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Research Article

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The Bahri–Coron Theorem for Fractional Yamabe-Type Problems

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Abstract: We study the following fractional Yamabe-type equation:

$$\begin{cases} A_s u = u^{\frac{n+2s}{n-2s}}, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

Here Ω is a regular bounded domain of $\mathbb{R}^n$, $n \geq 2$, and A_s, $s \in (0, 1)$, represents the fractional Laplacian operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary condition. We investigate the effect of the topology of Ω on the existence of solutions. Our result can be seen as the fractional counterpart of the Bahri–Coron theorem [3].

Keywords: Fractional PDE, Variational Method, Critical Exponent, Loss of Compactness

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1 Introduction

In this work, we consider the following fractional Yamabe-type equation:

$$\begin{cases} A_s u = u^{\frac{n+2s}{n-2s}}, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a regular bounded domain and A_s, $s \in (0, 1)$, represents the fractional Dirichlet Laplacian operator $(-\Delta)^s$ in Ω defined by using the spectrum of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary condition. It can be viewed as the nonlocal version of the Yamabe-type equation

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}}, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The fractional Laplacian has attracted the attention of a lot of researchers in the last years since it appears in numerous applications in diverse domains including medicine, biology, physics, modeling populations, mathematical finance and nonlocal diffusion; see [10] and the references [1, 7, 19, 38, 58] therein. The nonlocal character of the fractional Laplacian makes it difficult to handle. After the breakthrough work of

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Caffarelli–Silvestre [6] who provided to the fractional Laplacian a local interpretation in one more dimension, a large amount of studies were developed on problems involving the fractional Laplacian. Here we point out only some results related to equation (1.1). In [5], Cabré and Tan studied the subcritical cases, that is, equation (1.1) with subcritical nonlinearities in the particular case $s = \frac{1}{2}$. They transformed the equation in a local form as the Caffarelli–Silvestre extension and established the existence of positive solutions. For similar extensions, we refer to [4, 7, 18]. For more results concerning the subcritical cases, we refer to [14].

Motivated by the work of Pohozaev [13] on equation (1.2), Tan [17] proved that if Ω is a starshaped domain, equation (1.1) has no solutions in the case $s = \frac{1}{2}$. For such a non-existence result, the author used a Pohozaev-type identity.

The resemblance between (1.1) and (1.2) leads us to investigate the effect of the topology of Ω on the existence of solutions of (1.1). By assuming that Ω admits a non-trivial homological group with $\mathbb{Z}_2$ coefficients $H_k(\Omega, \mathbb{Z}_2)$ of an order $k \in \mathbb{N}^*$, we get a fractional analog of the Bahri–Coron existence result [3] in form of the following theorem.

Theorem 1.1. *If there exists $k \in \mathbb{N}^*$ such that $H_k(\Omega, \mathbb{Z}_2) \neq 0$, then equation* (1.1) *admits a solution.*

The proof of Theorem 1.1 hinges on the "critical points at infinity" method and the algebraic-topological tools of [3]. Nevertheless, the nonlocal properties of the fractional Laplacian involve many additional difficulties and require some novelties in the proofs.

In Section 2, we recall some known notations related to the operator A_s, and state the associated variational formulations including the local equivalent extension of problem (1.1). Section 3 will be devoted to an asymptotic expansion of the associated Euler–Lagrange functional J. In Section 4, we will apply Bahri–Coron topological tools and prove Theorem 1.1.

2 Local Equivalent Problem and Variational Structure

First, we recall some preliminaries related to the fractional Laplacian. Let $(e_k)_{k\in\mathbb{N}}$ be the basis of $L^2(\Omega)$ such that for any $k \in \mathbb{N}$ one has $\|e_k\|_{L^2(\Omega)} = 1$, $\langle e_k, e_\ell\rangle = 0$ for all $k \neq \ell$, and

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega, \\ \quad e_k = 0 & \text{on } \partial\Omega. \end{cases}$$

So $\lambda_k > 0$ for any $k \in \mathbb{N}$.

The fractional Laplacian A_s, $s \in (0, 1)$, is defined by

$$H_0^s(\Omega) \to H_0^{-s}(\Omega) \simeq H_0^s(\Omega),$$
$$u = \sum_{k=1}^{\infty} b_k e_k \mapsto A_s(u) = \sum_{k=1}^{\infty} b_k \lambda_k^s e_k,$$

where

$$H_0^s(\Omega) := \Big\{u = \sum_{k=1}^{\infty} b_k e_k \in L^2(\Omega) : \sum_{k=1}^{\infty} b_k^2 \lambda_k^s < \infty\Big\}$$

and $H_0^{-s}(\Omega)$ is the dual space of the Hilbert fractional Sobolev space $H_0^s(\Omega)$. Concerning the local equivalent problem to (1.1), we follow the results of [6] for $\Omega = \mathbb{R}^n$, and [5] for a bounded domain Ω; see also [4, 7, 15, 18]. Therefore, we consider the associated local problem on the half cylinder with base Ω:

$$C = \Omega \times [0, \infty) = \{(x, t) : x \in \Omega \text{ and } t \in [0, \infty)\}.$$

Let

$$C_{0L}^\infty(C) = \{v \in C^\infty(\overline{C}) : v = 0 \text{ on } \partial_L C\},$$

where $\partial_L C$ denotes the lateral boundary of C, which is defined by $\partial\Omega \times [0, \infty)$. Let $H^s_{0L}(C)$ be the Hilbert Sobolev space defined as the closure of $C^\infty_{0L}(C)$ with respect to

$$|v| = \Big(\int_C t^{1-2s}|\nabla v|^2\Big)^{\frac{1}{2}},$$

and equipped by the following inner product:

$$\langle v, w\rangle_{H^s_{0L}(C)} = \int_C t^{1-2s}\nabla v \nabla w \quad \text{for all } v, w \in H^s_{0L}(C).$$

Following [4, 18], we associate to any $u \in H^s_0(\Omega)$ the unique s-harmonic function, denoted by $s - h(u)$, in $H^s_{0L}(C)$, the unique solution of the following problem:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C,\\ v = 0 & \text{on } \partial_L C,\\ v = u & \text{on } \Omega \times \{0\}; \end{cases}$$

(see [4, 18] for the explicit expression of $s - h(u)$). It follows that A_s is expressed by the following map:

$$u = \sum_{k=1}^{\infty} b_k e_k \mapsto A_s(u) = \partial^s_\nu(s - h(u))/_{\Omega\times\{0\}},$$

where ν denotes the unit outward normal vector to C on $\Omega \times \{0\}$, and for any $v \in H^s_{0L}(C)$ and any $x \in \Omega$ we have

$$\partial^s_\nu(v)(x, 0) = -c_s \lim_{t\to 0^+} t\frac{\partial v}{\partial t}(x, t) \quad \text{and} \quad c_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1 - s)}.$$

In this way, problem (1.1) is equivalent to the following local problem:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C,\\ v > 0 & \text{in } C,\\ v = 0 & \text{on } \partial_L C,\\ \partial^s_\nu(v) = v^{\frac{n+2s}{n-2s}} & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.1}$$

Therefore, if v satisfies (2.1), then $u(x) = v(x, 0) := \operatorname{tr}(v)(x)$ for all $x \in \Omega$ is a solution of (1.1).

