in $H^s(\mathbb{R}^N)$, we take R large enough (to be fixed later) such that

$$\Big(\int_{\{u_n>R\}} u_n^{2_s^*}\Big)^{\frac{2_s^*-2}{2_s^*}} \le \frac{1}{2C_\gamma}.$$

Using the fact that $\varphi(u_n) \le u_n^\gamma$, it follows from (3.2) and (3.3) that

$$\|\varphi(u_n)\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le \tilde{C}_\gamma R^{2_s^*-2} \int_{\mathbb{R}^N} u_n^{2\gamma},$$

where $\tilde{C}_\gamma > 0$ is a constant. If $u_n \in L^{2\gamma}(\mathbb{R}^N)$, then, by letting $T \to \infty$, we get $u_n \in L^{2_s^*\gamma}(\mathbb{R}^N)$. By iteration, for any $p \ge 2$, $u_n \in L^p(\mathbb{R}^N)$. In (3.2), by letting $T \to \infty$ and using the fact $\varphi(u_n) \le u_n^\gamma$, we infer that

$$\|u_n\|^{2\gamma}_{L^{2_s^*\gamma}(\mathbb{R}^N)} \le C_\gamma \int_{\mathbb{R}^N} u_n^{2_s^*+2\gamma-2}.$$

Let $\gamma_1 = 2_s^*/2$ and $2_s^* + 2\gamma_{i+1} - 2 = 2_s^*\gamma_i$, $i = 1, 2, \dots$. Then

$$\gamma_{i+1} - 1 = \Big(\frac{2_s^*}{2}\Big)^i (\gamma_1 - 1)$$

and

$$\Big(\int_{\mathbb{R}^N} u_n^{\gamma_{i+1}2_s^*}\Big)^{\frac{1}{2_s^*(\gamma_{i+1}-1)}} \le C_{\gamma_{i+1}}^{\frac{1}{2(\gamma_{i+1}-1)}} \Big(\int_{\mathbb{R}^N} u_n^{\gamma_i 2_s^*}\Big)^{\frac{1}{2_s^*(\gamma_i-1)}},$$

where $C_{\gamma_{i+1}} = CS_s^2\gamma_{i+1}$. Let $K_i = (\int_{\mathbb{R}^N} u_n^{\gamma_i 2_s^*})^{\frac{1}{2_s^*(\gamma_i-1)}}$. Then for $\tau > 0$,

$$K_{\tau+1} \le \prod_{i=2}^{\tau} C_{\gamma_i}^{\frac{1}{2(\gamma_i-1)}} K_1.$$

After a simple calculation, we conclude that there exists a constant $C_0 > 0$, independent of τ and n, such that $K_{\tau+1} \le C_0 K_1$ for any $n \in N$. Hence,

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \le C_0 K_1 < \infty \quad \text{for all } n \in N.$$

The proof is completed. □

4 Proof of Theorem 1.1

In this section, we use the truncation approach to prove Theorem 1.1. First, we construct spike solutions of the truncation problem in some neighborhood of ground sate solutions to the limit problem. Second, by the elliptic estimate, we show that a solution of the truncation problem is indeed a solution of the original problem.

4.1 Truncation Problems

By Proposition 3.1, there exists $\bar{k} > 0$ such that

$$\sup_{u\in S_m} \|u\|_\infty < \bar{k}.$$

For any $k > \max_{t\in[0,\bar{k}]} f(t)$, let $f_k(t) = \min\{f(t), k\}$ $(t \in \mathbb{R})$. In the following, we consider the truncation problem

$$(-\Delta)^s u + V_\varepsilon(x)u = f_k(u), \quad u \in H^s_{V_\varepsilon}(\mathbb{R}^N). \tag{4.1}$$

Obviously, any solution u_ε of (4.1) is indeed a solution of the original problem (2.1) if $\|u_\varepsilon\|_\infty \le \bar{k}$. Now, we consider the corresponding limit equation

$$(-\Delta)^s u + mu = f_k(u), \quad u \in H^s(\mathbb{R}^N), \tag{4.2}$$

whose energy functional is given by

$$I_m^k(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + mu^2 - \int_{\mathbb{R}^N} F_k(u),$$

and the extended problem is

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\frac{1}{N_s}\lim_{t\to 0} t^{1-2s}\partial_t U(x,t) = -mU(x,0) + f_k(U(x,0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.3}$$

where $F_k(t) = \int_0^t f_k(t)\,dt$.

