Research Article

Bixiao Shi, Rongchang Liu, Duokui Yan* and Tiancheng Ouyang

Multiple Periodic Orbits Connecting a Collinear Configuration and a Double Isosceles Configuration in the Planar Equal-Mass Four-Body Problem

DOI: 10.1515/ans-2017-6028
Received January 11, 2017; revised June 18, 2017; accepted June 26, 2017

Abstract: By applying our variational method, we show that there exist 24 local action minimizers connecting two prescribed configurations: a collinear configuration and a double isosceles configuration in $H^1([0, 1], \chi)$ in the planar equal-mass four-body problem. Among the 24 local action minimizers, we prove that the one with the smallest action has no collision singularity and it can be extended to a periodic or quasi-periodic orbit. Furthermore, if all the 24 local action minimizers are free of collision, we show that they can generate sixteen different periodic orbits.

Keywords: Four-Body Problem, Variational Method, Topological Constraint, Free Boundary Value Problem

MSC 2010: 70F10, 70F16

Communicated by: Yiming Long

1 Introduction

Variational method is one of the important tools in studying periodic orbits in the N-body problem. Actually, back to the 1890s, Poincaré had tried to apply variational methods to periodic orbits in the N-body problem, but he did not succeed because of two major difficulties [7]: “One is the lack of coercivity due to the vanishing at infinity of the force fields. The other is the possible existence of collision: the Lagrangian action stays finite even when some of the bodies are colliding.” Until recently in 2000, Chenciner and Montgomery [8] considered the action minimizer over a suitable symmetric loop space and successfully overcame the two difficulties. In their celebrated paper [8], they showed the existence of the figure-eight orbit in the planar equal-mass three-body problem. Following their ideas of imposing symmetry constraints or topological constraints, many new periodic orbits have been discovered numerically and proved rigorously [3–5, 7–10].

Besides the existence of periodic orbits, the variational method has widely been applied to study the multiplicity of periodic orbits, see [11, 17–19] and the references therein. Instead of fixing energy, we fix two prescribed configurations and show the multiplicity of periodic orbits from a new perspective. Inspired by the works [1, 7, 8, 12, 13], we study the existence and multiplicity of periodic orbits connecting two fixed

Bixiao Shi, Rongchang Liu: School of Mathematics and System Sciences, Beihang University, Beijing 100191, P. R. China, e-mail: 13693213105@126.com, 12091021@buaa.edu.cn
***Corresponding author: Duokui Yan:** School of Mathematics and System Sciences, Beihang University, Beijing 100191, P. R. China, e-mail: duokuiyan@buaa.edu.cn
Tiancheng Ouyang: Department of Mathematics, Brigham Young University, Provo, UT 84602, USA, e-mail: ouyang@math.byu.edu

configurations in (1.1): a collinear configuration and a double isosceles configuration in the planar equal-mass four-body problem. Let $N = 4$ and $m_1 = m_2 = m_3 = m_4 = 1$. Let the row vectors $q_i(t) = (q_{ix}(t), q_{iy}(t)) \in \mathbb{R}^2$ $(i = 1, 2, 3, 4)$ be the trajectories of the body m_i. Set

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix},$$

which is a (4×2)-matrix path. The standard Lagrangian action functional is as follows:

$$\mathcal{A} = \int_0^1 [K(\dot{q}(t)) + U(q(t))]\, dt,$$

where

$$K(\dot{q}(t)) = \frac{1}{2}\sum_{i=1}^{4} m_i|\dot{q}_i(t)|^2 \quad \text{and} \quad U(q(t)) = \sum_{1\le i<j\le 4} \frac{m_i m_j}{|q_i(t) - q_j(t)|}.$$

For a given value of $\theta \in (0, \frac{\pi}{2})$, the two prescribed configurations are defined by

$$Q_{\text{start}} = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \\ 0 & -c_1 \\ 0 & c_1 - a_1 - b_1 \end{bmatrix}, \quad Q_{\text{end}} = \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \\ -c_2 & -\frac{a_2+b_2}{2} \\ c_2 & -\frac{a_2+b_2}{2} \end{bmatrix} R(\theta), \tag{1.1}$$

where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Without loss of generality, we assume the center of mass to be at the origin. That is, $q \in \chi$, where

$$\chi = \left\{ q \in \mathbb{R}_{4\times 2} : \sum_{i=1}^{4} m_i q_i = 0 \right\}.$$

