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Research Article

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Strong Comparison Principle for the Fractional p-Laplacian and Applications to Starshaped Rings

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Abstract: In the following, we show the strong comparison principle for the fractional p-Laplacian, i.e. we analyze

$$\begin{cases} (-\Delta)_p^s v + q(x)|v|^{p-2}v \geq 0 & \text{in } D, \\ (-\Delta)_p^s w + q(x)|w|^{p-2}w \leq 0 & \text{in } D, \\ v \geq w & \text{in } \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, $p > 1$, $D \subset \mathbb{R}^N$ is an open set, and $q \in L^\infty(\mathbb{R}^N)$ is a nonnegative function. Under suitable conditions on s, p and some regularity assumptions on v, w, we show that either $v \equiv w$ in $\mathbb{R}^N$ or $v > w$ in D. Moreover, we apply this result to analyze the geometry of nonnegative solutions in starshaped rings and in the half space.

Keywords: Fractional p-Laplacian, Strong Comparison Principle, Starshaped Superlevel Sets

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1 Introduction

In the following, we investigate an ordered pair of functions $v, w : \mathbb{R}^N \to \mathbb{R}$ which are the sub- and super-solution of the equation

$$(-\Delta)_p^s u + q(x)|u|^{p-2}u = g \quad \text{in } D, \tag{1.1}$$

where $s \in (0,1)$, $p > 1$, $q \in L^\infty(D)$ is a nonnegative function, $g \in L^{p'}(D)$ with $p' = \frac{p}{p-1}$ being the conjugate of p, and $(-\Delta)_p^s$ is the s-fractional p-Laplacian (up to a constant). Recall that for suitable (s,p) and some smoothness conditions on u we may write (see [11, Proposition 2.12])

$$(-\Delta)_p^s u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy, \quad x \in \mathbb{R}^N.$$

In order to derive a strong comparison principle for the fractional p-Laplacian we use a weak setting. We denote by $W^{s,p}(\mathbb{R}^N)$ as usual the fractional Sobolev space of order (s,p) given by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < \infty \right\},$$

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and for an open set $D \subset \mathbb{R}^N$ we denote

$$\mathcal{W}_0^{s,p}(D) := \{u \in W^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus D\}. \tag{1.2}$$

For an introduction into fractional Sobolev spaces, we refer to [6]. Finally, we also use the space

$$\tilde{W}^{s,p}(D) := \Big\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : \int_D \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < \infty\Big\} \tag{1.3}$$

to admit functions with a certain growth at infinity. Given an open set $D \subset \mathbb{R}^N$, $q \in L^\infty(D)$, a function $v \in \tilde{W}^{s,p}(D)$ is called a supersolution of (1.1) if for all nonnegative $\varphi \in \mathcal{W}_0^{s,p}(D)$ with compact support in $\mathbb{R}^N$ we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy + \int_D q|v|^{p-2}v\varphi\,dx \geq \int_D g\varphi\,dx. \tag{1.4}$$

Similarly, we call v a subsolution of (1.1) if $-v$ is a supersolution of (1.1). If v is a sub- and a supersolution of (1.1) *and* $v \in \mathcal{W}_0^{s,p}(D)$, then we call v a solution of (1.1). We note that indeed the left-hand side in (1.4) is well-defined as is shown in Lemma 2.4 below.

Equations involving the fractional Laplacian, that is, the case of $p = 2$, have been studied extensively in recent years (see, e.g., [1] and the references therein), whilst for its nonlinear counterpart there are still several unanswered questions. Existence of solutions and their regularity has been treated in [5, 10, 11, 15]. In particular, the question of existence of nontrivial solutions to problem (1.1) in the case $q = 0$ with nontrivial outside data has been studied in [5, 15]. Let us also mention [17], where the Rayleigh quotient associated to $(-\Delta)^s_p$ has been studied, and [14], which analyzes the obstacle problem associated with the fractional p-Laplacian. In this work, we prove a strong comparison principle for equations of type (1.1) and apply this to equations in starshaped rings and in the half space.

Theorem 1.1 (Strong Comparison Principle). *Let $s \in (0,1)$, let $p > 1$, let $D \subset \mathbb{R}^N$ be an open set, let $q \in L^\infty(D)$, $q \geq 0$, let $g \in L^{p'}(D)$, where $p' = \frac{p}{p-1}$, and let $v, w \in \tilde{W}^{s,p}(D)$ be such that v is a supersolution and w a subsolution of (1.1) with $v \geq w$. Assume one of the following conditions holds:*
(i) $\frac{1}{1-s} < p \leq 2$, *and $v \in L^\infty(\mathbb{R}^N)$ or $w \in L^\infty(\mathbb{R}^N)$.*
(ii) $p \geq 2$ *and for some $\alpha \in (0,1]$ with $\alpha(p-2) > sp-1$ we have $v \in C^\alpha_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$ or $w \in C^\alpha_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$.*
Then either $v = w$ a.e. in $\mathbb{R}^N$ or

$$\operatorname*{essinf}_K (v-w) > 0 \quad \textit{for all } K \Subset D.$$

The weak comparison principle for the fractional p-Laplacian with $q = 0$ goes back to [17] (see also [11, 15]). However, the validity of a strong comparison principle is already a delicate question in the case $s = 1$, i.e. the case of the classical p-Laplacian. We refer here to the works [18, 19]. Note that in the above nonlocal case neither v nor w need to be solutions, and indeed to achieve such a statement we strongly use the nonlocal structure of the fractional p-Laplacian. In the case $p = 2$, of course, the strong comparison principle follows from the strong maximum principle by linearity (see, e.g., [7]). But in general, when $p \neq 2$, the strong maximum principle for the fractional p-Laplacian does not imply the strong comparison principle due to the nonlinear structure of the operator. For the strong maximum principle and a Hopf-type lemma for the fractional p-Laplacian, we refer to the recent work [4].

For an application of Theorem 1.1, we investigate bounded nonnegative solutions of (1.1) in starshaped rings, i.e. we analyze

$$\begin{cases} (-\Delta)^s_p u + q(x)|u|^{p-2}u = 0 & \text{in } D = D_0 \setminus \overline{D}_1, \\ u = 0 & \text{on } \mathbb{R}^N \setminus D_0, \\ u = 1 & \text{on } D_1, \end{cases} \tag{1.5}$$

where $D_0, D_1 \subset \mathbb{R}^N$ are open sets with $0 \in \overline{D_1} \subset D_0$. For our main statement, we recall that a subset A of $\mathbb{R}^N$ is said *starshaped with respect to the point $\bar{x} \in A$* if for every $x \in A$ the segment $(1-s)\bar{x} + sx$, $s \in [0,1]$, is contained in A. If $\bar{x} = 0$ (as we can always assume up to a translation), we simply say that A is *starshaped*, meaning that for every $x \in A$ we have $sx \in A$ for $s \in [0, 1]$, or equivalently

$$A \text{ is starshaped if } sA \subseteq A \text{ for every } s \in [0, 1].$$

The set A is said *strictly starshaped* if 0 is in the interior of A and any ray starting from 0 intersects the boundary of A in only one point.

