

Research Article

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On Critical p -Laplacian Systems

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Abstract: We consider the critical p -Laplacian system

$$\begin{cases} -\Delta_p u - \frac{\lambda a}{p} |u|^{\alpha-2} u |v|^b = \mu_1 |u|^{p^*-2} u + \frac{\alpha \gamma}{p^*} |u|^{\alpha-2} u |v|^\beta, & x \in \Omega, \\ -\Delta_p v - \frac{\lambda b}{p} |u|^a |v|^{b-2} v = \mu_2 |v|^{p^*-2} v + \frac{\beta \gamma}{p^*} |u|^a |v|^{\beta-2} v, & x \in \Omega, \\ u, v \in D_0^{1,p}(\Omega), \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator defined on

$$D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\},$$

endowed with the norm $\|u\|_{D^{1,p}} := (\int_{\mathbb{R}^N} |\nabla u|^p dx)^{\frac{1}{p}}$, $N \geq 3$, $1 < p < N$, $\lambda, \mu_1, \mu_2 \geq 0$, $\gamma \neq 0$, $a, b, \alpha, \beta > 1$ satisfy $a + b = p$, $\alpha + \beta = p^* := \frac{Np}{N-p}$, the critical Sobolev exponent, Ω is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$ and $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\mathbb{R}^N)$. Under suitable assumptions, we establish the existence and nonexistence of a positive least energy solution of this system. We also consider the existence and multiplicity of the nontrivial nonnegative solutions.

Keywords: Nehari Manifold, p -Laplacian Systems, Least Energy Solutions, Critical Exponent**MSC 2010:** 35B33, 35J20, 58E05**Communicated by:** Zhi-Qiang Wang

1 Introduction

Equations and systems involving the p -Laplacian operator have been extensively studied in the recent years (see, e.g., [2, 3, 5, 7–10, 13, 16, 17, 19, 20, 22, 23, 26] and the references therein). In the present paper, we study the critical p -Laplacian system

$$\begin{cases} -\Delta_p u - \frac{\lambda a}{p} |u|^{\alpha-2} u |v|^b = \mu_1 |u|^{p^*-2} u + \frac{\alpha \gamma}{p^*} |u|^{\alpha-2} u |v|^\beta, & x \in \Omega, \\ -\Delta_p v - \frac{\lambda b}{p} |u|^a |v|^{b-2} v = \mu_2 |v|^{p^*-2} v + \frac{\beta \gamma}{p^*} |u|^a |v|^{\beta-2} v, & x \in \Omega, \\ u, v \in D_0^{1,p}(\Omega), \end{cases} \quad (1.1)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator defined on

$$D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\},$$

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endowed with the norm $\|u\|_{D^{1,p}} := (\int_{\mathbb{R}^N} |\nabla u|^p dx)^{\frac{1}{p}}$, $N \geq 3$, $1 < p < N$, $\lambda, \mu_1, \mu_2 \geq 0$, $\gamma \neq 0$, $a, b, \alpha, \beta > 1$ satisfy $a + b = p$, $\alpha + \beta = p^* := \frac{Np}{N-p}$, the critical Sobolev exponent, Ω is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$ and $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\mathbb{R}^N)$. The case for $p = 2$ was thoroughly investigated by Peng, Peng and Wang [21] recently; some uniqueness, synchronization and non-degenerated properties were verified there. Note that we allow the powers in the coupling terms to be unequal. We consider the two cases

(H1) $\Omega = \mathbb{R}^N$, $\lambda = 0$, $\mu_1, \mu_2 > 0$;

(H2) Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$, $\mu_1, \mu_2 = 0$, $\gamma = 1$.

Let

$$S := \inf_{u \in D_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{(\int_{\Omega} |u|^{p^*} dx)^{\frac{p}{p^*}}} \quad (1.2)$$

be the sharp constant of imbedding for $D_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ (see, e.g., [1]). Then S is independent of Ω and is attained only when $\Omega = \mathbb{R}^N$. In this case, a minimizer $u \in D^{1,p}(\mathbb{R}^N)$ satisfies the critical p -Laplacian equation

$$-\Delta_p u = |u|^{p^*-2} u, \quad x \in \mathbb{R}^N. \quad (1.3)$$

Damascelli, Merchán, Montoro and Sciunzi [14] recently showed that all solutions of (1.3) are radial and radially decreasing about some point in $\mathbb{R}^N$ when $\frac{2N}{N+2} \leq p < 2$. Vétois [25] considered a more general equation and extended the result to the case $1 < p < \frac{2N}{N+2}$. Sciunzi [24] extended this result to the case $2 < p < N$. By exploiting the classification results in [4, 18], we see that, for $1 < p < N$, all positive solutions of (1.3) are of the form

$$U_{\varepsilon, \gamma}(x) := \left[N \left(\frac{N-p}{p-1} \right)^{p-1} \right]^{\frac{N-p}{p^2}} \left(\frac{\varepsilon^{\frac{1}{p-1}}}{\varepsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}}} \right)^{\frac{N-p}{p}}, \quad \varepsilon > 0, y \in \mathbb{R}^N, \quad (1.4)$$

and

$$\int_{\mathbb{R}^N} |\nabla U_{\varepsilon, \gamma}|^p dx = \int_{\mathbb{R}^N} |U_{\varepsilon, \gamma}|^{p^*} dx = S^{\frac{N}{p}}. \quad (1.5)$$

In case (H1), the energy functional associated with system (1.1) is given by

$$I(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) - \frac{1}{p^*} \int_{\mathbb{R}^N} (\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta), \quad (u, v) \in D,$$

where $D := D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$, endowed with the norm $\|(u, v)\|_D^p = \|u\|_{D^{1,p}}^p + \|v\|_{D^{1,p}}^p$. In this case, (1.1) with $\alpha = \beta$ and $p = 2$ is well studied by Chen and Zou [11, 12]. Define

$$\mathcal{N} = \left\{ (u, v) \in D : u \neq 0, v \neq 0, \int_{\mathbb{R}^N} |\nabla u|^p = \int_{\mathbb{R}^N} \left(\mu_1 |u|^{p^*} + \frac{\alpha \gamma}{p^*} |u|^\alpha |v|^\beta \right), \int_{\mathbb{R}^N} |\nabla v|^p = \int_{\mathbb{R}^N} \left(\mu_2 |v|^{p^*} + \frac{\beta \gamma}{p^*} |u|^\alpha |v|^\beta \right) \right\}.$$

It is easy to see that $\mathcal{N} \neq \emptyset$ and that any nontrivial solution of (1.1) is in $\mathcal{N}$. By a nontrivial solution we mean a solution (u, v) such that $u \neq 0$ and $v \neq 0$. A solution is called a least energy solution if its energy is minimal among energies of all nontrivial solutions. A solution (u, v) is positive if $u > 0$ and $v > 0$, and semitrivial if it is of the form $(u, 0)$ with $u \neq 0$ or $(0, v)$ with $v \neq 0$. Set $A := \inf_{(u, v) \in \mathcal{N}} I(u, v)$, and note that

$$A = \inf_{(u, v) \in \mathcal{N}} \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) = \inf_{(u, v) \in \mathcal{N}} \frac{1}{N} \int_{\mathbb{R}^N} (\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta).$$

Consider the nonlinear system of equations

$$\begin{cases} \mu_1 k^{\frac{p^*-p}{p}} + \frac{\alpha \gamma}{p^*} k^{\frac{\alpha-p}{p}} l^{\frac{\beta}{p}} = 1, \\ \mu_2 l^{\frac{p^*-p}{p}} + \frac{\beta \gamma}{p^*} k^{\frac{\alpha}{p}} l^{\frac{\beta-p}{p}} = 1, \\ k > 0, l > 0. \end{cases} \quad (1.6)$$

Our main results in this case are the following.

Theorem 1.1. If (H1) holds and $\gamma < 0$, then $A = \frac{1}{N}(\mu_1^{-(N-p)/p} + \mu_2^{-(N-p)/p})S^{N/p}$ and A is not attained.

Theorem 1.2. Let (H1) and the following conditions hold:

(C1) $\frac{N}{2} < p < N$, $\alpha, \beta > p$ and

$$0 < \gamma \leq \frac{3p^2}{(3-p)^2} \min \left\{ \frac{\mu_1}{\alpha} \left(\frac{\alpha-p}{\beta-p} \right)^{\frac{\beta-p}{p}}, \frac{\mu_2}{\beta} \left(\frac{\beta-p}{\alpha-p} \right)^{\frac{\alpha-p}{p}} \right\}; \quad (1.7)$$

(C2) $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and

$$\gamma \geq \frac{Np^2}{(N-p)^2} \max \left\{ \frac{\mu_1}{\alpha} \left(\frac{p-\beta}{p-\alpha} \right)^{\frac{p-\beta}{p}}, \frac{\mu_2}{\beta} \left(\frac{p-\alpha}{p-\beta} \right)^{\frac{p-\alpha}{p}} \right\}. \quad (1.8)$$

Then $A = \frac{1}{N}(k_0 + l_0)S^{N/p}$ and A is attained by $(\sqrt[p]{k_0}U_{\varepsilon,\gamma}, \sqrt[p]{l_0}U_{\varepsilon,\gamma})$, where (k_0, l_0) satisfies (1.6) and

$$k_0 = \min \{k : (k, l) \text{ satisfies (1.6)}\}. \quad (1.9)$$

Theorem 1.3. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (H1) holds. If $\gamma > 0$, then A is attained by some (U, V) , where U and V are positive, radially symmetric and decreasing.

Theorem 1.4 (Multiplicity). Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (H1) holds. There exists

$$\gamma_1 \in \left(0, \frac{Np^2}{(N-p)^2} \max \left\{ \frac{\mu_1}{\alpha} \left(\frac{2-\beta}{2-\alpha} \right)^{\frac{2-\beta}{2}}, \frac{\mu_2}{\beta} \left(\frac{2-\alpha}{2-\beta} \right)^{\frac{2-\alpha}{2}} \right\} \right]$$

such that for any $\gamma \in (0, \gamma_1)$ there exists a solution $(k(\gamma), l(\gamma))$ of (1.6) satisfying

$$I(\sqrt[p]{k(\gamma)}U_{\varepsilon,\gamma}, \sqrt[p]{l(\gamma)}U_{\varepsilon,\gamma}) > A$$

and $(\sqrt[p]{k(\gamma)}U_{\varepsilon,\gamma}, \sqrt[p]{l(\gamma)}U_{\varepsilon,\gamma})$ is a (second) positive solution of (1.1).

