

Research Article

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A General Approximation Approach for the Simultaneous Treatment of Integral and Discrete Operators

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Abstract: In this paper, we give a unitary approach for the simultaneous study of the convergence of discrete and integral operators described by means of a family of linear continuous functionals acting on functions defined on locally compact Hausdorff topological groups. The general family of operators introduced and studied includes very well-known operators in the literature. We give results of uniform convergence and modular convergence in the general setting of Orlicz spaces. The latter result allows us to cover many other settings as the L^p -spaces, the interpolation spaces, the exponential spaces and many others.

Keywords: Approximation by Operators, Sampling Kantorovich Operators, Convolution Operators, Durrmeyer Generalized Sampling Series, Orlicz Spaces, Modular Convergence, Locally Compact Hausdorff Topological Groups

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1 Introduction

The aim of this paper is to give a unitary approach for the simultaneous study of discrete and integral operators. In order to do this, we need to introduce a general setting dealing with a family of linear continuous functionals acting on suitable functional spaces, consisting on functions defined in locally compact Hausdorff topological groups. More precisely, we introduce the following linear integral operators:

$$T_w f(z) = \int_H \chi_w(z - h_w(t)) \cdot L_{h_w(t)} f \, d\mu_H(t), \quad (1.1)$$

where H and G are locally compact Hausdorff topological groups provided with their regular Haar measures μ_H and μ_G , respectively, $f \in M(G)$ (the space of measurable functions on G) is such that the above integral defining T_w is well defined, $h_w : H \rightarrow h_w(H) \subset G$ are homeomorphisms, $(L_{h_w(t)})_{t \in H}$ is a suitable family of linear operators $L_{h_w(t)} : M(G) \rightarrow \mathbb{R}$, and $(\chi_w)_{w>0}$ is a family of kernel functionals $\chi_w : G \rightarrow \mathbb{R}$.

For the above family (1.1), we establish a uniform approximation result for the aliasing error $T_w f - f$ (Theorem 4.2), and also a modular convergence result in the general setting of Orlicz spaces (Theorem 4.7). As pointed out in the paper, this latter result implies convergence in L^p -spaces and in other particular Orlicz spaces as, for example, interpolation spaces, exponential spaces and others.

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In order to obtain the general result of Theorem 4.7 we first test the modular convergence for continuous functions with compact support (Theorem 4.3), we establish a modular continuity of family (1.1) (Proposition 4.5), and, using a density result (Proposition 4.6), we are able to prove the main result.

It is worthwhile noting that the operators (1.1) contain various families of discrete and integral operators, very well known in the literature. Among them, by putting $G = \mathbb{R}$, $H = \mathbb{Z}$ and $h_w(k) = t_k/w$, where $(t_k)_k$ is a suitable sequence of real numbers, there are the generalized sampling series (see [3, 7, 11, 17, 18])

$$T_w f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w} \right) f \left(\frac{t_k}{w} \right),$$

where $L_{t_k/w} f = f(\frac{t_k}{w})$, the Kantorovich sampling series (see [8, 23, 24, 52–54])

$$T_w f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w} \right) \frac{w}{t_{k+1} - t_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz,$$

where

$$L_{t_k/w} f = \frac{w}{t_{k+1} - t_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz,$$

and the Durrmeyer generalized sampling series (see [5])

$$T_w f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w} \right) \cdot w \int_{\mathbb{R}} \psi(wu - t_k) f(u) du, \quad x \in \mathbb{R},$$

where

$$L_{t_k/w} f := w \int_{\mathbb{R}} \psi(wu - t_k) f(u) du,$$

and ψ is a suitable kernel function.

Note that also perturbed versions of the above operators can be produced, as shown in Section 3.

Moreover, different choices of the topological groups, the corresponding Haar measures and the families h_w and $L_{h_w(t)}$ produce classical convolution operators (see [15]) and the Mellin convolution operators (see [1, 2]), as shown in Section 5, together with their multidimensional versions. Since each of the above operators plays an important role in approximation theory and in sampling and signal theory, it seems very useful to have at disposal a unifying approach. Note that the study of the generalized sampling operators has been started by P.L. Butzer and his school at Aachen (see [11–14, 16–18, 48]), while the Kantorovich sampling series has been introduced and studied in, e.g., [8, 52, 53]. Both are important from a mathematical point of view and also in signal reconstruction: indeed, the generalized sampling operators represent extensions of the classical sampling theorem, and their Kantorovich version has an important meaning in the applications in signal processing since, by replacing the value $f(\frac{t_k}{w})$ with the average of f in a small interval close to $\frac{t_k}{w}$ (i.e., with the value $w \int_{t_k/w}^{t_{k+1}/w} f(u) du$), they reduce the “time-jitter” errors. For related references on the above operators, including connections with the classical sampling, see, e.g., [6, 7, 10, 19, 20, 23–29, 31, 34–39, 46, 49–51, 55].

In Section 5, we will show that, for the above particular cases of operators, the assumptions of our approach are all satisfied.

In Section 6, we will present some graphical representations showing the approximation of the considered operators to the function f .

Finally, we point out that the case of Durrmeyer generalized sampling series is important since it represents a case of operators for which continuous functions with compact support are approximated modularly and not with respect to the norm convergence in the Orlicz space, i.e., for which Corollary 4.4 cannot be established and assumption (L3) (given in Section 4) becomes not trivial.

2 Preliminaries

We review some basic concepts and notations which will be used throughout the paper. Let $(H, +)$ be a locally compact Hausdorff topological group, with θ_H as neutral element. It is well known (see [41, 43]) that there exists a unique (up to a multiplicative constant) left (resp. right) translation invariant regular Haar measure μ_H (resp. ν_H). In general, if A is a Borel set of H and if $-A = \{-a \mid a \in A\}$, then $\mu_H(-A) = k\nu_H(A)$ for some nonzero constant $k \in \mathbb{R}$. The measures μ_H and ν_H coincide if and only if A is an *unimodular group*. In this case, one has $\mu_H(A) = \nu_H(A) = \mu_H(-A)$ for every Borel set $A \subset H$. Compact groups, abelian groups and discrete groups are well-known examples of unimodular groups.

Now, let $(G, +)$ be a locally compact abelian topological group. Let \mathcal{B} be a local base of its neutral element θ_G . Let $M(G)$ denote the set of measurable bounded functions $f : M \rightarrow \mathbb{R}$. By $C(G)$ (resp. $C_c(G)$) we denote the subsets of $M(G)$ consisting on uniformly continuous and bounded (resp. continuous with compact support) functions $f : G \rightarrow \mathbb{R}$. It is understood that $C(G)$ and $C_c(G)$ are equipped with the standard $\|\cdot\|_\infty$ -norm.

Let $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a continuous function. It is called a φ -function if the following conditions hold:

- (i) $\varphi(0) = 0$ and $\varphi(x) > 0$ for all $x > 0$;
- (ii) φ is non-decreasing on \mathbb{R}_0^+ ;
- (iii) $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$.

If a φ -function φ is chosen, the functional

$$I_G^\varphi : M(G) \rightarrow [0, +\infty] : f \mapsto \int_G \varphi(|f(z)|) d\mu_G(z)$$

is called *modular functional* on $M(G)$.

The Orlicz space $L^\varphi(G)$ is the subset of $M(G)$ consisting of those functions $f \in M(G)$ such that

$$I_G^\varphi(\lambda f) < +\infty$$

for some $\lambda > 0$, equipped with the (strong) norm

$$\|f\|_\varphi := \inf\{\lambda > 0 \mid I_G^\varphi(\lambda f) < +\infty\}.$$

The norm $\|\cdot\|_\varphi$ is called *Luxemburg norm*. Anyway, the most natural notion of convergence in $L^\varphi(G)$, which is weaker than the norm convergence, is the *modular convergence*: a sequence $(f_n) \subset L^\varphi(\mathbb{R})$ converges modularly to a function $f \in L^\varphi(\mathbb{R})$ if there exists a number $\lambda > 0$ such that

$$\lim_{n \rightarrow +\infty} I_G^\varphi[\lambda(f_n - f)] = 0.$$

It is not difficult to see that for a sequence to converge strongly in $L^\varphi(G)$ it is necessary and sufficient that the above limit is true for every $\lambda > 0$. However, in some important cases the modular convergence and the Luxemburg norm convergence on $L^\varphi(G)$ are equivalent: this happens if and only if the so-called Δ_2 -condition is fulfilled, namely if there exists a number $M > 0$ such that

$$\frac{\varphi(2x)}{\varphi(x)} \leq M \quad \text{for every } x > 0.$$

It can be shown that the Δ_2 -condition is satisfied if and only if

$$f \in E^\varphi(G) = \{f \in L^\varphi(G) \mid I_G^\varphi(\lambda f) < +\infty \text{ for every } \lambda > 0\}.$$

For references on Orlicz spaces, see, e.g., [7, 40, 42, 44, 45].

