DE GRUYTER Adv. Nonlinear Stud. 2018; 18(4): 785–797

Research Article

Razvan Gabriel Iagar* and Philippe Laurençot

Extinction for a Singular Diffusion Equation with Strong Gradient Absorption Revisited

https://doi.org/10.1515/ans-2018-0002
Received November 15, 2017; revised February 6, 2018; accepted February 7, 2018

Abstract: When $2N/(N+1) < p < 2$ and $0 < q < p/2$, non-negative solutions to the singular diffusion equation with gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N$$

vanish after a finite time. This phenomenon is usually referred to as finite-time extinction and takes place provided the initial condition u_0 decays sufficiently rapidly as $|x| \to \infty$. On the one hand, the optimal decay of u_0 at infinity guaranteeing the occurrence of finite-time extinction is identified. On the other hand, assuming further that $p - 1 < q < p/2$, optimal extinction rates near the extinction time are derived.

Keywords: Extinction, Optimal Rates, p-Laplacian Equation, Gradient Absorption, Strong Absorption

MSC 2010: 35K67, 35K92, 35B40

Communicated by: Ireneo Peral

1 Introduction

We study some properties related to the phenomenon of finite-time extinction of non-negative solutions to the initial value problem in $\mathbb{R}^N$ for the singular diffusion equation with gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$u(0) = u_0, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

when the exponents p and q satisfy

$$p_c := \frac{2N}{N+1} < p < 2, \quad 0 < q < \frac{p}{2}, \tag{1.3}$$

the p-Laplacian operator being given as usual by

$$\Delta_p u(t,x) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^N.$$

We also assume throughout the paper that the initial condition u_0 enjoys the following properties:

$$u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad u_0(x) \ge 0, \ x \in \mathbb{R}^N, \quad u_0 \not\equiv 0. \tag{1.4}$$

According to the analysis performed in [8, Section 6], the Cauchy problem (1.1)–(1.2), with initial condition satisfying (1.4), has a unique non-negative (viscosity) solution u, the notion of viscosity solutions being the

***Corresponding author: Razvan Gabriel Iagar,** Instituto de Ciencias Matemáticas (ICMAT), Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain; and Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania, e-mail: razvan.iagar@icmat.es
Philippe Laurençot, Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université de Toulouse, 31062 Toulouse Cedex 9, France, e-mail: philippe.laurencot@math.univ-toulouse.fr

one developed in [14] to handle the singularity of the diffusion, see [8, Definition 6.1]. It is also a weak solution by [8, Theorem 6.2]. Moreover, in the range of exponents (1.3), the phenomenon of extinction of the solution u in finite time occurs according to [8, Theorem 1.2 (iii)] provided that the initial condition u_0 decays sufficiently fast as $|x| \to \infty$. More precisely, it is shown that, if

$$u_0(x) \le C_0|x|^{-(p-Q)/(Q-p+1)}, \quad x \in \mathbb{R}^N, \tag{1.5}$$

for some $C_0 > 0$ and suitable $Q > 0$ (which is equal to q if $q > q_1 := \max\{p-1, N/(N+1)\}$ and is arbitrary in $(q_1, p/2)$ otherwise), then

$$T_e := \sup\{t \ge 0 : u(t) \not\equiv 0\} \tag{1.6}$$

is finite and positive. More recent works such as [10, 11] go further in characterizing how the finite-time extinction takes place, showing that (under suitable conditions on u_0) an even more striking phenomenon, the *instantaneous shrinking of the support*, takes place for $q \in (0, p-1)$. More precisely, the positivity set $\mathcal{P}(t)$ of u at time t defined by

$$\mathcal{P}(t) = \{x \in \mathbb{R}^N : u(t,x) > 0\} \tag{1.7}$$

is compact (and localized uniformly in t) for any $t \in (0, T_e)$, even if $u_0(x) > 0$ for any $x \in \mathbb{R}^N$. In [10], equation (1.1) with critical exponent $q = p-1$ is studied thoroughly and both optimal extinction rates and precise extinction profiles (in separate variable form) are given, provided that the initial condition u_0 is radially symmetric, radially non-increasing in $|x|$, and has an exponential spatial tail as $|x| \to \infty$. As a by-product, it is also shown that *simultaneous extinction* occurs both for $q = p-1$ and for $q \in (p-1, p/2)$, that is,

$$u(t,x) > 0 \quad \text{for any } (t,x) \in (0, T_e) \times \mathbb{R}^N.$$

However, it was noticed already in [11, Theorems 1.1 and 1.2] for the range $0 < q < p-1$ that the previous tail (1.5) is not optimal for finite-time extinction to take place. Our first main result is devoted to the identification of the optimal decay of u_0 as $|x| \to \infty$ guaranteeing the occurrence of this phenomenon.

Theorem 1.1 (Optimal Tail for Extinction). *Let u be a solution to the Cauchy problem* (1.1)–(1.2) *with exponents satisfying* (1.3) *and an initial condition u_0 satisfying* (1.4).

(a) *Assume further that*

$$u_0(x) \le C_0(1+|x|)^{-q/(1-q)}, \quad x \in \mathbb{R}^N, \tag{1.8}$$

for some $C_0 > 0$. Then the extinction time T_e of u defined in (1.6) *is positive and finite.*

(b) *If*

$$\lim_{|x|\to\infty} |x|^{q/(1-q)} u_0(x) = \infty, \quad u_0(x) > 0 \text{ for any } x \in \mathbb{R}^N, \tag{1.9}$$

then $T_e = \infty$ and $\mathcal{P}(t) = \mathbb{R}^N$ for any $t > 0$.

An obvious consequence of Theorem 1.1 is the optimality of the tail behavior (1.8) for finite-time extinction to occur. Furthermore, it strictly improves [8, Theorem 1.2 (iii)]. Indeed, since $p > q + Q$, the exponent Q being introduced in (1.5), it follows that $(p-Q)(1-q) > q(Q-p+1)$ or equivalently

$$\frac{p-Q}{Q-p+1} > \frac{q}{1-q}.$$

Consequently, the decay assumed in (1.5) is strictly faster than the optimal one (1.8). Let us also remark that we state Theorem 1.1 here for exponents p and q satisfying (1.3), but for the range of exponents $0 < q < p-1$, it is already proved in [11, Theorems 1.2 and 1.3].