Notice that

$$H^s_0(\Omega) = \{u = \operatorname{tr}(v) : v \in H^s_{0L}(C) \text{ with } \operatorname{div}(t^{1-2s}\nabla v) = 0 \text{ in } C\}.$$

In order to present the variational structure associated to (2.1), we introduce the following Hilbert space constructed by all s-harmonic functions in $H^s_{0L}(C)$: More precisely, let

$$\mathcal{H} = \{v \in H^s_{0L}(C) : \operatorname{div}(t^{1-2s}\nabla v) = 0 \text{ in } C\}.$$

For all $v \in \mathcal{H}$, we set

$$\|v\|^2 := |v|^2 = \int_C t^{1-2s}|\nabla v|^2\, dx\, dt = c_s^{-1} \int_{\Omega\times\{0\}} \partial^s_\nu v(x, 0).v(x, 0)\, dx,$$

and for all $v, w \in \mathcal{H}$, we set

$$\langle v, w\rangle = \langle v, w\rangle_{H^s_{0L}(C)} = c_s^{-1} \int_{\Omega\times\{0\}} \partial^s_\nu v(x, 0)w(x, 0)\, dx.$$

The first Euler–Lagrange functional is

$$I : \mathcal{H} \to \mathbb{R}, \quad v \mapsto \frac{c_s}{2}\|v\|^2 - \frac{n - 2s}{2n}\int_\Omega |v(x, 0)|^{\frac{2n}{n-2s}}\, dx$$

and its positive critical points are the unique solutions of (2.1). Since $\frac{2n}{n-2s}$ is the critical Sobolev exponent of the Sobolev trace embedding $v \in \mathcal{H} \mapsto \operatorname{tr}(v) \in L^p(\Omega)$ (which is continuous, but not compact for $p = \frac{2n}{n-2s}$), the functional I is of class C^1 and fails the Palais–Smale condition. Moreover, it is not lower bounded. Due to these considerable constraints, we will consider another functional as follows: Let Σ be the sphere of $\mathcal{H}$ defined by

$$\Sigma = \{v \in \mathcal{H} : \|v\| = c_s^{-1/2}\}.$$

Set

$$J_1 : \Sigma \to \mathbb{R}, \quad v \mapsto J_1(v) = \sup_{\lambda \geq 0} I(\lambda v).$$

Lemma 2.1. *For all $v \in \Sigma$, there exists a unique $\lambda(v) > 0$ such that $J_1(v) = I(\lambda(v)v)$.*

Proof. Let $v \in \Sigma$ such that

$$\frac{\partial}{\partial \lambda} I(\lambda v) = \lambda - \lambda^{\frac{n+2s}{n-2s}} \int_\Omega |v(x,0)|^{\frac{2n}{n-2s}}\, dx.$$

Therefore,

$$\frac{\partial}{\partial \lambda} I(\lambda v) = 0 \quad \Longleftrightarrow \quad \lambda = 0 \text{ or } \lambda = \lambda(v) := \frac{1}{\left(\int_\Omega |v(x,0)|^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{4s}}}.$$

Since $I(\lambda v)|_{\lambda=0} = 0$ and $I(\lambda v) \to -\infty$ as $\lambda \to +\infty$, the maximum of the map $\lambda \mapsto I(\lambda v)$ is achieved at $\lambda(v)$. □

Lemma 2.2. *Let $v \in \Sigma$. We have the following equivalence:*

$$v \text{ is a critical point of } J_1 \quad \Longleftrightarrow \quad \lambda(v)v \text{ is a critical point of } I.$$

Moreover, J_1 can be expressed for all $v \in \Sigma$ by

$$J_1(v) = \left(\frac{1}{2} - \frac{n-2s}{2n}\right) \frac{1}{\left(\int_\Omega |v(x,0)|^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{2s}}}.$$

Proof. Using Lemma 2.1, for all $v \in \Sigma$ we have

$$J_1'(v) = I'(\lambda(v)v)(v).\lambda'(v) + \lambda(v)I'(\lambda(v)v).$$

Observe that

$$\frac{\partial}{\partial \lambda} I(\lambda v) = I'(\lambda v)(v).$$

Therefore, we get $I'(\lambda(v)v)(v) = 0$, and hence

$$J_1'(v) = \lambda(v)I'(\lambda(v)v).$$

The expression of J_1 follows from the definition of $\lambda(v)$. □

The Sobolev trace embedding continuity implies that J_1 is lower bounded. But it is more convenient for us to work with

$$\left(\frac{1}{2} - \frac{n-2s}{2n}\right)^{\frac{-2s}{n-2s}} J_1^{\frac{2s}{n-2s}}.$$

Therefore, in what follows we will consider the Euler–Lagrange functional

$$J(v) = \frac{1}{\int_\Omega |v(x,0)|^{\frac{2n}{n-2s}}\, dx}, \quad v \in \Sigma.$$

By Lemma 2.2, if v is a positive critical point of J, then $\lambda(v)v$ is a solution of (2.1).

In Section 3, we introduce the almost solutions family of problem (2.1) and a useful expansion of J which provides the proof of Theorem 1.1.

3 Asymptotic Expansion

For $x, y \in \Omega$ and $t > 0$, let $\widetilde{G}((x,t),y)$ denote the s-harmonic extension of Green's function of the fractional Dirichlet Laplacian A_s. It satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \widetilde{G}(\cdot\,,y)) = 0 & \text{in } C,\\ \widetilde{G}(\cdot\,,y) = 0 & \text{on } \partial_L C,\\ \partial_\nu^s \widetilde{G}(\cdot\,,y) = \delta_y & \text{on } \Omega \times \{0\}. \end{cases}$$

We have

$$\widetilde{G}((x,t),y) = \frac{\hat{c}}{\|(x-y,t)\|^{n-2s}_{\mathbb{R}^{n+1}}} - \widetilde{H}((x,t),y),$$

where $\hat{c}$ is a fixed constant defined in (3.3) and $\widetilde{H}$ is the regular part of $\widetilde{G}$. The latter satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \widetilde{H}(\cdot\,,y)) = 0 & \text{in } C,\\ \widetilde{H}((x,t),y) = \dfrac{\hat{c}}{\|(x-y,t)\|^{n-2s}_{\mathbb{R}^{n+1}}} & \text{on } \partial_L C,\\ \partial_\nu^s \widetilde{H}(\cdot\,,y) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Following [8, 11, 12], we see that the family of functionals $\delta_{(a,\lambda)}$, $a \in \Omega$ and $\lambda > 0$, defined by

$$\delta_{(a,\lambda)}(x) = \frac{\lambda^{\frac{n-2s}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-2s}{2}}}, \quad x \in \mathbb{R}^n,$$

is the only solution of

$$\begin{cases} A_s u = c_0 u^{\frac{n+2s}{n-2s}} & \text{in } \mathbb{R}^n,\\ u > 0 & \text{in } \mathbb{R}^n,\\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where c_0 is a fixed positive constant which depends only n and s.

Let $\hat{\delta}_{(a,\lambda)}$ be the s-harmonic extension of $\delta_{(a,\lambda)}$ in $\mathbb{R}^{n+1}_+$. In what follows, it is more convenient to work with $\widetilde{\delta}_{(a,\lambda)}$, $a \in \Omega$ and $\lambda > 0$, defined by

$$\widetilde{\delta}_{(a,\lambda)} = \hat{\gamma}\hat{\delta}_{(a,\lambda)},$$

where

$$\hat{\gamma} = c_s^{\frac{-1}{2}} \|\hat{\delta}_{(a,\lambda)}\|^{-1}_{D^s(\mathbb{R}^{n+1}_+)} := \Big(c_s \int\limits_{\mathbb{R}^{n+1}} t^{1-2s}|\nabla \hat{\delta}_{(a,\lambda)}|^2\,dx\,dt\Big)^{\frac{-1}{2}}$$

is a fixed constant independent of a and λ. Therefore,

$$\|\widetilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)} = c_s^{\frac{-1}{2}},$$
$$\operatorname{tr}(\widetilde{\delta}_{(a,\lambda)}) = \hat{\gamma}\delta_{(a,\lambda)} \quad \text{on } \mathbb{R}^n$$

and

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \widetilde{\delta}_{(a,\lambda)}) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ \partial_\nu^s \widetilde{\delta}_{(a,y)} = \gamma_0 \widetilde{\delta}^{\frac{n+2s}{n-2s}}_{(a,y)} & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

where

$$\gamma_0 = c_0 \hat{\gamma}^{\frac{-4s}{n-2s}}.$$

The family $P\widetilde{\delta}_{(a,\lambda)}$, $a \in \Omega$ and $\lambda > 0$, of almost solutions of (2.1) is defined as the family of unique solutions of the following problem:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla P\widetilde{\delta}_{(a,\lambda)}) = 0 & \text{in } C,\\ P\widetilde{\delta}_{(a,\lambda)} = 0 & \text{on } \partial_L C,\\ \partial_\nu^s P\widetilde{\delta}_{(a,y)} = \partial_\nu^s \widetilde{\delta}_{(a,y)} = \gamma_0 \widetilde{\delta}^{\frac{n+2s}{n-2s}}_{(a,y)} & \text{on } \Omega \times \{0\}. \end{cases}$$