Lemma 4.1. *For any $k > \max_{t\in[0,\bar{k}]}$, (4.3) admits a positive least energy solution.*

Proof. It suffices to prove f_k satisfies the Berestycki–Lions type conditions. It is obvious that $f_k(t) = o(t)$ as $t \to 0$ and $\limsup_{t\to\infty} f_k(t)/t^p < C$ for some $C > 0$ and $p \in (1, 2_s^* - 1)$. Now, we show that there exists $T > 0$ such that $mT^2 < 2F_k(T)$. Indeed, taking any $u \in S_m$, by the Pohozăev identity

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 = N\int_{\mathbb{R}^N} F(u) - \frac{m}{2}u^2, \tag{4.4}$$

there exists $x_0 \in \mathbb{R}^N$ such that $F(u(x_0)) > \frac{m}{2}u^2(x_0)$. Obviously, $F_k(u(x)) \equiv F(u(x))$ for all $x \in \mathbb{R}^N$. Letting $T = u(x_0) > 0$, we get that $F_k(T) > \frac{m}{2}T^2$. ☐

As can be seen in [26], for every least energy solution $U(x,t)$ of (4.3), the trace $u = U(x,0)$ is a positive classical solution of (4.2). Moreover, $U(x,t)$ is a mountain pass solution. Denote by $\tilde{S}_m^k$ the set of least energy solutions U of (4.3) such that $U(x,0)$ attains its maximum at $0 \in \mathbb{R}^N$. Then $\tilde{S}_m^k$ is compact. Denote by E_m^k the energy of $u \in S_m^k$. Noting that $f_k(t) \le f(t)$ for any t, we get that $E_m^k \ge E_m$ because every solution is a mountain pass solution. On the other hand, it follows from $\sup_{u\in S_m}\|u\|_\infty < \bar{k}$ and the definition of f_k that $S_m \subset S_m^k$. So $E_m^k \le E_m$. Thus,

$$E_m^k = E_m \quad \text{for } k > \max_{t\in[0,\bar{k}]} f(t).$$

Lemma 4.2. *For $k > \max_{t\in[0,\bar{k}]} f(t)$, we have $S_m^k = S_m$.*

Proof. Obviously, $S_m \subset S_m^k$. Now, we claim that $S_m^k \subset S_m$. Let

$$T(u) = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2, \quad V(u) = \int_{\mathbb{R}^N} G(u), \quad V_k(u) = \int_{\mathbb{R}^N} G_k(u),$$

where $G(u) = F(u) - \frac{m}{2}u^2$ and $G_k(u) = F_k(u) - \frac{m}{2}u^2$. We consider the constraint minimization problems

$$M := \inf\{T(u) : V(u) \equiv 1, u \in H^s(\mathbb{R}^N)\} \tag{4.5}$$

and

$$M_k := \inf\{T(u) : V_k(u) \equiv 1, u \in H^s(\mathbb{R}^N)\}. \tag{4.6}$$

Let $\tilde{u}_k$ be a minimizer of (4.6). Then, by the Pohozăev's identity (4.4), $u_k = \tilde{u}_k(\frac{x}{\sigma}) \in S_m^k$, where $\sigma = (\frac{N-2s}{2N}M_k)^{\frac{1}{2s}}$. Moreover, u_k is a minimizer of $T(u)$ in

$$\Big\{u \in H^s(\mathbb{R}^N) : V_k(u) = \Big(\frac{N-2s}{2N}M_k\Big)^{\frac{N}{2s}}\Big\}$$

and

$$E_m^k = I_m^k(u_k) = \frac{1}{2}T(u_k) - V_k(u_k) = \frac{s}{N}\Big(\frac{N-2s}{2N}\Big)^{\frac{N-2s}{2s}} M_k^{\frac{N}{2s}}.$$