For the case (H2), we have the following theorem.

Theorem 1.5. If (H2) holds, $p \leq \sqrt{N}$ and

$$0 < \lambda < \frac{p}{(a^a b^b)^{\frac{1}{p}}} \lambda_1(\Omega),$$

where $\lambda_1(\Omega) > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω , then system (1.1) has a nontrivial nonnegative solution.

2 Proof of Theorem 1.1

Lemma 2.1. Assume that (H1) holds and $-\infty < \gamma < 0$. If A is attained by a couple $(u, v) \in \mathcal{N}$, then (u, v) is a critical point of I , i.e., (u, v) is a solution of (1.1).

Proof. Define

$$\begin{aligned} \mathcal{N}_1 &:= \left\{ (u, v) \in D : u \neq 0, v \neq 0, G_1(u, v) := \int_{\mathbb{R}^N} |\nabla u|^p - \int_{\mathbb{R}^N} \left(\mu_1 |u|^{p^*} + \frac{\alpha\gamma}{p^*} |u|^\alpha |v|^\beta \right) = 0 \right\}, \\ \mathcal{N}_2 &:= \left\{ (u, v) \in D : u \neq 0, v \neq 0, G_2(u, v) := \int_{\mathbb{R}^N} |\nabla v|^p - \int_{\mathbb{R}^N} \left(\mu_2 |v|^{p^*} + \frac{\beta\gamma}{p^*} |u|^\alpha |v|^\beta \right) = 0 \right\}. \end{aligned}$$

Obviously, $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$. Suppose that $(u, v) \in \mathcal{N}$ is a minimizer for I restricted to $\mathcal{N}$. It follows from the standard minimization theory that there exist two Lagrange multipliers $L_1, L_2 \in \mathbb{R}$ such that

$$I'(u, v) + L_1 G'_1(u, v) + L_2 G'_2(u, v) = 0.$$

Noticing that

$$\begin{aligned} I'(u, v)(u, 0) &= G_1(u, v) = 0, \\ I'(u, v)(0, v) &= G_2(u, v) = 0, \\ G'_1(u, v)(u, 0) &= -(p^* - p) \int_{\mathbb{R}^N} \mu_1 |u|^{p^*} + (p - \alpha) \int_{\mathbb{R}^N} \frac{\alpha \gamma}{p^*} |u|^\alpha |v|^\beta, \\ G'_1(u, v)(0, v) &= -\beta \int_{\mathbb{R}^N} \frac{\alpha \gamma}{p^*} |u|^\alpha |v|^\beta > 0, \\ G'_2(u, v)(u, 0) &= -\alpha \int_{\mathbb{R}^N} \frac{\beta \gamma}{p^*} |u|^\alpha |v|^\beta > 0, \\ G'_2(u, v)(0, v) &= -(p^* - p) \int_{\mathbb{R}^N} \mu_2 |v|^{p^*} + (p - \beta) \int_{\mathbb{R}^N} \frac{\beta \gamma}{p^*} |u|^\alpha |v|^\beta, \end{aligned}$$

we get that

$$\begin{cases} G'_1(u, v)(u, 0)L_1 + G'_2(u, v)(u, 0)L_2 = 0, \\ G'_1(u, v)(0, v)L_1 + G'_2(u, v)(0, v)L_2 = 0 \end{cases}$$

and

$$\begin{aligned} G'_1(u, v)(u, 0) + G'_1(u, v)(0, v) &= -(p^* - p) \int_{\mathbb{R}^N} |\nabla u|^p \leq 0, \\ G'_2(u, v)(u, 0) + G'_2(u, v)(0, v) &= -(p^* - p) \int_{\mathbb{R}^N} |\nabla v|^p \leq 0. \end{aligned}$$

We claim that $\int_{\mathbb{R}^N} |\nabla u|^p > 0$. Indeed, if $\int_{\mathbb{R}^N} |\nabla u|^p = 0$, then by (1.2) we have

$$\int_{\mathbb{R}^N} |u|^{p^*} \leq S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} |\nabla u|^p \right)^{\frac{p^*}{p}} = 0.$$

Thus, a desired contradiction comes out, $u \equiv 0$ almost everywhere in $\mathbb{R}^N$. Similarly, $\int_{\mathbb{R}^N} |\nabla v|^p > 0$. Hence,

$$\begin{aligned} |G'_1(u, v)(u, 0)| &= -G'_1(u, v)(u, 0) > G'_1(u, v)(0, v), \\ |G'_2(u, v)(0, v)| &= -G'_2(u, v)(0, v) > G'_2(u, v)(u, 0). \end{aligned}$$

Define the matrix

$$M := \begin{pmatrix} G'_1(u, v)(u, 0) & G'_2(u, v)(u, 0) \\ G'_1(u, v)(0, v) & G'_2(u, v)(0, v) \end{pmatrix}.$$

Then

$$\det(M) = |G'_1(u, v)(u, 0)| \cdot |G'_2(u, v)(0, v)| - G'_1(u, v)(0, v) \cdot G'_2(u, v)(u, 0) > 0,$$

which means that $L_1 = L_2 = 0$. □

Proof of Theorem 1.1. It is standard to see that $A > 0$. By (1.4), we know that $\omega_{\mu_i} := \mu_i^{(p-N)/p^2} U_{1,0}$ satisfies $-\Delta_p u = \mu_i |u|^{p^*-2} u$ in $\mathbb{R}^N$, where $i = 1, 2$. Set $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$ and

$$(u_R(x), v_R(x)) = (\omega_{\mu_1}(x), \omega_{\mu_2}(x + Re_1)),$$

where R is a positive number. Then $v_R \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $v_R \rightharpoonup 0$ weakly in $L^{p^*}(\mathbb{R}^N)$ as $R \rightarrow +\infty$. Hence,

$$\begin{aligned} \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} u_R^\alpha v_R^\beta dx &= \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} u_R^\alpha v_R^{\frac{\alpha}{p^*-1}} v_R^{\frac{\beta^*(\beta-1)}{p^*-1}} dx \\ &\leq \lim_{R \rightarrow +\infty} \left(\int_{\mathbb{R}^N} u_R^{p^*-1} v_R dx \right)^{\frac{\alpha}{p^*-1}} \left(\int_{\mathbb{R}^N} v_R^{p^*} dx \right)^{\frac{\beta-1}{p^*-1}} \\ &= 0. \end{aligned}$$

Therefore, for $R > 0$ sufficiently large, the system

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_R|^p dx = \int_{\mathbb{R}^N} \mu_1 u_R^{p^*} dx = t_R^{\frac{p^*-p}{p}} \int_{\mathbb{R}^N} \mu_1 u_R^{p^*} dx + t_R^{\frac{\alpha-p}{p}} s_R^{\frac{\beta}{p}} \int_{\mathbb{R}^N} \frac{\alpha\gamma}{p^*} u_R^\alpha v_R^\beta dx, \\ \int_{\mathbb{R}^N} |\nabla v_R|^p dx = \int_{\mathbb{R}^N} \mu_2 v_R^{p^*} dx = s_R^{\frac{p^*-p}{p}} \int_{\mathbb{R}^N} \mu_2 v_R^{p^*} dx + t_R^{\frac{\alpha}{p}} s_R^{\frac{\beta-p}{p}} \int_{\mathbb{R}^N} \frac{\beta\gamma}{p^*} u_R^\alpha v_R^\beta dx \end{cases}$$

has a solution (t_R, s_R) with

$$\lim_{R \rightarrow +\infty} (|t_R - 1| + |s_R - 1|) = 0.$$

Furthermore, $(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R) \in \mathcal{N}$. Then, by (1.5), we obtain that

$$\begin{aligned} A &= \inf_{(u,v) \in \mathcal{N}} I(u, v) \leq I(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R) \\ &= \frac{1}{N} \left(t_R \int_{\mathbb{R}^N} |\nabla u_R|^p dx + s_R \int_{\mathbb{R}^N} |\nabla v_R|^p dx \right) \\ &= \frac{1}{N} \left(t_R \mu_1^{-\frac{N-p}{p}} + s_R \mu_2^{-\frac{N-p}{p}} \right) S^{\frac{N}{p}}, \end{aligned}$$

which implies that $A \leq \frac{1}{N} (\mu_1^{-(N-p)/p} + \mu_2^{-(N-p)/p}) S^{N/p}$.

For any $(u, v) \in \mathcal{N}$,

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \leq \mu_1 \int_{\mathbb{R}^N} |u|^{p^*} dx \leq \mu_1 S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{p^*}{p}}.$$

Therefore, $\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \mu_1^{-(N-p)/p} S^{N/p}$. Similarly,

$$\int_{\mathbb{R}^N} |\nabla v|^p dx \geq \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}.$$

Then $A \geq \frac{1}{N} (\mu_1^{-(N-p)/p} + \mu_2^{-(N-p)/p}) S^{N/p}$. Hence,

$$A = \frac{1}{N} \left(\mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}} \right) S^{\frac{N}{p}}. \quad (2.1)$$

Suppose by contradiction that A is attained by some $(u, v) \in \mathcal{N}$. Then $(|u|, |v|) \in \mathcal{N}$ and $I(|u|, |v|) = A$. By Lemma 2.1, we see that $(|u|, |v|)$ is a nontrivial solution of (1.1). By the strong maximum principle, we may assume that $u > 0$, $v > 0$, and so $\int_{\mathbb{R}^N} u^\alpha v^\beta dx > 0$. Then

$$\int_{\mathbb{R}^N} |\nabla u|^p dx < \mu_1 \int_{\mathbb{R}^N} |u|^{p^*} dx \leq \mu_1 S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{p^*}{p}},$$

which yields that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx > \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}}.$$

Similarly,

$$\int_{\mathbb{R}^N} |\nabla v|^p dx > \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}.$$

Therefore,

$$A = I(u, v) = \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) dx > \frac{1}{N} \left(\mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}} \right) S^{\frac{N}{p}},$$

which contradicts (2.1). $\square$

3 Proof of Theorem 1.2

Proposition 3.1. Assume that $c, d \in \mathbb{R}$ satisfy

$$\begin{cases} \mu_1 c^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} c^{\frac{\alpha-p}{p}} d^{\frac{\beta}{p}} \geq 1, \\ \mu_2 d^{\frac{p^*-p}{p}} + \frac{\beta\gamma}{p^*} c^{\frac{\alpha}{p}} d^{\frac{\beta-p}{p}} \geq 1, \\ c > 0, \quad d > 0. \end{cases} \quad (3.1)$$

If $\frac{N}{2} < p < N$, $\alpha, \beta > p$ and (1.7) holds, then $c + d \geq k + l$, where $k, l \in \mathbb{R}$ satisfy (1.6).