Next, we introduce the best constant of Sobolev. Let

$$\imath : H^s_{0L}(C) \to L^{\frac{2n}{n-2s}}(\Omega), \quad v \mapsto \operatorname{tr}(v),$$

be the Sobolev trace embedding. The best constant of Sobolev is given by

$$S = \frac{\|\operatorname{tr} \widetilde{\delta}_{(a,\lambda)}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}}{\|\widetilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)}} = c_s^{\frac{1}{2}} \|\operatorname{tr} \widetilde{\delta}_{(a,\lambda)}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}$$

since

$$\|\widetilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)} = c_s^{\frac{-1}{2}}.$$

Notice that S is independent of a and λ; see [19].

Observe that

$$\inf_{v\in\Sigma} J(v) = c_s^{\frac{n}{n-2s}} S^{\frac{-2n}{n-2s}} =: \widetilde{S}.$$

Therefore,

$$\widetilde{S} = \Big(\|\operatorname{tr} \widetilde{\delta}_{(a,\lambda)}\|^{\frac{2n}{n-2s}}_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}\Big)^{-1} = \frac{1}{\int_{\mathbb{R}^n} (\operatorname{tr} \widetilde{\delta}_{(a,\lambda)})^{\frac{2n}{n-2s}}\, dx}. \tag{3.1}$$

Remark 3.1. The equation $\widetilde{S} = \gamma_0$ holds. Indeed,

$$\|\widetilde{\delta}_{(a,\lambda)}\|^2_{D^s(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \widetilde{\delta}_{(a,\lambda)}|^2 \, dx\, dt = c_s^{-1} \int_{\mathbb{R}^n} \partial^s_\nu \widetilde{\delta}_{(a,\lambda)}(x,0) \widetilde{\delta}_{(a,\lambda)}(x,0)\, dx.$$

Thus,

$$c_s^{-1} = c_s^{-1} \gamma_0 \int_{\mathbb{R}^n} \widetilde{\delta}_{(a,\lambda)}(x,0)^{\frac{2n}{n-2s}}\, dx,$$

and therefore

$$\gamma_0 = \frac{1}{\int_{\mathbb{R}^n} (\operatorname{tr} \widetilde{\delta}_{(a,\lambda)})^{\frac{2n}{n-2s}}\, dx}.$$

Remark 3.2. $\widetilde{S}$ is achieved in the case of $\Omega = \mathbb{R}^n$. But for a bounded domain Ω, $\widetilde{S}$ is never achieved by J; see [9].

In order to give the expansion of the Euler–Lagrange functional J associated to problem (2.1), we introduce the following notations: Let K be a compact set in Ω. For any $\lambda > 0$ and $p \in \mathbb{N}^*$, we define

$$g_{\lambda,p} : \Delta_{p-1} \times K^p \to \Sigma, \quad (\alpha, a) \mapsto c_s^{\frac{-1}{2}} \frac{\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}}{\|\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\|},$$

where $\Delta_{p-1} = \{\alpha = (\alpha_1, \dots, \alpha_p) : \alpha_i \in (0,1) \text{ for all } i, \text{ and } \sum_{i=1}^p \alpha_i = 1\}$ and $a = (a_1, \dots, a_p)$. Without loss of generality, we can assume that $a_i \neq a_j$ for all $i \neq j$. In what follows, we set $d_a := \min_{i\neq j} |a_i - a_j|$. We then have the following expansion of the functional $J \circ g_{\lambda,p}$.

Proposition 3.3. *Let $p \in \mathbb{N}^*$ and $\lambda > 0$. For all $(\alpha, a) \in \Delta_{p-1} \times K^p$, we have*

$$\begin{aligned} J(g_{\lambda,p}(\alpha,a)) = \widetilde{S} \frac{(\sum_{i=1}^p \alpha_i^2)^{\frac{n}{n-2s}}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}} \Bigg\{ 1 - \frac{c_2}{\lambda^{n-2s}} \Bigg[\sum_{j=1}^p \Bigg(\frac{\alpha_j^2}{\sum_{i=1}^p \alpha_i^2} - \frac{2\alpha_j^{\frac{2n}{n-2s}}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}} \Bigg) \widetilde{H}((a_j,0),a_j) \\ + \sum_{k\neq j} \Bigg(\frac{2\alpha_k \alpha_j^{\frac{n+2s}{n-2s}}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}} - \frac{\alpha_j \alpha_k}{\sum_{i=1}^p \alpha_i^2} \Bigg) \widetilde{G}((a_i,0),a_j) \Bigg] + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big) \Bigg\}. \end{aligned}$$

Here,

$$c_2 = \frac{n}{n-2s} \widetilde{S} c_1 \hat{\gamma}^{\frac{n+2s}{n-2s}} c',$$

where c_1 is defined in Lemma 3.4 and

$$c' = \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{\frac{n+2s}{2}}}.$$

The proof of Proposition 3.3 requires the following lemma.

Lemma 3.4. *For all $a \in K \Subset \Omega$ and $\lambda > 0$, we have*

$$P\widetilde{\delta}_{(a,\lambda)}(x,t) = \widetilde{\delta}_{(a,\lambda)}(x,t) - c_1 \frac{\widetilde{H}((x,t),a)}{\lambda^{\frac{n-2s}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2-2s}{2}}}\Big)$$

for all $x \in \Omega$ and $t > 0$. Here,

$$c_1 := \widetilde{S}\hat{\gamma}^{\frac{n+2s}{n-2s}}\hat{c}c'.$$

Proof. Let $\varphi_{(a,\lambda)} := \widetilde{\delta}_{(a,\lambda)} - P\widetilde{\delta}_{(a,\lambda)}$. It satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\varphi_{(a,\lambda)}) = 0 & \text{in } C, \\ \varphi_{(a,\lambda)} = \widetilde{\delta}_{(a,\lambda)} & \text{on } \partial_L C, \\ \partial_\nu^s \varphi_{(a,\lambda)} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Therefore, the functional

$$\phi_{(a,\lambda)} = \varphi_{(a,\lambda)} - c_1 \frac{\widetilde{H}(\cdot\,,a)}{\lambda^{\frac{n-2s}{2}}}$$

satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\phi_{(a,\lambda)}) = 0 & \text{in } C, \\ \phi_{(a,\lambda)} = \widetilde{\delta}_{(a,\lambda)} - \dfrac{c_1\hat{c}}{\lambda^{\frac{n-2s}{2}}\|(x-a,t)\|^{n-2s}} & \text{on } \partial_L C, \\ \partial_\nu^s \phi_{(a,\lambda)} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Thus, by the maximum principle we get

$$\|\phi_{(a,\lambda)}\|_{L^\infty(C)} \le \Big\|\widetilde{\delta}_{(a,\lambda)} - \frac{c_1\hat{c}}{\lambda^{\frac{n-2s}{2}}\|(x-a,t)\|^{n-2s}}\Big\|_{L^\infty(\partial_L C)}.$$