Similarly, we obtain $E_m = \frac{s}{N}(\frac{N-2s}{2N})^{\frac{N-2s}{2s}} M^{\frac{N}{2s}}$. Since $E_m = E_m^k$, we have $M = M_k$ for $k > \max_{t\in[0,\bar{k}]} f(t)$. Now, we claim that $\tilde{u}_k$ is also a minimizer of (4.5). As a consequence, $u_k = \tilde{u}_k(\frac{x}{\sigma}) \in S_m$, which means $S_m^k \subset S_m$. Since $\tilde{u}_k$ is a minimizer of (4.6), we have

$$T(\tilde{u}_k) = M_k = M \tag{4.7}$$

and

$$T(\tilde{u}_k) = M_k V_k(\tilde{u}_k) = M_k(V_k(\tilde{u}_k))^{\frac{N-2s}{N}} \le M(V(\tilde{u}_k))^{\frac{N-2s}{N}}. \tag{4.8}$$

Now, it suffices to show that $V(\tilde{u}_k) = 1$. Let $\tilde{w}_k = \tilde{u}_k(\lambda\,\cdot)$ such that $V(\tilde{w}_k) = \lambda^{-N}V(\tilde{u}_k) = 1$, where $\lambda = (V(\tilde{u}_k))^{\frac{1}{N}}$. Then we get

$$T(\tilde{w}_k) = \lambda^{-N+2s}T(\tilde{u}_k) = (V(\tilde{u}_k))^{\frac{-N+2s}{N}}T(\tilde{u}_k) \ge M.$$

So we have $T(\tilde{u}_k) \ge M(V(\tilde{u}_k))^{\frac{N-2s}{N}}$, which together with (4.8) yields $T(\tilde{u}_k) = M(V(\tilde{u}_k))^{\frac{N-2s}{N}}$. It follows from (4.7) that $V(\tilde{u}_k) = 1$. The proof is completed. □

Corollary 4.3. *For $k > \max_{t\in[0,\bar{k}]} f(t)$, $\tilde{S}_m^k = \tilde{S}_m$.*

Remark 4.4. By Proposition 2.5, up to translations, $\tilde{S}_m^k$ is compact.

4.2 Completion of Proof of Theorem 1.1

Step 1. We show the existence of solutions to the truncation problem (4.1). By Corollary 4.3, for fixed $k > \max_{t\in[0,\bar{k}]} f(t)$, we have $\tilde{S}_m^k = \tilde{S}_m$. In the following, we construct a solution of (4.1) in some neighborhood of $\tilde{S}_m$. Precisely, define a set of approximation solutions by

$$N_\varepsilon(\rho) = \Big\{\phi_\varepsilon\Big(\cdot - \frac{x_\varepsilon}{\varepsilon}\Big)U\Big(\cdot - \frac{x_\varepsilon}{\varepsilon}, t\Big) + w : x_\varepsilon \in M^\delta,\ U \in \tilde{S}_m,\ w \in H_{V_\varepsilon},\ \delta > 0, \|w\|_{V_\varepsilon} \le \rho\Big\},$$

where $\phi\colon \mathbb{R}^N \to [0,1]$ is defined as a smooth cut-off function satisfying

$$\phi(x) = \begin{cases} 1, & |x| \le \delta, \\ 0, & |x| \ge 2\delta, \end{cases}$$

and $M^\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, A) \le \delta\}$. By Lemma 4.1, f_k satisfies all the assumptions in [26, Theorem 1.1], which implies that (4.1) admits a solution u_ε, where $u_\varepsilon = U_\varepsilon(\cdot, 0)$ and $U_\varepsilon \in N_\varepsilon(\rho)$ is a solution of (2.5) where $f(U(x,0))$ is replaced by $f_k(U(x,0))$. Moreover, there exists a maximum point x_ε of u_ε such that $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon x_\varepsilon, \mathcal{M}) = 0$. As a consequence of [26, Proposition 3.1], $\|u_\varepsilon(\cdot + x_\varepsilon) \to u(\cdot + z_0)\|_{H^s_{V_\varepsilon}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$, where $u \in S_m^k = S_m$ and $z_0 \in \mathbb{R}^N$.