Once it is known that finite-time extinction takes place, a further important step in understanding the extinction mechanism is to identify the behavior of the solution u to the Cauchy problem (1.1)–(1.2) as $t \to T_e$, where T_e is the extinction time defined in (1.6). To this end, a first point is to determine the extinction rate, that is, the precise (optimal) space and time scales in which $u(t)$ vanishes as $t \to T_e$. This is the second main result of the present note. Before stating it, let us introduce the exponents

$$\alpha := \frac{p-q}{p-2q}, \quad \beta := \frac{q-p+1}{p-2q},$$

which will be used throughout the paper.

Theorem 1.2 (Optimal Extinction Rate). *Assume that $p \in (p_c, 2)$ and $p - 1 < q < p/2$. Let u be the solution to the Cauchy problem* (1.1)–(1.2) *with an initial condition u_0 satisfying* (1.4) *as well as the decay property*

$$0 \le u_0(x) \le K_0 |x|^{-(p-q)/(q-p+1)}, \quad x \in \mathbb{R}^N, \tag{1.10}$$

for some $K_0 > 0$. Then there exist two positive constants c_∞ and C_∞ (depending on N, p, q, and the initial condition), such that

$$c_\infty (T_e - t)^\alpha \le \|u(t)\|_\infty \le C_\infty (T_e - t)^\alpha, \quad t \in (T_e/2, T_e). \tag{1.11}$$

Furthermore, there are two positive constants c_1 and C_1 (depending on N, p, q, and the initial condition), such that

$$c_1 (T_e - t)^{\alpha - N\beta} \le \|u(t)\|_1 \le C_1 (T_e - t)^{\alpha - N\beta}, \quad t \in (T_e/2, T_e). \tag{1.12}$$

The proof of these optimal bounds near extinction is very clear-cut, elementary and based on a rather simple *energy technique*, and its application is thus likely to extend beyond (1.1). For instance, we refer the interested reader to the companion paper [9] where a related approach allows us to derive optimal extinction rates for a fast diffusion equation with zero-order strong absorption.

Let us point out here that the range of application of Theorem 1.2 is narrower than that of Theorem 1.1, as we have to impose two further restrictions. The first one is related to the decay at infinity of the initial condition u_0, which is required to be much faster than the optimal one (1.8) identified in Theorem 1.1. As a consequence, we do not know whether, for initial conditions satisfying (1.8) but not (1.10), the outcome of Theorem 1.2 remains valid. The second restriction is related to the range of the exponent q which is restricted to the smaller interval $(p-1, p/2)$ in Theorem 1.2. This assumption is seemingly only technical and some arguments in that direction are the following: on the one hand, for the critical case $q = p - 1$, the extinction rate (1.11) is already proved in [10] for radially symmetric initial data, though by a completely different technique. In addition, an optimal upper bound near the extinction is derived for the L^2-norm of u. On the other hand, when $q \in (0, p-1)$, the behavior near the extinction time is studied in [11, Proposition 5.1]. Although we show the validity of the lower bound in (1.11) in that case as well, we are unfortunately only able to obtain upper bounds of the form $C(\varepsilon)(T_e - t)^{\alpha - \varepsilon}$ without a suitable control on the behavior of $C(\varepsilon)$ as $\varepsilon \to 0$. A proof of the upper bound in (1.11) when $q \in (0, p-1)$ might however require a different approach. Indeed, in that case, as we previously mentioned, *instantaneous shrinking* takes place, that is, the support of $u(t)$ is compact for all $t \in (0, T_e)$, and identifying the optimal rate of shrinking of the support might be a helpful piece of information.

A further comment is that the analysis carried out herein leaves aside the case $(p, q) \in (1, p_c) \times (0, 1)$, as it is likely to require a different approach to be studied precisely. Indeed, on the one hand, extinction is no longer driven only by the absorption term for that range of parameters as it also takes place for non-negative solutions to the singular diffusion equation

$$\partial_t u - \Delta_p u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \tag{1.13}$$

when $p \in (1, p_c)$ (see [15, Chapter 11]). On the other hand, as a byproduct of the results on the fast diffusion equation $\partial_t u - \Delta u^m = 0$ in $(0, \infty) \times \mathbb{R}^N$, $m \in (0, 1)$, in [15, Chapter 7] and the one-to-one correspondence between radially symmetric solutions to the fast diffusion equation and to (1.13) given in [12], equation (1.13) features a continuum of extinction rates when $p \in (1, p_c)$. Though the additional absorption term reinforces the extinction phenomenon, it is yet unclear whether it brings some rigidity in the dynamics by selecting a specific extinction rate.

We finally mention that optimal extinction rates have also been studied for the related fast diffusion equation with zero-order strong absorption

$$\partial_t u - \Delta u^m + u^q = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,$$

for exponents $m \in ((N-2)_+/N, 1]$ and $q \in (0, 1)$ but, unlike the present contribution, many works focus on the one-dimensional case $N = 1$ (see [2, 3, 5–7]). Extinction rates in the general N-dimensional case are only studied in [4] for $m = 1$ and a restricted class of initial conditions, and in the companion paper [9] for $m \in ((N-2)_+/N, 1)$ and $q \in (m, 1)$.

2 Preliminary Results

In this section, we collect some definitions and estimates that will be used in the proofs of Theorems 1.1 and 1.2 and are already available in the literature. In particular, we put the emphasis on the notions of solution, subsolution, and supersolution to (1.1), as we have to work in the viscosity framework and this requires some special attention when dealing with singular equations.

2.1 Notions of Subsolution and Supersolution

We recall here for the sake of completeness (according to [8, Definition 6.1]) the notions of subsolution and supersolution that we use in the sequel. They are to be understood in the *viscosity sense* and follow the general (abstract) approach developed in [13, 14], where the class of admissible functions for comparison is reduced in order to cope well with the singular diffusion featured in (1.1). In order to introduce the class of admissible functions for comparison, let $\mathcal{F}_p$ be the set of functions $\xi \in C^2([0,\infty))$ such that

$$\xi(0) = \xi'(0) = \xi''(0) = 0, \quad \xi''(r) > 0 \text{ for all } r > 0, \quad \lim_{r\to 0} |\xi'(r)|^{p-2}\xi''(r) = 0. \tag{2.1}$$

Notice that l'Hospital's rule and (2.1) entail that

$$\lim_{r\to 0} \frac{|\xi'(r)|^{p-1}}{r} = 0. \tag{2.2}$$

As a simple example of a function in the class $\mathcal{F}_p$, any power $\xi(r) = r^\sigma$ can be taken, provided $\sigma > p/(p-1)$. We next define the class $\mathcal{A}$ of admissible comparison functions. A function $\psi \in C^2((0,\infty)\times\mathbb{R}^N)$ belongs to $\mathcal{A}$ if, for any point $(t_0, x_0) \in (0,\infty)\times\mathbb{R}^N$ such that $\nabla\psi(t_0,x_0) = 0$, there exist $\delta > 0$, a function $\xi \in \mathcal{F}_p$ and a modulus of continuity $\omega \in C([0,\infty))$ with $\omega(t)/t \to 0$ as $t \to 0$ enjoying the following property: for any $(t,x) \in (t_0-\delta, t_0+\delta)\times B_\delta(x_0)$, there holds

$$|\psi(t,x) - \psi(t_0,x_0) - \partial_t\psi(t_0,x_0)(t-t_0)| \le \xi(|x-x_0|) + \omega(|t-t_0|).$$

With this construction, we now define viscosity subsolutions and supersolutions.