From another part, using the property

$$((x,t),y) \mapsto \frac{\hat{c}}{\|(x-y,t)\|^{n-2s}}$$

of Green's function on $\mathbb{R}^{n+1}_+$, which is the singular part of $\widetilde{G}$, we have

$$\begin{aligned} \widetilde{\delta}_{(a,\lambda)}(x,t) &= \int_{\mathbb{R}^n} \frac{\hat{c}}{\|(x-y,t)\|^{n-2s}} \partial_\nu^s(\widetilde{\delta}_{(a,\lambda)}(y,0))\,dy \\ &= \widetilde{S} \int_{\mathbb{R}^n} \frac{\hat{c}}{\|(x-y,t)\|^{n-2s}} \widetilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}(y,0)\,dy. \end{aligned}$$

Recall that

$$\widetilde{\delta}_{(a,\lambda)}(y,0) = \operatorname{tr}(\widetilde{\delta}_{(a,\lambda)})(y) = \hat{\gamma}\delta_{(a,\lambda)}(y) \quad \text{for all } y \in \mathbb{R}^n.$$

Therefore,

$$\widetilde{\delta}_{(a,\lambda)}(x,t) = \widetilde{S}\hat{\gamma}^{\frac{n+2s}{n-2s}}\hat{c} \int_{\mathbb{R}^n} \frac{1}{\|(x-y,t)\|^{n-2s}} \frac{\lambda^{\frac{n+2s}{2}}}{(1+\lambda^2|y-a|^2)^{\frac{n+2s}{2}}}\,dy.$$

A change of variables $z = \lambda(y-a)$ yields

$$\widetilde{\delta}_{(a,\lambda)}(x,t) = \widetilde{S}\hat{\gamma}^{\frac{n+2s}{n-2s}}\hat{c}\frac{1}{\lambda^{\frac{n-2s}{2}}} \int_{\mathbb{R}^n} \frac{1}{\|(x-a-\frac{z}{\lambda},t)\|^{n-2s}} \frac{dz}{(1+|z|^2)^{\frac{n+2s}{2}}}.$$

For any $x \in \partial\Omega$ and $t \ge 0$, we expand

$$\frac{1}{\|(x-a-\frac{z}{\lambda},t)\|^{n-2s}}$$

around $x - a$. Using the fact that $|x - a| \geq d_a > 0$ for all $x \in \partial\Omega$, we get

$$\widetilde{\delta}_{(a,\lambda)}(x,t) = \widetilde{S}\hat{\gamma}^{\frac{n+2s}{n-2s}}\hat{c}\frac{1}{\lambda^{\frac{n-2s}{2}}}\frac{1}{\|(x-a,t)\|^{n-2s}}\int_{\mathbb{R}^n}\frac{dz}{(1+|z|^2)^{\frac{n+2s}{2}}}\,dy + O\Big(\frac{1}{\lambda^{\frac{n+2-2s}{2}}}\Big)$$
$$= \frac{c_1}{\lambda^{\frac{n-2s}{2}}\|(x-a,t)\|^{n-2s}} + O\Big(\frac{1}{\lambda^{\frac{n+2-2s}{2}}}\Big).$$

This concludes the proof of Lemma 3.4. □

Proof of Proposition 3.3. We have

$$J(g_{\lambda,p}(\alpha,a)) = c_s^{\frac{n}{n-2s}}\frac{\|\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\|^{\frac{2n}{n-2s}}}{\int_\Omega(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}(x,0))^{\frac{2n}{n-2s}}\,dx} = c_s^{\frac{n}{n-2s}}\frac{N}{D}$$

and

$$N^{\frac{n-2s}{n}} = c_s^{-1}\int_\Omega \frac{\partial^s}{\partial\nu}\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big).\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)\,dx$$
$$= c_s^{-1}\widetilde{S}\Big[\sum_{i=1}^p \alpha_i^2\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_i,\lambda)} + \sum_{j\neq i}\alpha_i\alpha_j\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_j,\lambda)}\Big].$$

Using Lemma 3.4, we have

$$\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_i,\lambda)} = \int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}} - \frac{c_1}{\lambda^{\frac{n-2s}{2}}}\int_\Omega \widetilde{H}((x,0),a_i)\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}} + O\Big(\frac{1}{\lambda^{\frac{n+2-2s}{2}}}\int_{\mathbb{R}^n}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big)$$
$$= \widetilde{S}^{-1} + O\Big(\frac{1}{\lambda^n}\Big) - \frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}}{\lambda^{n-2s}}\Big[\int_{\mathbb{R}^n}\widetilde{H}\Big(\Big(a_i+\frac{z}{\lambda},0\Big),a_i\Big)\frac{dz}{(1+|z|^2)^{\frac{n+2s}{2}}} + O\Big(\frac{1}{\lambda^{2s}}\Big)\Big] + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big)$$
$$= \widetilde{S}^{-1} - \frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\widetilde{H}((a_i,0),a_i) + O\Big(\frac{1}{\lambda^{n-2s}}\Big). \tag{3.2}$$

Moreover, for any $i \neq j$ we have

$$\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_j,\lambda)} = \int_{\mathbb{R}^n}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_j,\lambda)} + O\Big(\frac{1}{\lambda^n}\Big).$$

Using again Lemma 3.4, we have

$$\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_j,\lambda)} = \int_{\mathbb{R}^n}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\widetilde{\delta}_{(a_j,\lambda)} - \frac{c_1}{\lambda^{\frac{n-2s}{2}}}\int_{\mathbb{R}^n}\widetilde{H}((x,0),a_j)\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}} + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big)$$
$$= \int_{\mathbb{R}^n}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\widetilde{\delta}_{(a_j,\lambda)} - \frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\widetilde{H}((a_j,0),a_j) + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big).$$

Using a similar computation as in [1], we have

$$\int_{\mathbb{R}^n}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\widetilde{\delta}_{(a_j,\lambda)} = \frac{\hat{\gamma}_0^{\frac{2n}{n-2s}}c'}{\lambda^{n-2s}|a_i-a_j|^{n-2s}} + O\Big(\frac{1}{(\lambda|a_i-a_j|)^{n-s}}\Big).$$

Therefore,

$$\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}P\widetilde{\delta}_{(a_j,\lambda)} = \frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\Big(\frac{\hat{\gamma}/c_1}{|a_i-a_j|^{n-2s}} - \widetilde{H}((a_j,0),a_j)\Big) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big).$$

For

$$\hat{c} = (\widetilde{S}c')^{\frac{-1}{2}}\hat{\gamma}^{\frac{-2n}{n-2s}}, \tag{3.3}$$

which is a universal constant, we have $\hat{\gamma}/c_1 = \hat{c}$. Thus,

$$\int_\Omega \widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}} P\widetilde{\delta}_{(a_j,\lambda)} = \frac{c_1 \hat{\gamma}^{\frac{n+2s}{n-2s}} c'}{\lambda^{n-2s}} \widetilde{G}((a_i,0),a_j) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big). \tag{3.4}$$

From (3.2) and (3.4) we deduce

$$N^{\frac{n-2s}{n}} = c_s^{-1}\Big[\sum_{i=1}^p \alpha_i^2 - \frac{\widetilde{S} c_1 \hat{\gamma}^{\frac{n+2s}{n-2s}} c'}{\lambda^{n-2s}}\Big(\sum_{i=1}^p \alpha_i^2 \widetilde{H}((a_i,0),a_i) - \sum_{j\neq i} \alpha_i\alpha_j \widetilde{G}((a_i,0)a_j)\Big)\Big] + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big),$$

and thus

$$N = c_s^{\frac{-n}{n-2s}}\Big(\sum_{i=1}^p \alpha_i^2\Big)^{\frac{n}{n-2s}}\Big[1 - \frac{n}{n-2s}\frac{\widetilde{S} c_1 \hat{\gamma}^{\frac{n+2s}{n-2s}} c'}{(\sum_{i=1}^p \alpha_i^2)\lambda^{n-2s}}\Big(\sum_{i=1}^p \alpha_i^2 \widetilde{H}((a_i,0),a_i) - \sum_{j\neq i}\alpha_i\alpha_j\widetilde{G}((a_i,0)a_j)\Big)\Big] + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big). \tag{3.5}$$