Let $\vec{a} = (a_1, b_1, c_1, a_2, b_2, c_2)$. Given $\theta \in (0, \frac{\pi}{2})$ and $\vec{a} \in \mathbb{R}^6$, the position matrices Q_{start} and Q_{end} in (1.1) are fixed. We set $P(Q_{\text{start}}, Q_{\text{end}})$ to be the path space connecting the two fixed ends Q_{start} and Q_{end}:

$$\mathrm{P}(Q_{\text{start}}, Q_{\text{end}}) := \{q(t) \in H^1([0, 1], \chi) : q(0) = Q_{\text{start}},\ q(1) = Q_{\text{end}}\}.$$

It is known that there exists an action minimizer $\mathcal{P}$ connecting the two fixed ends, which satisfies

$$\mathcal{A}(\mathcal{P}) = \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A}.$$

In general, the minimizer $\mathcal{P}$ is not a part of a periodic solution. In order to find a periodic or quasi-periodic solution, we consider the following free boundary value problem:

$$\operatorname*{localmin}_{\{\vec{a}\in\mathbb{R}^6\}} \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A}.$$

For each fixed value of $\theta \in (0, \frac{\pi}{2})$, it is interesting to understand the existence and multiplicity of periodic orbits connecting Q_{start} and Q_{end} defined by (1.1). Actually, we can show the existence of an action minimizer $\mathcal{P}_0$, which minimizes $\mathcal{A}$ over $\vec{a} \in \mathbb{R}^6$. The following theorem implies that $\mathcal{P}_0$ is a classical solution and also a part of a periodic or quasi-periodic orbit. Its proof follows by Theorem 2.1, Theorem 3.1 and Theorem 4.1. One of the main difficulties in proving Theorem 1.1 is to exclude possible triple collisions in $\mathcal{P}_0$. For this purpose, a local deformation argument is introduced in Lemma 3.4, which discusses all possible central configurations case by case.

Theorem 1.1. *For any given $\theta \in (0, \frac{\pi}{2})$, there exists a noncollision minimizing path $\mathcal{P}_0 \equiv \mathcal{P}_0$ $(t \in [0, 1])$, which satisfies*

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{\vec{a}\in\mathbb{R}^6\}} \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A}.$$

Furthermore, if $\frac{\theta}{\pi}$ is rational, the minimizer $\mathcal{P}_0$ can be extended to a periodic orbit. Otherwise, it can be extended to a quasi-periodic orbit.

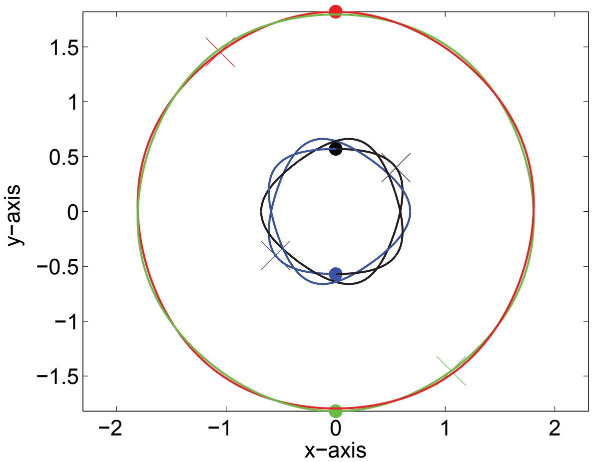


Figure 1. Periodic orbit corresponding to the minimizing path $\mathcal{P}_0$ at $\theta = \frac{\pi}{5}$. The four dots represent a collinear configuration Q_{start}, and the four crosses represent a diamond configuration Q_{end} in $\mathcal{P}_0$.

By taking $\theta = \frac{\pi}{5}$ in (1.1), a sample picture of the minimizer $\mathcal{P}_0$ ($t \in [0, 1]$) and its periodic extension is given in Figure 1.