Definition 2.1. Let $T > 0$.

(a) An upper semicontinuous function $u : (0,T)\times\mathbb{R}^N \mapsto \mathbb{R}$ is a *viscosity subsolution* to (1.1) if, for any $\psi \in \mathcal{A}$ and $(t_0,x_0) \in (0,T)\times\mathbb{R}^N$ such that $u - \psi$ has a local maximum at (t_0,x_0), there holds

$$\partial_t\psi(t_0,x_0) \le \begin{cases} \Delta_p\psi(t_0,x_0) - |\nabla\psi(t_0,x_0)|^q & \text{if } \nabla\psi(t_0,x_0) \ne 0, \\ 0 & \text{if } \nabla\psi(t_0,x_0) = 0. \end{cases}$$

(b) A lower semicontinuous function $u : (0,T)\times\mathbb{R}^N \mapsto \mathbb{R}$ is a *viscosity supersolution* to (1.1) if $-u$ is a viscosity subsolution to (1.1).

(c) A continuous function $u : (0,T)\times\mathbb{R}^N \mapsto \mathbb{R}$ is a *viscosity solution* to (1.1) in $(0,T)\times\mathbb{R}^N$ when it is at the same time a viscosity subsolution and a viscosity supersolution.

An immediate consequence of Definition 2.1 is that special attention shall be paid to critical points (with respect to the space variable) of subsolutions and supersolutions, this fact being obviously related to the singular behavior of the p-Laplacian operator at critical points of u when $p \in (1,2)$. The main abstract results concerning viscosity subsolutions and supersolutions are contained in [14]. More precisely, the comparison principle is stated in [14, Theorem 3.9] and the stability property with respect to uniform limits is [14, Theorem 6.1], both of them being valid in a more general setting encompassing equation (1.1). As we shall see below in Lemma 3.2, this specific notion of viscosity subsolution and supersolution requires some care to be properly handled.

2.2 Useful Estimates for the Solutions to (1.1)–(1.2)

We gather in this short section several results established in [8] which are needed later on. First of all, the well-posedness of (1.1)–(1.2) in the framework of viscosity solutions (in the sense of Definition 2.1) and the finite-time extinction for exponents (p,q) satisfying (1.3) are proved in [8, Theorems 1.1 and 1.2 (iii)].

Proposition 2.2. *Assume that the initial condition u_0 and (p,q) satisfy* (1.4) *and* (1.3), *respectively. Then there exists a unique non-negative viscosity solution to the Cauchy problem* (1.1)–(1.2). *Moreover, if u_0 enjoys the decay condition* (1.5), *then there exists $T_e \in (0,\infty)$ depending only on N, p, q, and u_0 such that $u(t,x) \equiv 0$ for $(t,x) \in [T_e,\infty) \times \mathbb{R}^N$.*

A fundamental technical tool in proving precise results about solutions to (1.1), in particular our optimal bounds in Theorem 1.2, are the optimal gradient estimates derived in [8]. We state here the ones that apply for exponents as in (1.3), which are originally proved in [8, Theorem 1.3 (iii)] and [8, Theorem 1.7].

Proposition 2.3. *Let (p,q) be as in* (1.3) *and let u be a (viscosity) solution to* (1.1)–(1.2) *with initial condition u_0 satisfying* (1.4). *Then:*

(a) *There is a positive constant $C_2 > 0$ depending only on N, p, q, and u_0 such that, for $t > 0$, $s \in [0,t)$, and $x \in \mathbb{R}^N$,*
$$|\nabla u(t,x)| \le C_2 \|u(s)\|_\infty^{1/q} (t-s)^{-1/q}. \tag{2.3}$$

(b) *Assume further that $p-1 < q < p/2$. Then there is a positive constant $C > 0$ depending only on N, p, q, and u_0 such that, for $t \in (0,T_e)$ and $x \in \mathcal{P}(t)$,*
$$|\nabla u^{-(q-p+1)/(p-q)}(t,x)| \le C\big[1 + \|u_0\|_\infty^{(p-2q)/p(p-q)} t^{-1/p}\big], \tag{2.4}$$
recalling that the positivity set $\mathcal{P}(t)$ of u at time t is defined in (1.7).

We finally recall the following Gagliardo–Nirenberg inequality.

Proposition 2.4. *There is a positive constant $C > 0$ depending only on N such that, for $w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$,*
$$\|w\|_\infty \le C \|\nabla w\|_\infty^{N/(N+1)} \|w\|_1^{1/(N+1)}. \tag{2.5}$$

After this preparation, we are in a position to carry out the proofs of Theorems 1.1 and 1.2.

3 Optimal Tail for Extinction

In this section, we prove Theorem 1.1. The technique of the proof is based on constructing suitable supersolutions with finite-time extinction, on the one hand, and subsolutions which are positive everywhere, on the other hand. We thus need two preparatory, technical lemmas. As already explained in the Introduction, Theorem 1.1 is already proved in [11] in the range $0 < q < p-1$, so that the novelty of this section is the fact that we handle the case $q \in [p-1, p/2)$.

3.1 Construction of a Viscosity Supersolution

We devote this subsection to the construction of a viscosity supersolution to (1.1), in the sense of Definition 2.1. It requires a different analysis at points where the spatial gradient of the supersolution vanishes. As in [1] for $p=2$ and $q \in (0,1)$ and in [11] for $p \in (p_c, 2)$ and $q \in (0, p-1]$, we look for a supersolution in self-similar form.