Next, we expand D. For any $i = 1,\dots,p$, let $B_i = B(a_i, \min(\frac{d_a}{2},\ell))$ be the ball of center a_i and radius $\min(\frac{d_a}{2},\ell)$, where $\ell = d(K,\partial\Omega)$. Then

$$D = \int_\Omega \Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}} = \int_{\cup B_i}\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}} + \int_{\Omega\setminus\cup B_i}\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}}.$$

On B_i, we have

$$\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)} = \alpha_i\widetilde{\delta}_{(a_i,\lambda)} + \sum_{k\neq i}\alpha_k P\widetilde{\delta}_{(a_k,\lambda)} - \alpha_i\varphi_{(a_i,\lambda)},$$

where $\varphi_{(a_i,\lambda)} = \widetilde{\delta}_{(a_i,\lambda)} - P\widetilde{\delta}_{(a_i,\lambda)}$. Observe that

$$\begin{aligned}\Big(\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2s}} &= \alpha_i^{\frac{2n}{n-2s}}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}} + \frac{2n}{n-2s}\alpha_i^{\frac{n+2s}{n-2s}}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big(\sum_{k\neq i}\alpha_k P\widetilde{\delta}_{(a_k,\lambda)} - \alpha_i\varphi_{(a_i,\lambda)}\Big)\\ &\quad + O\Big[\Big(\sum_{k\neq i}\alpha_k P\widetilde{\delta}_{(a_k,\lambda)} - \alpha_i\varphi_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}}\\ &\quad + (\alpha_i\widetilde{\delta}_{(a_i,\lambda)})^{\frac{4s}{n-2s}}\inf\Big((\alpha_i\widetilde{\delta}_{(a_i,\lambda)})^2, \Big(\sum_{k\neq i}\alpha_k P\widetilde{\delta}_{(a_k,\lambda)} - \alpha_i\varphi_{(a_i,\lambda)}\Big)^2\Big)\Big].\end{aligned}$$

Therefore,

$$\int_{B_i}\Big(\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2s}} = \alpha_i^{\frac{2n}{n-2s}}\int_{B_i}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}} + \frac{2n}{n-2s}\alpha_i^{\frac{n+2s}{n-2s}}\int_{B_i}\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big(\sum_{k\neq i}\alpha_k P\widetilde{\delta}_{(a_k,\lambda)} - \alpha_i\varphi_{(a_i,\lambda)}\Big) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big).$$

A computation similar to (3.2)–(3.5) yields

$$\begin{aligned}\int_{B_i}\Big(\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2s}} &= \alpha_i^{\frac{2n}{n-2s}}\widetilde{S}^{-1} - \frac{2n}{n-2s}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}} c'\alpha_i^{\frac{2n}{n-2s}}}{\lambda^{n-2s}}\widetilde{H}((a_i,0),a_i)\\ &\quad + \frac{2n}{n-2s}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}} c'\alpha_i^{\frac{n+2s}{n-2s}}}{\lambda^{n-2s}}\sum_{k\neq i}\widetilde{G}((a_i,0),a_k) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big).\end{aligned}$$

From another part,

$$\int_{\Omega\setminus\bigcup_{i=1}^p B_i}\Big(\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2s}} \leq \int_{\Omega\setminus\bigcup_{i=1}^p B_i} c\sum_{k=1}^p \alpha_k^{\frac{2n}{n-2s}}(\widetilde{\delta}_{(a_k,\lambda)} - \varphi_{(a_k,\lambda)})^{\frac{2n}{n-2s}}.$$

Using Lemma 3.4, we have

$$\int_{\Omega\setminus\bigcup_{i=1}^p B_i}\Big(\sum_{k=1}^p \alpha_k P\widetilde{\delta}_{(a_k,\lambda)}\Big)^{\frac{2n}{n-2s}} \le c\sum_{k=1}^p \int_{\Omega\setminus\bigcup_{i=1}^p B_i} \widetilde{\delta}_{(a_k,\lambda)}^{\frac{2n}{n-2s}} + O\Big(\frac{1}{\lambda^n}\Big) \le c\frac{1}{(\lambda d_a)^n} + O\Big(\frac{1}{\lambda^n}\Big).$$

Thus,

$$\begin{aligned} D &= \widetilde{S}^{-1}\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}} - \frac{2n}{n-2s}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\Big(\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}\widetilde{H}((a_i,0),a_i) - \sum_{k\neq i}\alpha_i^{\frac{n+2s}{n-2s}}\alpha_k\widetilde{G}((a_i,0),a_k)\Big) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big)\\ &= \widetilde{S}^{-1}\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}\Big[1 - \frac{2n}{n-2s}\frac{\widetilde{S}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\Big(\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}\widetilde{H}((a_i,0),a_i) - \sum_{k\neq i}\alpha_i^{\frac{n+2s}{n-2s}}\alpha_k\widetilde{G}((a_i,0),a_k)\Big)\Big]\\ &\quad + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big). \end{aligned} \tag{3.6}$$

Therefore, from (3.5) and (3.6) we get

$$\begin{aligned} J(g_{\lambda,p}(\alpha,a)) = \widetilde{S}\frac{(\sum_{i=1}^p \alpha_i^2)^{\frac{n}{n-2s}}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}}\Big\{1 - \frac{n}{n-2s}\frac{\widetilde{S}}{\sum_{i=1}^p \alpha_i^2}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\\ \times\Big[\sum_{j=1}^p \alpha_j^2\widetilde{H}((a_j,0),a_j) - \sum_{k\neq j}\alpha_j\alpha_k\widetilde{G}((a_j,0),a_k)\Big] + \frac{2n}{n-2s}\frac{\widetilde{S}}{\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}}}\frac{c_1\hat{\gamma}^{\frac{n+2s}{n-2s}}c'}{\lambda^{n-2s}}\\ \times\Big[\sum_{j=1}^p \alpha_j^{\frac{2n}{n-2s}}\widetilde{H}((a_j,0),a_j) - \sum_{k\neq j}\alpha_j^{\frac{2n}{n-2s}}\alpha_k\widetilde{G}((a_j,0),a_k)\Big]\Big\} + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big). \end{aligned}$$

This concludes the proof of Proposition 3.3. □

The above expansion is of course useful when λd_a is very large. The next proposition provides an estimate of J without involving λd_a.

Proposition 3.5. *For any $p \in \mathbb{N}^*$ and $\lambda > 0$, for all $(\alpha, a) \in \Delta_{p-1}\times K^p$ we have*

$$J(g_{\lambda,p}(\alpha,a)) \le \widetilde{S}^{\frac{n}{n-2s}}\Big\{\frac{\int_\Omega(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)})^{\frac{2n}{n-2s}}}{\int_\Omega(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)})^{\frac{2n}{n-2s}}}\Big\}^{\frac{1}{2}}\Big(\sum_{i=1}^p\int_\Omega \beta_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}}\Big)^{\frac{2s}{n-2s}},$$

where

$$\beta_i = \frac{\alpha_i\widetilde{\delta}_{(a_i,\lambda)}}{\sum_{k=1}^p \alpha_k\widetilde{\delta}_{(a_k,\lambda)}}.$$

Proof. Following the proof of Proposition 3.3, we have

$$\begin{aligned} N^{\frac{n-2s}{n}} &= c_s^{-1}\widetilde{S}\int_\Omega\Big(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big)\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)\\ &\le c_s^{-1}\widetilde{S}\Big[\int_\Omega\Big(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big)^{\frac{2n}{n+2s}}\Big]^{\frac{n+2s}{2n}}\Big[\int_\Omega\Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}}\Big]^{\frac{n-2s}{2n}}. \end{aligned}$$