Proof. Let $y = c + d$, $x = \frac{c}{d}$, $y_0 = k + l$ and $x_0 = \frac{k}{l}$. By (3.1) and (1.6), we have that

$$\begin{aligned} y^{\frac{p^*-p}{p}} &\geq \frac{(x+1)^{\frac{p^*-p}{p}}}{\mu_1 x^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} x^{\frac{\alpha-p}{p}}} := f_1(x), & y_0^{\frac{p^*-p}{p}} &= f_1(x_0), \\ y^{\frac{p^*-p}{p}} &\geq \frac{(x+1)^{\frac{p^*-p}{p}}}{\mu_2 + \frac{\beta\gamma}{p^*} x^{\frac{\alpha}{p}}} := f_2(x), & y_0^{\frac{p^*-p}{p}} &= f_2(x_0). \end{aligned}$$

Thus,

$$\begin{aligned} f_1'(x) &= \frac{\alpha\gamma(x+1)^{\frac{p^*-2p}{p}} x^{\frac{\alpha-2p}{p}}}{pp^*(\mu_1 x^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} x^{\frac{\alpha-p}{p}})^2} \left[-\frac{p^*(p^*-p)\mu_1}{\alpha\gamma} x^{\frac{\beta}{p}} + \beta x - (\alpha - p) \right], \\ f_2'(x) &= \frac{\beta\gamma(x+1)^{\frac{p^*-2p}{p}}}{pp^*(\mu_2 + \frac{\beta\gamma}{p^*} x^{\frac{\alpha}{p}})^2} \left[(\beta - p)x^{\frac{\alpha}{p}} - \alpha x^{\frac{\alpha-p}{p}} + \frac{p^*(p^*-p)\mu_2}{\beta\gamma} \right]. \end{aligned}$$

Let $x_1 = (\frac{p\alpha\gamma}{p^*(p^*-p)\mu_1})^{p/(\beta-p)}$, $x_2 = \frac{\alpha-p}{\beta-p}$ and

$$\begin{aligned} g_1(x) &= -\frac{p^*(p^*-p)\mu_1}{\alpha\gamma} x^{\frac{\beta}{p}} + \beta x - (\alpha - p), \\ g_2(x) &= (\beta - p)x^{\frac{\alpha}{p}} - \alpha x^{\frac{\alpha-p}{p}} + \frac{p^*(p^*-p)\mu_2}{\beta\gamma}. \end{aligned}$$

It follows from (1.7) that

$$\begin{aligned} \max_{x \in (0, +\infty)} g_1(x) &= g_1(x_1) = (\beta - p) \left(\frac{p\alpha\gamma}{p^*(p^*-p)\mu_1} \right)^{\frac{p}{\beta-p}} - (\alpha - p) \leq 0, \\ \min_{x \in (0, +\infty)} g_2(x) &= g_2(x_2) = -p \left(\frac{\alpha-p}{\beta-p} \right)^{\frac{\alpha-p}{p}} + \frac{p^*(p^*-p)\mu_2}{\beta\gamma} \geq 0. \end{aligned}$$

That is, $f_1(x)$ is strictly decreasing in $(0, +\infty)$ and $f_2(x)$ is strictly increasing in $(0, +\infty)$. Hence,

$$\begin{aligned} y^{\frac{p^*-p}{p}} &\geq \max\{f_1(x), f_2(x)\} \geq \min_{x \in (0, +\infty)} (\max\{f_1(x), f_2(x)\}) \\ &= \min_{\{f_1=f_2\}} (\max\{f_1(x), f_2(x)\}) = y_0^{\frac{p^*-p}{p}}, \end{aligned}$$

where $\{f_1 = f_2\} := \{x \in (0, +\infty) : f_1(x) = f_2(x)\}$. □

Remark 3.1. From the proof of Proposition 3.1 it is easy to see that system (1.6), under the assumption of Proposition 3.1, has only one real solution $(k, l) = (k_0, l_0)$, where (k_0, l_0) is defined as in (1.9).

Define the functions

$$\begin{cases} F_1(k, l) := \mu_1 k^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} k^{\frac{\alpha-p}{p}} l^{\frac{\beta}{p}} - 1, & k > 0, l \geq 0, \\ F_2(k, l) := \mu_2 l^{\frac{p^*-p}{p}} + \frac{\beta\gamma}{p^*} k^{\frac{\alpha}{p}} l^{\frac{\beta-p}{p}} - 1, & k \geq 0, l > 0, \\ l(k) := \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{\beta}{p}} k^{\frac{p-\alpha}{p}} (1 - \mu_1 k^{\frac{p^*-p}{p}})^{\frac{p}{\beta}}, & 0 < k \leq \mu_1^{-\frac{p}{p^*-p}}, \\ k(l) := \left(\frac{p^*}{\beta\gamma}\right)^{\frac{p}{\alpha}} l^{\frac{p-\beta}{p}} (1 - \mu_2 l^{\frac{p^*-p}{p}})^{\frac{p}{\alpha}}, & 0 < l \leq \mu_2^{-\frac{p}{p^*-p}}. \end{cases} \quad (3.2)$$

Then $F_1(k, l(k)) \equiv 0$ and $F_2(k(l), l) \equiv 0$.

Lemma 3.2. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and $\gamma > 0$. Then

$$F_1(k, l) = 0, \quad F_2(k, l) = 0, \quad k, l > 0, \quad (3.3)$$

has a solution (k_0, l_0) such that

$$F_2(k, l(k)) < 0 \quad \text{for all } k \in (0, k_0), \quad (3.4)$$

that is, (k_0, l_0) satisfies (1.9). Similarly, (3.3) has a solution (k_1, l_1) such that

$$F_1(k(l), l) < 0 \quad \text{for all } l \in (0, l_1), \quad (3.5)$$

that is,

$$(k_1, l_1) \text{ satisfies (1.6) and } l_1 = \min\{l : (k, l) \text{ is a solution of (1.6)}\}.$$

Proof. We only prove the existence of (k_0, l_0) . It follows from $F_1(k, l) = 0$, $k, l > 0$, that

$$l = l(k) \quad \text{for all } k \in (0, \mu_1^{-\frac{p}{p^*-p}}).$$

Substituting this into $F_2(k, l) = 0$, we have

$$\mu_2 \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{\alpha}{\beta}} (1 - \mu_1 k^{\frac{p^*-p}{p}})^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{p^*} k^{\frac{(p^*-p)\alpha}{p\beta}} - \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{p-\beta}{\beta}} k^{-\frac{(p^*-p)(p-\alpha)}{p\beta}} (1 - \mu_1 k^{\frac{p^*-p}{p}})^{\frac{p-\beta}{\beta}} = 0. \quad (3.6)$$

By setting

$$f(k) := \mu_2 \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{\alpha}{\beta}} (1 - \mu_1 k^{\frac{p^*-p}{p}})^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{p^*} k^{\frac{(p^*-p)\alpha}{p\beta}} - \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{p-\beta}{\beta}} k^{-\frac{(p^*-p)(p-\alpha)}{p\beta}} (1 - \mu_1 k^{\frac{p^*-p}{p}})^{\frac{p-\beta}{\beta}},$$

the existence of a solution of (3.6) in $(0, \mu_1^{-\frac{p}{p^*-p}})$ is equivalent to $f(k) = 0$ possessing a solution in $(0, \mu_1^{-\frac{p}{p^*-p}})$. Since $\alpha, \beta < p$, we get that

$$\lim_{k \rightarrow 0^+} f(k) = -\infty, \quad f\left(\mu_1^{-\frac{p}{p^*-p}}\right) = \frac{\beta\gamma}{p^*} \mu_1^{-\frac{\alpha}{\beta}} > 0,$$

which implies that there exists $k_0 \in (0, \mu_1^{-\frac{p}{p^*-p}})$ such that $f(k_0) = 0$ and $f(k) < 0$ for $k \in (0, k_0)$. Let $l_0 = l(k_0)$. Then (k_0, l_0) is a solution of (3.3) and (3.4) holds. $\square$

Remark 3.2. From $\frac{2N}{N+2} < p < \frac{N}{2}$ and $\alpha, \beta < p$ we get that $2 < p^* < 2p$. It can be seen from $\frac{N}{2} < p < N$ and $\alpha, \beta > p$ that $2 < 2p < p^*$.

Lemma 3.3. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (1.8) holds. Let (k_0, l_0) be the same as in Lemma 3.2. Then

$$(k_0 + l_0)^{\frac{p^*-p}{p}} \max\{\mu_1, \mu_2\} < 1$$

and

$$F_2(k, l(k)) < 0 \quad \text{for all } k \in (0, k_0), \quad F_1(k(k), l) < 0 \quad \text{for all } l \in (0, l_0). \quad (3.7)$$

Proof. Recalling (3.2), we obtain that

$$\begin{aligned} l'(k) &= \left(\frac{p^*}{\alpha\gamma}\right)^{\frac{p}{\beta}} \frac{p}{\beta} \left(k^{\frac{p-\alpha}{p}} - \mu_1 k^{\frac{\beta}{p}}\right)^{\frac{p-\beta}{\beta}} \left(\frac{p-\alpha}{p} k^{-\frac{\alpha}{p}} - \frac{\mu_1 \beta}{p} k^{\frac{\beta-p}{p}}\right) \\ &= \left(\frac{p^* \mu_1}{\alpha\gamma}\right)^{\frac{p}{\beta}} k^{\frac{p-p^*}{\beta}} (\mu_1^{-1} - k^{\frac{p^*-p}{p}})^{\frac{p-\beta}{\beta}} \left(\frac{p-\alpha}{\mu_1 \beta} - k^{\frac{p^*-p}{p}}\right), \\ l'\left(\left(\frac{p-\alpha}{\mu_1 \beta}\right)^{\frac{p}{p^*-p}}\right) &= l'\left(\mu_1^{-\frac{p}{p^*-p}}\right) = 0, \\ l'(k) &> 0 \quad \text{for } k \in \left(0, \left(\frac{p-\alpha}{\mu_1 \beta}\right)^{\frac{p}{p^*-p}}\right), \\ l'(k) &< 0 \quad \text{for } k \in \left(\left(\frac{p-\alpha}{\mu_1 \beta}\right)^{\frac{p}{p^*-p}}, \mu_1^{-\frac{p}{p^*-p}}\right). \end{aligned}$$