Lemma 3.1. *Assume p and q satisfy* (1.3). *There are* $\bar{a} > 0$ *and* $\bar{b} > 0$ *such that, for any* $(a, b) \in (\bar{a}, \infty) \times (\bar{b}, \infty)$, *the function W defined by*

$$W(t, x) = (T - t)^{\alpha} f(|x|(T - t)^{\beta}), \quad (t, x) \in (0, T) \times \mathbb{R}^N, \tag{3.1}$$

$$f(y) = (a + by^{\theta})^{-\gamma}, \quad y \in [0, \infty), \tag{3.2}$$

with exponents

$$\alpha = \frac{p - q}{p - 2q}, \quad \beta = \frac{q - p + 1}{p - 2q}, \quad \theta = \frac{p}{p - 1}, \quad \gamma = \frac{(p - 1)q}{p(1 - q)}$$

is a (classical) supersolution to (1.1) *in* $(0, \infty) \times (\mathbb{R}^N \setminus \{0\})$.

Proof. Let $(t, x) \in (0, \infty) \times (\mathbb{R}^N \setminus \{0\})$. We set $y = |x|(T - t)^{\beta}$ and note that

$$f'(y) = -\gamma b\theta (a + by^{\theta})^{-\gamma-1} y^{\theta-1},$$

$$f''(y) = -\gamma b\theta (a + by^{\theta})^{-\gamma-1} y^{\theta-2}\left[\theta - 1 - \theta(\gamma + 1)\frac{by^{\theta}}{a + by^{\theta}}\right].$$

After direct and straightforward (but rather long) calculations we obtain

$$\begin{aligned}\mathcal{L}W(t, x) &:= \partial_t W(t, x) - \Delta_p W(t, x) + |\nabla W(t, x)|^q \\ &= (T - t)^{\alpha-1}\left[-\alpha f(y) - \beta y f'(y) - (p - 1)(|f'|^{p-2} f'')(y) - \frac{N - 1}{y}(|f'|^{p-2} f')(y) + |f'(y)|^q\right] \\ &= (T - t)^{\alpha-1}(a + by^{\theta})^{-\gamma-1}(H_1(y) + H_2(y)),\end{aligned}$$

where

$$\begin{aligned}H_1(y) &= -\alpha a + (\gamma b\theta)^{p-1}\left[N - 1 + (p - 1)(\theta - 1) - (p - 1)\theta(\gamma + 1)\frac{by^{\theta}}{a + by^{\theta}}\right] y^{(\theta-1)(p-1)-1}(a + by^{\theta})^{(\gamma+1)(2-p)} \\ &= -\alpha a + (\gamma b\theta)^{p-1}\left[N - p(\gamma + 1)\frac{by^{\theta}}{a + by^{\theta}}\right](a + by^{\theta})^{(\gamma+1)(2-p)},\end{aligned}$$

since $(\theta - 1)(p - 1) = 1$ and $(p - 1)\theta = p$, and

$$H_2(y) = (\gamma b\theta)^q y^{q(\theta-1)}(a + by^{\theta})^{(1-q)(\gamma+1)} + (\beta\gamma\theta - \alpha)by^{\theta}.$$

Since

$$\gamma\beta\theta - \alpha = -\frac{1}{1 - q} < 0$$

and

$$q(\theta - 1) + \theta(1 - q)(\gamma + 1) = \theta,$$

we obtain that

$$\begin{aligned}H_2(y) &\geq (\gamma b\theta)^q y^{q(\theta-1)}(by^{\theta})^{(1-q)(\gamma+1)} - \frac{b}{1 - q} y^{\theta} \\ &= \frac{b}{1 - q} y^{\theta}[(1 - q)(\gamma\theta)^q b^{(1-q)\gamma} - 1] \\ &\geq \frac{(\gamma\theta)^q}{2} b^{1+(1-q)\gamma} y^{\theta} \geq 0,\end{aligned} \tag{3.3}$$

provided

$$b^{(1-q)\gamma} \geq \frac{2}{(1 - q)(\gamma\theta)^q}. \tag{3.4}$$

In order to estimate the term $H_1(y)$ we split the range $(0, \infty)$ of y into two regions, one close to the origin and another far from the origin. Let thus $y_0 > 0$ to be determined later and consider first $y \in (0, y_0]$. Then,

$$H_1(y) \geq (\gamma b\theta)^{p-1}\left[N - (\gamma + 1)p\frac{by_0^{\theta}}{a}\right](a + by^{\theta})^{(2-p)(\gamma+1)} - a\alpha. \tag{3.5}$$

If we require $a > 0$, $b > 0$, and $y_0 > 0$ to satisfy

$$\frac{Na}{2p(\gamma+1)} \geq b y_0^\theta, \tag{3.6}$$

then we infer from (3.5) that

$$\begin{aligned} H_1(y) &\geq (\gamma b\theta)^{p-1}\frac{N}{2}(a+by^\theta)^{(2-p)(\gamma+1)} - a\alpha \\ &\geq \frac{N(\gamma\theta)^{p-1}}{2} b^{p-1} a^{(2-p)(\gamma+1)} - a\alpha \\ &\geq \alpha a^{(2-p)(\gamma+1)}\Big[\frac{N(\gamma\theta)^{p-1}}{2\alpha} b^{p-1} - a^{1-(2-p)(\gamma+1)}\Big] \geq 0, \end{aligned} \tag{3.7}$$

provided that

$$\frac{N(\gamma\theta)^{p-1}}{2\alpha} b^{p-1} \geq a^{1-(2-p)(\gamma+1)}. \tag{3.8}$$

Let us turn our attention to the complementary region $y > y_0$. We use the obvious bound $by^\theta/(a+by^\theta) < 1$ to find

$$H_1(y) \geq -(\gamma b\theta)^{p-1} p(\gamma+1)(a+by^\theta)^{(2-p)(\gamma+1)} - a\alpha,$$

hence, putting $L := p(\gamma+1)(\gamma\theta)^{p-1}$, we further deduce from (3.3) that

$$\begin{aligned} (H_1+H_2)(y) &\geq \frac{(\gamma\theta)^q}{2} b^{1+(1-q)\gamma} y^\theta - a\alpha - L b^{p-1}(a+by^\theta)^{(2-p)(\gamma+1)} \\ &\geq \frac{(\gamma\theta)^q}{4} b^{(1-q)\gamma} b y_0^\theta - a\alpha + \frac{(\gamma\theta)^q}{4} b^{1+(1-q)\gamma} y^\theta - L b^{p-1+(2-p)(\gamma+1)} y^{\theta-(p-2q)/(1-q)}\Big[1+\frac{a}{by_0^\theta}\Big]^{(2-p)(\gamma+1)} \\ &\geq b^{1+(1-q)\gamma} y^{\theta-(p-2q)/(1-q)}\Big[\frac{(\gamma\theta)^q}{4} y^{(p-2q)/(1-q)} - L\Big(1+\frac{a}{by_0^\theta}\Big)^{(2-p)(\gamma+1)} b^{(q-p+1)\gamma}\Big], \end{aligned}$$

provided that

$$\frac{(\gamma\theta)^q}{4\alpha} b^{(1-q)\gamma+1} y_0^\theta \geq a. \tag{3.9}$$