Orlicz spaces arise as a natural generalization of L^p spaces, and in fact if $1 \leq p < \infty$ and $\varphi(x) = x^p$, then the Orlicz space generated by I_G^φ is exactly the Lebesgue space $L^p(G)$. The function $\varphi(x) = x^p$ satisfies the Δ_2 -condition, hence modular convergence is equivalent to the strong Luxemburg one, and they are in turn

equivalent to the standard L^p -norm. However, there are many other examples of Orlicz spaces which play an important role in many different situations like PDEs and functional analysis. We mention the so-called exponential spaces [33], where the φ -function generating the Orlicz space is given by $\varphi_\alpha(x) = \exp(x^\alpha) - 1$ for a certain number $\alpha > 0$. The exponential space $L^{\varphi_\alpha}(G)$ is an example in which the modular convergence is not equivalent to the Luxemburg one, and indeed φ_α does not satisfy the Δ_2 -condition. Another interesting example is furnished by the so-called interpolation spaces or Zygmund spaces [9, 30, 47]. The generating φ -function is given by $\varphi_{\alpha,\beta}(x) = x^\alpha \ln^\beta(e + x)$ for fixed $\alpha \geq 1$ and $\beta > 0$. The function $\varphi_{\alpha,\beta}$ satisfies the Δ_2 -condition, hence in $L^{\varphi_{\alpha,\beta}}(G)$ the modular and the Luxemburg convergence are equivalent.

3 Basic Assumptions and Examples

Let H and G be locally compact Hausdorff topological groups with regular Haar measures μ_H and μ_G , respectively. We require that G is abelian (but, a priori, this assumption is not required on H). Assume further that for every $w > 0$ there exists a map $h_w : H \rightarrow G$ which restricts to a homeomorphism from H to $h_w(H)$. Along this section and Section 4 we will make assumptions on both the family $(L_{h_w(t)})_{t \in H}$ and on the kernel functions $(\chi_w)_{w > 0}$. In Section 5, we will show several examples of operators for which these assumptions are all satisfied.

For every $w > 0$, let $(L_{h_w(t)})_{t \in H}$ be a family of linear operators $L_{h_w(t)} : M(G) \rightarrow \mathbb{R}$ such that the following conditions hold:

(L1) When restricted to $L^\infty(G)$, the operators $L_{h_w(t)}$ are *uniformly bounded*, i.e., if

$$\tilde{L}_{h_w(t)} := L_{h_w(t)} \upharpoonright_{L^\infty(G)} : L^\infty(G) \rightarrow \mathbb{R},$$

then

$$\|\tilde{L}_{h_w(t)}\| := \sup_{\|f\|_\infty \leq 1} |L_{h_w(t)}f| \leq Y < \infty$$

for some positive constant Y and for every $w > 0$ and $t \in H$.

(L2) If $f \in C(G)$, then the family $(L_{h_w(t)})_{t \in H}$ *preserves the continuity of f* in the sense that for every $\varepsilon > 0$ there exist $\bar{w}(\varepsilon) > 0$ and a set $B_\varepsilon \in \mathcal{B}$ such that if $z - h_w(t) \in B_\varepsilon$ and $w > \bar{w}$, then $|L_{h_w(t)}f - f(z)| < \varepsilon$.

Further, let $(\chi_w)_{w > 0} \subset L^1(G)$ be a family of kernel functionals $\chi_w : G \rightarrow \mathbb{R}$ which is uniformly bounded in $L^1(G)$ (i.e., $\|\chi_w\|_1 < \Gamma$ for some constant $\Gamma > 0$ and for every $w > 0$).

Let us introduce some notation which will be useful from now on. For $w > 0$ and $z \in G$, if $A \subset G$ is a measurable set, we define

$$A_{w,z} := \{t \in H \mid z - h_w(t) \in A\},$$

$$A_w := \{t \in H \mid h_w(t) \in A\},$$

$$Y_w(A) := \mu_H(A_w).$$

The family $(\chi_w)_w$ is assumed to be chosen in such a way that

(χ_1) $t \mapsto \chi_w(z - h_w(t)) \in L^1(H)$ for every $z \in G$ and $w > 0$;

(χ_2) for every $w > 0$ and $z \in G$,

$$\int_H \chi_w(z - h_w(t)) d\mu_H(t) = 1;$$

(χ_3) for every $w > 0$,

$$M := \sup_{z \in G} \int_H |\chi_w(z - h_w(t))| d\mu_H(t) < \infty;$$

(χ_4) if $w > 0$, $z \in G$ and $B \in \mathcal{B}$, then

$$\lim_{w \rightarrow +\infty} \int_{H \setminus B_{w,z}} |\chi_w(z - h_w(t))| d\mu_H(t) = 0$$

uniformly with respect to $z \in G$;

(χ_5) for every $\varepsilon > 0$ and for every compact set $K \subset G$, there exist a symmetric compact set $C \subset G$ such that, for every $t \in K_w$, we have

$$\int_{G \setminus C} \Upsilon_w(K) |\chi_w(z - h_w(t))| d\mu_G(z) < \varepsilon$$

for sufficiently large $w > 0$.

For $w > 0$, we study the operator $T_w : M(G) \rightarrow R$ defined by

$$T_w f(z) = \int_H \chi_w(z - h_w(t)) \cdot L_{h_w(t)} f d\mu_H(t), \tag{3.1}$$

where $f \in M(G)$ is such that the above integral defining T_w is well defined.

There are a lot of known examples of operators which can be expressed in the form (3.1). We describe some (but certainly not all) kinds of operators which are included in the general setting introduced by (3.1).

(1) If $G = \mathbb{R}$, $H = \mathbb{Z}$ with the counting measure $d\mu_H(t)$, then (3.1) becomes a series, namely

$$\sum_{k \in \mathbb{Z}} \chi_w(z - h_w(k)) \cdot L_{h_w(k)} f.$$

This type of series has been introduced in [54]. According to the form of the operators $L_{h_w(k)}$, we obtain various types of the so-called *sampling series*.

If, for example, $h_w(k) = t_k/w \in \mathbb{R}$, and $(t_k)_k$ is an increasing sequence of real numbers such that

$$\lim_{k \rightarrow \pm\infty} t_k = \pm\infty, \quad \delta < t_{k+1} - t_k < \Delta \quad (\delta, \Delta > 0),$$

we can define $L_{h_w(k)} : M(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$L_{h_w(k)} f = L_{t_k/w} f = f\left(\frac{t_k}{w}\right).$$

The operator $T_w f$ translates into the *generalized sampling series*, namely

$$T_w^{(1)} f(x) = \sum_{k \in \mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}\right) f\left(\frac{t_k}{w}\right).$$

Instead, we can define $L_{t_k/w} f : M(\mathbb{R}) \rightarrow \mathbb{R}$ as

$$L_{t_k/w} f = \frac{w}{t_{k+1} - t_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz.$$

In this case, operators (3.1) give rise to the *Kantorovich sampling series*

$$T_w^{(2)} f(x) = \sum_{k \in \mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}\right) \frac{w}{t_{k+1} - t_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz.$$

In $T_w^{(2)} f$ the sampling values $f(t_k/w)$ are replaced by averages, with the effect of reducing the so-called time-jitter errors and of regularizing the series, allowing to study the convergence in a wider sense (see [8, 21, 22, 52]).