By $U(\ell)$, $\ell \in \mathbb{R}$, we denote the superlevel sets of a function $u : \mathbb{R}^N \to \mathbb{R}$:

$$U(\ell) := \{u \geq \ell\} = \{x \in \mathbb{R}^N : u(x) \geq \ell\}.$$

Theorem 1.2. *Let $s \in (0, 1)$, let $p > 1$, and let $D = D_0 \setminus \overline{D}_1$, where $D_0, D_1 \subset \mathbb{R}^N$ are open bounded sets such that $0 \in D_1$ and $\overline{D}_1 \subset D_0$. Let $q : D \to [0, \infty)$ such that the following conditions hold:*

(A1) *q is a bounded Borel-function.*

(A2) *For all $t \geq 1$ and $x \in \mathbb{R}^N$ such that $tx \in D$, we have $t^{sp} q(tx) \geq q(x)$.*

Moreover, let u be a continuous bounded weak solution of (1.5) *such that* $0 \leq u \leq 1$, *and assume D_0 and D_1 are starshaped. Then the superlevel sets $U(\ell)$ of u are starshaped for $\ell \in (0, 1)$.*

If in addition D_0 and D_1 are strictly starshaped sets and

(i) $\frac{1}{1-s} < p \leq 2$ or

(ii) *$p \geq 2$ and $u \in C^{\alpha}_{\mathrm{loc}}(D)$ for some $\alpha \in (0, 1]$ with $\alpha(p-2) > sp - 1$,*

then the superlevel sets $U(\ell)$ of u are strictly starshaped for $\ell \in (0, 1)$.

Remark 1.3. (i) The starshapedness of superlevel sets is indeed a consequence of the weak comparison principle, hence the assumptions are rather general in this case. To prove the strict starshapedness of superlevel sets, however, we need the strong comparison principle and hence stronger assumptions on u, s, and p in view of Theorem 1.1. We note that in the case $q \equiv 0$, existence and local Hölder regularity of solutions of (1.5) has been discussed in [5], so Theorem 1.2 can be applied for $p \in (0, 1)$, $s < \frac{p-1}{p}$, and for any $p \geq 2$, $s < \frac{1}{p}$.

(ii) In the case $p = 2$, neither the bounds on s nor the regularity assumption on u are necessary (see [12]).

Let us close this introduction with the following further result in half spaces (see also [2, 3, 8] for similar results).

Theorem 1.4. *Let $s \in (0, 1)$, $p > 1$ and set $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_1 > 0\}$. Moreover, let $q \in L^\infty(\mathbb{R}^N_+)$, $q \geq 0$, and let $u \in \mathcal{W}^{s,p}_0(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ be a nonnegative continuous function which satisfies*

$$(-\Delta)^s_p u + q(x)|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \lim_{|x|\to\infty} u(x) = 0.$$

If q is increasing in the direction of x_1, i.e. $q(x + te_1) \geq q(x)$ for all $x \in \Omega$, $t \geq 0$, and

(i) $\frac{1}{1-s} < p \leq 2$ or

(ii) *$p \geq 2$ and $u \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N_+)$ for some $\alpha \in (0, 1]$ with $\alpha(p-2) > sp - 1$,*

then $u \equiv 0$ on $\mathbb{R}^N$.

The article is organized as follows: In Section 2, we give some basic properties on the involved function spaces and useful elementary inequalities. In Section 3, we give the proof of a variant of a weak comparison principle and then prove Theorem 1.1. The proofs of Theorem 1.2 and 1.4 are given in Section 4.

2 Preliminaries and Notation

We will use the following notation: For subsets $D, U \subset \mathbb{R}^N$, we denote by $D^c := \mathbb{R}^N \setminus D$ the complement of D in $\mathbb{R}^N$, and we write $\mathrm{dist}(D, U) := \inf\{|x - y| : x \in D,\ y \in U\}$. If $D = \{x\}$ is a singleton, we write $\mathrm{dist}(x, U)$ in place of $\mathrm{dist}(\{x\}, U)$. The notation $U \Subset D$ means that $\overline{U}$ is compact and contained in D. For $U \subset \mathbb{R}^N$ and $r > 0$, we consider $B_r(U) := \{x \in \mathbb{R}^N : \mathrm{dist}(x, U) < r\}$, and we let, as usual $B_r(x) = B_r(\{x\})$ be the open ball in $\mathbb{R}^N$

centered at $x \in \mathbb{R}^N$ with radius $r > 0$. For any subset $M \subset \mathbb{R}^N$, we denote by $1_M : \mathbb{R}^N \to \mathbb{R}$ the characteristic function of M, and by $\operatorname{diam}(M)$ the diameter of M. If M is measurable, $|M|$ denotes the Lebesgue measure of M. For the unit ball, we will use in particular $\omega_N := |B_1(0)|$. Moreover, if $w : M \to \mathbb{R}$ is a function, we let $w^+ = \max\{w, 0\}$ and $w^- = -\min\{w, 0\}$ denote the positive and negative part of w, respectively.

2.1 Some Elementary Inequalities

We use the notation $a^{*q} := |a|^{q-1}a$ for any $a \in \mathbb{R}$, $q > 0$. Note that for $a \geq 0$ we have $a^{*q} = a^q$, and for $a < 0$ we have $a^{*q} = -|a|^q$. Moreover, we have the following elementary inequalities of this function.

Lemma 2.1 (see [11, Section 2.2]). *For all $b \geq 0$, $q > 0$,*

$$(a+b)^q \leq \max\{1, 2^{q-1}\}(a^q + b^q) \quad \text{if } a \geq 0. \tag{2.1}$$

$$(a+b)^{*q} - a^{*q} \geq 2^{1-q} b^q \quad \text{if } a \in \mathbb{R},\ q \geq 1. \tag{2.2}$$

Lemma 2.2. *Let $M, q > 0$. Then there are $C_{M,1}, C_{M,2} > 0$ such that for all $a \in [-M, M]$, $b \geq 0$,*

$$a^{*q} - (a-b)^{*q} \leq C_{M,1} \max\{b, b^q\}, \tag{2.3}$$

$$(a+b)^{*q} - a^{*q} \geq C_{M,2} \min\{b, b^q\}. \tag{2.4}$$

Proof. Inequality (2.3) is shown in [11, Section 2.2]. Moreover, if $q \geq 1$, then (2.4) follows from (2.2) and indeed no bound on a is needed. For $q \in (0, 1)$, fix $M > 0$ as stated and let $a \in [-M, M]$. Note that the map $t \mapsto (t + a)^{*q} - t^{*q}$ satisfies, for $t \in [-1, 1]$,

$$(t+1)^{*q} - t^{*q} = q \int_t^{t+1} |v|^{q-1}\, dv \geq q \max\{|t|, |t+1|\}^{q-1} \geq q 2^{q-1}$$

since $q < 1$. Hence for $b > \max\{0, |a|\}$ we have

$$\frac{(a+b)^{*q} - a^{*q}}{b^q} = \left(\frac{a}{b} + 1\right)^{*q} - \left(\frac{a}{b}\right)^{*q} \geq q 2^{q-1}.$$

And for $0 \leq b \leq |a|$, using again that $q < 1$, we have

$$(a+b)^{*q} - a^{*q} = qb \int_0^1 |a + vb|^{q-1}\, dv \geq q(|a| + |b|)^{q-1} b \geq q 2^{q-1} M^{q-1} b.$$

□

Lemma 2.3 (see [16, Lemma 2]). *For all $q \in (0, 1]$, there is $C > 0$ such that*

$$|(a+b)^{*q} - a^{*q}| \leq C|b|^q \quad \text{for all } a, b \in \mathbb{R}. \tag{2.5}$$

2.2 Function Spaces and Their Properties

In the following, we let $s \in (0, 1)$, $p > 1$, and $D \subset \mathbb{R}^N$ open. We let $\mathcal{W}_0^{s,p}(D)$ and $\tilde{W}^{s,p}(D)$ as introduced in (1.2) and (1.3), respectively. Moreover, we set formally for functions $u, v : \mathbb{R}^N \to \mathbb{R}$,

$$\langle u, v\rangle_{s,p} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x) - u(y))^{*(p-1)}(v(x) - v(y))}{|x-y|^{N+sp}}\, dx\, dy.$$

Recall that $\mathcal{W}_0^{s,p}(D)$ is a Banach space with the norm

$$\|u\|_{W^{s,p}} := \left(\|u\|_{L^p(D)}^p + \langle u, u\rangle_{s,p}\right)^{\frac{1}{p}},$$

and that it corresponds to the completion of $C_c^\infty(D)$ with respect to this norm (see, e.g., [9]).