We now choose

$$a = \lambda b y_0^\theta, \tag{3.10}$$

with $\lambda > 0$ to be specified later. Then

$$(H_1+H_2)(y) \geq b^{1+(1-q)\gamma} y^{\theta-(p-2q)/(1-q)}\Big[\frac{(\gamma\theta)^q}{4} y_0^{(p-2q)/(1-q)} - L(1+\lambda)^{(2-p)(\gamma+1)} b^{(q-p+1)\gamma}\Big] \geq 0, \tag{3.11}$$

provided

$$y_0^{(p-2q)/(1-q)} \geq \frac{4L(1+\lambda)^{(2-p)(\gamma+1)}}{(\gamma\theta)^q} b^{(q-p+1)\gamma}. \tag{3.12}$$

Summarizing, according to (3.3), (3.7), and (3.11), we have established that $(H_1+H_2)(y) \geq 0$ for all $y \in (0,\infty)$ as soon as the parameters a, b, y_0, and λ satisfy (3.4), (3.6), (3.8), (3.9), and (3.10). We end the proof by choosing the parameters b, λ, and y_0 in order to ensure the compatibility of all the conditions we had to impose along the way in the estimates. First of all, we set

$$\lambda = \frac{2p(\gamma+1)}{N},$$

which implies the validity of (3.6). Moreover, from (3.4) and (3.9) we have to choose $b > 0$ such that

$$b^{(1-q)\gamma} \geq \max\Big\{\frac{2}{(1-q)(\gamma\theta)^q}, \frac{4\lambda\alpha}{(\gamma\theta)^q}\Big\}. \tag{3.13}$$

Finally, inserting (3.10) into (3.8), we readily deduce that

$$\frac{N(\gamma\theta)^{p-1}\lambda^{\gamma(2-p)-p+1}}{2\alpha} b^{\gamma(2-p)} \geq y_0^{(p-2q)/(1-q)}. \tag{3.14}$$

Let us notice that, since $2-p > q-p+1$, the conditions (3.12), (3.13), and (3.14) can be met simultaneously by choosing $b > 0$ sufficiently large, which ends the proof. □

Now, let $T > 0$, $a > \bar{a}$, and $b > \bar{b}$, and consider the function W defined by (3.1)–(3.2). With the aim of showing that W is a viscosity supersolution to (1.1) in the sense of Definition 2.1, let $\psi \in \mathcal{A}$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$ be such that $W - \psi$ has a local minimum at (t_0, x_0). Since both W and ψ belong to $C^1([0, T] \times \mathbb{R}^N)$, this property implies that

$$\partial_t W(t_0, x_0) = \partial_t \psi(t_0, x_0) \quad \text{and} \quad \nabla W(t_0, x_0) = \nabla \psi(t_0, x_0). \tag{3.15}$$

Since $\nabla W(t_0, x_0) \neq 0$ when $x_0 \neq 0$, Lemma 3.1 and (3.15) guarantee that the condition to be a viscosity supersolution is fulfilled if $x_0 \neq 0$. No information is provided by Lemma 3.1 if $x_0 = 0$. In that case, we might actually face a problem. Indeed, for W to meet the requirement of viscosity solutions when $W - \psi$ has a local minimum at $(t_0, 0)$ for some $t_0 \in (0, T)$, the inequality $\partial_t \psi(t_0, 0) \geq 0$ has to be satisfied according to Definition 2.1. However, recalling (3.15), we realize that

$$\partial_t \psi(t_0, 0) = \partial_t W(t_0, 0) = -\alpha (T - t_0)^{\alpha - 1} a^{-\gamma} < 0,$$

and an apparent contradiction. This is in fact an artificial problem: there *does not exist* any admissible function ψ such that $W - \psi$ attains a local minimum at a point $(t_0, 0)$ as the following lemma shows.

Lemma 3.2. *Let $T > 0$, $a > 0$, and $b > 0$ and consider the function W defined by* (3.1)–(3.2)*, the exponents p and q still satisfying* (1.3)*. Let $\psi \in \mathcal{A}$ and assume that $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$ is a local minimum for $W - \psi$. Then $x_0 \neq 0$.*

Proof. To obtain a contradiction, assume that $x_0 = 0$. On the one hand, since $W \in C^1([0, T] \times \mathbb{R}^N)$, we have $\nabla \psi(t_0, 0) = \nabla W(t_0, 0) = 0$. On the other hand, $\psi \in \mathcal{A}$ and there exist a function $\xi \in \mathcal{F}_p$, a modulus of continuity $\omega \in C([0, \infty))$, $\omega \geq 0$, and a sufficiently small $\delta > 0$ such that, for $(t, x) \in (t_0 - \delta, t_0 + \delta) \times B_\delta(0)$,

$$|\psi(t, x) - \psi(t_0, 0) - \partial_t \psi(t_0, 0)(t - t_0)| \leq \xi(|x|) + \omega(|t - t_0|). \tag{3.16}$$

In particular for $t = t_0$, inequality (3.16) becomes

$$|\psi(t_0, x) - \psi(t_0, 0)| \leq \xi(|x|) \quad \text{for } x \in B_\delta(0).$$

Furthermore, since $(t_0, 0)$ is a local minimum of $W - \psi$, we realize that

$$W(t_0, 0) - W(t_0, x) \leq \psi(t_0, 0) - \psi(t_0, x) \leq \xi(|x|) \quad \text{for } x \in B_\delta(0). \tag{3.17}$$

Taking into account equations (3.1)–(3.2) defining W, we infer from (3.17) that

$$(T - t_0)^\alpha \big[a^{-\gamma} - \big(a + b|x|^\theta (T - t_0)^{\theta\beta}\big)^{-\gamma}\big] \leq \xi(|x|) \quad \text{for } x \in B_\delta(0),$$

hence, as $|x| \to 0$,

$$\frac{(T - t_0)^\alpha}{(a + b|x|^\theta (T - t_0)^{\theta\beta})^\gamma} \Big[\frac{b\gamma}{a} (T - t_0)^{\theta\beta} |x|^\theta + o(|x|^\theta)\Big] \leq \xi(|x|).$$