A more general class of sampling series is given by the so-called *Durrmeyer generalized sampling series*. Let $\psi \in L^1(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} \psi(u) du = 1,$$

and let

$$L_{h_w(k)} f = L_{t_k/w} f := w \int_{\mathbb{R}} \psi(wu - t_k) f(u) du.$$

We now have the series

$$(T_w^{(3)}f)(x) = \sum_{k \in \mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}\right) \cdot L_{t_k/w}f = \sum_{k \in \mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}\right) \cdot w \int_{\mathbb{R}} \psi(wu - t_k)f(u) \, du, \quad x \in \mathbb{R}.$$

Perturbations can be added to the family of operators $L_{h_w(k)}$ to emulate the effect of the so-called time-jitter errors: so, for instance we can take $L_{h_w(k)} = f(t_k/w + j_k(w))$, where $\lim_{w \rightarrow +\infty} j_k(w) = 0$ uniformly with respect to $k \in \mathbb{Z}$, to obtain a "time-jitter" generalized sampling series; see, e.g. [14, 53]. We can also produce series analogous to $T_w^{(2)}f(x)$ and $T_w^{(3)}f(x)$ by adding suitable time-jitter perturbations.

(2) If $H = G = \mathbb{R}$, then we retrieve some interesting integral operators together with their Kantorovich versions. For instance, we can take $h_w(t) = t$ and $L_{h_w(t)}f = L_t f = f(t)$, and (3.1) translates into a convolution operator, namely

$$T_w^{(4)}f(x) = \int_{\mathbb{R}} \chi_w(x - t)f(t) \, dt.$$

However, other types of integral operators can be generated: for example, if we take $h_w(t) = t$ and

$$L_{h_w(t)} = \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) \, du,$$

we obtain

$$T_w^{(5)}f(x) = \int_{\mathbb{R}} \chi_w(x - t) \left(\frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) \, du \right) dt,$$

which is a Kantorovich version of a convolution operator. Note that, whenever $f \in L_{loc}^1(\mathbb{R})$, the factor

$$\frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) \, du$$

converges, as $w \rightarrow +\infty$, to $f(t)$ for a.a. $t \in \mathbb{R}$ (by the Lebesgue–Besicovich theorem). This fact testifies that, asymptotically, the operator $T_w^{(5)}f$ can be compared to the standard convolution operator.

As another example, let us now consider $H = G = \mathbb{R}^+$. In this case, the group operation is the product, and the only regular Haar measure on \mathbb{R}^+ (up to multiplicative constants) is the logarithmic measure $d\mu(t) = \frac{dt}{t}$. We can then set $h_w(t) = t$ and $L_{h_w(t)}f = L_t f = f(t)$. Under these assumptions, we are left with

$$T_w^{(6)}f(x) = \int_0^\infty \chi_w\left(\frac{x}{t}\right)f(t) \frac{dt}{t},$$

which is the Mellin convolution operator.

Moreover, we can modify the operators $L_{h_w(t)}f$ by taking

$$L_{h_w(t)}f = \frac{1}{2 \ln(1 + 1/w)} \int_{t^{\frac{w}{w+1}}}^{t^{\frac{w+1}{w}}} f(u) \frac{du}{u},$$

and obtain a Kantorovich version of the Mellin convolution, namely

$$T_w^{(7)}f(x) = \int_0^\infty \chi_w\left(\frac{x}{t}\right) \left(\frac{1}{2 \ln(1 + 1/w)} \int_{t^{\frac{w}{w+1}}}^{t^{\frac{w+1}{w}}} f(u) \frac{du}{u} \right) \frac{dt}{t}.$$

Note that, for large values of $w > 0$, the factor $\frac{1}{2 \ln(1+1/w)}$ can be replaced simply by $\frac{w}{2}$.

Finally, the setting of locally compact topological groups allows us to cover also the multidimensional version of the above operators.

4 Approximation Results

In this section, we state and prove the results concerning the convergence of (3.1). Besides Theorem 4.2, which proves the convergence of (3.1) to f uniformly for $f \in C(G)$, Theorem 4.7 proves the convergence of (3.1) to f in a subspace \mathcal{Y} of an Orlicz space $L^\varphi(G)$, where, from now on, φ is a convex φ -function. We will achieve this result by density, and by a number of preliminary (but important) facts which need to be proved. In the course of the discussion, we will also state two more assumptions on the family of operators $(L_{h_w(t)})_{t \in H}$, which will in turn both restrict the admissible families $(L_{h_w(t)})_{t \in H}$ and define the subspace \mathcal{Y} in which the main convergence result can be proved.

Proposition 4.1. *Let $f \in L^\infty(G)$. Then the operator T_w is well defined for every $w > 0$, and in fact*

$$|T_w f(z)| \leq M\gamma \|f\|_\infty.$$

Proof. Let us take $f \in L^\infty(G)$. Then

$$|T_w f(z)| \leq \int_H |\chi_x(z - h_w(t))| \cdot |L_{h_w(t)} f| \, d\mu_H(t) \leq \gamma \|f\|_\infty \int_H |\chi_w(z - h_w(t))| \, d\mu_H(t) \leq M\gamma \|f\|_\infty. \quad \square$$

Now we prove the first result of convergence.

Theorem 4.2. *Let $f \in C(G)$. Then $T_w f$ converges uniformly to f , i.e.,*

$$\lim_{w \rightarrow +\infty} \|T_w f - f\|_\infty = 0.$$

Proof. Let $f \in C(G)$. We have

$$|T_w f(z) - f(z)| \leq \int_H |\chi_w(z - h_w(t))| \cdot |L_{h_w(t)} f - f(z)| \, d\mu_H(t) =: A.$$

Fix $\varepsilon > 0$, and let $B \in \mathcal{B}$ be such that if $z - h_w(t) \in B$, then $|L_{h_w(t)} f - f(z)| < \varepsilon$, for sufficiently large $w > 0$, by property (L2). Now,

$$A = \int_{B_{w,z}} |\chi_w(z - h_w(t))| \cdot |L_{h_w(t)} f - f(z)| \, d\mu_H(t) + \int_{H \setminus B_{w,z}} |\chi_w(z - h_w(t))| \cdot |L_{h_w(t)} f - f(z)| \, d\mu_H(t) =: A_1 + A_2.$$

We have

$$A_1 = \int_{B_{w,z}} |\chi_w(z - h_w(t))| \cdot |L_{h_w(t)} f - f(z)| \, d\mu_H(t) \leq \varepsilon M$$

by using property (χ_3) . Now, concerning A_2 , we have

$$A_2 = \int_{H \setminus B_{w,z}} |\chi_w(z - h_w(t))| \cdot |L_{h_w(t)} f - f(z)| \, d\mu_H(t) \leq (\gamma + 1) \|f\|_\infty \int_{H \setminus B_{w,z}} |\chi_w(z - h_w(t))| \, d\mu_H(t).$$

The proof now follows since ε is arbitrarily chosen and by using property (χ_4) . □

Several examples show that assumptions (L1)–(L2) alone do not guarantee the convergence of $T_w f$ in Orlicz spaces, even in the case when $f \in C_c(G)$. What we need is an assumption which describes the behavior of the family $(L_{h_w(t)})$ outside the support of a function $f \in C_c(G)$. We state this assumption as follows:

(L3) Let $f \in C_c(G)$ and let $\text{Supp } f = \tilde{K} \subset G$. We assume that there exist a compact set $K \supset \tilde{K}$ and a constant $\alpha > 0$ such that for every $\varepsilon > 0$ there exists $\bar{w} > 0$ such that

$$\int_{H \setminus K_w} \varphi(\alpha |L_{h_w(t)} f|) \, d\mu_H(t) < \varepsilon$$

for every $w > \bar{w}$.

With the help of this property, we can prove the following theorem.

Theorem 4.3. *Let $f \in C_c(G)$ and let (L3) be valid. Then for every $\lambda < \frac{\alpha}{M}$ we have*

$$\lim_{w \rightarrow +\infty} I^\varphi(T_w f - f) = 0, \quad (4.1)$$

that is, $T_w f$ converges modularly to f in $L^\varphi(G)$.