Lemma 2.4. *The map* $\langle\cdot,\cdot\rangle_{s,p}:\tilde W^{s,p}(D)\times\mathcal{W}_0^{s,p}(D)\to\mathbb{R}$ *is well-defined.*

Proof. We have by Hölder's inequality with $q=\frac{p}{p-1}$,

$$
\begin{aligned}
|\langle u,v\rangle_{s,p}| &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+sp}}\,dx\,dy\\
&=\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+sp}}\,dx\,dy+\int_{\mathbb{R}^N\setminus D}\int_D\frac{|u(x)-u(y)|^{p-1}|v(x)|}{|x-y|^{N+sp}}\,dx\,dy\\
&\le\Big(\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{[\frac{N}{q}+s(p-1)]q}}\,dx\,dy\Big)^{\frac1q}\Big(\int_D\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{[\frac{N}{p}+s]p}}\,dx\,dy\Big)^{\frac1p}\\
&\qquad+\Big(\int_{\mathbb{R}^N\setminus D}\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{[\frac{N}{q}+s(p-1)]q}}\,dx\,dy\Big)^{\frac1q}\Big(\int_{\mathbb{R}^N\setminus D}\int_D\frac{|v(x)|^p}{|x-y|^{[\frac{N}{p}+s]p}}\,dx\,dy\Big)^{\frac1p}\\
&\le 2\Big(\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{\frac1q}\|v\|_{W^{s,p}(\mathbb{R}^N)}<\infty.
\end{aligned}
$$

□

Lemma 2.5. *We have that* $\tilde W^{s,p}(D)$ *is a vector space with the following properties:*
(i) $C_c^{0,1}(\mathbb{R}^N)\subset W^{s,p}(\mathbb{R}^N)\subset\tilde W^{s,p}(D)$.
(ii) *If* $u\in\tilde W^{s,p}(D)$, *then* $u^\pm\in\tilde W^{s,p}(D)$.

Proof. The fact that $\tilde W^{s,p}(D)$ is a vector space follows from (2.1). Moreover, we have $C_c^{0,1}(\mathbb{R}^N)\subset W^{s,p}(\mathbb{R}^N)$ (see, e.g., [9]), and $u\in W^{s,p}(\mathbb{R}^N)\subset\tilde W^{s,p}(D)$ is trivial. For the second statement, note that we have $|u|\in\tilde W^{s,p}(D)$ since

$$|u(x)-u(y)|\ge\big||u|(x)-|u|(y)\big|\quad\text{for all } x,y\in\mathbb{R}^N.$$

Hence $2u^\pm=|u|\pm u\in\tilde W^{s,p}(D)$. □

Lemma 2.6. *Let D be bounded and* $u\in\tilde W^{s,p}(D)$ *with* $u=0$ *on* $\mathbb{R}^N\setminus D$. *Then* $u\in\mathcal{W}_0^{s,p}(D)$.

Proof. Since D is bounded, we have $u\in L^p(D)$. Moreover,

$$
\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy &=\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\int_{\mathbb{R}^N\setminus D}\int_D\frac{|u(x)|^p}{|x-y|^{N+sp}}\,dx\,dy\\
&\le\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\int_{\mathbb{R}^N}\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\\
&=2\int_D\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy,
\end{aligned}
$$

which is bounded by assumption. □

An immediate consequence of Lemmas 2.6 and 2.5 is the following corollary.

Corollary 2.7. *Let D be bounded and* $u\in\tilde W^{s,p}(D)$ *with* $u\ge0$ *on* $\mathbb{R}^N\setminus D$. *Then* $u^-\in\mathcal{W}_0^{s,p}(D)$.

In the following, we also say that $v\in\tilde W^{s,p}(D)$ satisfies (in weak sense)

$$(-\Delta)_p^s v\ge g\quad\text{in } D$$

for $g\in[\mathcal{W}_0^{s,p}(D)]'$, the dual of $\mathcal{W}_0^{s,p}(D)$, if for all nonnegative $\varphi\in\mathcal{W}_0^{s,p}(D)$ with compact support in $\mathbb{R}^N$ we have

$$\langle u,\varphi\rangle_{s,p}\ge\int_\Omega g(x)\varphi(x)\,dx.$$

Similarly, we use "≤" and "=".

Lemma 2.8 (cf. [11, Lemma 2.9]). *Let $t > 0$ and let $u \in \tilde{W}^{s,p}(D)$ satisfy $(-\Delta)^s_p u = g$ in D for some $g \in [\mathcal{W}^{s,p}_0(D)]'$. Then the function $v : \mathbb{R}^N \to \mathbb{R}$, $v(x) = u(tx)$ satisfies $v \in \tilde{W}^{s,p}(t^{-1}D)$ and*

$$(-\Delta)^s_p v = t^{sp} g(t\,\cdot) \quad \text{on } t^{-1}D.$$

Proof. Let $\varphi \in \mathcal{W}^{s,p}_0(t^{-1}D)$. Then clearly $\varphi(\frac{\cdot}{t}) \in \mathcal{W}^{s,p}_0(D)$ and

$$\begin{aligned}\langle v, \varphi\rangle_{s,p} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^{*(p-1)}(\varphi(\frac{x}{t})-\varphi(\frac{y}{t}))}{|\frac{x}{t}-\frac{y}{t}|^{N+sp}}\,dx\,dy \\ &= t^{-N+sp}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^{*(p-1)}(\varphi(\frac{x}{t})-\varphi(\frac{y}{t}))}{|x-y|^{N+sp}}t^{-2N}\,dx\,dy \\ &= t^{-N+sp}\int_D g(x)\varphi\Big(\frac{x}{t}\Big)\,dx \\ &= \int_{t^{-1}D} t^{sp} g(tx)\varphi(x)\,dx.\end{aligned}$$

□

3 Comparison Principles

The following is a slight variant of the weak maximum principle presented in [11, Proposition 2.10], [15, Lemma 6], and [17, Lemma 9].

Lemma 3.1. *Let $D \subset \mathbb{R}^N$ be an open set, let $q \in L^\infty(D)$, $q \geq 0$, and $g \in L^{p'}(D)$, where $p' = \frac{p}{p-1}$. If $v, w \in \tilde{W}^{s,p}(D)$ are super- and subsolution, respectively, of $(-\Delta)^s_p u + q(x)u^{*(p-1)} = g$ such that $v \geq w$ in $\mathbb{R}^N \setminus D$ and*

$$\liminf_{|x|\to\infty}(v(x)-w(x)) \geq 0,$$

then $v \geq w$ a.e. in $\mathbb{R}^N$.