In order to find local action minimizers connecting Q_{start} and Q_{end} in (1.1), we define several closed subsets in $\mathbb{R}^6$ and minimize $\mathcal{A}$ over $\vec{a}$ in these subsets. So 24 subsets are defined as follows. We assume $a_2 \geq b_2$ and $c_2 \geq 0$ in (1.1), which is equivalent to fix a partial order of the bodies on the configuration Q_{end}. For the collinear configuration Q_{start}, it has 24 different orders. Hence, 24 different subsets Γ_i ($i = 1, 2 \ldots, 24$) can be defined by setting $a_2 \geq b_2$, $c_2 \geq 0$ in Q_{end} and fixing the order of the four bodies in Q_{start}. For example, Γ_1 is defined by

$$\Gamma_1 = \{\vec{a} : a_1 \leq b_1 \leq -c_1 \leq c_1 - a_1 - b_1,\ a_2 \geq b_2,\ c_2 \geq 0\},$$

which implies that the y-components of the four bodies at $t = 0$ satisfy

$$q_{1y}(0) \leq q_{2y}(0) \leq q_{3y}(0) \leq q_{4y}(0).$$

By Theorem 2.1, for each Γ_i ($i = 1, 2, \ldots, 24$), there exists an action minimizer $\mathcal{P}_{\Gamma_i} \in H^1([0, 1], \chi)$ connecting Q_{start} and Q_{end} in (1.1):

$$\mathcal{A}(\mathcal{P}_{\Gamma_i}) = \inf_{\{\vec{a} \in \Gamma_i\}} \inf_{\{q(t) \in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A}.$$

It is clear that the 24 local action minimizers are all different. However, some of them may contain collisions on Q_{start} or Q_{end}. Furthermore, different action minimizers can be extended to the same solution of the N-body problem. By applying the first variation formulas and analyzing the equivalence relation between action minimizers $\mathcal{P}_{\Gamma_i}$ ($i = 1, 2, \ldots, 24$), we show the following:

Theorem 1.2. *Assume that $\theta \in (0, \frac{\pi}{2})$ and $\frac{\theta}{\pi}$ is rational. If all the minimizers $\mathcal{P}_{\Gamma_i}(i = 1, 2, \ldots, 24)$ are classical solutions of the N-body problem, then there are sixteen different periodic orbits connecting Q_{start} and Q_{end} defined in* (1.1).

The proof of Theorem 1.2 can be found in Theorem 5.1. As a numerical evidence, we draw the motions of all the sixteen periodic orbits for $\theta = \frac{\pi}{5}$.

Structure of the paper. The paper is organized as follows. Section 2 introduces a general coercivity result and shows the existence of 24 local action minimizers. Section 3 shows that the minimizer $\mathcal{P}_0$ is free of collision. In Section 4, we prove that $\mathcal{P}_0$ can be extended to a periodic or quasi-periodic orbit. In Section 5, we show that under appropriate assumptions, the 24 local action minimizers can be extended to sixteen nontrivial periodic orbits. Numerical evidences for $\theta = \frac{\pi}{5}$ are presented in the end.

2 Variational Settings and Coercivity

In this section, we introduce a general coercivity result (Theorem 2.1) of the Lagrangian action functional $\mathcal{A}$ under structural prescribed boundary conditions in the N-body problem. Let $\chi = \{q \in \mathbb{R}_{N\times d} : \sum_{i=1}^{N} m_i q_i = 0\}$. We set

$$Q_{\text{start}} = \begin{bmatrix} q_1(a_1,\dots,a_k) \\ \vdots \\ q_N(a_1,\dots,a_k) \end{bmatrix}, \quad Q_{\text{end}} = \begin{bmatrix} q_1(b_1,\dots,b_s) \\ \vdots \\ q_N(b_1,\dots,b_s) \end{bmatrix},$$

where $q_i \in \mathbb{R}^d$ ($i = 1, 2, \dots, N$, $d = 1, 2,$ or3) are row vectors, and $Q_{\text{start}}, Q_{\text{end}} \in \chi$. Our variational argument is a two-step minimizing procedure. First, we consider a fixed-end boundary value problem, which is also known as the Bolza problem. For given values of $a_1, \dots, a_k$ and $b_1, \dots, b_s$, the two matrices Q_{start} and Q_{end} are fixed. There exists an action minimizer $\mathcal{P}$, which satisfies

$$\mathcal{A}(\mathcal{P}) = \inf_{\{q(t)\in P(Q_{\text{start}},Q_{\text{end}})\}} \mathcal{A} = \inf_{\{q(t)\in P(Q_{\text{start}},Q_{\text{end}})\}} \int_0^1 [K(\dot{q}(t)) + U(q(t))]\,dt,$$

where $\mathrm{P}(Q_{\text{start}}, Q_{\text{end}})$ is defined as follows:

$$\mathrm{P}(Q_{\text{start}}, Q_{\text{end}}) := \{q(t) \in H^1([0,1],\chi) : q(0) = Q_{\text{start}},\ q(1) = Q_{\text{end}}\}.$$