Proof. Theorem 4.2 tells us that for every $\lambda > 0$ we have

$$\lim_{w \rightarrow +\infty} I^\varphi(\lambda \|T_w f - f\|_\infty) = 0. \quad (4.2)$$

Next, let $\tilde{K} = \text{Supp } f$ and let $K \supset \tilde{K}$ and $\alpha > 0$ such that (L3) holds. We apply the Vitali convergence theorem to the family of functions $(\varphi(T_w f(\cdot)))_w$. For $\varepsilon > 0$ and $K \subset G$ as above, there exists a symmetric compact set $C \subset G$ such that (χ_5) holds, i.e.,

$$\int_{G \setminus C} \Upsilon_w(K) |\chi_w(z - h_w(t))| d\mu_G(z) < \varepsilon$$

for every $t \in K_w$ and for every sufficiently large $w > 0$.

Using the Jensen inequality and the Fubini–Tonelli theorem, we have

$$\begin{aligned} J &= \int_{G \setminus C} \varphi(\lambda |T_w f(z)|) d\mu_G(z) \\ &= \int_{G \setminus C} \varphi\left(\lambda \left| \int_H \chi_w(z - h_w(t)) \cdot L_{h_w(t)} f d\mu_H(t) \right|\right) d\mu_G(z) \\ &\leq \frac{1}{\Upsilon_w(K)M} \int_H \left(\varphi(\lambda M |L_{h_w(t)} f|) \int_{G \setminus C} |\chi_w(z - h_w(t))| \Upsilon_w(K) d\mu_G(z) \right) d\mu_H(t) \\ &\leq \frac{1}{\Upsilon_w(K)M} \left[\int_{K_w} \cdots + \int_{H \setminus K_w} \cdots \right] =: J_1 + J_2. \end{aligned}$$

Now,

$$\begin{aligned} J_1 &= \frac{1}{\Upsilon_w(K)M} \int_{K_w} \left(\varphi(\lambda M |L_{h_w(t)} f|) \int_{G \setminus C} |\chi_w(z - h_w(t))| \Upsilon_w(K) d\mu_G(z) \right) d\mu_H(t) \\ &\leq \frac{1}{\Upsilon_w(K)M} \cdot \Upsilon_w(K) \varphi(\lambda M \Upsilon \|f\|_\infty) \cdot \varepsilon \\ &= \frac{1}{M} \cdot \varphi(\lambda M \Upsilon \|f\|_\infty) \cdot \varepsilon. \end{aligned}$$

Concerning J_2 , choose $\lambda > 0$ such that $\lambda M < \alpha$. We have, using (L3),

$$J_2 \leq \frac{1}{M} \|\chi_w\|_1 \varepsilon$$

for sufficiently large $w > 0$. In conclusion, for sufficiently large $w > 0$,

$$J \leq (\varphi(\lambda M \Upsilon \|f\|_\infty) + \|\chi_w\|_1) \frac{\varepsilon}{M} < (\varphi(\lambda M \Upsilon \|f\|_\infty) + \Gamma) \frac{\varepsilon}{M},$$

where $\Gamma > 0$ is an upper bound for the family $(\chi_w)_{w>0}$ in $L^1(G)$. Moreover, for every measurable set $A \subset G$ with $\mu_G(A) < \infty$ we have

$$\int_A \varphi(\lambda |T_w f(z)|) d\mu_G(z) \leq \mu_G(A) \varphi(\lambda M \Upsilon \|f\|_\infty).$$

Hence, for every $\varepsilon > 0$ it suffices to take

$$\delta \leq \frac{\varepsilon}{\varphi(\lambda M \Upsilon \|f\|_\infty)}$$

to have

$$\int_A \varphi(\lambda|T_w f(z)|) d\mu_G(z) < \varepsilon$$

if $\mu_G(A) < \delta$. This shows that the Vitali convergence theorem can be applied to the functions $\varphi(T_w f(\cdot))$, and, using (4.2), the proof follows at once. \square

The limit (4.1) is valid when $\lambda M < \alpha$. If the family $\{L_{h_w(t)}\}$ satisfies $L_{h_w(t)} f = 0$ whenever $t \notin K_w$, then assumption (L3) is not necessary, and there is no reason for choosing a particular α in the proof of Theorem 4.3. Hence the operators $T_w f$ converge to f in the stronger Luxemburg norm. We state this result as follows.

Corollary 4.4. *Let $f \in C_c(G)$. Let $\tilde{K} = \text{Supp } f$ and let $K \supset \tilde{K}$ be a compact set in G which contains \tilde{K} . Assume that $L_{h_w(t)} f = 0$ for sufficiently large $w > 0$ and for every $t \notin K_w$. Then*

$$\lim_{w \rightarrow +\infty} \|T_w f - f\|_\varphi = 0.$$

We point out that if one wants to extend the convergence in Orlicz spaces to a family larger than $C_c(G)$, then one has technical difficulties to face. Let us show what happens: estimating $I^\varphi(\lambda T_w f)$ ($\lambda > 0$), we obtain

$$I^\varphi(\lambda T_w f) \leq \frac{1}{M} \int_H \left[\varphi(M\lambda|L_{h_w(t)} f|) \int_G |\chi_w(z - h_w(t))| d\mu_G(z) \right] d\mu_H(t) \leq \frac{\|\chi_w\|_1}{M} \int_H \varphi(M\lambda|L_{h_w(t)} f|) d\mu_H(t) \quad (4.3)$$

by using again the Jensen inequality and the Fubini–Tonelli theorem. It is now clear that if we want to achieve a desired result of convergence in Orlicz space for a function $f \in L^\varphi(G)$, the last inequality in (4.3) should be compared to $I^\varphi(\lambda\beta f)$ for some $\beta > 0$, in order to have a modular continuity of the operators. In general, this is not possible, and in fact one does not even know if the values $I^\varphi(\lambda T_w f)$ exist. Hence we need an additional assumption which determines a restriction on both the family $\{L_{h_w(t)}\}$ and the functions $f \in L^\varphi(G)$ for which the convergence can be actually obtained. The assumption reads as follows:

(L4) Let φ be a convex φ -function. There exists a subspace $\mathcal{Y} \subset L^\varphi(G)$ with $C_c^\infty(G) \subset \mathcal{Y}$ such that for every $f \in \mathcal{Y}$ and for every $\lambda > 0$ there exist constants $c = c(\lambda, f)$, $\beta = \beta(\lambda, f) > 0$ satisfying

$$\limsup_{w \rightarrow +\infty} \|\chi_w\|_1 \int_H \varphi(\lambda|L_{h_w(t)} f|) d\mu_H(t) \leq c I^\varphi(\lambda\beta f).$$

Later, we will discuss some examples for which (L4) is satisfied.

The first consequence of assumption (L4) is the fact that T_w maps \mathcal{Y} into $L^\varphi(G)$; indeed, we have the following proposition.

Proposition 4.5. *Let φ be a convex φ -function. Let (L3) and (L4) be satisfied. Then for every $f \in \mathcal{Y}$ and $\lambda > 0$ we have*

$$I^\varphi(\lambda T_w f) \leq \frac{c}{M} I^\varphi(\lambda\beta M f)$$

for sufficiently large $w > 0$. In particular, $T_w : \mathcal{Y} \rightarrow L^\varphi(G)$ for sufficiently large $w > 0$.

We now need a density result (see, e.g., [4]).

Proposition 4.6. *The set $C_c^\infty(G)$ is dense in $L^\varphi(G)$ with respect to the modular convergence.*

Theorem 4.7. *Let $f \in \mathcal{Y}$ and let (L1)–(L4) be valid. Then there exists $\lambda > 0$ such that*

$$\lim_{w \rightarrow +\infty} I^\varphi(\lambda(T_w f - f)) = 0,$$

i.e., $T_w f$ converges modularly to f as $w \rightarrow +\infty$.