Proof. First assume D is bounded and set $u(x) = v(x) - w(x)$ and

$$Q(x,y) := (p-1)\int_0^1 |w(x)-w(y)+t(u(x)-u(y))|^{p-2}\,dt, \quad P(x) = (p-1)\int_0^1 |w(x)+tu(x)|^{p-2}\,dt.$$

Note that $P \geq 0$ on $\mathbb{R}^N$, and $Q(x,y) = Q(y,x) \geq 0$ for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Moreover, if $P(x) = 0$, then $w(x) = 0 = u(x)$, and if $Q(x,y) = 0$, then $w(x) = w(y)$ and $u(x) = u(y)$. Note that $u \geq 0$ on $\mathbb{R}^N \setminus D$, and hence $u^- \in \mathcal{W}^{s,p}_0(D)$ due to Corollary 2.7. Note that for any $\varphi \in \mathcal{W}^{s,p}_0(D)$, $\varphi \geq 0$, we have

$$\begin{aligned}-\int_D q(x)P(x)u(x)\varphi(x)\,dx &= -\int_D q(x)v^{*(p-1)}(x)\varphi(x) + q(x)w^{*(p-1)}(x)\varphi(x)\,dx \\ &\leq \langle v, \varphi\rangle_{s,p} - \langle w, \varphi\rangle_{s,p} \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big[(v(x)-v(y))^{*(p-1)} - (w(x)-w(y))^{*(p-1)}\big](\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(x,y)}{|x-y|^{N+sp}}(u(x)-u(y))(\varphi(x)-\varphi(y))\,dx\,dy.\end{aligned}$$

Since D is bounded, we have $u^- \in \mathcal{W}^{s,p}_0(D)$. Hence, since

$$[u(x)-u(y)][u^-(x)-u^-(y)] = -2u^+(x)u^-(y) - (u^-(x)-u^-(y))^2 \leq 0$$

for $x, y \in \mathbb{R}^N$, we have with $\varphi = u^-$,

$$-\|q^-\|_{L^\infty(D)} \int_D u^-(x)^2 P(x)\,dx \le -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(x,y)}{|x-y|^{N+sp}}(u^-(x)-u^-(y))^2\,dx\,dy$$
$$\le -\int_D u^-(x)^2 \int_{\mathbb{R}^N\setminus D} \frac{2Q(x,y)}{|x-y|^{N+sp}}\,dy\,dx \le 0.$$

Since $q \ge 0$, this implies

$$0 = \int_D u^-(x)^2 \int_{\mathbb{R}^N\setminus D} \frac{Q(x,y)}{|x-y|^{N+sp}}\,dy\,dx,$$

which by an argumentation as in [17, Lemma 9] is only possible if $u^- = 0$ a.e. Indeed, we have for a.e. $(x, y) \in D \times (\mathbb{R}^N \setminus D)$ either $u^-(x) = 0$ or $Q(x, y) = 0$, but in the latter case we have $u(x) = u(y)$ as mentioned above. Since $x \in \operatorname{supp} u^-$ and $y \in \mathbb{R}^N \setminus D$, we have $u(x) \le 0 \le u(y)$, so that $u(x) = 0 = u(y)$. Hence also in the latter case it follows that $u^-(x) = 0$. Hence the claim follows for bounded sets.

If D is unbounded, then since

$$\liminf_{|x|\to\infty}(v(x) - w(x)) \ge 0,$$

we find for every $\epsilon > 0$ a number $R > 0$ such that $v_\epsilon \ge w$ in $\mathbb{R}^N \setminus D_R$, where $v_\epsilon := v + \epsilon$ and $D_R := B_R(0) \cap D$. Moreover, for $\varphi \in \mathcal{W}_0^{s,p}(D)$, $\varphi \ge 0$, with compact support in $\mathbb{R}^N$, using that $q \ge 0$ and $t \mapsto t^{*(p-1)}$ is increasing, we have

$$\begin{aligned}
\langle v_\epsilon, \varphi\rangle_{s,p} &+ \int_D q(x) v_\epsilon^{*(p-1)}(x)\varphi(x)\,dx \\
&\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)+\epsilon-(v(y)+\epsilon))^{*(p-1)}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy + \int_D q(x)v^{*(p-1)}(x)\varphi(x)\,dx \\
&= \langle v, \varphi\rangle_{s,p} + \int_D q(x)v^{*(p-1)}(x)\varphi(x)\,dx \\
&\ge \int_D g(x)\varphi(x)\,dx,
\end{aligned}$$

hence v_ϵ is a supersolution of $(-\Delta)^s_p u + q(x)u^{*(p-1)} = g$ in D_R, while w is a subsolution of this equation in D_R. The first part gives $v + \epsilon = v_\epsilon \ge w$ in $\mathbb{R}^N$, and since $\epsilon > 0$ is arbitrary, this finishes the proof. □

Remark 3.2. We note that almost with the same proof it is possible to show the following: Let $D \subset \mathbb{R}^N$ be an open bounded set, let $q \in L^\infty(D)$, $q \ge 0$, and assume $v, w \in \tilde{W}^{s,p}(D)$ satisfy in weak sense

$$\begin{aligned}
(-\Delta)^s_p v - (-\Delta)^s_p w &\ge -q(x)(v-w)^{*(p-1)} && \text{in } D,\\
v &\ge w && \text{in } \mathbb{R}^N \setminus D.
\end{aligned}$$

Then $v \ge w$ a.e. in $\mathbb{R}^N$.

Lemma 3.3 (see [11, Lemma 2.8]). *Let $D \subset \mathbb{R}^N$ be an open bounded set, let $K \Subset \mathbb{R}^N \setminus D$, and let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then for any $v \in \tilde{W}^{s,p}(D)$ we have in weak sense*

$$(-\Delta)^s_p(v + h1_K) = (-\Delta)^s_p v + H \quad \text{in } D,$$

with

$$H(x) = 2\int_K \frac{((v(x)-v(y))-h(y))^{*(p-1)} - (v(x)-v(y))^{*(p-1)}}{|x-y|^{N+sp}}\,dy$$

for a.e. $x \in D$.

Lemma 3.4. *Let $D \subset \mathbb{R}^N$ be an open bounded set and let $K \Subset \mathbb{R}^N \setminus D$ with $|K| > 0$.*
(i) *If $p \geq 2$, then there is $C > 0$ such that for any $v \in \tilde{W}^{s,p}(D)$ and any $\delta \in (0, 1]$ we have in weak sense*

$$(-\Delta)_p^s(v - \delta 1_K) \geq (-\Delta)_p^s v + C\delta^{p-1},$$
$$(-\Delta)_p^s(v + \delta 1_K) \leq (-\Delta)_p^s v - C\delta^{p-1}. \tag{3.1}$$

(ii) *If $p \in (1, 2)$, then for any $M > 0$ there is $C > 0$ such that for any $v \in \tilde{W}^{s,p}(D)$ with $\|v\|_{L^\infty(\mathbb{R}^N)} \leq M$ and any $\delta \in (0, 1]$ we have in weak sense*

$$(-\Delta)_p^s(v - \delta 1_K) \geq (-\Delta)_p^s v + C\delta,$$
$$(-\Delta)_p^s(v + \delta 1_K) \leq (-\Delta)_p^s v - C\delta. \tag{3.2}$$

Proof. By Lemma 3.3, we have in weak sense

$$(-\Delta)_p^s(v - \delta 1_K) = (-\Delta)_p^s v + 2\int_K \frac{(v(x) - v(y) + \delta)^{*(p-1)} - (v(x) - v(y))^{*(p-1)}}{|x - y|^{N+sp}}\,dy.$$

We will start by showing the first inequalities in (3.1) and (3.2).