We have

$$\begin{aligned} \int_\Omega\Big(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{n+2s}{n-2s}}\Big)^{\frac{2n}{n+2s}} &\le \Big[\int_\Omega\Big(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}}\Big]^{\frac{n-2s}{n+2s}}\Big[\int_\Omega\Big(\sum_{i=1}^p \beta_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{8ns}{n^2-4s^2}}\Big)^{\frac{n+2s}{4s}}\Big]^{\frac{4s}{n+2s}}\\ &\le \Big[\int_\Omega\Big(\sum_{i=1}^p \alpha_i\widetilde{\delta}_{(a_i,\lambda)}\Big)^{\frac{2n}{n-2s}}\Big]^{\frac{n-2s}{n+2s}}\Big[\sum_{i=1}^p\int_\Omega \beta_i\widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}}\Big]^{\frac{4s}{n+2s}}. \end{aligned}$$

Therefore,

$$N \leq c_s^{-\frac{n}{n-2s}} \widetilde{S}^{\frac{n}{n-2s}} \Big[\int_\Omega \Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2s}} \Big]^{\frac{1}{2}} \Big[\int_\Omega \Big(\sum_{i=1}^p \alpha_i \widetilde{\delta}_{(a_i,\lambda)} \Big)^{\frac{2n}{n-2s}} \Big]^{\frac{1}{2}} \Big[\sum_{i=1}^p \int_\Omega \beta_i \widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}} \Big]^{\frac{2s}{n-2s}}.$$

Hence Proposition 3.5 follows. □

Corollary 3.6. *For any $p \in \mathbb{N}^*$ and $\varepsilon > 0$, there exists $\lambda_1 = \lambda(p, \varepsilon)$ such that for all $\lambda \geq \lambda_1$ and $(\alpha, a) \in \Delta_{p-1} \times K^p$ we have*

$$J(g_{\lambda,p}(\alpha, a)) \leq (p + \varepsilon)^{\frac{4s}{n-2s}} \widetilde{S}.$$

Proof. Using (3.1), we have

$$\int_\Omega \sum_{i=1}^p \beta_i \widetilde{\delta}_{(a_i,\lambda)}^{\frac{2n}{n-2s}} \leq p\widetilde{S}^{-1} + O\Big(\frac{1}{\lambda^n}\Big).$$

Therefore, by Proposition 3.5 we obtain

$$J(g_{\lambda,p}(\alpha, a)) \leq \widetilde{S}^{\frac{n}{n-2s}} \left\{ \frac{\int_\Omega (\sum_{i=1}^p \alpha_i \widetilde{\delta}_{(a_i,\lambda)})^{\frac{2n}{n-2s}}}{\int_\Omega (\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda)})^{\frac{2n}{n-2s}}} \right\}^{\frac{1}{2}} \Big(p^{\frac{2s}{n-2s}} \widetilde{S}^{\frac{-2s}{n-2s}} + O\Big(\frac{1}{\lambda^{\frac{2ns}{n-2s}}}\Big)\Big).$$

Thus, Corollary 3.6 follows from the above estimate and Lemma 3.4. □

Corollary 3.7. *There exist $p_0 \in \mathbb{N}^*$ and $\lambda_0 > 0$ such that for all $(\alpha, a) \in \Delta_{p_0-1} \times K^{p_0}$ and all $\lambda \geq \lambda_0$ we have*

$$J(g_{\lambda,p_0}(\alpha, a)) \leq p_0^{\frac{4s}{n-2s}} \widetilde{S}.$$

Proof. We divide the proof into two steps.

Step 1: We assume that there exists an index i, $1 \leq i \leq p$ for $p \geq 2$, such that α_i is small enough.

Without loss of generality, we may assume that $\alpha_1 \ll 1$. According to the proof of Proposition 3.3, we have

$$N^{\frac{n-2s}{n}} = c_s^{-1} \Big[O(\alpha_1) + (1 - \alpha_1)^2 \int_\Omega \Big(\sum_{i=2}^p \frac{\alpha_i}{(1 - \alpha_1)} \partial_\lambda^s (P\widetilde{\delta}_{(a_i,\lambda)}) \Big) \Big(\sum_{i=2}^p \frac{\alpha_i}{(1 - \alpha_1)} (P\widetilde{\delta}_{(a_i,\lambda)}) \Big) \Big].$$

Moreover,

$$D = O(\alpha_1) + (1 - \alpha_1)^{\frac{2n}{n-2s}} \int_\Omega \Big(\sum_{i=2}^p \frac{\alpha_i}{(1 - \alpha_1)} (P\widetilde{\delta}_{(a_i,\lambda)}) \Big)^{\frac{2n}{n-2s}}.$$

Thus,

$$J(g_{\lambda,p}(\alpha, a)) \leq J\Big[g_{\lambda,p-1}\Big(\Big(\frac{\alpha_2}{1 - \alpha_1}, \dots, \frac{\alpha_p}{1 - \alpha_1} \Big), (a_2, \dots, a_p) \Big) \Big] + O(\alpha_1).$$

Therefore, Corollary 3.7 follows for α_1 small enough by using Corollary 3.6.

Step 2: We assume that α_i, for all i, $1 \leq i \leq p$, is lower bounded by a fixed constant $\varepsilon_0 > 0$.

Assume that $d_a = |a_{i_0} - a_{j_0}|$. If λd_a is large enough, by using the expansion of Proposition 3.3, there exists $d_1 > 0$ such that if $d_a \leq d_1$, we have

$$J(g_{\lambda,p}(\alpha, a)) \leq p^{\frac{2s}{n-2s}} \widetilde{S}.$$

This is a consequence of the fact that $\widetilde{G}((a_{i_0}, 0), a_{j_0}) \to +\infty$ as $|a_{i_0} - a_{j_0}| \to 0$.

From another part, if λd_a is upper bounded, by using Lemma 3.4, Proposition 3.5 and the fact that

$$\int_\Omega \frac{\widetilde{\delta}_{(a_{i_0},\lambda)}}{\widetilde{\delta}_{(a_{i_0},\lambda)} + \varepsilon_0 \widetilde{\delta}_{(a_{j_0},\lambda)}} \widetilde{\delta}_{(a_{j_0},\lambda)}^{\frac{2n}{n-2s}} \leq \widetilde{S}^{-1}(1 - \varepsilon_0),$$

Corollary 3.7 follows in this case.

Lastly, if $d_a \geq d_1$, using the fact that there exist $\tilde{c} > 0$ and $\tilde{\gamma} > 0$ such that

$$\widetilde{H}((a,0),a) \leq \tilde{c} \quad \text{for all } a \in K,$$
$$\widetilde{G}((a,0),b) \leq \tilde{\gamma} \quad \text{for all } a,b \in K,$$

from Proposition 3.3 we derive that there exist $\bar{c} > 0$ and $\bar{\gamma} > 0$ such that

$$J(g_{\lambda,p}(\alpha,a)) \leq p^{\frac{2s}{n-2s}}\Big(\widetilde{S} + \frac{2}{\lambda^{n-2s}}(\bar{c} - p\gamma)\Big) + O\Big(\frac{1}{(\lambda d_a)^{n-s}}\Big).$$

Therefore, by taking p_0 such that $\bar{c} - p\gamma < 0$ and λ large enough, Corollary 3.7 follows. □

4 Proof of Theorem 1.1

We provide the proof of Theorem 1.1 by prescribing the loss of compactness of our problem. Since the exponent $\frac{2n}{n-2s}$ is the critical exponent for the Sobolev trace embedding, the functional J associated to problem (2.1) fails to satisfy the Palais–Smale condition. Arguing as [16], by the following proposition we describe the Palais–Smale sequences.