Proof. Take $f \in \mathcal{Y}$ and fix $\varepsilon > 0$. We can find a constant $\bar{\lambda} > 0$ such that for every $\varepsilon > 0$ there exists $g \in C_c^\infty(G)$ with

$$I^\varphi(\bar{\lambda}(f - g)) < \varepsilon.$$

Moreover,

$$I^\varphi(\lambda(T_w g - g)) < \varepsilon$$

for every $\lambda < \frac{\alpha}{M}$ and for sufficiently large $w > 0$ by Theorem 4.3. Choose $\lambda \leq \min\{\frac{\alpha}{3M}, \frac{\bar{\lambda}}{3}, \frac{\bar{\lambda}}{3\beta M}\}$. Then, for sufficiently large $w > 0$,

$$\begin{aligned} I^\varphi(\lambda(T_w f - f)) &\leq I^\varphi(3\lambda(T_w f - T_w g)) + I^\varphi(3\lambda(T_w g - g)) + I^\varphi(3\lambda(f - g)) \\ &\leq \frac{c}{M} I^\varphi(\bar{\lambda}(f - g)) + I^\varphi(3\lambda(T_w g - g)) + I^\varphi(\bar{\lambda}(f - g)) \\ &\leq \left(\frac{c}{M} + 2\right) \cdot \varepsilon. \end{aligned}$$

The proof follows easily since ε can be chosen arbitrarily. \square

5 Examples and Applications

In this section, we apply the previous results to some operators which can be generated by (3.1). Some of the examples which we will show below have been introduced in Section 3.

We first test the validity of assumptions (L1)–(L4) for the series $T_w^{(1)}f(x)$ and $T_w^{(2)}f(x)$ since they are very well known in the theory of sampling series.

We start with the operator

$$T_w^{(1)}f(x) = \sum_{k \in \mathbb{Z}} \chi_w\left(x - \frac{t_k}{w}\right) f\left(\frac{t_k}{w}\right).$$

In this case, $H = \mathbb{Z}$, $G = \mathbb{R}$, $h_w(t_k) = t_k/w$ and $L_{t_k/w}f = f(t_k/w)$, where $(t_k)_{k \in \mathbb{Z}}$ is an increasing sequence of real numbers such that

$$\lim_{k \rightarrow \pm\infty} t_k = \pm\infty, \quad \delta < \Delta_k := t_{k+1} - t_k < \Delta \quad (\delta, \Delta > 0).$$

Clearly, the family $(L_{t_k/w})_{k \in \mathbb{Z}}$ satisfies (L1) and (L2), and in fact $\|\tilde{L}_{t_k/w}\| = 1$ for every $k \in \mathbb{Z}$ and $w > 0$. Assumption (L3) is satisfied as well, and in particular if $f \in C_c(\mathbb{R})$ and $K = [-\gamma, \gamma] = \text{Supp}(f)$, then $L_{t_k/w}f \equiv 0$ whenever $t_k/w \notin K$. It follows that the stronger version of Theorem 4.3 (i.e., Corollary 4.4) is valid. Concerning (L4), we first observe that

$$\limsup_{w \rightarrow +\infty} w \|\chi_w\|_1 \sum_{k \in \mathbb{Z}} \varphi\left(\lambda \left| f\left(\frac{t_k}{w}\right) \right| \right) \leq \limsup_{w \rightarrow +\infty} \frac{w \|\chi_w\|_1}{\delta} \sum_{k \in \mathbb{Z}} \frac{\Delta_k}{w} \varphi\left(\lambda \left| f\left(\frac{t_k}{w}\right) \right| \right). \quad (5.1)$$

Now, the sum in the right-hand side of (5.1) is a Riemann sum, and

$$\limsup_{w \rightarrow +\infty} \sum_{k \in \mathbb{Z}} \frac{\Delta_k}{w} \varphi\left(\lambda \left| f\left(\frac{t_k}{w}\right) \right| \right) = I^\varphi(\lambda f)$$

whenever $f \in E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ ($BV^\varphi(\mathbb{R})$ is the set of those functions such that $\varphi(\lambda|f|) \in BV(\mathbb{R})$ for every $\lambda > 0$); see, e.g., [7, 32]. It follows that the limit in (5.1) is finite and gives exactly (L4) (with $\beta = 1$) when $f \in E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ and

$$\limsup_{w \rightarrow +\infty} w \|\chi_w\|_1 < +\infty.$$

The finiteness of the above limit can be easily achieved if we consider a single function $\chi \in L^1(\mathbb{R})$ and define the family of kernels as $\chi_w(x) = \chi(wx)$ so that $\chi_w(x - t_k/w) := \chi(wx - t_k)$, which is a common situation for the generalized sampling series. In view of these considerations, we can state the following theorem.

Theorem 5.1. *Let $f \in E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ and let the family $(\chi_w)_{w>0}$ satisfy (χ_1) – (χ_5) together with the additional assumption*

$$\limsup_{w \rightarrow +\infty} w \|\chi_w\|_1 < +\infty.$$

Then there exists a number $\lambda > 0$ such that

$$\lim_{w \rightarrow +\infty} I^\varphi(\lambda(T_w^{(1)}f - f)) = 0.$$

Note that a sufficient condition for f to belong to $E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ is that $f \in R(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$, where $R(\mathbb{R})$ is the space of absolutely Riemann-integrable functions on \mathbb{R} .

Now we consider the Kantorovich sampling series

$$T_w^{(2)} f(x) = \sum_{k \in \mathbb{Z}} \chi_w \left(x - \frac{t_k}{w} \right) \frac{w}{t_{k+1} - t_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz,$$

where $(t_k)_{k \in \mathbb{Z}}$ is a sequence of real numbers which satisfies the properties listed above. In this case, we have $h_w(k) = t_k/w$ and

$$L_{t_k/w} f = \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz, \quad \Delta_k = t_{k+1} - t_k.$$

Assumption (L1) is easily satisfied since

$$\|L_{t_k/w}\| = \sup_{\|f\|_\infty \leq 1} \left| \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz \right| = 1.$$

For (L2), if $f \in C(\mathbb{R})$, then in particular f is continuous at an arbitrary point $x \in \mathbb{R}$. Now, for every $\varepsilon > 0$ let $\gamma > 0$ be such that $|f(x) - f(z)| \leq \varepsilon$ whenever $|x - z| \leq \gamma$. It follows that

$$|L_{t_k/w} f - f(x)| = \left| \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz - f(x) \right| \leq \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(z) - f(x)| dz.$$

Let $\bar{w} = \frac{2\Delta}{\gamma}$ and $B_\varepsilon = B(0, \frac{\gamma}{2})$. The interval $(t_k/w, t_{k+1}/w)$ has length less than $\frac{\gamma}{2}$ as soon as $w > \bar{w}$, hence if $w > \bar{w}$ and $z \in (t_k/w, t_{k+1}/w)$, then $|x - z| \leq \gamma$, thus $|f(x) - f(z)| \leq \varepsilon$. This means that, for every $\varepsilon > 0$,

$$|L_{t_k/w} f - f(x)| \leq \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(z) - f(x)| dz \leq \varepsilon$$

whenever $|x - t_k/w| \leq \frac{\gamma}{2}$ and $w > \bar{w} = \frac{2\Delta}{\gamma}$, i.e., assumption (L2) is satisfied. In order to check (L3), suppose $\text{Supp}(f) = \tilde{K} = [\tilde{\gamma}, \tilde{\gamma}]$. Let $\gamma = \tilde{\gamma} + \Delta$ and $K = [-\gamma, \gamma]$. If $t_k/w \notin K$, then, for every $w > 0$,

$$\int_{t_k/w}^{t_{k+1}/w} f(z) dz = 0.$$

This implies that also in this case the stronger Corollary 4.4 holds. Now, it remains to understand what (L4) means. Using the convexity of φ , we can write

$$\begin{aligned} \|\chi_w\|_1 \sum_{k \in \mathbb{Z}} \varphi \left(\lambda \left| \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(z) dz \right| \right) &\leq \frac{w \|\chi_w\|_1}{\delta} \sum_{k \in \mathbb{Z}} \int_{t_k/w}^{t_{k+1}/w} \varphi(\lambda |f(z)|) dz \\ &= \frac{w \|\chi_w\|_1}{\delta} \int_{\mathbb{R}} \varphi(\lambda |f(z)|) dz \\ &= \frac{w \|\chi_w\|_1}{\delta} I^\varphi(\lambda f). \end{aligned}$$

The above relation implies that (L4) is satisfied by every function $f \in L^\varphi(\mathbb{R})$ as soon as

$$\limsup_{w \rightarrow +\infty} w \|\chi_w\|_1 < \infty,$$

as in the previous example. We can now state the following theorem.