Consequently, recalling that $\theta = p/(p-1)$, we have

$$0 < \frac{b\gamma (T - t_0)^{\alpha + \theta\beta}}{a^{\gamma + 1}} \leq \liminf_{r \to 0} \frac{\xi(r)}{r^{p/(p-1)}}. \tag{3.18}$$

Next, since $\xi \in \mathcal{F}_p$, we infer from (2.2) that

$$\lim_{r \to 0} \frac{\xi'(r)}{r^{1/(p-1)}} = 0,$$

and a further application of l'Hospital's rule gives

$$\lim_{r \to 0} \frac{\xi(r)}{r^{p/(p-1)}} = 0,$$

thereby contradicting (3.18). Therefore, we cannot have $x_0 = 0$, ending the proof. □

Combining Lemma 3.1 and Lemma 3.2, we infer from the discussion preceding the statement of Lemma 3.2 that, for $T > 0$, $a > \bar{a}$, and $b > \bar{b}$, the function W defined in (3.1)–(3.2) is a viscosity supersolution to (1.1) in the whole $(0, T) \times \mathbb{R}^N$ in the sense of Definition 2.1. Summarizing, we have established the following result.

Corollary 3.3. *Assume that p and q satisfy* (1.3). *For $T > 0$, $a > \bar{a}$, and $b > \bar{b}$, the function W defined in* (3.1)–(3.2) *is a viscosity supersolution to* (1.1) *in* $(0, T) \times \mathbb{R}^N$.

A similar construction (already performed in [11]) gives us a subsolution to (1.1) that will be used for comparison from below in order to show positivity and non-extinction when u_0 satisfies (1.9). We recall it here for the sake of completeness.

Lemma 3.4. *Assume that p and q satisfy* (1.3). *There exists $b_0 > 0$ depending only on p and q such that, given $T > 0$ and $b \in (0, b_0)$, there is $A(b, T) > 0$ depending only on N, p, q, b, and T such that the function*

$$w(t, x) := (T - t)^{1/(1-q)}(a + b|x|^\theta)^{-\gamma}, \quad \theta = \frac{p}{p-1}, \; \gamma = \frac{q(p-1)}{(1-q)p},$$

is a subsolution to (1.1) *in* $(0, T) \times \mathbb{R}^N$ *provided* $a > A(b, T)$.

Proof. The proof is totally identical to that of [11, Lemma 6.1]. In fact, in the quoted reference, it is assumed that $0 < q < p - 1$, but a simple inspection of the proof shows that it works identically for any $q \in (0, p/2)$. □

3.2 Proof of Theorem 1.1

With these constructions, we are now in a position to prove the optimality of the spatial decay (1.8) for finite-time extinction to take place.

Proof of Theorem 1.1. The proof is divided into two parts.

Extinction with Optimal Tail. Let u be a solution to the Cauchy problem (1.1)–(1.2) with an initial condition u_0 satisfying (1.8) and consider $a > \bar{a}$ and $b > \bar{b}$. For $T > 0$, it follows from Corollary 3.3 that

$$W(t, x) = (T - t)^\alpha (a + b(T - t)^{\beta\theta}|x|^\theta)^{-\gamma}, \quad (t, x) \in (0, T) \times \mathbb{R}^N,$$

is a supersolution to (1.1) in $(0, T) \times \mathbb{R}^N$, the parameters θ and γ being given as usual by

$$\theta = \frac{p}{p-1} > 1, \quad \gamma = \frac{q(p-1)}{(1-q)p}.$$

For $x \in \mathbb{R}^N$,

$$\begin{aligned} W(0, x) &= T^\alpha (a + bT^{\beta\theta}|x|^\theta)^{-\gamma} \\ &= T^{\alpha-\beta\theta\gamma} b^{-\gamma} \Big[\frac{a}{bT^{\beta\theta}} + |x|^\theta\Big]^{-\gamma} \\ &= T^{1/(1-q)} b^{-\gamma} \Big[\frac{a}{bT^{\beta\theta}} + |x|^\theta\Big]^{-\gamma}, \end{aligned}$$

since $\alpha - \beta\theta\gamma = 1/(1 - q)$. Choose in a first step $T > 0$ sufficiently large such that

$$\frac{a}{bT^{\beta\theta}} < 1.$$

Then, taking into account that $\theta > 1$ and the elementary inequality $1 + |x|^\theta \le (1 + |x|)^\theta$ for any $x \in \mathbb{R}^N$, we further infer from (1.8) that

$$\begin{aligned} W(0, x) &\ge T^{1/(1-q)} b^{-\gamma} (1 + |x|^\theta)^{-\gamma} \\ &\ge T^{1/(1-q)} b^{-\gamma} (1 + |x|)^{-q/(1-q)} \\ &\ge \frac{T^{1/(1-q)} b^{-\gamma}}{C_0} C_0 (1 + |x|)^{-q/(1-q)} \\ &\ge \frac{T^{1/(1-q)} b^{-\gamma}}{C_0} u_0(x) \ge u_0(x), \end{aligned}$$

provided we take $T > 0$ sufficiently large such that

$$T^{1/(1-q)} b^{-\gamma} \geq C_0.$$

Thus, for T sufficiently large, we deduce from the comparison principle that

$$u(t,x) \leq W(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^N.$$

It is immediate to conclude that this implies extinction in finite time for u, with an extinction time $T_e \leq T$.

Non-Extinction with Slower Tail. Let us now consider a solution u to the Cauchy problem (1.1)–(1.2) with an initial condition u_0 satisfying (1.9). Then the non-extinction in finite time and the positivity for any $t > 0$ (that is, $\mathcal{P}(t) = \mathbb{R}^N$ for any $t > 0$) follow from the same proof as in [11, Section 6], which applies identically also for the range $q \in [p-1, p/2)$. Thus, optimality of the tail in (1.8) is proved. □

4 Optimal Extinction Rates

This section is devoted to the proof of Theorem 1.2. We thus assume from now on that the exponents p and q satisfy (1.3) as well as $q > p - 1$. Assume also that u_0 satisfies (1.4) and (1.10) for some constant $K_0 > 0$. Throughout this section, C and C_i, $i \geq 1$, denote positive constants depending only on N, p, q, and u_0. Dependence upon additional parameters will be indicated explicitly.