From

$$l''(\bar{k}) = \frac{p-\beta}{\beta} \left(\frac{p^* \mu_1}{\alpha\gamma}\right)^{\frac{p}{\beta}} \bar{k}^{\frac{p-2\beta-\alpha}{\beta}} (\mu_1^{-1} - \bar{k}^{\frac{p^*-p}{p}})^{\frac{p-2\beta}{\beta}} \left[\left(\frac{p-\alpha}{\mu_1 \beta} - \bar{k}^{\frac{p^*-p}{p}}\right)^2 - (\mu_1^{-1} - \bar{k}^{\frac{p^*-p}{p}}) \left(\frac{\alpha(p-\alpha)}{\mu_1 \beta(p-\beta)} - \bar{k}^{\frac{p^*-p}{p}}\right)\right] = 0$$

and $\bar{k} \in ((\frac{p-\alpha}{\mu_1 \beta})^{p/(p^*-p)}, \mu_1^{-p/(p^*-p)})$, we have $\bar{k} = (\frac{p(p-\alpha)}{(2p-p^*)\mu_1 \beta})^{p/(p^*-p)}$. Then, by (1.8), we get that

$$\begin{aligned} \min_{k \in (0, \mu_1^{-p/(p^*-p)})} l'(k) &= \min_{k \in ((\frac{p-\alpha}{\mu_1 \beta})^{p/(p^*-p)}, \mu_1^{-p/(p^*-p)})} l'(k) = l'(\bar{k}) \\ &= -\left(\frac{p^*(p^*-p)\mu_1}{p\alpha\gamma}\right)^{\frac{p}{\beta}} \left(\frac{p-\beta}{p-\alpha}\right)^{\frac{p-\beta}{\beta}} \\ &\geq -1. \end{aligned}$$

Therefore, $l'(k) > -1$ for $k \in (0, \mu_1^{-p/(p^*-p)})$ with

$$k \neq \left(\frac{p(p-\alpha)}{(2p-p^*)\mu_1 \beta}\right)^{\frac{p}{p^*-p}},$$

which implies that $l(k) + k$ is strictly increasing on $[0, \mu_1^{-p/(p^*-p)}]$. Noticing that $k_0 < \mu_1^{-p/(p^*-p)}$, we have

$$\mu_1^{-\frac{p}{p^*-p}} = l\left(\mu_1^{-\frac{p}{p^*-p}}\right) + \mu_1^{-\frac{p}{p^*-p}} > l(k_0) + k_0 = l_0 + k_0,$$

that is, $\mu_1(k_0 + l_0)^{(p^*-p)/p} < 1$. Similarly, $\mu_2(k_0 + l_0)^{(p^*-p)/p} < 1$. To prove (3.7), by Lemma 3.2 it suffices to show that $(k_0, l_0) = (k_1, l_1)$. It follows from (3.4) and (3.5) that $k_1 \geq k_0$ and $l_0 \geq l_1$. Suppose by contradiction that $k_1 > k_0$. Then $l(k_1) + k_1 > l(k_0) + k_0$. Hence, $l_1 + k(l_1) = l(k_1) + k_1 > l(k_0) + k_0 = l_0 + k(l_0)$. Following the arguments as in the beginning of the current proof, we have that $l + k(l)$ is strictly increasing for $l \in [0, \mu_2^{-p/(p^*-p)}]$. Therefore, $l_1 > l_0$, which contradicts $l_0 \geq l_1$. Then $k_1 = k_0$, and similarly $l_0 = l_1$. $\square$

Remark 3.3. For any $\gamma > 0$, condition (1.8) always holds for dimension N large enough.

Proposition 3.4. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (1.8) holds. Then

$$\begin{cases} k + l \leq k_0 + l_0, \\ F_1(k, l) \geq 0, & F_2(k, l) \geq 0, \\ k, l \geq 0, & (k, l) \neq (0, 0), \end{cases} \quad (3.8)$$

has a unique solution $(k, l) = (k_0, l_0)$.

Proof. Obviously, (k_0, l_0) satisfies (3.8). Suppose that $(\tilde{k}, \tilde{l})$ is any solution of (3.8) and, without loss of generality, assume that $\tilde{k} > 0$. We claim that $\tilde{l} > 0$. In fact, if $\tilde{l} = 0$, then $\tilde{k} \leq k_0 + l_0$ and $F_1(\tilde{k}, 0) = \mu_1 \tilde{k}^{(p^*-p)/p} - 1 \geq 0$. Thus,

$$1 \leq \mu_1 \tilde{k}^{\frac{p^*-p}{p}} \leq \mu_1(k_0 + l_0)^{\frac{p^*-p}{p}},$$

a contradiction to Lemma 3.3.

Suppose by contradiction that $\tilde{k} < k_0$. It can be seen that $k(l)$ is strictly increasing on $(0, (\frac{p-\beta}{\mu_2\alpha})^{p/(p^*-p)})$ and strictly decreasing on

$$\left[\left(\frac{2-\beta}{\mu_2\alpha} \right)^{\frac{p}{p^*-p}}, \mu_2^{-\frac{p}{p^*-p}} \right] \quad \text{and} \quad k(0) = k(\mu_2^{-\frac{p}{p^*-p}}) = 0.$$

Since $0 < \tilde{k} < k_0 = k(l_0)$, there exist $0 < l_1 < l_2 < \mu_2^{-p/(p^*-p)}$ such that $k(l_1) = k(l_2) = \tilde{k}$ and

$$F_2(\tilde{k}, l) < 0 \iff \tilde{k} < k(l) \iff l_1 < l < l_2. \quad (3.9)$$

It follows from $F_1(\tilde{k}, \tilde{l}) \geq 0$ and $F_2(\tilde{k}, \tilde{l}) \geq 0$ that $\tilde{l} \geq l(\tilde{k})$ and $\tilde{l} \leq l_1$ or $\tilde{l} \geq l_2$. By (3.7), we see $F_2(\tilde{k}, l(\tilde{k})) < 0$. By (3.9), we get that $l_1 < l(\tilde{k}) < l_2$. Therefore, $\tilde{l} \geq l_2$.

On the other hand, set $l_3 := k_0 + l_0 - \tilde{k}$. Then $l_3 > l_0$ and, moreover,

$$k(l_3) + k_0 + l_0 - \tilde{k} = k(l_3) + l_3 > k(l_0) + l_0 = k_0 + l_0,$$

that is, $k(l_3) > \tilde{k}$. By (3.9), we have $l_1 < l_3 < l_2$. Since $\tilde{k} + \tilde{l} \leq k_0 + l_0$, we obtain that $\tilde{l} \leq k_0 + l_0 - \tilde{k} = l_3 < l_2$. This contradicts $\tilde{l} \geq l_2$. $\square$

Proof of Theorem 1.2. Recalling (1.4) and (1.6), we see that $(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y}) \in \mathcal{N}$ is a nontrivial solution of (1.1), and

$$A \leq I(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y}) = \frac{1}{N}(k_0 + l_0)S^{\frac{N}{p}}. \quad (3.10)$$

Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequence for A , i.e., $I(u_n, v_n) \rightarrow A$ as $n \rightarrow \infty$. Define

$$c_n = \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} \quad \text{and} \quad d_n = \left(\int_{\mathbb{R}^N} |v_n|^{p^*} dx \right)^{\frac{p}{p^*}}.$$

Then

$$\begin{aligned} Sc_n &\leq \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} \left(\mu_1 |u_n|^{p^*} + \frac{\alpha\gamma}{p^*} |u_n|^\alpha |v_n|^\beta \right) dx \\ &\leq \mu_1 c_n^{\frac{p^*}{p}} + \frac{\alpha\gamma}{p^*} c_n^{\frac{\alpha}{p}} d_n^{\frac{\beta}{p}}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} Sd_n &\leq \int_{\mathbb{R}^N} |\nabla v_n|^p dx = \int_{\mathbb{R}^N} \left(\mu_2 |v_n|^{p^*} + \frac{\beta\gamma}{p^*} |u_n|^\alpha |v_n|^\beta \right) dx \\ &\leq \mu_2 d_n^{\frac{p^*}{p}} + \frac{\beta\gamma}{p^*} c_n^{\frac{\alpha}{p}} d_n^{\frac{\beta}{p}}. \end{aligned} \quad (3.12)$$

Dividing both sides of these inequalities by Sc_n and Sd_n , respectively, and denoting

$$\tilde{c}_n = \frac{c_n}{S^{\frac{p}{p^*-p}}}, \quad \tilde{d}_n = \frac{d_n}{S^{\frac{p}{p^*-p}}},$$

we deduce that

$$\mu_1 \tilde{c}_n^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} \tilde{c}_n^{\frac{\alpha-p}{p}} \tilde{d}_n^{\frac{\beta}{p}} \geq 1, \quad \mu_2 \tilde{d}_n^{\frac{p^*-p}{p}} + \frac{\beta\gamma}{p^*} \tilde{c}_n^{\frac{\alpha}{p}} \tilde{d}_n^{\frac{\beta-p}{p}} \geq 1,$$

that is, $F_1(\tilde{c}_n, \tilde{d}_n) \geq 0$ and $F_2(\tilde{c}_n, \tilde{d}_n) \geq 0$. Then, for $\frac{N}{2} < p < N$ and $\alpha, \beta > p$, Proposition 3.1 and Remark 3.1 ensure that $\tilde{c}_n + \tilde{d}_n \geq k + l = k_0 + l_0$, whereas for $\frac{2N}{N+2} < p < \frac{N}{2}$ and $\alpha, \beta < p$ Proposition 3.4 guarantees that $\tilde{c}_n + \tilde{d}_n \geq k_0 + l_0$. Therefore,

$$c_n + d_n \geq (k_0 + l_0)S^{\frac{p}{p^*-p}} = (k_0 + l_0)S^{\frac{N-p}{p}}. \quad (3.13)$$

Noticing that $I(u_n, v_n) = \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u_n|^p + |\nabla v_n|^p)$, by (3.10)–(3.12) we have

$$S(c_n + d_n) \leq NI(u_n, v_n) = NA + o(1) \leq (k_0 + l_0)S^{\frac{N}{p}} + o(1).$$