Step 2. We show that u_ε is indeed a solution of the original problem (2.1). By Step 1, $u_\varepsilon(\cdot + x_\varepsilon) \to u(\cdot + z_0)$ strongly in $H^s(\mathbb{R}^N)$ as $\varepsilon \to 0$. Then, in a similar manner as in Proposition 3.1, we know that there exists $\varepsilon_0 > 0$ such that $\sup_{\varepsilon\le\varepsilon_0}\|u_\varepsilon\|_\infty < \infty$. From (2.3) and the fact that $\int_{\mathbb{R}^N} P_s(x,1)\,dx = 1$, it follows that for any $x \in \mathbb{R}^N$, $t \in \mathbb{R}^+$,

$$|U_\varepsilon(x,t)| \le \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} P_s(x,t)\,dx = \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}.$$

So $\sup_{\varepsilon\le\varepsilon_0}\|U_\varepsilon\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^+)} < \infty$. Let $w_\varepsilon(\cdot) = u_\varepsilon(\cdot + x_\varepsilon)$. Then $w_\varepsilon(0) = \|u_\varepsilon\|_\infty$. To conclude the proof, it suffices to show that for $\varepsilon > 0$ small enough, $w_\varepsilon(0) < \bar{k}$. Indeed, $w_\varepsilon = U_\varepsilon(x + x_\varepsilon, 0)$ and $U_\varepsilon(x + x_\varepsilon, t)$ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U_\varepsilon(x + x_\varepsilon, t)) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{1}{N_s}\lim\limits_{t\to 0} t^{1-2s}\partial_t U_\varepsilon(x + x_\varepsilon, t) = -V_\varepsilon(x + x_\varepsilon)U_\varepsilon(x + x_\varepsilon, 0) + f_k(U_\varepsilon(x + x_\varepsilon, 0)) & \text{in } \mathbb{R}^N. \end{cases}$$

Then, by virtue of Proposition 2.6, for some $\alpha \in (0,1)$ and $C > 0$ (independent of ε),

$$\|U(\cdot + x_\varepsilon, \cdot)\|_{C^\alpha(\overline{\Omega_{1/2}})} \le C(\|U_\varepsilon\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^+)} + \|f_k(U_\varepsilon(x + x_\varepsilon, 0))\|_{L^p(B_1^N(0))}),$$

where $\overline{\Omega_{1/2}} = \overline{B^N_{1/2}(0)} \times [0, 1/2]$. It follows from $\sup_{\varepsilon\le\varepsilon_0} \|U_\varepsilon\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^+)} < \infty$ that

$$\sup_{\varepsilon\le\varepsilon_0} \|w_\varepsilon\|_{C^\alpha(B^N_{1/2}(0))} < \infty.$$

This implies that $\{w_\varepsilon\}$ is uniformly bounded and equicontinuous concerning ε in $B^N_1(0)$. By the Arzelà–Ascoli theorem, $w_\varepsilon(\cdot) \to u(\cdot + z_0)$ uniformly in $B^N_1(0)$ and then $\|u_\varepsilon\|_\infty = w_\varepsilon(0) < \bar{k}$ holds uniformly for sufficiently small $\varepsilon > 0$. Therefore, for $\varepsilon > 0$ small, $f_k(u_\varepsilon(x)) \equiv f(u_\varepsilon(x))$ for $x \in \mathbb{R}^N$, which means that $u_\varepsilon(x)$ is a solution of the original problem (2.1). Let $v_\varepsilon(\cdot) = u_\varepsilon(\cdot/\varepsilon)$ and $y_\varepsilon = \varepsilon x_\varepsilon$. Then v_ε is a solution of (1.1), whose maximum point is y_ε, satisfying $\lim_{\varepsilon\to 0} \operatorname{dist}(y_\varepsilon, \mathcal{M}) = 0$. The proof is completed.

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