We begin with the proof of the lower bound, which relies on the derivation of a functional inequality for the L^∞-norm of u. Exploiting this functional inequality requires the following preliminary result.

Lemma 4.1. *Let $T > 0$ and a function $h : [0,T] \to [0,\infty)$ such that*

$$\mu(t) := \inf_{s\in[0,t]} \{h(s)\} > 0, \quad t \in (0,T), \quad h(T) = 0, \tag{4.1}$$

and

$$\delta(t-s)h(t)^m \leq h(s), \quad 0 < s < t < T, \tag{4.2}$$

for some $m \in (0,1)$ and $\delta > 0$. Then

$$h(t) \geq \left(\frac{\delta^{1-m}}{2}\right)^{1/(1-m)^2} (T-t)^{1/(1-m)}, \quad t \in [0,T].$$

Proof. Fix $t \in (0,T)$ and $\tau \in (t,T)$. Introducing the sequence $(t_i)_{i\geq 0}$ defined by

$$t_i := \frac{t}{2^i} + \left(1 - \frac{1}{2^i}\right)\tau, \quad i \geq 0,$$

we observe that

$$t = t_0 < t_i < t_{i+1} < \tau, \quad i \geq 1, \quad \lim_{i\to\infty} t_i = \tau. \tag{4.3}$$

Since $t_{i+1} - t_i = (\tau - t)/2^{i+1}$ for $i \geq 0$, we infer from (4.2) that

$$\frac{\delta(\tau - t)}{2^{i+1}} h(t_{i+1})^m \leq h(t_i), \quad i \geq 0. \tag{4.4}$$

By an induction argument, we deduce from (4.4) that

$$h(t) = h(t_0) \geq \frac{(\delta(\tau - t))^{\Sigma_i}}{2^{\sigma_i}} h(t_i)^{m^i}, \quad i \geq 1, \tag{4.5}$$

where

$$\Sigma_i := \sum_{j=0}^{i-1} m^j = \frac{1-m^i}{1-m} \quad \text{and} \quad \sigma_i := \sum_{j=0}^{i-1} (j+1)m^j = \frac{1-(i+1)m^i + i m^{i+1}}{(1-m)^2}$$

for $i \geq 1$. We then infer from (4.1), (4.3), and (4.5) that

$$h(t) \geq \frac{(\delta(\tau - t))^{\Sigma_i}}{2^{\sigma_i}} \mu(\tau)^{m^i}, \quad i \geq 1.$$

Owing to the positivity of $\mu(\tau)$, we may pass to the limit as $i \to \infty$ in the previous inequality to obtain

$$h(t) \geq \left(\frac{(\delta(\tau - t))^{1-m}}{2} \right)^{1/(1-m)^2}.$$

We then let $\tau \to T$ in the previous inequality to complete the proof. □

Proof of Theorem 1.2. The proof is divided into three parts.

L^∞-Lower Bound. By [8, Lemma 5.1], there exists $C_1 > 0$ such that

$$\|u(t)\|_1 \leq C_1 \|u(t)\|_\infty^\nu, \tag{4.6}$$

with

$$\nu := \frac{(N+1)(q_* - q)}{p - q}, \quad q_* := p - \frac{N}{N+1}.$$

Let $t > 0$ and $s \in (0, t)$. We infer from the Gagliardo–Nirenberg inequality (2.5) and the estimates (2.3) and (4.6) that

$$\begin{aligned} \|u(t)\|_\infty &\leq C \|\nabla u(t)\|_\infty^{N/(N+1)} \|u(t)\|_1^{1/(N+1)} \\ &\leq C C_1^{1/(N+1)} C_2^{N/(N+1)} \|u(s)\|_\infty^{N/q(N+1)} \|u(t)\|_\infty^{\nu/(N+1)} (t-s)^{-N/q(N+1)}, \end{aligned}$$

from which, taking into account that

$$1 - \frac{\nu}{N+1} = 1 - \frac{q_* - q}{p - q} = \frac{N}{(N+1)(p-q)},$$

we derive that

$$(t-s) \|u(t)\|_\infty^{q/(p-q)} \leq C_3 \|u(s)\|_\infty.$$

Let T_e be the extinction time of u. Since $q < p - q$, it follows from the properties of u prior to the extinction time that we are in a position to apply Lemma 4.1 (with $h(t) = \|u(t)\|_\infty$, $T = T_e$, and $m = q/(p-q) < 1$) and obtain the claimed lower bound.

Upper Bounds. Let $t \in (T_e/2, T_e)$. Recalling the gradient estimate (2.4) stated in Proposition 2.3, we see that the right-hand side of (2.4) is bounded as $t \in (T_e/2, T_e)$ and we further obtain

$$|\nabla u(t,x)| \leq C_5 u(t,x)^{1/(p-q)}, \quad (t,x) \in (T_e/2, T_e) \times \mathbb{R}^N. \tag{4.7}$$

Moreover, it follows from [8, (5.5)] and (1.10) that

$$0 \leq u(t,x) \leq C_4 |x|^{-(p-q)/(q-p+1)}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{4.8}$$

Now, integrating (1.1) over $(t, T_e) \times \mathbb{R}^N$ and using (4.7) as well as the property $\|u(T_e)\|_1 = 0$, we find

$$\|u(t)\|_1 = \int_t^{T_e} \int_{\mathbb{R}^N} |\nabla u(s,x)|^q \, dx \, ds \leq C_5^q \int_t^{T_e} \int_{\mathbb{R}^N} |u(s,x)|^{q/(p-q)} \, dx \, ds. \tag{4.9}$$

Since $p > 2q$, we have $q/(p-q) \in (0,1)$. We infer from (4.8) and Hölder's inequality that, for any $R \in (0,\infty)$ and $s \in (t, T_e)$,

$$\begin{aligned} \int_{\mathbb{R}^N} |u(s,x)|^{q/(p-q)} \, dx &= \int_{B_R(0)} |u(s,x)|^{q/(p-q)} \, dx + \int_{\mathbb{R}^N \setminus B_R(0)} |u(s,x)|^{q/(p-q)} \, dx \\ &\leq \left[\int_{B_R(0)} u(s,x) \, dx \right]^{q/(p-q)} \left(\int_{B_R(0)} dx \right)^{(p-2q)/(p-q)} + C \int_R^\infty r^{N-1-q/(q-p+1)} \, dr \\ &\leq C \left[\|u(s)\|_1^{q/(p-q)} R^{N(p-2q)/(p-q)} + R^{N-q/(q-p+1)} \right], \end{aligned}$$

where, in order to derive the last inequality, we took into account that, since $p-1<q<p/2$ and $p>p_c$,

$$\begin{aligned} N-\frac{q}{q-p+1} &= \frac{(N-1)q-N(p-1)}{q-p+1} \\ &< \frac{(N-1)p-2N(p-1)}{2(q-p+1)} \\ &= \frac{(N+1)(p_c-p)}{2(q-p+1)} < 0. \end{aligned}$$