Case 1: $p \geq 2$. Then by inequality (2.2) we have

$$2\int_K \frac{(v(x) - v(y) + \delta)^{*(p-1)} - (v(x) - v(y))^{*(p-1)}}{|x - y|^{N+sp}}\,dy \geq 2^{3-p}\delta^{*(p-1)}\int_K \frac{1}{|x - y|^{N+sp}}\,dy.$$

Hence the first inequality in (3.1) holds with $C_1 = 2^{3-p}|K| \sup_{x\in D, y\in K}|x - y|^{-N-sp}$.

Case 2: $p \in (1, 2)$. Then with (2.4) we have

$$2\int_K \frac{(v(x) - v(y) + \delta)^{*(p-1)} - (v(x) - v(y))^{*(p-1)}}{|x - y|^{N+sp}}\,dy \geq C_{M,2}\delta\int_K \frac{1}{|x - y|^{N+sp}}\,dy.$$

Hence the first inequality in (3.2) holds with $C_2 = C_{M,2}|K| \sup_{x\in D, y\in K}|x - y|^{-N-sp}$.

For the second inequalities in (3.1) and (3.2), note that

$$\begin{aligned}(-\Delta)_p^s(v - \delta 1_K) &= (-\Delta)_p^s v + 2\int_K \frac{(v(x) - v(y) - \delta)^{*(p-1)} - (v(x) - v(y))^{*(p-1)}}{|x - y|^{N+sp}}\,dy\\ &= (-\Delta)_p^s v - 2\int_K \frac{(v(x) - v(y) - \delta + \delta)^{*(p-1)} - (v(x) - v(y) - \delta)^{*(p-1)}}{|x - y|^{N+sp}}\,dy.\end{aligned}$$

Hence this part follows similarly. □

Lemma 3.5. *Let $D \subset \mathbb{R}^N$ be an open bounded set and let $s \in (0, 1)$, $p > 1$, and $f \in C_c^2(D)$.*
(i) *If $\frac{1}{1-s} < p \leq 2$, then for any $v \in \tilde{W}^{s,p}(D)$ there is $C > 0$ such that for all $a \in [-1, 1]$ in weak sense*

$$|(-\Delta)_p^s(v - af) - (-\Delta)_p^s v| \leq |a|^{p-1}C \quad \text{in } \operatorname{supp} f. \tag{3.3}$$

(ii) *If $p \geq 2$, then for any $v \in \tilde{W}^{s,p}(D) \cap C_{\mathrm{loc}}^\alpha(D) \cap L^\infty(\mathbb{R}^N)$, $\alpha \in (0, 1]$ with $\alpha(p - 2) > sp - 1$, there is $C > 0$ such that for all $a \in [-1, 1]$ in weak sense*

$$|(-\Delta)_p^s(v - af) - (-\Delta)_p^s v| \leq |a|C \quad \text{in } \operatorname{supp} f. \tag{3.4}$$

Proof. Let $s \in (0, 1)$ and $a \in [-1, 1]$, fix $f \in C_c^2(D)$, $R = 2\operatorname{diam}(D) + 1$, $U := \operatorname{supp} f$, and

$$K := \sum_{|\alpha|\leq 2} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^N)}.$$

Moreover, let $v \in \tilde{W}^{s,p}(D)$. If $p \in (\frac{1}{1-s}, 2]$, then we have for any $\varphi \in \mathcal{W}_0^{s,p}(U)$, $\varphi \geq 0$, and a constant $C_1 > 0$

given by (2.5),

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{((v(x)-v(y)-a(f(x)-f(y)))^{*(p-1)}-(v(x)-v(y))^{*(p-1)})(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\Big|\\
&\qquad\le 2C_1|a|^{p-1}\int_{\mathbb{R}^N}\varphi(x)\int_{\mathbb{R}^N}\frac{|f(x)-f(x+y)|^{p-1}}{|y|^{N+sp}}\,dx\,dy\\
&\qquad\le 2C_1|a|^{p-1}K^{p-1}\int_U \varphi(x)\Big(\int_{B_R(0)}|y|^{-N+(1-s)p-1}\,dy+\int_{\mathbb{R}^N\setminus B_R(0)}|y|^{-N-sp}\,dy\Big)\,dx\\
&\qquad\le 2C_1N\omega_N|a|^{p-1}K^{p-1}\int_U\varphi(x)\Big(\int_0^R r^{(1-s)p-2}\,dr+\int_R^\infty r^{-1-sp}\,dr\Big)\,dx\\
&\qquad\le 2|a|^{p-1}C_1K^{p-1}N\omega_N\Big(\frac{1}{(1-s)p-1}+\frac{2}{sp}\Big)\int_U\varphi(x)\,dx,
\end{aligned}$$

where $\omega_N=|B_1(0)|$. Hence (3.3) holds with

$$C=2C_1K^{p-1}N\omega_N\Big(\frac{1}{(1-s)p-1}+\frac{2}{sp}\Big).$$

Next let $p\ge 2$, assume additionally that $v\in L^\infty(\mathbb{R}^N)\cap C^\alpha_{\mathrm{loc}}(D)$ for some $\alpha\in(0,1)$ with $\alpha(p-2)>sp-1$, and set for $x,y\in\mathbb{R}^N$,

$$Q(x,y):=(p-1)\int_0^1|v(x)-v(y)-at(f(x)-f(y))|^{p-2}\,dt.$$

Note that $Q(x,y)=Q(y,x)\ge 0$ for any $x,y\in\mathbb{R}^N$. Moreover, there is

$$C_2=C_2(p,\|v\|_{L^\infty(\mathbb{R}^N)},f)\quad\text{and}\quad C_3=C_3(p,\|v\|_{C^{s+\epsilon}(D)},f)$$

such that

$$Q(x,y)\le C_2\qquad\text{for } x,y\in\mathbb{R}^N, \tag{3.5}$$

$$Q(x,y)\le C_3|x-y|^{\alpha(p-2)}\qquad\text{for } x,y\in U, \tag{3.6}$$

where we used that $\overline{U}\subset D$. Moreover, we have $\operatorname{dist}(\operatorname{supp} f,D^c)=\delta>0$. Fix $\varphi\in\mathcal{W}^{s,p}_0(U)$, $\varphi\ge 0$. Then, since

$$(v(x)-v(y)-a(f(x)-f(y)))^{*(p-1)}-(v(x)-v(y))^{*(p-1)}=-aQ(x,y)(f(x)-f(y)),$$

we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{((v(x)-v(y)-a(f(x)-f(y)))^{*(p-1)}-(v(x)-v(y))^{*(p-1)})(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\Big|\\
&\qquad=|a|\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f(x)-f(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}Q(x,y)\,dx\,dy\Big|\\
&\qquad\le 2|a|\int_U\varphi(x)\Big|\lim_{\epsilon\to0}\int_{B^c_\epsilon(x)}\frac{(f(x)-f(y))Q(x,y)}{|x-y|^{N+sp}}\,dy\Big|\,dx.
\end{aligned}$$