Theorem 5.2. *Let the family $(\chi_w)_{w>0}$ satisfy (χ_1) – (χ_5) together with the assumption that*

$$\limsup_{w \rightarrow +\infty} w \|\chi_w\|_1 < \infty.$$

Let $f \in L^\varphi(\mathbb{R})$. Then there exists a number $\lambda > 0$ such that

$$\lim_{w \rightarrow +\infty} I^\varphi(\lambda(T_w^{(2)}f - f)) = 0.$$

We now move our attention to cases in which (3.1) is actually an integral operator. So, let $H = G = \mathbb{R}$. Apart from the convolution operator $T_w^{(4)}f$ obtained in Section 3, we focus our attention on a Kantorovich version of the convolution operator, namely $T_w^{(5)}f$ (see again Section 3). So, let $h_w(t) = t$, and set

$$L_{h_w(t)}f := \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) du.$$

We have

$$T_w^{(5)}f(x) = \int_{\mathbb{R}} \chi_w(x-t) \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) du.$$

It is fairly clear that (L1) is satisfied, and in fact $\|L_{h_w(t)}\| \equiv 1$.

Assumption (L2) is valid as well: indeed, arguing as above, let f be continuous at a point $x \in \mathbb{R}$, choose $\varepsilon > 0$, and let $\gamma > 0$ be such that $|f(x) - f(u)| < \varepsilon$ whenever $|x - u| < \varepsilon$. Set $B_\varepsilon = B(0, \frac{\gamma}{2})$ and $\bar{w} = \frac{\gamma}{\varepsilon}$. Then if $|x - t| < \frac{\gamma}{2}$, $w > \bar{w}$ and $u \in (t - \frac{1}{w}, t + \frac{1}{w})$, we have $|x - u| < |x - t| + |t - u| < \frac{\gamma}{2} + \frac{2}{w} < \gamma$, hence $|f(x) - f(u)| < \varepsilon$. It follows that

$$|L_{h_w(t)}f - f(x)| \leq \frac{w}{2} \int_{t-1/w}^{t+1/w} |f(u) - f(x)| du \leq \varepsilon.$$

This proves the validity of (L2). To check (L3), let $f \in C_c(\mathbb{R})$ and assume that $\text{Supp}(f) = [-\tilde{\gamma}, \tilde{\gamma}]$. Choose $\gamma = \tilde{\gamma} + 2$, and set $K = [-\gamma, \gamma]$. Then, for sufficiently large $w > 0$ and $t \notin K$, one has

$$\int_{t-1/w}^{t+1/w} f(u) du = 0.$$

This implies that (L3) is valid, and in particular that for every $\varepsilon > 0$, if $\text{Supp}(f) = [\tilde{\gamma}, \tilde{\gamma}]$, $\gamma = \tilde{\gamma} + 2$ and $K = [-\gamma, \gamma]$, then

$$\int_{t \notin K} \varphi \left(\lambda \left| \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) du \right| \right) dt = 0$$

for every $\lambda > 0$ and $w > 0$. In this case, Corollary (4.1) is valid, hence $T_w^{(4)}f$ converges to f in the Luxemburg norm in $L^\varphi(\mathbb{R})$. It remains to understand what (L4) means. We have

$$\begin{aligned} \|\chi_w\|_1 \int_{\mathbb{R}} \varphi \left(\lambda \left| \frac{w}{2} \int_{t-1/w}^{t+1/w} f(u) du \right| \right) dt &\leq \frac{w \|\chi_w\|_1}{2} \int_0^{2/w} \left[\int_{\mathbb{R}} \varphi \left(\lambda \left| f \left(s + t - \frac{1}{w} \right) \right| \right) ds \right] dt \\ &= \frac{w \|\chi_w\|_1}{2} \int_0^{2/w} I^\varphi(\lambda f) dt \\ &= \|\chi_w\|_1 I^\varphi(\lambda f) \end{aligned}$$

by using the Jensen inequality, the Fubini theorem and some change of variables. This means that (L4) is satisfied for every $f \in L^\varphi(\mathbb{R})$, and we note that in this case no further assumptions are required on the kernels since $\|\chi_w\|_1 \leq \Gamma < +\infty$ for every $w > 0$ by assumption. So we can state the following theorem.

Theorem 5.3. *Let the family $(\chi_w)_{w>0}$ satisfy (χ_1) – (χ_5) . Then if $f \in L^p(\mathbb{R})$, there exists a number $\lambda > 0$ such that*

$$\lim_{w \rightarrow +\infty} I^p(\lambda(T_w^{(5)}f - f)) = 0.$$

We now discuss operators $T_w^{(3)}f$ which provide an example for which assumption (L3) is valid but Corollary 4.4 is not, i.e., the family $\{L_{h_w(k)}\}$ does not satisfy $L_{h_w(t)}f = 0$ when $t \notin K_w$ (and f has compact support). For simplicity, assume $G = \mathbb{R}$, $H = \mathbb{Z}$, $t = k$, $w = n$ and $h_n(t) = \frac{k}{n}$. Let $\psi \in L^1(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} \psi(u) \, du = 1,$$

and let

$$L_{k/n}f := n \int_{\mathbb{R}} \psi(nu - k)f(u) \, du.$$

Choose a function $\chi \in L^1(\mathbb{R})$ as above, and define $\chi_n(x) = \chi(nx)$. We now have the series

$$(T_n^{(3)}f)(x) = \sum_{k \in \mathbb{Z}} \chi(nx - k) \cdot L_{k/n}f = \sum_{k \in \mathbb{Z}} \chi(nx - k) \cdot n \int_{\mathbb{R}} \psi(nu - k)f(u) \, du, \quad x \in \mathbb{R}.$$

This is the Durrmeyer generalized sampling series.

It is easy to show that if f has compact support K and if $[-\gamma, \gamma] \supset K$, then in general

$$L_{k/n}f = n \int_{-\gamma}^{\gamma} \psi(nu - k)f(u) \, du \neq 0.$$

This implies that Corollary 4.4 does not hold in general, and only the weaker Theorem 4.3 concerning modular convergence can be established.

Now, we check the properties (L_i) , $i = 1, \dots, 4$.

Assumption (L1) is easily satisfied, and in fact

$$\|L_{k/n}f\| \leq n\|f\|_{\infty} \int_{\mathbb{R}} |\psi(nu - k)| \, du = \|f\|_{\infty} \cdot \|\psi\|_1,$$

hence

$$\|L_{k/n}\|_{\infty} = \|\psi\|_1$$

for every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Now, concerning (L2), the conservation of continuity, let us set

$$K_n(x) = n\psi(nx).$$

Then

$$\|K_n\|_1 = \|\psi\|_1, \quad n \in \mathbb{N}.$$

We can rewrite

$$L_{k/n}f = \int_{\mathbb{R}} K_n\left(u - \frac{k}{n}\right)f(u) \, du.$$

This shows that $L_{k/n}$ is the restriction of the classical convolution

$$(L_n f)(v) = \int_{\mathbb{R}} K_n(u - v)f(u) \, du$$

to the values $v = \frac{k}{n}$. Now, if f is uniformly continuous and bounded, it is well known [15] that

$$\lim_{n \rightarrow \infty} \|L_n f - f\|_{\infty} = 0.$$

It follows that for every $\varepsilon > 0$ there exists a number $\bar{n} \in \mathbb{N}$ such that for every $n > \bar{n}$ and $v \in \mathbb{R}$ one has

$$|(L_n f)(v) - f(v)| \leq \frac{\varepsilon}{2}.$$

By setting $v = \frac{k}{n}$ ($n > \bar{n}$), one obtains

$$\left| L_{k/n} f - f\left(\frac{k}{n}\right) \right| \leq \frac{\varepsilon}{2}, \quad k \in \mathbb{Z}.$$

Now, fix $n > \bar{n}$. There exists $\delta > 0$ such that if $|x - \frac{k}{n}| < \delta$, then $|f(x) - f(\frac{k}{n})| < \frac{\varepsilon}{2}$. Hence, if $|x - \frac{k}{n}| < \delta$, we have

$$|L_{k/n} f - f(x)| \leq \left| (L_n f)\left(\frac{k}{n}\right) - f\left(\frac{k}{n}\right) \right| + \left| f\left(\frac{k}{n}\right) - f(x) \right| < \varepsilon.$$

This shows that (L2) is valid if f is uniformly continuous and bounded.