If one wants $\mathcal{P}$ to be a part of a periodic solution, the boundaries must be special and they should meet certain structural prescribed boundary conditions (SPBC). Hence, we introduce a second minimizing procedure. Instead of fixing the boundaries, we free several parameters on the boundaries $q(0) = Q_{\text{start}}$ and $q(1) = Q_{\text{end}}$. The Lagrangian action functional is then minimized over these parameters. The resulting minimizing path may be extended to a periodic or quasi-periodic solution. There are mainly three challenges to show the existence of such classical solutions. The first one is the coercivity of the Lagrangian action functional under the boundary constraints. The second one is to show the minimizer is collision-free on the boundaries. The third one is whether the minimizing path can be extended to a periodic or quasi-periodic solution. With an appropriate choice of SPBC, all the three challenges can be resolved. A general coercivity theorem [1] is introduced here to resolve the first challenge. For the reader's convenience, a proof of the theorem is also given. The other two challenges will be resolved in Section 3 and Section 4.

Theorem 2.1. *Let*

$$Q_{\text{start}} = \begin{bmatrix} q_1(a_1,\dots,a_k) \\ \vdots \\ q_N(a_1,\dots,a_k) \end{bmatrix}, \quad Q_{\text{end}} = \begin{bmatrix} q_1(b_1,\dots,b_s) \\ \vdots \\ q_N(b_1,\dots,b_s) \end{bmatrix},$$

where $Q_{\text{start}}, Q_{\text{end}} \in \chi$, $q_i \in \mathbb{R}^d$, $i = 1, \dots, N$, and $a_1, \dots, a_k, b_1, \dots, b_s$ are independent variables. The matrix Q_{start} is linear with respect to a_i ($i = 1, 2, \dots, k$) and Q_{end} is linear with respect to b_j ($j = 1, 2, \dots, s$). Let $(a_1, \dots, a_k) \in \mathcal{S}_1$, $(b_1, \dots, b_s) \in \mathcal{S}_2$, where $\mathcal{S}_1 \subset \mathbb{R}^k$ and $\mathcal{S}_2 \subset \mathbb{R}^s$ are closed subsets. Set $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}$. Assume that

$$\{Q_{\text{start}} : (a_1,\dots,a_k) \in \mathbb{R}^k\} \cap \{Q_{\text{end}} : (b_1,\dots,b_s) \in \mathbb{R}^s\} = \{\vec{0}\}.$$

Then there exist a path sequence $\{\mathcal{P}_{n_l}\}$ and a minimizer $\mathcal{P}_0$ in $H^1([0,1],\chi)$ such that for each n_l,

$$\mathcal{A}(\mathcal{P}_{n_l}) = \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi),\, a_i=a_{i_{n_l}},\, b_j=b_{j_{n_l}}\ (i=1,\dots,k,\, j=1,\dots,s)\}} \mathcal{A},$$

$$\begin{aligned}\mathcal{A}(\mathcal{P}_0) &= \inf_{\{(a_1,\dots,a_k,b_1,\dots,b_s)\in\mathcal{S}\}} \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi)\}} \mathcal{A} \\ &= \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi),\, a_i=a_{i_0},\, b_j=b_{j_0}\ (i=1,\dots,k,\, j=1,\dots,s)\}} \mathcal{A}.\end{aligned}$$

For $t \in [0,1]$, $\mathcal{P}_{n_l}(t)$ converges to $\mathcal{P}_0(t)$ uniformly. In particular,

$$\lim_{n_l\to\infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l\to\infty} b_{j_{n_l}} = b_{j_0}, \quad i = 1, \dots, k,\ j = 1, \dots, s.$$

Proof. Note that $L = K + U \geq 0$, hence there exists some $M_0 \geq 0$ such that

$$\inf_{\{(a_1,\dots,a_k,b_1,\dots,b_s)\in\mathcal{S}\}} \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi)\}} \mathcal{A} = M_0.$$