We now follow closely the lines of the proof of [13, Lemma 3.8] to show that

$$\sup_{x\in D}\Big|\lim_{\epsilon\to0}\int_{B^c_\epsilon(x)}\frac{(f(x)-f(y))Q(x,y)}{|x-y|^{N+sp}}\,dy\Big|\le C_4 \tag{3.7}$$

for a constant $C_4 = C_4(D, N, p, f, \|v\|_{C^\alpha(D)}, \|v\|_{L^\infty(\mathbb{R}^N)}) > 0$. Clearly, once (3.7) is shown, this finishes the proof of (3.4). To see (3.7), fix $x \in U$ and note that $B_\delta(x) \subset D$. Let $\epsilon \in (0, \delta)$. Moreover, since $f \in C_c^2(D)$, we have

$$|2f(x) - f(x-y) - f(x+y)| \le K|y|^2 \quad \text{for all } x, y \in \mathbb{R}^N,$$

and from (3.6) and (2.1) there is $K_2 > 0$ such that

$$|Q(x, x-y) - Q(x, x+y)| \le K_2|y|^{\alpha(p-2)} \quad \text{for all } x \in \operatorname{supp} f,\ y \in B_\delta(0).$$

Hence with (3.5) and (3.6), we have

$$\begin{aligned}
\Bigg|\int_{B_\epsilon^c(x)} \frac{(f(x)-f(y))Q(x,y)}{|x-y|^{N+sp}}\,dy\Bigg| &= \Bigg|\int_{B_\epsilon^c(0)} \frac{(f(x)-f(x\pm y))Q(x,x\pm y)}{|y|^{N+sp}}\,dy\Bigg| \\
&= \Bigg|\frac{1}{2}\int_{B_\epsilon^c(0)} \frac{(2f(x)-f(x+y)-f(x-y))Q(x,x+y)}{|y|^{N+sp}}\,dy \\
&\quad + \frac{1}{2}\int_{B_\epsilon^c(0)} (f(x)-f(x-y))(Q(x,x+y)-Q(x,x-y))|y|^{-N-sp}\,dy\Bigg| \\
&\le \frac{C_3K}{2}\int_{B_\delta(0)\setminus B_\epsilon(0)} |y|^{2-N-sp+\alpha(p-2)}\,dy + C_2 4K\int_{B_\delta^c(0)} |y|^{-N-sp}\,dy \\
&\quad + \frac{K_2K}{2}\int_{B_\delta(0)\setminus B_\epsilon(0)} |y|^{1+\alpha(p-2)-N-sp}\,dy \\
&\le \frac{C_3}{2}NK_1\omega_N\int_0^\delta \rho^{1-sp+\alpha(p-2)}\,d\rho + C_2K4N\omega_N\int_\delta^\infty \rho^{-1-sp}\,d\rho \\
&\quad + \frac{K_2K}{2}N\omega_N\int_0^\delta \rho^{\alpha(p-2)-sp}\,d\rho \\
&= \frac{C_3KN\omega_N\delta^{2-sp+\alpha(p-2)}}{2(2-sp+\alpha(p-2))} + \frac{C_2K4}{sp}N\omega_N\delta^{-sp} + \frac{K_2KN\omega_N\delta^{1-2s+\alpha(p-2)}}{2(1-sp+\alpha(p-2))} \\
&=: C_4 < \infty,
\end{aligned}$$

where we have used $\alpha(p-2) > sp - 1$. □

Lemma 3.6. *Let $D \subset \mathbb{R}^N$ be an open bounded set, let $K \Subset \mathbb{R}^N \setminus D$ with $|K| > 0$, $\delta \in (0, 1]$, $s \in (0, 1)$, and $p > 1$. Moreover, fix $f \in C_c^2(D)$ with $0 \le f \le 1$.*

(i) *If $\frac{1}{1-s} < p \le 2$, then for any $v \in \tilde{W}^{s,p}(D) \cap L^\infty(\mathbb{R}^N)$ there is $a_0, b > 0$ such that in weak sense for all $a \in (0, a_0]$,*

$$\begin{aligned}
(-\Delta)_p^s(v - af - \delta 1_K) &\ge (-\Delta)_p^s v + b && \text{in } \operatorname{supp} f, \\
(-\Delta)_p^s v - b &\ge (-\Delta)_p^s(v + af + \delta 1_K) && \text{in } \operatorname{supp} f.
\end{aligned} \tag{3.8}$$

(ii) *If $p \ge 2$, then for any $v \in \tilde{W}^{s,p}(D) \cap C^\alpha(D) \cap L^\infty(\mathbb{R}^N)$, $\alpha \in (0, 1]$ with $\alpha(p-2) > sp - 1$, there is $a_0, b > 0$ such that (3.8) holds in weak sense for all $a \in (0, a_0]$.*

Proof. Fix v, s, p as stated. By Lemma 3.4, we have in all cases

$$(-\Delta)_p^s(v - af - \delta 1_K) \ge (-\Delta)_p^s(v - af) + C\min\{\delta, \delta^{p-1}\}$$

for some $C > 0$. Moreover, by Lemma 3.5 we have for some $\tilde{C} > 0$ in weak sense

$$(-\Delta)_p^s(v - af) \ge (-\Delta)_p^s v - \max\{a, a^{p-1}\}\tilde{C}, \quad a \in (0, 1].$$

Hence we may fix $a_0 = a_0(\delta) \in (0, 1]$ such that $b = -\max\{a_0, a_0^{p-1}\}\tilde{C} + C\min\{\delta, \delta^{p-1}\} > 0$. This shows the first inequality in (3.8). The second inequality in (3.8) follows similarly. □

Proof of Theorem 1.1. We start with the case $v \in L^\infty(\mathbb{R}^N)$. Assume there is $M \Subset \mathbb{R}^N$ with $0 < |M| < |D|$ and $\delta := \operatorname{essinf}_M(v - w) > 0$. Without loss of generality, we may assume $\delta \le 1$. Fix $K \Subset D \setminus M$ and let $f \in C_c^2(D \setminus M)$ be given with $f \equiv 1$ in K and $0 \le f \le 1$. Let $a_0, b > 0$ be given by Lemma 3.6 and fix $a \in (0, a_0]$. Furthermore, let $u_a := v - af - \delta 1_M$ and note that $u \in \tilde{W}^{s,p}(\operatorname{supp} f)$. Then, assuming for (ii) in addition $v \in C^\alpha_{\mathrm{loc}}(D)$ for $\alpha \in (0, 1]$ as stated, we have in weak sense in $\operatorname{supp} f$ by Lemma 3.6,

$$\begin{aligned}(-\Delta)^s_p u_a &\ge (-\Delta)^s_p v + b\\ &\ge -q(x)v^{*(p-1)} + b\\ &\ge -q(x)u_a^{*(p-1)} + b + q(x)((v - af)^{*(p-1)} - v^{*(p-1)})\\ &\ge -q(x)u_a^{*(p-1)} + b - \|q\|_{L^\infty(D)}(v^{*(p-1)} - (v - a)^{*(p-1)}).\end{aligned}$$