Combining this with (3.13), we get that $c_n + d_n \rightarrow (k_0 + l_0)S^{(N-p)/p}$ as $n \rightarrow \infty$. Thus,

$$A = \lim_{n \rightarrow \infty} I(u_n, v_n) \geq \lim_{n \rightarrow \infty} \frac{1}{N} S(c_n + d_n) = \frac{1}{N} (k_0 + l_0)S^{\frac{N}{p}}.$$

Hence,

$$A = \frac{1}{N} (k_0 + l_0)S^{\frac{N}{p}} = I(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y}). \quad \square$$

4 Proofs of Theorems 1.3 and 1.4

For (H1) holding and $\gamma > 0$, define

$$A' := \inf_{(u,v) \in \mathcal{N}'} I(u, v),$$

where

$$\mathcal{N}' := \left\{ (u, v) \in D \setminus \{(0, 0)\} : \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) = \int_{\mathbb{R}^N} (\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta) \right\}.$$

It follows from $\mathcal{N} \subset \mathcal{N}'$ that $A' \leq A$. By the Sobolev inequality, we see that $A' > 0$. Consider

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2} u + \frac{\alpha \gamma}{p^*} |u|^{\alpha-2} u |v|^\beta, & x \in B(0, R), \\ -\Delta_p v = \mu_2 |v|^{p^*-2} v + \frac{\beta \gamma}{p^*} |u|^\alpha |v|^{\beta-2} v, & x \in B(0, R), \\ u, v \in H_0^1(B(0, R)), \end{cases}$$

where $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}$. Define

$$\mathcal{N}'(R) := \left\{ (u, v) \in H(0, R) \setminus \{(0, 0)\} : \int_{B(0, R)} (|\nabla u|^p + |\nabla v|^p) = \int_{B(0, R)} (\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta) \right\} \quad (4.1)$$

and

$$A'(R) := \inf_{(u,v) \in \mathcal{N}'(R)} I(u, v),$$

where $H(0, R) := H_0^1(B(0, R)) \times H_0^1(B(0, R))$. For $\varepsilon \in [0, \min\{\alpha, \beta\} - 1]$, consider

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2-2\varepsilon} u + \frac{(\alpha - \varepsilon)\gamma}{p^* - 2\varepsilon} |u|^{\alpha-2-\varepsilon} u |v|^{\beta-\varepsilon}, & x \in B(0, 1), \\ -\Delta_p v = \mu_2 |v|^{p^*-2-2\varepsilon} v + \frac{(\beta - \varepsilon)\gamma}{p^* - 2\varepsilon} |u|^{\alpha-\varepsilon} |v|^{\beta-2-\varepsilon} v, & x \in B(0, 1), \\ u, v \in H_0^1(B(0, 1)). \end{cases} \quad (4.2)$$

Define

$$\begin{cases} I_\varepsilon(u, v) := \frac{1}{p} \int_{B(0, 1)} (|\nabla u|^p + |\nabla v|^p) - \frac{1}{p^* - 2\varepsilon} \int_{B(0, 1)} (\mu_1 |u|^{p^*-2\varepsilon} + \mu_2 |v|^{p^*-2\varepsilon} + \gamma |u|^{\alpha-\varepsilon} |v|^{\beta-\varepsilon}), \\ \mathcal{N}'_\varepsilon := \left\{ (u, v) \in H(0, 1) \setminus \{(0, 0)\} : G_\varepsilon(u, v) := \int_{B(0, 1)} (|\nabla u|^p + |\nabla v|^p) \right. \\ \left. - \int_{B(0, 1)} (\mu_1 |u|^{p^*-2\varepsilon} + \mu_2 |v|^{p^*-2\varepsilon} + \gamma |u|^{\alpha-\varepsilon} |v|^{\beta-\varepsilon}) = 0 \right\} \end{cases} \quad (4.3)$$

and

$$A_\varepsilon := \inf_{(u,v) \in \mathcal{N}'_\varepsilon} I_\varepsilon(u, v).$$

Lemma 4.1. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$. For $\varepsilon \in (0, \min\{\alpha, \beta\} - 1)$, there holds

$$A_\varepsilon < \min \left\{ \inf_{(u,0) \in \mathcal{N}'_\varepsilon} I_\varepsilon(u, 0), \inf_{(0,v) \in \mathcal{N}'_\varepsilon} I_\varepsilon(0, v) \right\}.$$

Proof. From $\min\{\alpha, \beta\} \leq \frac{p^*}{2}$ it is easy to see that $2 < p^* - 2\varepsilon < p^*$. Then we may assume that u_i is a least energy solution of

$$-\Delta_p u = \mu_i |u|^{p^*-2-2\varepsilon} u, \quad u \in H_0^1(B(0, 1)), \quad i = 1, 2.$$

Therefore,

$$I_\varepsilon(u_1, 0) = a_1 := \inf_{(u,0) \in \mathcal{N}'_\varepsilon} I_\varepsilon(u, 0), \quad I_\varepsilon(0, u_2) = a_2 := \inf_{(0,v) \in \mathcal{N}'_\varepsilon} I_\varepsilon(0, v).$$

We claim that, for any $s \in \mathbb{R}$, there exists a unique $t(s) > 0$ such that $(\sqrt[p]{t(s)}u_1, \sqrt[p]{t(s)}su_2) \in \mathcal{N}'_\varepsilon$. In fact,

$$\begin{aligned} t(s)^{\frac{p^*-p-2\varepsilon}{p}} &= \frac{\int_{B(0,1)} (|\nabla u_1|^p + |s|^p |\nabla u_2|^p)}{\int_{B(0,1)} (\mu_1 |u_1|^{p^*-2\varepsilon} + \mu_2 |su_2|^{p^*-2\varepsilon} + \gamma |u_1|^{\alpha-\varepsilon} |su_2|^{\beta-\varepsilon})} \\ &= \frac{qa_1 + qa_2 |s|^p}{qa_1 + qa_2 |s|^{p^*-2\varepsilon} + |s|^{\beta-\varepsilon} \int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}}, \end{aligned}$$

where $q := \frac{p(p^*-2\varepsilon)}{p^*-p-2\varepsilon} = \frac{p(Np-2\varepsilon+2\varepsilon p)}{p^2-2\varepsilon N+2\varepsilon p} \rightarrow N$ as $\varepsilon \rightarrow 0$. Noticing that $t(0) = 1$, we have

$$\lim_{s \rightarrow 0} \frac{t'(s)}{|s|^{\beta-\varepsilon-2}s} = - \frac{(\beta-\varepsilon) \int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1},$$

that is,

$$t'(s) = - \frac{(\beta-\varepsilon) \int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1} |s|^{\beta-\varepsilon-2} s (1 + o(1)) \quad \text{as } s \rightarrow 0.$$

Then

$$t(s) = 1 - \frac{\int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1} |s|^{\beta-\varepsilon} (1 + o(1)) \quad \text{as } s \rightarrow 0,$$

and so,

$$t(s)^{\frac{p^*-2\varepsilon}{p}} = 1 - \frac{\int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}}{pa_1} |s|^{\beta-\varepsilon} (1 + o(1)) \quad \text{as } s \rightarrow 0.$$

Since $\frac{1}{p} - \frac{1}{q} = \frac{1}{p^*-2\varepsilon}$, we have

$$\begin{aligned} A_\varepsilon &\leq I_\varepsilon(\sqrt[p]{t(s)}u_1, \sqrt[p]{t(s)}su_2) \\ &= \left(\frac{1}{p} - \frac{1}{p^*-2\varepsilon}\right) \left(qa_1 + qa_2 |s|^{p^*-2\varepsilon} + |s|^{\beta-\varepsilon} \int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon}\right) t^{\frac{p^*-2\varepsilon}{p}} \\ &= a_1 - \left(\frac{1}{p} - \frac{1}{q}\right) |s|^{\beta-\varepsilon} \int_{B(0,1)} \gamma |u_1|^{\alpha-\varepsilon} |u_2|^{\beta-\varepsilon} + o(|s|^{\beta-\varepsilon}) \\ &< a_1 = \inf_{(u,0) \in \mathcal{N}'_\varepsilon} I_\varepsilon(u, 0) \quad \text{as } |s| \text{ is small enough.} \end{aligned}$$

Similarly, $A_\varepsilon < \inf_{(0,v) \in \mathcal{N}'_\varepsilon} I_\varepsilon(0, v)$. □

Noticing the definition of ω_{μ_i} in the proof of Theorem 1.1, similarly to Lemma 4.1, we obtain that

$$\begin{aligned} A' &< \min\left\{\inf_{(u,0) \in \mathcal{N}'} I(u, 0), \inf_{(0,v) \in \mathcal{N}'} I(0, v)\right\} \\ &= \min\{I(\omega_{\mu_1}, 0), I(0, \omega_{\mu_2})\} \\ &= \min\left\{\frac{1}{N} \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}}, \frac{1}{N} \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}\right\}. \end{aligned} \quad (4.4)$$

Proposition 4.2. For any $\varepsilon \in (0, \min\{\alpha, \beta\} - 1)$, system (4.2) has a classical positive least energy solution $(u_\varepsilon, v_\varepsilon)$ and $u_\varepsilon, v_\varepsilon$ are radially symmetric decreasing.