Now, we check (L3). Let f have as support the compact set $K \subset \mathbb{R}$. Let $\gamma > 0$ be such that $[-\gamma, \gamma] \supset K$. It suffices to show that for every $\varepsilon > 0$ there exists a compact set $[-M_n, M_n] \supset K$ such that

$$\sum_{|k| > M_n} \varphi(|L_{k/n} f|) = \sum_{|k| > M_n} \varphi\left(n \left| \int_{\mathbb{R}} \psi(nu - k) f(u) du \right| \right) < \varepsilon$$

for all sufficiently large $n \in \mathbb{N}$. The summands in the above series can be estimated by

$$\varphi\left(n \left| \int_{\mathbb{R}} \psi(nu - k) f(u) du \right| \right) \leq \frac{1}{\|\psi\|_1} \int_{\mathbb{R}} |\psi(t)| \varphi\left(\|\psi\|_1 f\left(\frac{k+t}{n}\right)\right) dt.$$

Set

$$I_n = \sum_{k \in \mathbb{Z}} \varphi\left(\|\psi\|_1 f\left(\frac{k+t}{n}\right)\right).$$

Now, for fixed $n \in \mathbb{N}$ the series I_n has only a finite number of nonzero summands, namely those which are given by the k 's such that

$$-n\gamma - t < k < n\gamma - t,$$

hence they lie in an interval of length at most $2n\gamma$. It follows that

$$I_n = \sum_{k \in \mathbb{Z}} \varphi\left(\|\psi\|_1 f\left(\frac{k+t}{n}\right)\right) \leq 2n\gamma \varphi(\|\psi\|_1 \cdot \|f\|_\infty).$$

This shows that I_n converges totally, and

$$\frac{1}{\|\psi\|_1} \int_{\mathbb{R}} |\psi(t)| \left[\sum_{k \in \mathbb{Z}} \varphi\left(\|\psi\|_1 f\left(\frac{k+t}{n}\right)\right) \right] dt \leq 2n\gamma \varphi(\|\psi\|_1 \cdot \|f\|_\infty).$$

This implies that for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a number $M_n \in \mathbb{N}$ such that

$$\sum_{|k| > M_n} \varphi(|L_{k/n} f|) < \varepsilon.$$

It remains to prove (L4). As before, set $K_n(x) = n\psi(nx)$. Then

$$L_{k/n} f = \int_{\mathbb{R}} K_n\left(u - \frac{k}{n}\right) f(u) du.$$

We have

$$\|\chi_n\|_1 \cdot \sum_{k \in \mathbb{Z}} \varphi(\lambda |L_{k/n} f|) \leq n \|\chi_n\|_1 \cdot \sum_{k \in \mathbb{Z}} \frac{1}{n} \varphi\left(\lambda \left| \int_{\mathbb{R}} K_n\left(u - \frac{k}{n}\right) f(u) du \right| \right).$$

The quantity $n\|\chi_n\|_1$ equals $\|\chi\|_1$, and the right-hand side of the above inequality is a Riemann sum of the function $\varphi(\lambda|L_n f(v)|)$ for $v = \frac{k}{n}$, where, as before,

$$(L_n f)(v) = \int_{\mathbb{R}} K_n(u - v)f(u) \, du.$$

If $f \in E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ ($BV^\varphi(\mathbb{R})$ is the set of those functions such that $\varphi(\lambda|f|) \in BV(\mathbb{R})$ for every $\lambda > 0$), then, since $L_n f$ is a convolution, $L_n f \in E^\varphi(\mathbb{R}) \cap BV^\varphi(\mathbb{R})$ as well, hence

$$\limsup_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}} \frac{1}{n} \varphi\left(\lambda \left| \int_{\mathbb{R}} K_n\left(u - \frac{k}{n}\right) f(u) \, du \right|\right) = I^\varphi(\lambda L_n f).$$

It remains to show that there exist numbers $\beta, c > 0$ such that

$$I^\varphi(\lambda L_n f) \leq c I^\varphi(\lambda \beta f).$$

We have

$$\begin{aligned} I^\varphi(\lambda L_n f) &= \int_{\mathbb{R}} \varphi\left(\lambda \left| \int_{\mathbb{R}} K_n(u - v)f(u) \, du \right|\right) \, dv \\ &\leq \frac{1}{\|K_n\|_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(\lambda \|K_n\|_1 |f(u)|) \, du \cdot |K_n(u - v)| \, dv \\ &\leq \int_{\mathbb{R}} \varphi(\lambda \|K_n\|_1 |f(u)|) \, du \\ &= \int_{\mathbb{R}} \varphi(\lambda \|\psi\|_1 |f(u)|) \, du \\ &= I^\varphi(\lambda \|\psi\|_1 f). \end{aligned}$$

We have proved (L4) (with $c = 1$ and $\beta = \|\psi\|_1$).

6 Some Graphical Examples

This section provides some graphical representations of the convergence of the operators we have studied in the previous sections. In all the examples below, the convergence must be interpreted as to be in the L^p setting.

Although the prototypical example for a kernel is obtained from the Fejer kernel function

$$F(x) = \frac{1}{2} \operatorname{sinc}^2\left(\frac{x}{2}\right),$$

where

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0, \end{cases}$$

it will be convenient for computational purposes to take a kernel with compact support over \mathbb{R} . Well-known examples of such kernels are those arising from linear combinations of the so-called B -splines functions of order $n \in \mathbb{N}$, namely

$$M_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{n}{2} + x - 1\right)_+^{n-1},$$

where the symbol $(\cdot)_+$ denotes the positive part. In Figure 1, we represent the graphs of the functions $M_3(x)$, $M_4(x)$ and $M(x) = 4M_3(x) - 3M_4(x)$. It is easy to see that $M(x)$ satisfies all the assumptions required in Section 3.

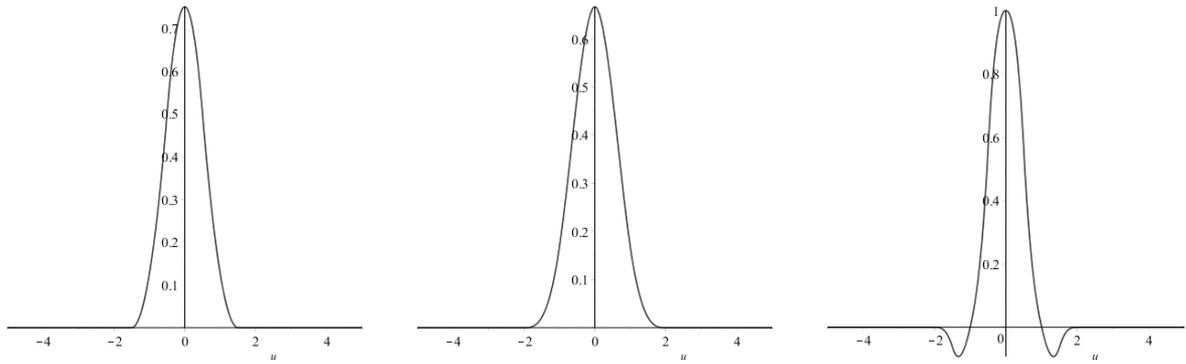


Figure 1: The graphs of $M_3(x)$, $M_4(x)$ and $M(x)$ for $-5 \leq x \leq 5$.

We first consider the “Kantorovich-type generalized sampling series”

$$(T_w^{(2)} f)(x) = \sum_{k \in \mathbb{Z}} M(wx - k)w \int_{k/w}^{(k+1)/w} f(u) du,$$

where

$$f(u) = \begin{cases} \frac{40}{u^2}, & u < -5, \\ -1, & -5 \leq u < -3, \\ 2, & -3 \leq u < -2, \\ -\frac{1}{2}, & -2 \leq u < -1, \\ \frac{3}{2}, & -1 \leq u < 0, \\ 1, & 0 \leq u < 1, \\ -\frac{1}{2}, & 1 \leq u < 2, \\ -\frac{2}{u^5}, & u \geq 2. \end{cases}$$

The graphs in Figure 2 represent the approximation of $T_w^{(2)} f(x)$ for $w = 5, 15, 40$.