By (2.5) if $p \le 2$, or by (2.3) with $M = \|v\|_{L^\infty(\mathbb{R}^N)}$, there is $C > 0$ depending only on p, and if $p > 2$ on M, such that

$$(v^{*(p-1)} - (v - a)^{*(p-1)}) \le C \max\{a, a^{p-1}\}.$$

It follows that

$$b - \|q\|_{L^\infty(D)}(v^{*(p-1)} - (v - a)^{*(p-1)}) \ge b - \|q\|_{L^\infty(D)} C \max\{a, a^{p-1}\}.$$

Hence we may choose $a_1 \in (0, a_0]$ such that $b - \|q\|_{L^\infty(D)} C \max\{a_1, a_1^{p-1}\} > 0$. Then u_{a_1} satisfies

$$(-\Delta)^s_p u_{a_1} + q(x)u_{a_1}^{*(p-1)} \ge g \quad \text{in } \operatorname{supp} f,$$

and $u_{a_1} \ge w$ in $\mathbb{R}^N \setminus \operatorname{supp} f$. Lemma 3.1 implies $u_{a_1} \ge w$ a.e. in $\operatorname{supp} f$, and hence $v \ge w +_{a_1} f$ in $\operatorname{supp} f$. In particular,

$$v \ge w +_{a_1} \quad \text{in } K.$$

Since K and f were chosen arbitrarily, this shows

$$\operatorname*{essinf}_K (v - w) > 0 \quad \text{for all } K \Subset D \setminus M.$$

Since $0 < |M| < |D|$, we may fix $\tilde{M} \Subset D \setminus M$ with $\tilde{\delta} := \operatorname{essinf}_{\tilde{M}}(v - w) > 0$. By repeating the above argument now with $D \setminus \tilde{M}$ in place of $D \setminus M$, this shows the claim for case (i) and for case (ii) with $v \in C^\alpha_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$.

If $w \in C^\alpha_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$, we note that Lemma 3.6 implies also the existence of $a_0, b > 0$ such that

$$(-\Delta)^s_p(w + af + \delta 1_K) \le (-\Delta)^s_p w - b$$

for all $a \in (0, a_0]$. Proceeding as in the first case, we find $v \ge w + af$ in $\operatorname{supp} f$, and since f was chosen arbitrarily, this finishes the proof similarly to the first case. □

Remark 3.7. We note that, as for the weak maximum principle, it is also with almost the same proof possible to show the following: Let $D \subset \mathbb{R}^N$ be an open set, let $q \in L^\infty(D)$, $q \ge 0$, $s \in (0, 1)$, $p > 1$, and assume that $v, w \in \tilde{W}^{s,p}(D)$ satisfy in weak sense

$$\begin{aligned}(-\Delta)^s_p v - (-\Delta)^s_p w &\ge -q(x)(v - w)^{*(p-1)} && \text{in } D,\\ v &\ge w && \text{in } \mathbb{R}^N.\end{aligned}$$

If one of the two conditions

(i) $\frac{1}{1-s} < p \le 2$ and $v \in L^\infty(\mathbb{R}^N)$ or $w \in L^\infty(\mathbb{R}^N)$, or

(ii) $p \ge 2$ and for some $\alpha \in (0, 1]$ with $\alpha(p - 2) > sp - 1$ we have $v \in C^\alpha_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$ or $w \in C^s_{\mathrm{loc}}(D) \cap L^\infty(\mathbb{R}^N)$,

holds, then either $v \equiv w$ a.e. in $\mathbb{R}^N$ or $\operatorname{essinf}_K(v - w) > 0$ for all $K \Subset D$.

Corollary 3.8 (Strong Maximum Principle). *Let $D \subset \mathbb{R}^N$ be an open set and let $f : D \times \mathbb{R} \to [0, \infty)$ be such that $f(x, 0) = 0$ for all $x \in D$ and*

$$\int_K f(x, v(x))\varphi(x)\,dx < \infty \quad \text{for all } K \Subset D \text{ and } v, \varphi \in L^p(K),\ v, \varphi \ge 0.$$

Moreover, let $v \in \tilde{W}^{s,p}(D)$ be a nonnegative function satisfying in weak sense

$$(-\Delta)_p^s v \geq f(x, v) \quad \text{in } D.$$

If $p > \frac{1}{1-s}$, then either $v \equiv 0$ in $\mathbb{R}^N$ or $\operatorname{essinf}_K v > 0$ for all $K \Subset \Omega$.

Proof. Since $0 \in W^{s,p}(\mathbb{R}^N) \cap C_c^{0,1}(\mathbb{R}^N)$ satisfies $(-\Delta)_p^s 0 = 0 \leq f(x, v) \leq (-\Delta)_p^s v$, an application of Theorem 1.1 proves the claim. □

4 Starshaped Superlevel Sets

As in [12], we use the following observation.

Lemma 4.1 (see [12, Lemma 3.1]). *Let $u : \mathbb{R}^N \to \mathbb{R}$ such that $M = \max_{\mathbb{R}^N} u = u(0)$. Then the superlevel sets $U(\ell)$, $\ell \in \mathbb{R}$, of u are all (strictly) starshaped if and only if $u(tx) \leq u(x)$ ($u(tx) < u(x)$) for every $x \in \mathbb{R}^N$ and every $t > 1$.*

Theorem 4.2. *Let $D = D_0 \setminus \overline{D}_1$ with $D_0, D_1 \subset \mathbb{R}^N$ open bounded sets such that $0 \in D_1$ and $\overline{D}_1 \subset D_0$. Moreover, let $b_0, b_1 \in L^\infty(\mathbb{R}^N) \cap \tilde{W}^{s,p}(\mathbb{R}^N)$ such that $b_0 \equiv 0$ on ∂D_0 and $b_1 \equiv 1$ on ∂D_1, let $q \in L^\infty(D)$ be a nonnegative function, and let $g \in L^{p'}(D)$ with $p' = \frac{p}{p-1}$ such that both functions satisfy* (A2)*, i.e. the following condition holds:*
(i) *For all $t \geq 1$ and $x \in \mathbb{R}^N$ such that $tx \in D_0 \setminus \overline{D}_1$, we have $t^{sp} q(tx) \geq q(x)$ and $t^{sp} g(tx) \geq g(x)$.*
Let $u \in \tilde{W}^{s,p}(\mathbb{R}^N)$ be a continuous solution of

$$\begin{cases} (-\Delta)_p^s u + q(x)|u|^{p-2}u = -g & \text{in } D, \\ u = b_0 & \text{on } D_0^c, \\ u = b_1 & \text{on } D_1, \end{cases}$$

such that $0 \leq u \leq 1$ in D. If $b_0|_{D_0^c}$ and $b_1|_{D_1}$ have starshaped superlevel sets, then the following holds:
(i) *If D_0 and D_1 are starshaped sets, then the superlevel sets $U(\ell)$ of u are starshaped for $\ell \in (0, 1)$.*
(ii) *If D_0 and D_1 are strictly starshaped sets, $0 < u < 1$ in D and*
 (a) $\frac{1}{1-s} < p \leq 2$ or
 (b) *$p \geq 2$ and $u \in C^\alpha_{\mathrm{loc}}(D)$ for some $\alpha \in (0, 1]$ with $\alpha(p-2) > sp - 1$,*
 then the superlevel sets $U(\ell)$ of u are strictly starshaped for $\ell \in (0, 1)$.