Proof. It is standard to see that $A_\varepsilon > 0$. For $(u, v) \in \mathcal{N}'_\varepsilon$ with $u \geq 0$ and $v \geq 0$, we denote by (u^*, v^*) its Schwartz symmetrization. By the properties of the Schwartz symmetrization and $\gamma > 0$, we get that

$$\int_{B(0,1)} (|\nabla u^*|^p + |\nabla v^*|^p) \leq \int_{B(0,1)} (\mu_1 |u^*|^{p^*-2\varepsilon} + \mu_2 |v^*|^{p^*-2\varepsilon} + \gamma |u^*|^{\alpha-\varepsilon} |v^*|^{\beta-\varepsilon}).$$

Obviously, there exists $t^* \in (0, 1]$ such that $(\sqrt[p]{t^*}u^*, \sqrt[p]{t^*}v^*) \in \mathcal{N}'_\varepsilon$. Therefore,

$$\begin{aligned} I_\varepsilon(\sqrt[p]{t^*}u^*, \sqrt[p]{t^*}v^*) &= \left(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\right)t^* \int_{B(0,1)} (|\nabla u^*|^p + |\nabla v^*|^p) \\ &\leq \frac{p^* - 2\varepsilon - p}{p(p^* - 2\varepsilon)} \int_{B(0,1)} (|\nabla u|^p + |\nabla v|^p) \\ &= I_\varepsilon(u, v). \end{aligned} \quad (4.5)$$

Therefore, we may choose a minimizing sequence $(u_n, v_n) \in \mathcal{N}'_\varepsilon$ of A_ε such that $(u_n, v_n) = (u_n^*, v_n^*)$ and $I_\varepsilon(u_n, v_n) \rightarrow A_\varepsilon$ as $n \rightarrow \infty$. By (4.5), we see that u_n, v_n are uniformly bounded in $H_0^1(B(0, 1))$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u_\varepsilon, v_n \rightharpoonup v_\varepsilon$ weakly in $H_0^1(B(0, 1))$. Since $H_0^1(B(0, 1)) \hookrightarrow L^{p^*-2\varepsilon}(B(0, 1))$ is compact, we deduce that

$$\begin{aligned} \int_{B(0,1)} (\mu_1|u_\varepsilon|^{p^*-2\varepsilon} + \mu_2|v_\varepsilon|^{p^*-2\varepsilon} + \gamma|u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon}) &= \lim_{n \rightarrow \infty} \int_{B(0,1)} (\mu_1|u_n|^{p^*-2\varepsilon} + \mu_2|v_n|^{p^*-2\varepsilon} + \gamma|u_n|^{\alpha-\varepsilon}|v_n|^{\beta-\varepsilon}) \\ &= \frac{p(p^* - 2\varepsilon)}{p^* - 2\varepsilon - p} \lim_{n \rightarrow \infty} I_\varepsilon(u_n, v_n) \\ &= \frac{p(p^* - 2\varepsilon)}{p^* - 2\varepsilon - p} A_\varepsilon > 0, \end{aligned}$$

which implies that $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$. Moreover, $u_\varepsilon \geq 0, v_\varepsilon \geq 0$ are radially symmetric. Noticing that

$$\int_{B(0,1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) \leq \lim_{n \rightarrow \infty} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p),$$

we get that

$$\int_{B(0,1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) \leq \int_{B(0,1)} (\mu_1|u_\varepsilon|^{p^*-2\varepsilon} + \mu_2|v_\varepsilon|^{p^*-2\varepsilon} + \gamma|u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon}).$$

Then there exists $t_\varepsilon \in (0, 1]$ such that $(\sqrt[p]{t_\varepsilon}u_\varepsilon, \sqrt[p]{t_\varepsilon}v_\varepsilon) \in \mathcal{N}'_\varepsilon$, and therefore

$$\begin{aligned} A_\varepsilon &\leq I_\varepsilon(\sqrt[p]{t_\varepsilon}u_\varepsilon, \sqrt[p]{t_\varepsilon}v_\varepsilon) \\ &= \left(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\right)t_\varepsilon \int_{B(0,1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) \\ &\leq \lim_{n \rightarrow \infty} \frac{p^* - 2\varepsilon - p}{p(p^* - 2\varepsilon)} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p) \\ &= \lim_{n \rightarrow \infty} I_\varepsilon(u_n, v_n) = A_\varepsilon, \end{aligned}$$

which yields that $t_\varepsilon = 1, (u_\varepsilon, v_\varepsilon) \in \mathcal{N}'_\varepsilon, I(u_\varepsilon, v_\varepsilon) = A_\varepsilon$ and

$$\int_{B(0,1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) = \lim_{n \rightarrow \infty} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p).$$

That is, $u_n \rightarrow u_\varepsilon, v_n \rightarrow v_\varepsilon$ strongly in $H_0^1(B(0, 1))$. It follows from the standard minimization theory that there exists a Lagrange multiplier $L \in \mathbb{R}$ satisfying

$$I'_\varepsilon(u_\varepsilon, v_\varepsilon) + LG'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0.$$

Since $I'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = G_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ and

$$G'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = -(p^* - 2\varepsilon - p) \int_{B(0,1)} (\mu_1|u_\varepsilon|^{p^*-2\varepsilon} + \mu_2|v_\varepsilon|^{p^*-2\varepsilon} + \gamma|u_\varepsilon|^{\alpha-\varepsilon}|v_\varepsilon|^{\beta-\varepsilon}) < 0,$$

we get that $L = 0$, and so $I'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$. By $A_\varepsilon = I(u_\varepsilon, v_\varepsilon)$ and Lemma 4.1, we have $u_\varepsilon \neq 0$ and $v_\varepsilon \neq 0$. Since $u_\varepsilon, v_\varepsilon \geq 0$ are radially symmetric decreasing, by the regularity theory and the maximum principle, we obtain that $(u_\varepsilon, v_\varepsilon)$ is a classical positive least energy solution of (4.2). $\square$

Proof of Theorem 1.3. We claim that

$$A'(R) \equiv A' \quad \text{for all } R > 0. \quad (4.6)$$

Indeed, assume $R_1 < R_2$. Since $\mathcal{N}'(R_1) \subset \mathcal{N}'(R_2)$, we get that $A'(R_2) \leq A'(R_1)$. On the other hand, for every $(u, v) \in \mathcal{N}'(R_2)$, define

$$(u_1(x), v_1(x)) := \left(\left(\frac{R_2}{R_1} \right)^{\frac{N-p}{p}} u \left(\frac{R_2}{R_1} x \right), \left(\frac{R_2}{R_1} \right)^{\frac{N-p}{p}} v \left(\frac{R_2}{R_1} x \right) \right).$$

Then it is easy to see that $(u_1, v_1) \in \mathcal{N}'(R_1)$. Thus, we have

$$A'(R_1) \leq I(u_1, v_1) = I(u, v) \quad \text{for all } (u, v) \in \mathcal{N}'(R_2),$$

which means that $A'(R_1) \leq A'(R_2)$. Hence, $A'(R_1) = A'(R_2)$. Obviously, $A' \leq A'(R)$. Let $(u_n, v_n) \in \mathcal{N}'$ be a minimizing sequence of A' . We assume that $u_n, v_n \in H_0^1(B(0, R_n))$ for some $R_n > 0$. Therefore, $(u_n, v_n) \in \mathcal{N}'(R_n)$ and

$$A' = \lim_{n \rightarrow \infty} I(u_n, v_n) \geq \lim_{n \rightarrow \infty} A'(R_n) = A'(R),$$

which completes the proof of the claim.

By recalling (4.1) and (4.3), for every $(u, v) \in \mathcal{N}'(1)$, there exists $t_\varepsilon > 0$ with $t_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$ such that $(\sqrt[p]{t_\varepsilon} u, \sqrt[p]{t_\varepsilon} v) \in \mathcal{N}'_\varepsilon$. Then

$$\limsup_{\varepsilon \rightarrow 0} A_\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon(\sqrt[p]{t_\varepsilon} u, \sqrt[p]{t_\varepsilon} v) = I(u, v) \quad \text{for all } (u, v) \in \mathcal{N}'(1).$$

It follows from (4.6) that

$$\limsup_{\varepsilon \rightarrow 0} A_\varepsilon \leq A'(1) = A'. \quad (4.7)$$

According to Proposition 4.2, we may let $(u_\varepsilon, v_\varepsilon)$ be a positive least energy solution of (4.2), which is radially symmetric decreasing. By (4.3) and the Sobolev inequality, we have

$$A_\varepsilon = \frac{p^* - 2\varepsilon - 2}{2(p^* - 2\varepsilon)} \int_{B(0,1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) \geq C > 0 \quad \text{for all } \varepsilon \in \left(0, \frac{\min\{\alpha, \beta\} - 1}{2}\right], \quad (4.8)$$

where C is independent of ε . Then it follows from (4.7) that $u_\varepsilon, v_\varepsilon$ are uniformly bounded in $H_0^1(B(0, 1))$. We may assume that $u_\varepsilon \rightharpoonup u_0, v_\varepsilon \rightharpoonup v_0$, up to a subsequence, weakly in $H_0^1(B(0, 1))$. Hence, (u_0, v_0) is a solution of

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2} u + \frac{\alpha \gamma}{p^*} |u|^{\alpha-2} u |v|^\beta, & x \in B(0, 1), \\ -\Delta_p v = \mu_2 |v|^{p^*-2} v + \frac{\beta \gamma}{p^*} |u|^\alpha |v|^{\beta-2} v, & x \in B(0, 1), \\ u, v \in H_0^1(B(0, 1)). \end{cases}$$

Suppose by contradiction that $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty$ is uniformly bounded. Then, by the dominated convergent theorem, we get that

$$\lim_{\varepsilon \rightarrow 0} \int_{B(0,1)} u_\varepsilon^{p^*-2\varepsilon} = \int_{B(0,1)} u_0^{p^*}, \quad \lim_{\varepsilon \rightarrow 0} \int_{B(0,1)} v_\varepsilon^{p^*-2\varepsilon} = \int_{B(0,1)} v_0^{p^*}, \quad \lim_{\varepsilon \rightarrow 0} \int_{B(0,1)} u_\varepsilon^{\alpha-\varepsilon} v_\varepsilon^{\beta-\varepsilon} = \int_{B(0,1)} u_0^\alpha v_0^\beta.$$

Combining these with $I'_\varepsilon(u_\varepsilon, v_\varepsilon) = I'(u_0, v_0)$, similarly to the proof of Proposition 4.2, we see that $u_\varepsilon \rightarrow u_0, v_\varepsilon \rightarrow v_0$ strongly in $H_0^1(B(0, 1))$. It follows from (4.8) that $(u_0, v_0) \neq (0, 0)$ and, moreover, $u_0 \geq 0, v_0 \geq 0$. Without loss of generality, we may assume that $u_0 \neq 0$. By the strong maximum principle, we obtain that $u_0 > 0$ in $B(0, 1)$. By the Pohozaev identity, we have a contradiction

$$0 < \int_{\partial B(0,1)} (|\nabla u_0|^p + |\nabla v_0|^p)(x \cdot \nu) d\sigma = 0,$$

where ν is the outward unit normal vector on $\partial B(0, 1)$. Hence, $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Let

$$K_\varepsilon := \max\{u_\varepsilon(0), v_\varepsilon(0)\}.$$

Since $u_\varepsilon(0) = \max_{B(0,1)} u_\varepsilon(x)$ and $v_\varepsilon(0) = \max_{B(0,1)} v_\varepsilon(x)$, we see that $K_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Setting

$$U_\varepsilon(x) := K_\varepsilon^{-1} u_\varepsilon(K_\varepsilon^{-a_\varepsilon} x), \quad V_\varepsilon(x) := K_\varepsilon^{-1} v_\varepsilon(K_\varepsilon^{-a_\varepsilon} x), \quad a_\varepsilon := \frac{p^* - p - p\varepsilon}{p},$$

we have

$$\max\{U_\varepsilon(0), V_\varepsilon(0)\} = \max\left\{\max_{x \in B(0, K_\varepsilon^{a_\varepsilon})} U_\varepsilon(x), \max_{x \in B(0, K_\varepsilon^{a_\varepsilon})} V_\varepsilon(x)\right\} = 1, \quad (4.9)$$

and $(U_\varepsilon, V_\varepsilon)$ is a solution of

$$\begin{cases} -\Delta_p U_\varepsilon = \mu_1 U_\varepsilon^{p^*-2\varepsilon-1} + \frac{(\alpha-\varepsilon)\gamma}{p^*-2\varepsilon} U_\varepsilon^{\alpha-1-\varepsilon} V_\varepsilon^{\beta-\varepsilon}, & x \in B(0, K_\varepsilon^{a_\varepsilon}), \\ -\Delta_p V_\varepsilon = \mu_2 V_\varepsilon^{p^*-2\varepsilon-1} + \frac{(\beta-\varepsilon)\gamma}{p^*-2\varepsilon} U_\varepsilon^{\alpha-\varepsilon} V_\varepsilon^{\beta-1-\varepsilon}, & x \in B(0, K_\varepsilon^{a_\varepsilon}). \end{cases}$$