Proposition 4.1. *Assume that* (2.1) *has no solution. Let* $(v_k)_k$ *be a sequence in* $\Sigma^+ := \{v \in \Sigma : v \geq 0\}$ *such that* $J(v_k) \to c$ *and* $\partial J(v_k) \to 0$. *There exists* $p \in \mathbb{N}^*$ *and a subsequence of* $(v_k)_k$ *denoted again* $(v_k)_k$ *such that* $v_k \in V(p,\varepsilon_k)$, *where* $\varepsilon_k \to 0$ *as* $k \to +\infty$, *and*

$$V(p,\varepsilon) = \Big\{u \in \Sigma^+ : \text{there exist } (a_1,\dots,a_p) \in \Omega^p,\ (\lambda_1,\dots,\lambda_p) \in \Big[\frac{1}{\varepsilon},\infty\Big)^p \text{ and } (\alpha_1,\dots,\alpha_p) \in \mathbb{R}_+^p$$
$$\text{such that } \Big\|u - \frac{1}{\sqrt{c_s}}\frac{\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}}{\|\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\|} < \varepsilon\Big\| < \varepsilon \text{ with } \lambda_i d(a_i,\partial\Omega) > \frac{1}{\varepsilon} \text{ and } \varepsilon_{ij} < \varepsilon \text{ for all } i \neq j\Big\}.$$

Here,

$$\varepsilon_{ij} = \frac{1}{(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2)^{\frac{n-2s}{2}}}.$$

Remark 4.2. If $(v_k)_k \subset V(p,\varepsilon_k)$, where $\varepsilon_k \to 0$, then

$$J(v_k) \to p^{\frac{2s}{n-2s}}\widetilde{S}.$$

The following proposition gives suitable parameters for $V(p,\varepsilon)$. The proof is similar to the one of [3, Proposition 7].

Proposition 4.3. *Let* $p \in \mathbb{N}^*$. *There exists* $\varepsilon > 0$ *such that for all* $u \in V(p,\varepsilon)$ *the minimization problem*

$$\inf_{\alpha_i,a_i,\lambda_i}\Big\|u - \sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\Big\|$$

admits a unique solution $(\bar{\alpha},\bar{a},\bar{\lambda})$ *modulo a parametrization on the indices set. Therefore, we write*

$$u = \sum_{i=1}^p \bar{\alpha}_i P\widetilde{\delta}_{(\bar{a}_i,\bar{\lambda}_i)} + v$$

for any $u \in V(p,\varepsilon)$. *Here,* $v \in \mathcal{H}$ *and* $\|v\| < \varepsilon$.

Next, we will use the following notations: For any $p \in \mathbb{N}^*$, we set

$$b_p = p^{\frac{2n}{n-2s}}\widetilde{S}, \quad B_p(K) = \Big\{\sum_{i=1}^p \alpha_i\delta_{a_i} : (\alpha_1,\dots,\alpha_p) \in \Delta_{p-1},\ (a_1,\dots,a_p) \in K^p\Big\}, \quad W_p = J_{b_{p+1}},$$

where

$$J_c = \{u \in \Sigma : u \geq 0 \text{ and } J(u) < c\} \quad \text{for all } c > 0.$$

Observe that

$$W_0 = J_{\widetilde{S}} = \emptyset \quad \text{and} \quad B_0(K) = \emptyset.$$

Following the Bahri–Brezis deformation lemma [2, Lemma 17], we have that J_{c_2} retracts by deformation on J_{c_1} for any $b_p < c_1 < c_2 \leq b_{p+1}$, provided J has no critical point.

Proof of Theorem 1.1. Under the assumption of Theorem 1.1, there exists a cycle K of dimension k in Ω such that $[K]$, the class of K in $H_k(\Omega, \mathbb{Z}_2)$, is not trivial. We introduce the following lemma.

Lemma 4.4. *Assume that* (2.1) *has no solution. There exists a group of homomorphisms*

$$\widetilde{g}_{1*} : H_*(B_1(K), B_0(K)) \to H_*(W_1, W_0), \quad * \in \mathbb{N},$$

such that $\widetilde{g}_{1k}([B_1(K), B_0(K)]) \neq 0$. Here $H_*(M, N)$ denotes the homology group associated to the topological pair (M, N) if $M \subset N$.

Proof. For $\lambda > 0$ large enough and $\varepsilon > 0$ small enough, we consider the following continuous mappings:

$$\pi_1 : B_1(K) \to K, \qquad \delta_a \mapsto a,$$

$$g_{1,\lambda} : K \to J_{\widetilde{S}+\varepsilon} \cap V(1, \varepsilon), \qquad a \mapsto \frac{1}{\sqrt{c_S}} \frac{P\widetilde{\delta}_{(a,\lambda)}}{\|P\widetilde{\delta}_{(a,\lambda)}\|},$$

where $g_{1,\lambda}$ is well defined by using Corollary 3.6. Consider the natural injection

$$\imath_1 : J_{\widetilde{S}+\varepsilon} \to W_1.$$

Recall that a continuous mapping $f : M \to N$ between two manifolds M and N induces a group of homomorphisms $(f_*)_{*\in\mathbb{N}}$ such that

$$f_* : H_*(M) \to H_*(N), \quad [c] \mapsto [f(c)].$$

Let

$$\widetilde{g}_1 = \imath_1 \circ g_{1,\lambda} \circ \pi_1.$$

Then $(\widetilde{g}_{1*})_{*\in\mathbb{N}}$ is a non-zero group. Indeed,

$$\widetilde{g}_{1*} = \imath_{1*} \circ g_{1,\lambda*} \circ \pi_{1*}.$$

Observe that π_{1*} is an isomorphism since π_1 is an homeomorphism. Furthermore, $\imath_{1*}$ is an isomorphism since $\imath_1$ is an homotopy equivalence (under the assumption that J has no critical point). Lastly, we claim that $g_{1,\lambda*}([K]) \neq 0$ for $* = k$. Indeed, if $g_{1,\lambda*}([K]) = 0$, then $(P_* \circ g_{1,\lambda*})([K]) = 0$, where

$$P : J_{\widetilde{S}+\varepsilon} \cap V(1, \varepsilon) \to \Omega, \quad u = \frac{\bar{\alpha} P\widetilde{\delta}_{(\bar{a},\bar{\lambda})} + v}{\|\bar{\alpha} P\widetilde{\delta}_{(\bar{a},\bar{\lambda})} + v\|} \mapsto \bar{a}.$$

Observe that $P \circ g_{1,\lambda}(K) = K$. We therefore have $[K] = 0$ in $H_*(\Omega)$, which is absurd. Hence our claim is valid. It is now easy to check that

$$\widetilde{g}_{1*}(\pi_{1*}^{-1}([K])) = \widetilde{g}_{1*}([B_1(K)]) \neq 0 \quad \text{for } * = k.$$

The proof of Lemma 4.4 follows. □

Next, we extend the result of Lemma 4.4 for any $p \geq 1$.

Lemma 4.5. *Assume that* (2.1) *has no solution. For any $p \in \mathbb{N}^*$, there exists a group of homomorphisms*

$$\widetilde{g}_{p*} : H_*(B_p(K), B_{p-1}(K)) \to H_*(W_p, W_{p-1}), \quad * \in \mathbb{N},$$

such that $\widetilde{g}_{pd}([B_p(K), B_{p-1}(K)]) \neq 0$, where $d = kp + p - 1$.