We next optimize in R with the choice

$$\|u(s)\|_1^{q/(p-q)} R^{N-Nq/(p-q)} = R^{N-q/(q-p+1)},$$

or equivalently

$$R = \|u(s)\|_1^{-(q-p+1)/[(N+1)(p-q)-N]}.$$

Substituting this choice of R in the previous inequality leads us to

$$\int_{\mathbb{R}^N} |u(s,x)|^{q/(p-q)}\,dx \le C\|u(s)\|_1^{\omega}, \tag{4.10}$$

with

$$\omega := \frac{q}{p-q} - \frac{N(p-2q)(q-p+1)}{(p-q)[(N+1)(p-q)-N]}.$$

We observe after straightforward calculations that, since $p-1<q<p/2<q_*=p-N/(N+1)$, there holds

$$1-\omega = \frac{p-2q}{(N+1)(p-q)-N} > 0.$$

Now, combining (4.9) and (4.10) gives

$$\|u(t)\|_1 \le C_6 \int_t^{T_e} \|u(s)\|_1^{\omega}\,ds, \qquad t\in(T_e/2,T_e). \tag{4.11}$$

It readily follows from (1.1) and the non-negativity of u that $s\mapsto\|u(s)\|_1$ is non-increasing and we infer from (4.11) that

$$\|u(t)\|_1 \le C_6(T_e-t)\|u(t)\|_1^{\omega}, \qquad t\in(T_e/2,T_e).$$

Therefore, since $\|u(t)\|_1\neq 0$ for $t\in(T_e/2,T_e)$, we have

$$\|u(t)\|_1 \le C_7(T_e-t)^{[(N+1)(p-q)-N]/(p-2q)}, \qquad t\in(T_e/2,T_e), \tag{4.12}$$

and we have established the upper bound in (1.12). It next follows from (4.7) and the Gagliardo–Nirenberg inequality (2.5) that, for $t\in(T_e/2,T_e)$,

$$\begin{aligned} \|u(t)\|_\infty &\le C\|\nabla u(t)\|_\infty^{N/(N+1)}\|u(t)\|_1^{1/(N+1)} \\ &\le C\|u(t)\|_\infty^{N/(N+1)(p-q)}\|u(t)\|_1^{1/(N+1)}, \end{aligned}$$

hence

$$\|u(t)\|_\infty \le C\|u(t)\|_1^{(p-q)/[(N+1)(p-q)-N]}. \tag{4.13}$$

Gathering (4.12) and (4.13), we readily obtain the upper bound in (1.11), as desired.

L^1-Lower Bound. We are left with proving the L^1-lower bound for $t\in(T_e/2,T_e)$. But this is an immediate consequence of (4.13) and the L^∞-lower bound in (1.11), which has been already proved. □

Funding: The first author is supported by the ERC Starting Grant GEOFLUIDS 633152. Part of this work was done while the first author enjoyed the hospitality and support of the Institut de Mathématiques de Toulouse, France.

References

[1] S. Benachour, Ph. Laurençot, D. Schmitt and Ph. Souplet, Extinction and non-extinction for viscous Hamilton–Jacobi equations in $\mathbb{R}^N$, *Asymptot. Anal.* **31** (2002), no. 3–4, 229–246.

[2] X.-Y. Chen, H. Matano and M. Mimura, Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption, *J. Reine Angew. Math.* **459** (1995), 1–36.

[3] R. Ferreira and J. L. Vázquez, Extinction behaviour for fast diffusion equations with absorption, *Nonlinear Anal.* **43** (2001), no. 8, 943–985.

[4] A. Friedman and M. A. Herrero, Extinction properties of semilinear heat equations with strong absorption, *J. Math. Anal. Appl.* **124** (1987), no. 2, 530–546.

[5] V. A. Galaktionov, M. A. Herrero and J. J. L. Velázquez, The structure of solutions near an extinction point in a semilinear heat equation with strong absorption: A formal approach, in: *Nonlinear Diffusion Equations and Their Equilibrium States* (Gregynog 1989), Progr. Nonlinear Differential Equations Appl. 7, Birkhäuser, Boston (1992), 215–236.

[6] V. A. Galaktionov and J. L. Vázquez, Extinction for a quasilinear heat equation with absorption. I. Technique of intersection comparison, *Comm. Partial Differential Equations* **19** (1994), no. 7–8, 1075–1106.

[7] M. A. Herrero and J. J. L. Velázquez, Approaching an extinction point in one-dimensional semilinear heat equations with strong absorption, *J. Math. Anal. Appl.* **170** (1992), no. 2, 353–381.

[8] R. G. Iagar and Ph. Laurençot, Positivity, decay, and extinction for a singular diffusion equation with gradient absorption, *J. Funct. Anal.* **262** (2012), no. 7, 3186–3239.

[9] R. G. Iagar and Ph. Laurençot, Optimal extinction rates for the fast diffusion equation with strong absorption, preprint (2017), http://arxiv.org/abs/1711.07237.

[10] R. G. Iagar and Ph. Laurençot, Self-similar extinction for a diffusive Hamilton–Jacobi equation with critical absorption, *Calc. Var. Partial Differential Equations* **56** (2017), no. 3, Article ID 77.

[11] R. G. Iagar, Ph. Laurençot and C. Stinner, Instantaneous shrinking and single point extinction for viscous Hamilton–Jacobi equations with fast diffusion, *Math. Ann.* **368** (2017), no. 1–2, 65–109.

[12] R. G. Iagar, A. Sánchez and J. L. Vázquez, Radial equivalence for the two basic nonlinear degenerate diffusion equations, *J. Math. Pures Appl. (9)* **89** (2008), no. 1, 1–24.

[13] H. Ishii and P. Souganidis, Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor, *Tohoku Math. J. (2)* **47** (1995), no. 2, 227–250.

[14] M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the p-Laplace diffusion equation, *Comm. Partial Differential Equations* **22** (1997), no. 3–4, 381–411.

[15] J. L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*, Oxford Lecture Ser. Math. Appl., Oxford University Press, Oxford, 2006.