Since

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla U_\varepsilon(x)|^p dx &= K_\varepsilon^{a_\varepsilon(N-p)-p} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(y)|^p dy \\ &= K_\varepsilon^{-(N-p)\varepsilon} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^p dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^p dx, \end{aligned}$$

we see that $\{(U_\varepsilon, V_\varepsilon)\}_{n \geq 1}$ is bounded in D . By elliptic estimates, we get that, up to a subsequence,

$$(U_\varepsilon, V_\varepsilon) \rightarrow (U, V) \in D$$

uniformly in every compact subset of $\mathbb{R}^N$ as $\varepsilon \rightarrow 0$, and (U, V) is a solution of (1.1), that is, $I'(U, V) = 0$. Moreover, $U \geq 0$, $V \geq 0$ are radially symmetric decreasing. By (4.9), we have $(U, V) \neq (0, 0)$, and so $(U, V) \in \mathcal{N}'$. Thus,

$$\begin{aligned} A' \leq I(U, V) &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} (|\nabla U|^p + |\nabla V|^p) dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{B(0, K_\varepsilon^{a_\varepsilon})} (|\nabla U_\varepsilon|^p + |\nabla V_\varepsilon|^p) dx \\ &= \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\right) \int_{B(0, K_\varepsilon^{a_\varepsilon})} (|\nabla U_\varepsilon|^p + |\nabla V_\varepsilon|^p) dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\right) \int_{B(0, 1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) dx \\ &= \liminf_{\varepsilon \rightarrow 0} A_\varepsilon. \end{aligned}$$

It follows from (4.7) that $A' \leq I(U, V) \leq \liminf_{\varepsilon \rightarrow 0} A_\varepsilon \leq A'$, which means that $I(U, V) = A'$. By (4.4), we get that $U \neq 0$ and $V \neq 0$. The strong maximum principle guarantees that $U > 0$ and $V > 0$. Since $(U, V) \in \mathcal{N}$, we have $I(U, V) \geq A \geq A'$. Therefore,

$$I(U, V) = A = A', \quad (4.10)$$

that is, (U, V) is a positive least energy solution of (1.1) with (H1) holding, which is radially symmetric decreasing. This completes the proof. $\square$

Remark 4.1. If (H1) and (C2) hold, then it can be seen from Theorems 1.2 and 1.3 that $(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y})$ is a positive least energy solution of (1.1), where (k_0, l_0) is defined by (1.9) and $U_{\varepsilon,y}$ is defined by (1.4).

Proof of Theorem 1.4. To prove the existence of $(k(\gamma), l(\gamma))$ for $\gamma > 0$ small, recalling (3.2), we denote $F_i(k, l, \gamma)$ by $F_i(k, l)$, $i = 1, 2$, in this proof. Let $k(0) = \mu_1^{-p/(p^*-p)}$ and $l(0) = \mu_2^{-p/(p^*-p)}$. Then

$$F_1(k(0), l(0), 0) = F_2(k(0), l(0), 0) = 0.$$

Obviously, we have

$$\begin{aligned}\partial_k F_1(k(0), l(0), 0) &= \frac{p^* - p}{p} \mu_1 k^{\frac{p^*-2p}{p}} > 0, \\ \partial_l F_1(k(0), l(0), 0) &= \partial_k F_2(k(0), l(0), 0) = 0, \\ \partial_l F_2(k(0), l(0), 0) &= \frac{p^* - p}{p} \mu_2 l^{\frac{p^*-2p}{p}} > 0,\end{aligned}$$

which implies that

$$\det \begin{pmatrix} \partial_k F_1(k(0), l(0), 0) & \partial_l F_1(k(0), l(0), 0) \\ \partial_k F_2(k(0), l(0), 0) & \partial_l F_2(k(0), l(0), 0) \end{pmatrix} > 0.$$

By the implicit function theorem, we see that $k(\gamma), l(\gamma)$ are well defined and of class C^1 in $(-\gamma_2, \gamma_2)$ for some $\gamma_2 > 0$, and $F_1(k(\gamma), l(\gamma), \gamma) = F_2(k(\gamma), l(\gamma), \gamma) = 0$. Then $(\sqrt[p]{k(\gamma)}U_{\varepsilon,y}, \sqrt[p]{l(\gamma)}U_{\varepsilon,y})$ is a positive solution of (1.1). Noticing that

$$\lim_{\gamma \rightarrow 0} (k(\gamma) + l(\gamma)) = k(0) + l(0) = \mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}},$$

we obtain that there exists $\gamma_1 \in (0, \gamma_2]$ such that

$$k(\gamma) + l(\gamma) > \min \left\{ \mu_1^{-\frac{N-p}{p}}, \mu_2^{-\frac{N-p}{p}} \right\} \quad \text{for all } \gamma \in (0, \gamma_1).$$

It follows from (4.4) and (4.10) that

$$\begin{aligned}I \left(\sqrt[p]{k(\gamma)}U_{\varepsilon,y}, \sqrt[p]{l(\gamma)}U_{\varepsilon,y} \right) &= \frac{1}{N} (k(\gamma) + l(\gamma)) S^{\frac{N}{p}} \\ &> \min \left\{ \frac{1}{N} \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}}, \frac{1}{N} \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}} \right\} \\ &> A' = A = I(U, V),\end{aligned}$$

that is, when (H1) is satisfied, $(\sqrt[p]{k(\gamma)}U_{\varepsilon,y}, \sqrt[p]{l(\gamma)}U_{\varepsilon,y})$ is a different positive solution of (1.1) with respect to (U, V) . $\square$

5 Proof of Theorem 1.5

In this section, we consider the case (H2).

Proposition 5.1. Let $q, r > 1$ satisfy $q + r \leq p^*$, and set

$$\begin{aligned}S_{q,r}(\Omega) &= \inf_{\substack{u,v \in W_0^{1,p}(\Omega) \\ u,v \neq 0}} \frac{\int_{\Omega} (|\nabla u|^p + |\nabla v|^p) dx}{\left(\int_{\Omega} |u|^q |v|^r dx \right)^{\frac{p}{q+r}}}, \\ S_{q+r}(\Omega) &= \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{q+r} dx \right)^{\frac{p}{q+r}}}.\end{aligned}$$

Then

$$S_{q,r}(\Omega) = \frac{q+r}{(q^q r^r)^{\frac{1}{q+r}}} S_{q+r}(\Omega). \quad (5.1)$$

Moreover, if u_0 is a minimizer for $S_{q+r}(\Omega)$, then $(q^{1/p}u_0, r^{1/p}u_0)$ is a minimizer for $S_{q,r}(\Omega)$.

Proof. For $u \neq 0$ in $W_0^{1,p}(\Omega)$ and $t > 0$, taking $v = t^{-1/p}u$ in the first quotient gives

$$S_{q,r}(\Omega) \leq \left[t^{\frac{r}{q+r}} + t^{-\frac{q}{q+r}} \right] \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{q+r} dx \right)^{\frac{p}{q+r}}},$$

and minimizing the right-hand side over u and t shows that $S_{q,r}(\Omega)$ is less than or equal to the right-hand side of (5.1). For $u, v \neq 0$ in $W_0^{1,p}(\Omega)$, let $w = t^{1/p}v$, where

$$t^{\frac{q+r}{p}} = \frac{\int_{\Omega} |u|^{q+r} dx}{\int_{\Omega} |v|^{q+r} dx}.$$

Then $\int_{\Omega} |u|^{q+r} dx = \int_{\Omega} |w|^{q+r} dx$, and hence

$$\int_{\Omega} |u|^q |w|^r dx \leq \int_{\Omega} |u|^{q+r} dx = \int_{\Omega} |w|^{q+r} dx$$

by the Hölder inequality, so

$$\begin{aligned} \frac{\int_{\Omega} (|\nabla u|^p + |\nabla v|^p) dx}{\left(\int_{\Omega} |u|^q |v|^r dx \right)^{\frac{p}{q+r}}} &= \frac{\int_{\Omega} (t^{\frac{r}{q+r}} |\nabla u|^p + t^{-\frac{q}{q+r}} |\nabla w|^p) dx}{\left(\int_{\Omega} |u|^q |w|^r dx \right)^{\frac{p}{q+r}}} \\ &\geq t^{\frac{r}{q+r}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^{q+r} dx \right)^{\frac{p}{q+r}}} + t^{-\frac{q}{q+r}} \frac{\int_{\Omega} |\nabla w|^p dx}{\left(\int_{\Omega} |w|^{q+r} dx \right)^{\frac{p}{q+r}}} \\ &\geq \left[t^{\frac{r}{q+r}} + t^{-\frac{q}{q+r}} \right] S_{q+r}(\Omega). \end{aligned}$$

The last expression is greater than or equal to the right-hand side of (5.1), so minimizing over (u, v) gives the reverse inequality. $\square$

By Proposition 5.1,

$$S_{a,b}(\Omega) = \frac{p}{(a^a b^b)^{\frac{1}{p}}} \lambda_1(\Omega), \quad S_{\alpha,\beta} = \frac{p^*}{(\alpha^{\alpha} \beta^{\beta})^{\frac{1}{p^*}}} S, \quad (5.2)$$

where $\lambda_1(\Omega) > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω . When (H2) is satisfied, we will obtain a nontrivial nonnegative solution of system (1.1) for $\lambda < S_{a,b}(\Omega)$. Consider the C^1 -functional

$$\Phi(w) = \frac{1}{p} \int_{\Omega} [|\nabla u|^p + |\nabla v|^p - \lambda(u^+)^a (v^+)^b] dx - \frac{1}{p^*} \int_{\Omega} (u^+)^{\alpha} (v^+)^{\beta} dx, \quad w \in W,$$

where $W = D_0^{1,p}(\Omega) \times D_0^{1,p}(\Omega)$ with the norm given by $\|w\|^p = |\nabla u|_p^p + |\nabla v|_p^p$ for $w = (u, v)$, $|\cdot|_p$ denotes the norm in $L^p(\Omega)$ and $u^{\pm}(x) = \max\{\pm u(x), 0\}$ are the positive and negative parts of u , respectively. If w is a critical point of Φ ,

$$0 = \Phi'(w)(u^-, v^-) = \int_{\Omega} (|\nabla u^-|^p + |\nabla v^-|^p) dx,$$

and hence $(u^-, v^-) = 0$, so $w = (u^+, v^+)$ is a nonnegative weak solution of (1.1) with (H2) holding.