Proof. We proceed as in the proof of [12, Theorem 1.8]. Let D_0, D_1, b_0, b_1, and u be given as for (i). Note that by assumption it follows that $u \in L^\infty(\mathbb{R}^N)$. Denote for any $t > 1$ and any function $v : \mathbb{R}^N \to \mathbb{R}$,

$$v_t(x) = v(tx) \quad x \in \mathbb{R}^N.$$

Thanks to Lemma 4.1, the starshapedness of the level sets of u is equivalent to

$$u \geq u_t \quad \text{in } \mathbb{R}^N \text{ for } t > 1.$$

Observe that since the superlevel sets of b_0 and b_1 are starshaped and $0 \leq u \leq 1$ in D, we have $u \geq u_t$ in $\mathbb{R}^N \setminus D_0$ and in $t^{-1}\overline{D}_1$, and

$$u(x) \geq u_t(x) \quad \text{for } x \in D_0 \setminus (t^{-1}D_0) \text{ and } x \in \overline{D}_1 \setminus (t^{-1}\overline{D}_1).$$

Put $D_t = (t^{-1}D_0) \setminus \overline{D}_1$. It remains to investigate u_t in D_t. Note that since D_0 is bounded, D_t is empty for t large enough. By Lemma 2.8, we get (in weak sense)

$$(-\Delta)_p^s u_t = t^{sp}\big[(-\Delta)_p^s u\big]_t = -t^{sp} q(tx) u_t^{*(p-1)} - t^{sp} g(tx) \leq -q(x) u_t^{*(p-1)} + g(x) \quad \text{in } D_t,$$

where we used that $u_t \geq 0$ in $\mathbb{R}^N$. Hence Lemma 3.1 implies $u \geq u_t$ in $\mathbb{R}^N$. This proves (i).

If in addition u, D_0, and D_1 satisfy the assumptions of (ii), then observe that with the same argument as above Theorem 1.1 yields either $u_t \equiv u$ in $\mathbb{R}^N$ or $u > u_t$ in D_t. Since $u \equiv u_t$ in $\mathbb{R}^N$ is not possible for $t > 1$ due to the strict inequality $0 < u < 1$ in D, we must have $u > u_t$ in D_t for all $t > 1$. This proves (ii). □

Proof of Theorem 1.2. If D_0 and D_1 are starshaped, then the result follows immediately from Theorem 4.2 (i). Hence let D_0 and D_1 be strictly starshaped. Since the functions $v \equiv 1$ satisfies $(-\Delta)_p^s v + q(x)|v|^{p-2}v = q(x) \geq 0$ in D and $v \geq u$, $v \not\equiv u$ in D^c, Theorem 1.1 implies $u < 1$ in D. In a similar way, the function $w \equiv 0$ satisfies $(-\Delta)_p^s w + q(x)|w|^{p-2}w = 0$ in D and $u \geq w$, $u \not\equiv w$ in D^c, Theorem 1.1 implies $u > 0$ in D. Hence the claim follows from Theorem 4.2 (ii) with $b_0 \equiv 0$ and $b_1 \equiv 1$. □

Proof of Theorem 1.4. Let q, u be as stated and assume $u \not\equiv 0$ on $\mathbb{R}^N$. Then by the strong comparison principle we must have $u > 0$ on $\mathbb{R}^N_+$ since 0 is solution of $(-\Delta)_p^s u + q(x)u^{*(p-1)} = 0$ and $u \geq 0$. For $t \geq 0$, denote $u_t(x) := u(x + te_1)$ for $x \in \mathbb{R}^N$ and $H_t := \{x \in \mathbb{R}^N_+ : x_1 > t\}$. Then we have in weak sense for all $t \geq 0$,

$$(-\Delta)_p^s u + q(x)u^{*(p-1)} \geq (-\Delta)_p^s u_t(x) + q(x)u_t^{*(p-1)} \quad \text{on } H_t$$

by the assumptions on q. Hence for all $t \geq 0$, since $\lim_{|x|\to\infty}(u(x) - u_t(x)) = 0$, Lemma 3.1 implies $u_t \leq u$ on Ω_t. We state the following claim:

$$\text{for all } t > 0, \text{ we have } u_t < u \text{ on } H_t. \tag{4.1}$$

Fix $t > 0$ and assume by contradiction that $u_t \equiv u$ on $\mathbb{R}^N$. But then $u \equiv 0$ on $\{x \in \mathbb{R}^N : 0 < x_1 < t\}$, which is a contradiction to the fact that $u > 0$ on $\mathbb{R}^N_+$. Hence $u_t \not\equiv u$ on $\mathbb{R}^N$, and Theorem 1.1 implies (4.1) since $t > 0$ is arbitrary.

Note that (4.1) implies that u is strictly decreasing in x_1, but since $u \geq 0$, $u = 0$ on $(\mathbb{R}^N_+)^c$ and u is continuous, this is a contradiction, and hence we must have $u \equiv 0$ on $\mathbb{R}^N$ as claimed. □

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References

[1] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lect. Notes Unione Mat. Ital. 20, Unione Matematica Italiana, Bologna, 2016.
[2] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, *Adv. Math.* **308** (2017), 404–437.
[3] W. Chen, C. Li and G. Li, Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions, *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Article ID 29.
[4] L. M. Del Pezzo and A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, *J. Differential Equations* **263** (2017), no. 1, 765–778.
[5] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), no. 5, 1279–1299.
[6] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.
[7] M. M. Fall and S. Jarohs, Overdetermined problems with fractional Laplacian, *ESAIM Control Optim. Calc. Var.* **21** (2015), no. 4, 924–938.
[8] M. M. Fall and T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half-space, *Commun. Contemp. Math.* **18** (2016), no. 1, Article ID 1550012.
[9] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monogr. Stud. Math. 24, Pitman, Boston, 1985.
[10] A. Iannizzotto, S. Liu, K. Perera and M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, *Adv. Calc. Var.* **9** (2016), no. 2, 101–125.
[11] A. Iannizzotto, S. Mosconi and M. Squassina, Global Hölder regularity for the fractional p-Laplacian, *Rev. Mat. Iberoam.* **32** (2016), no. 4, 1353–1392.
[12] S. Jarohs, T. Kulczycki and P. Salani, Starshape of the superlevel sets of solutions to equations involving the fractional Laplacian in starshaped rings, preprint (2017), https://arxiv.org/abs/1701.02367.
[13] S. Jarohs and T. Weth, On the strong maximum principle for nonlocal operators, preprint (2017), https://arxiv.org/abs/1702.08767.
[14] J. Korvenpää, T. Kuusi and G. Palatucci, Hölder continuity up to the boundary for a class of fractional obstacle problems, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 3, 355–367.

[15] J. Korvenpää, T. Kuusi and G. Palatucci, The obstacle problem for nonlinear integro-differential operators, *Calc. Var. Partial Differential Equations* **55** (2016), no. 3, Article ID 63.
[16] E. Lindgren, Hölder estimates for viscosity solutions of equations of fractional p-Laplace type, *NoDEA Nonlinear Differential Equations Appl.* **23** (2016), no. 5, Article ID 55.
[17] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differential Equations* **49** (2014), no. 1–2, 795–826.
[18] P. Pucci and J. Serrin, The strong maximum principle revisited, *J. Differential Equations* **196** (2004), no. 1, 1–66.
[19] P. Roselli and B. Sciunzi, A strong comparison principle for the p-Laplacian, *Proc. Amer. Math. Soc.* **135** (2007), no. 10, 3217–3224.