In the next example, we consider the Durrmeyer generalized sampling series $T_n^{(3)} f(x)$, in the form as in Section 5. For computational convenience, we take $\psi(u) = F(u)$, and define

$$T_n^{(3)} f(x) = \sum_{k \in \mathbb{Z}} M(nx - k) \cdot n \int_{\mathbb{R}} F(nu - k) f(u) du,$$

where

$$f(x) = \begin{cases} \frac{1}{u^2}, & x < -1, \\ -1, & -1 \leq x < 0, \\ 2, & 0 \leq x < 2, \\ -\frac{3}{u^3}, & x \geq 2. \end{cases}$$

The graphs in Figure 3 show the approximation of $T_n^{(3)} f(x)$ as $n = 5, 10, 20$.

The last example takes into account the operator $T_w^{(7)} f$ introduced in Section 3. In this case, however, we cannot take kernels based on the function $M(u)$ as before because of the base space \mathbb{R}^+ and the measure $d\mu(t) = \frac{dt}{t}$. Suitable kernel functions in this case are given by

$$\mathcal{M}_w(u) = \begin{cases} wu^w, & 0 < u < 1, \\ 0, & \text{otherwise.} \end{cases}$$

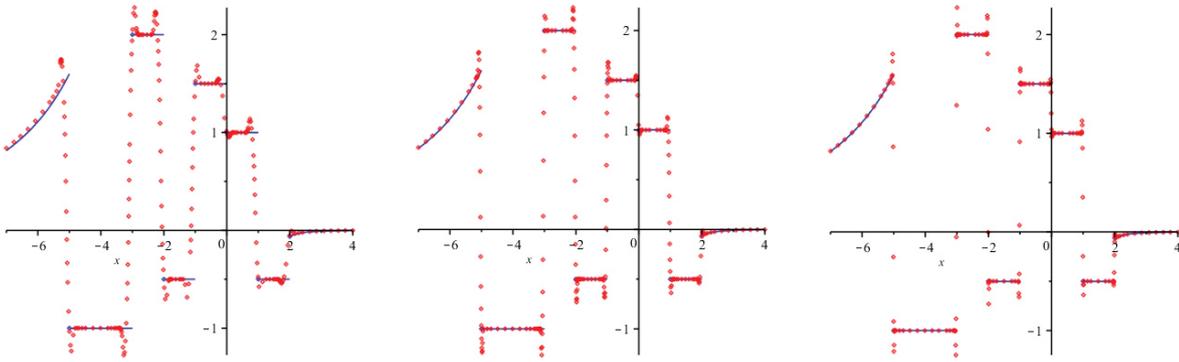


Figure 2: The graphs of the functions $T_5^{(2)}f(x)$, $T_{15}^{(2)}f(x)$ and $T_{40}^{(2)}f(x)$ (red) compared to the graph of $f(x)$ (blue).

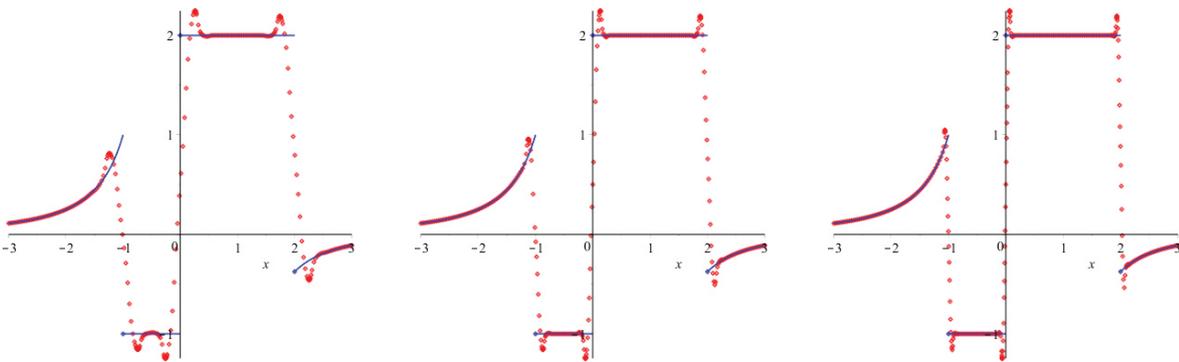


Figure 3: The graphs of the functions $T_5^{(3)}f(x)$, $T_{10}^{(3)}f(x)$ and $T_{20}^{(3)}f(x)$ (red) compared to the graph of $f(x)$ (blue).

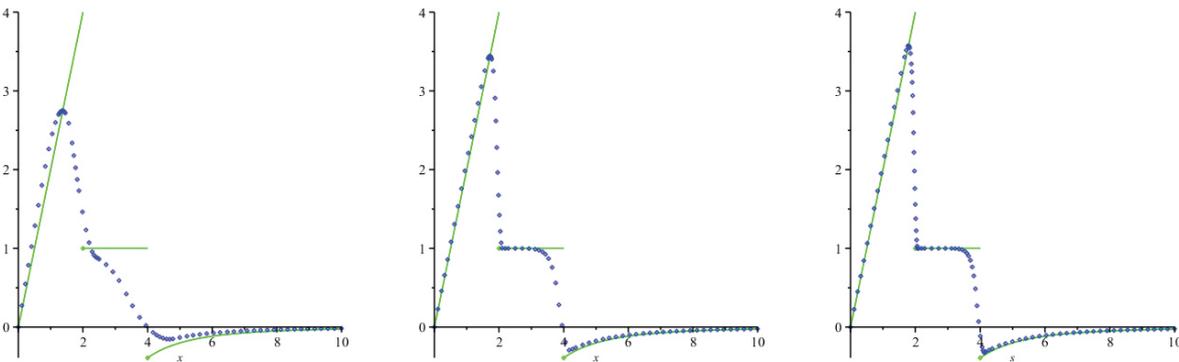


Figure 4: The graphs of the functions $T_5^{(7)}f(x)$, $T_{20}^{(7)}f(x)$ and $T_{30}^{(7)}f(x)$ (blue) compared to the graph of $f(x)$ (green).

Next, we consider the operators

$$T_w^{(7)}f(x) = \int_0^\infty \mathcal{M}_w\left(\frac{x}{t}\right) \frac{1}{2 \ln(1 + 1/w)} \left(\int_{t^{-\frac{w}{w+1}}}^{t^{\frac{w+1}{w}}} f(u) \frac{du}{u} \right) \frac{dt}{t},$$

where

$$f(x) = \begin{cases} 2x, & 0 \leq x < 2, \\ 1, & 2 \leq x < 4, \\ -\frac{25}{x^3}, & x \geq 4. \end{cases}$$

In Figure 4, we represent the approximation of the functions $T_w^{(7)}f(x)$ for $w = 5, 20$ and 30 , respectively.

Some Concluding Remarks. (i) Note that, apart from the case of the operators $T_w^{(3)}f$ (Durrmeyer generalized sampling series), where we use a uniform convergence result, for the other cases the same approximation results still hold by using the pointwise convergence. And indeed Theorem 4.2 can be established by similar reasonings for continuous functions with respect to the pointwise convergence.

(ii) We point out that when we deal with the discontinuous function f , the correct way to interpret the approximation results in the above graphical examples is, e.g., to consider approximation with respect to the L^p -norm (i.e., $\varphi(u) = u^p$). On the other hand, when we use continuous functions, the above graphs can be analogously plotted to show pointwise or uniform approximation of the considered operators to the function f .

(iii) The theory introduced and developed here represents a unified approach in order to study the convergence of several classes of operators, among them there are integral and discrete operators. In particular, we cover the cases of the generalized sampling series (with its "time-jitter" versions), the sampling Kantorovich ones, the Durrmeyer generalized sampling series and the cases of convolution operators (classical, in the Mellin sense and also of Kantorovich type).

(iv) The previous theory, set in a general Orlicz space with a general φ -function, represents also a unifying approach to formulate the approximation results in several particular cases of Orlicz spaces, interesting by themselves. Among them, for example, there are, as already mentioned in Section 2, the L^p -spaces ($p \geq 1$), the exponential spaces, the interpolation spaces or "Zygmund spaces", and many others. The latter are very important in the interpolation theory and in the theory of PDEs.

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