Proposition 5.2. *If $0 \neq c < S_{\alpha,\beta}^{N/p}/N$ and $\lambda < S_{a,b}(\Omega)$, then every $(PS)_c$ sequence of Φ has a subsequence that converges weakly to a nontrivial critical point of Φ .*

Proof. Let $\{w_j\}$ be a $(PS)_c$ sequence. Then

$$\begin{aligned} \Phi(w_j) &= \frac{1}{p} \int_{\Omega} [|\nabla u_j|^p + |\nabla v_j|^p - \lambda(u_j^+)^a (v_j^+)^b] dx - \frac{1}{p^*} \int_{\Omega} (u_j^+)^{\alpha} (v_j^+)^{\beta} dx \\ &= c + o(1) \end{aligned}$$

and

$$\begin{aligned}\Phi'(w_j)w_j &= \int_{\Omega} [|\nabla u_j|^p + |\nabla v_j|^p - \lambda(u_j^+)^a(v_j^+)^b] dx - \int_{\Omega} (u_j^+)^a(v_j^+)^b dx \\ &= o(\|w_j\|),\end{aligned}\quad (5.3)$$

so

$$\frac{1}{N} \int_{\Omega} [|\nabla u_j|^p + |\nabla v_j|^p - \lambda(u_j^+)^a(v_j^+)^b] dx = c + o(\|w_j\| + 1). \quad (5.4)$$

Since the integral on the left-hand side is greater than or equal to $(1 - \frac{\lambda}{S_{a,b}(\Omega)})\|w_j\|^p$, $\lambda < S_{a,b}(\Omega)$ and $p > 1$, it follows that $\{w_j\}$ is bounded in W . So a renamed subsequence converges to some w weakly in W , strongly in $L^s(\Omega) \times L^t(\Omega)$ for all $1 \leq s, t < p^*$ and a.e. in Ω . Then $w_j \rightarrow w$ strongly in $W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega)$ for all $1 \leq q, r < p$ by Boccardo and Murat [6, Theorem 2.1], and hence $\nabla w_j \rightarrow \nabla w$ a.e. in Ω for a further subsequence. It then follows that w is a critical point of Φ .

Suppose $w = 0$. Since $\{w_j\}$ is bounded in W and converges to zero in $L^p(\Omega) \times L^p(\Omega)$, equation (5.3) and the Hölder inequality give

$$o(1) = \int_{\Omega} (|\nabla u_j|^p + |\nabla v_j|^p) dx - \int_{\Omega} (u_j^+)^a(v_j^+)^b dx \geq \|w_j\|^p \left(1 - \frac{\|w_j\|^{p^*-p}}{S_{\alpha,\beta}^{\frac{p^*}{p}}}\right).$$

If $\|w_j\| \rightarrow 0$, then $\Phi(w_j) \rightarrow 0$, contradicting $c \neq 0$, so this implies

$$\|w_j\|^p \geq S_{\alpha,\beta}^{\frac{N}{p}} + o(1)$$

for a renamed subsequence. Then (5.4) gives

$$c = \frac{\|w_j\|^p}{N} + o(1) \geq \frac{S_{\alpha,\beta}^{\frac{N}{p}}}{N} + o(1),$$

contradicting $c < S_{\alpha,\beta}^{N/p}/N$. $\square$

Recall (1.4) and (1.5) and let $\eta : [0, \infty) \rightarrow [0, 1]$ be a smooth cut-off function such that $\eta(s) = 1$ for $s \leq \frac{1}{4}$ and $\eta(s) = 0$ for $s \geq \frac{1}{2}$; set

$$u_{\varepsilon,\rho}(x) = \eta\left(\frac{|x|}{\rho}\right)U_{\varepsilon,0}(x)$$

for $\rho > 0$. We have the following estimates for $u_{\varepsilon,\rho}$ (see [15, Lemma 3.1]):

$$\int_{\mathbb{R}^N} |\nabla u_{\varepsilon,\rho}|^p dx \leq S^{\frac{N}{p}} + C\left(\frac{\varepsilon}{\rho}\right)^{\frac{N-p}{p-1}}, \quad (5.5)$$

$$\int_{\mathbb{R}^N} u_{\varepsilon,\rho}^p dx \geq \begin{cases} \frac{1}{C}\varepsilon^p \log\left(\frac{\rho}{\varepsilon}\right) - C\varepsilon^p & \text{if } N = p^2, \\ \frac{1}{C}\varepsilon^p - C\rho^p\left(\frac{\varepsilon}{\rho}\right)^{\frac{N-p}{p-1}} & \text{if } N > p^2, \end{cases} \quad (5.6)$$

$$\int_{\mathbb{R}^N} u_{\varepsilon,\rho}^{p^*} dx \geq S^{\frac{N}{p}} - C\left(\frac{\varepsilon}{\rho}\right)^{\frac{N}{p-1}}, \quad (5.7)$$

where $C = C(N, p)$. We will make use of these estimates in the proof of our last theorem.

Proof of Theorem 1.5. In view of (5.2),

$$\Phi(w) \geq \frac{1}{p} \left(1 - \frac{\lambda}{S_{a,b}(\Omega)}\right) \|w\|^p - \frac{1}{p^* S_{\alpha,\beta}^{\frac{p^*}{p}}} \|w\|^{p^*},$$

so the origin is a strict local minimizer of Φ . We may assume without loss of generality that $0 \in \Omega$. Fix $\rho > 0$ so small that $\Omega \supset B_\rho(0) \supset \text{supp } u_{\varepsilon,\rho}$, and let $w_\varepsilon = (\alpha^{1/p} u_{\varepsilon,\rho}, \beta^{1/p} u_{\varepsilon,\rho}) \in W$. Note that

$$\Phi(Rw_\varepsilon) = \frac{R^p}{p} (p^* |\nabla u_{\varepsilon,\rho}|_p^p - \lambda \alpha^{\frac{a}{p}} \beta^{\frac{b}{p}} |u_{\varepsilon,\rho}|_p^p) - \frac{R^{p^*}}{p^*} \alpha^{\frac{a}{p}} \beta^{\frac{b}{p}} |u_{\varepsilon,\rho}|_{p^*}^{p^*} \rightarrow -\infty$$

as $R \rightarrow +\infty$ and fix $R_0 > 0$ so large that $\Phi(R_0 w_\varepsilon) < 0$. Then let

$$\Gamma = \{\gamma \in C([0, 1], W) : \gamma(0) = 0, \gamma(1) = R_0 w_\varepsilon\}$$

and set

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi(\gamma(t)) > 0.$$

By the mountain pass theorem, Φ has a $(PS)_c$ sequence $\{w_j\}$.

Since $t \mapsto tR_0 w_\varepsilon$ is a path in Γ ,

$$c \leq \max_{t \in [0, 1]} \Phi(tR_0 w_\varepsilon) = \frac{1}{N} \left(\frac{p^* |\nabla u_{\varepsilon,\rho}|_p^p - \lambda (\alpha^a \beta^b)^{\frac{1}{p}} |u_{\varepsilon,\rho}|_p^p}{(\alpha^a \beta^b)^{\frac{1}{p^*}} |u_{\varepsilon,\rho}|_{p^*}^p} \right)^{\frac{N}{p}} =: \frac{1}{N} S_\varepsilon^{\frac{N}{p}}. \quad (5.8)$$

By (5.5)–(5.7),

$$S_\varepsilon \leq \frac{p^* S^p + \frac{\lambda (\alpha^a \beta^b)^{\frac{1}{p}}}{C} \varepsilon^p \log \varepsilon + O(\varepsilon^p)}{(\alpha^a \beta^b)^{\frac{1}{p^*}} (S^p + O(\varepsilon^{\frac{p^2}{p-1}}))^{\frac{p-1}{p}}} = S_{\alpha,\beta} - \left(\frac{\lambda \alpha^{\frac{a}{p} - \frac{a}{p^*}} \beta^{\frac{b}{p} - \frac{b}{p^*}}}{C S^{p-1}} |\log \varepsilon| + O(1) \right) \varepsilon^p$$

if $N = p^2$, and

$$S_\varepsilon \leq \frac{p^* S^{\frac{N}{p}} - \frac{\lambda (\alpha^a \beta^b)^{\frac{1}{p}}}{C} \varepsilon^p + O(\varepsilon^{\frac{N-p}{p-1}})}{(\alpha^a \beta^b)^{\frac{1}{p^*}} (S^{\frac{N}{p}} + O(\varepsilon^{\frac{N}{p-1}}))^{\frac{N-p}{N}}} = S_{\alpha,\beta} - \left(\frac{\lambda \alpha^{\frac{a}{p} - \frac{a}{p^*}} \beta^{\frac{b}{p} - \frac{b}{p^*}}}{C S^{\frac{N-p}{p}}} + O(\varepsilon^{\frac{N-p^2}{p-1}}) \right) \varepsilon^p$$

if $N > p^2$, so $S_\varepsilon < S_{\alpha,\beta}$ if $\varepsilon > 0$ is sufficiently small. So $c < S_{\alpha,\beta}^{N/p}/N$ by (5.8), and hence a subsequence of $\{w_j\}$ converges weakly to a nontrivial critical point of Φ by Proposition 5.2, which then is a nontrivial nonnegative solution of (1.1) with (H2) holding. $\square$

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