Proof. Let ρ_p be the group of permutations of order p. The group ρ_p acts on $K^p \times \Delta_{p-1}$ by

$$\sigma((a_1, \dots, a_p), (\alpha_1, \dots, \alpha_p)) = ((a_{\sigma(1)}, \dots, a_{\sigma(p)}), (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}))$$

for any $\sigma \in \rho_p$. We denote by $K^p \times_{\rho_p} \Delta_{p-1}$ the associated quotient space. Set

$$\pi_p : B_p(K) \to K^p \times_{\rho_p} \Delta_{p-1}, \quad \sum_{i=1}^{p} \alpha_i \delta_{a_i} \mapsto ((a_{\sigma(1)}, \dots, a_{\sigma(p)}), (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)})).$$

For

$$S_p = \{(a_1, \dots, a_p) \in K^p : \text{there exists } i \neq j \text{ such that } a_i = a_j\},$$

we define an ρ_p equivariant neighborhood T_p of S_p in K^p. Observe that the projection π_p induces a map of pairs

$$\pi_p : (B_p(K), B_{p-1}(K)) \to (K^p \times_{\rho_p} \Delta_{p-1}, S_p \times \Delta_{p-1} \cup_{\rho_p} K^p \times \partial\Delta_{p-1}),$$

which is again an homeomorphism. Define the natural injection by

$$j_p : (K^p \times_{\rho_p} \Delta_{p-1}, S_p \times \Delta_{p-1} \cup_{\rho_p} K^p \times \partial\Delta_{p-1}) \to (K^p \times_{\rho_p} \Delta_{p-1}, \overline{T}_p \times \Delta_{p-1} \cup_{\rho_p} K^p \times \partial\Delta_{p-1}).$$

This is an homotopy equivalence since $\overline{T}_p$ retracts by deformation on S_p. Next, we denote

$$K_0^p := K^p \setminus T_p,$$

and we define the natural injection

$$\psi_p : (K_0^p \times_{\rho_p} \Delta_{p-1}, \partial(K_0^p \times_{\rho_p} \Delta_{p-1})) \to (K^p \times_{\rho_p} \Delta_{p-1}, \overline{T}_p \times \Delta_{p-1} \cup_{\rho_p} K^p \times \partial\Delta_{p-1}),$$

which is an homotopy equivalence. For any $* \in \mathbb{N}$, let

$$\varphi_{p*} : H_*(B_p(K), B_{p-1}(K)) \to H_*(K_0^p \times_{\rho_p} \Delta_{p-1}, \partial(K_0^p \times_{\rho_p} \Delta_{p-1}))$$

be the homomorphism defined by

$$\varphi_{p*} = \psi_{p*}^{-1} \circ j_{p*} \circ \pi_{p*}.$$

Let

$$g_{p,\lambda*} : H_*(K_0^p \times_{\rho_p} \Delta_{p-1}, \partial(K_0^p \times_{\rho_p} \Delta_{p-1})) \to H_*(J_{p^{\frac{2s}{n-2s}} \tilde{S}+\varepsilon} \cap V(p, \varepsilon), J_{p^{\frac{2s}{n-2s}} \tilde{S}} \cap V(p, \varepsilon))$$

be the homomorphism induced by $g_{p,\lambda}$, which is well defined by using Corollary 3.6 and the proof of Corollary 3.7. Let

$$\imath_{p*} : H_*(J_{p^{\frac{2s}{n-2s}} \tilde{S}+\varepsilon} \cap V(p, \varepsilon), J_{p^{\frac{2s}{n-2s}} \tilde{S}} \cap V(p, \varepsilon)) \to H_*(W_p, W_{p-1})$$

be the homomorphism induced by the trivial injection. The required group of homomorphisms $(\tilde{g}_{p*})_{*\in\mathbb{N}}$ is given by

$$\tilde{g}_{p*} = \imath_{p*} \circ g_{p,\lambda*} \circ \varphi_{p*}.$$

We now consider the following commutative diagram for all $* \geq 2$:

$$\begin{array}{ccc} H_*(B_p(K), B_{p-1}(K)) & \xrightarrow{\tilde{g}_{p*}} & H_*(W_p, W_{p-1}) \\ \downarrow \partial & & \downarrow \partial \\ H_*(B_{p-1}(K), B_{p-2}(K)) & \xrightarrow[\tilde{g}_{(p-1)*}]{} & H_*(W_{p-1}, W_{p-2}). \end{array}$$

Here ∂ denotes the connecting homomorphism. Following [3, (23)–(26)], we derive that for any $p \geq 2$,

$$\tilde{g}_{(p-1)*}([B_{p-1}(K), B_{p-2}(K)]) \neq 0.$$

Then

$$\tilde{g}_{p*}([B_p(K), B_{p-1}(K)]) \neq 0.$$

In the case $p - 1 = 1$, observe that $\tilde{g}_{1*}$ is exactly the one constructed by Lemma 4.4. Therefore, the fact that $\tilde{g}_{1*}([B_1(K), B_0(K)]) \neq 0$ implies the result of Lemma 4.5. □

Now, if we suppose that (2.1) has no solution, by Lemma 4.5 we get

$$\widetilde{g}_{p*}\big([B_p(K), B_{p-1}(K)]\big) \neq 0 \text{ in } H_*(W_p, W_{p-1}) \quad \text{for all } p \geq 1.$$

But this is impossible since from Corollary 3.7 we have

$$(\imath_{p_0} \circ g_{\lambda,p_0})\big(K_0^{p_0} \times_{\rho_{p_0}} \Delta_{p_0-1}, \partial(K_0^{p_0} \times_{\rho_{p_0}} \Delta_{p_0-1})\big) \subset (W_{p_0-1}, W_{p_0-1})$$

for some $p_0 \in \mathbb{N}^*$, and thus

$$\widetilde{g}_{p_0*}\big([B_{p_0}(K), B_{p_0-1}(K)]\big) = 0 \quad \text{in } H_*(W_{p_0}, W_{p_0-1}).$$

This finishes the proof of Theorem 1.1. □

References

[1] A. Bahri, *Critical Points at Infinity in Some Variational Problems*, Pitman Res. Notes Math. Ser. 182, Longman Scientific & Technical, Harlow, 1989.

[2] A. Bahri and H. Brezis, Elliptic differential equations involving the sobolev critical exponent on manifolds, in: *Topics in Geometry: In Memory of Joseph D'Atri*, Progr. Nonlinear Differential Equations Appl. 20, Birkhäuser, Basel (1996), 1–100.

[3] A. Bahri and J.-M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, *Comm. Pure Appl. Math.* **41** (1988), no. 3, 253–294.

[4] C. Brändle, E. Colorado, A. de Pablo and U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), no. 1, 39–71.

[5] X. Cabré and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. Math.* **224** (2010), no. 5, 2052–2093.

[6] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), no. 7–9, 1245–1260.

[7] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, Regularity of radial extremal solutions for some non-local semilinear equations, *Comm. Partial Differential Equations* **36** (2011), no. 8, 1353–1384.

[8] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.* **59** (2006), no. 3, 330–343.

[9] A. Cotsiolis and N. K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.* **295** (2004), no. 1, 225–236.

[10] H. Hajaiej, L. Molinet, T. Ozawa and B. Wang, Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations, in: *Harmonic Analysis and Nonlinear Partial Differential Equations*, RIMS Kôkyûroku Bessatsu B26, Research Institute for Mathematical Sciences, Kyoto (2011), 159–175.

[11] Y. Y. Li, Remark on some conformally invariant integral equations: The method of moving spheres, *J. Eur. Math. Soc. (JEMS)* **6** (2004), no. 2, 153–180.

[12] Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* **80** (1995), no. 2, 383–417.

[13] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Dokl.* **6** (1965), 1408–1411.

[14] K. Sharaf, An infinite number of solutions for an elliptic problem with power nonlinearity, *Differential Integral Equations* **30** (2017), no. 1–2, 133–144.

[15] P. R. Stinga and J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), no. 11, 2092–2122.

[16] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984), no. 4, 511–517.

[17] J. Tan, The Brezis–Nirenberg type problem involving the square root of the Laplacian, *Calc. Var. Partial Differential Equations* **42** (2011), no. 1–2, 21–41.

[18] J. Tan, Positive solutions for non local elliptic problems, *Discrete Contin. Dyn. Syst.* **33** (2013), no. 2, 837–859.

[19] J. Xiao, A sharp Sobolev trace inequality for the fractional-order derivatives, *Bull. Sci. Math.* **130** (2006), no. 1, 87–96.