The proof follows by the Arzelà–Ascoli theorem. Basically, we can find a sequence $\mathcal{P}_n$, such that the action of the sequence $\mathcal{A}(\mathcal{P}_n)$ approaches M_0. Then we show the uniform boundedness and equicontinuity of the sequence. Hence, by the Arzelà–Ascoli theorem, there is a subsequence $\mathcal{P}_{n_l}$ which converges uniformly to a minimizer $\mathcal{P}_0$. Note that there exist sequences a_{i_n} and b_{j_n} such that the minimum action value M_0 can be reached by a path sequence $\mathcal{P}_n \in H^1([0,1],\chi)$, which satisfies

$$\mathcal{A}(\mathcal{P}_n) = \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi),\, a_i=a_{i_n},\, b_j=b_{j_n}\ (i=1,\dots,k,\, j=1,\dots,s)\}} \mathcal{A},$$

and $\mathcal{A}(\mathcal{P}_n) \in [M_0, M_0 + \frac{1}{2n}]$. It is clear that $\mathcal{A}(\mathcal{P}_n) \in [M_0, M_0 + 1]$ for all $n \in \mathbb{N}$.

Next, we show the path sequence $\{\mathcal{P}_n\}$ is uniformly bounded. Let $q^{(n)}(t)$ be the position matrix path for $\{\mathcal{P}_n\}$. We rewrite Q_{start} and Q_{end} as $(dN \times 1)$-vectors:

$$\widetilde{Q_{\text{start}}} = \begin{bmatrix} q_1^T(a_1,\dots,a_k) \\ \vdots \\ q_N^T(a_1,\dots,a_k) \end{bmatrix}, \quad \widetilde{Q_{\text{end}}} = \begin{bmatrix} q_1^T(b_1,\dots,b_s) \\ \vdots \\ q_N^T(b_1,\dots,b_s) \end{bmatrix}.$$

Similarly, we can rewrite $q^{(n)}(t)$ as a $(dN \times 1)$-vector path $\tilde{q}^{(n)}(t)$. By assumption, the two linear spaces satisfy

$$\{Q_{\text{start}} : (a_1,\dots,a_k) \in \mathbb{R}^k\} \cap \{Q_{\text{end}} : (b_1,\dots,b_s) \in \mathbb{R}^s\} = \{\vec{0}\}.$$

Hence, $U_k \equiv \{\widetilde{Q_{\text{start}}} : (a_1,\dots,a_k) \in \mathbb{R}^k\}$ is a k-dimensional linear space, $V_s \equiv \{\widetilde{Q_{\text{end}}} : (b_1,\dots,b_s) \in \mathbb{R}^s\}$ is a s-dimensional linear space, and $U_k \cap V_s = \{0\}$. Let $\{u_1,\dots,u_k\}$ be an orthonormal basis of U_k and let $\{v_1,\dots,v_s\}$ be an orthonormal basis of V_s.

For any nonzero vectors $\vec{u} \in U_k$ and $\vec{v} \in V_s$, there exist constants g_i, h_j $(1 \leq i \leq k,\ 1 \leq j \leq s)$ such that

$$\frac{\vec{u}}{|\vec{u}|} = g_1 u_1 + \cdots + g_k u_k, \quad \sum_{i=1}^{k} g_i^2 = 1,$$

and

$$\frac{\vec{v}}{|\vec{v}|} = h_1 v_1 + \cdots + h_s v_s, \quad \sum_{j=1}^{s} h_j^2 = 1.$$

Note that g_i and h_j $(1 \leq i \leq k, 1 \leq j \leq s)$ satisfy $\sum_{i=1}^{k} g_i^2 = \sum_{j=1}^{s} h_j^2 = 1$. So they are on a compact set. It follows that the inner product of $\frac{\vec{u}}{|\vec{u}|}$ and $\frac{\vec{v}}{|\vec{v}|}$,

$$\left\langle \frac{\vec{u}}{|\vec{u}|}, \frac{\vec{v}}{|\vec{v}|} \right\rangle = \sum_{1\leq i\leq k,\, 1\leq j\leq s} g_i h_j \langle u_i, v_j \rangle = \cos(\vec{u}, \vec{v}),$$

can reach its maximum K_0. If $K_0 = 1$, there exist two vectors

$$\vec{u} \in U_k = \{\widetilde{Q_{\text{start}}} : (a_1,\dots,a_k) \in \mathbb{R}^k\} \quad \text{and} \quad \vec{v} \in V_s = \{\widetilde{Q_{\text{end}}} : (b_1,\dots,b_s) \in \mathbb{R}^s\}$$

such that $\frac{\vec{u}}{|\vec{u}|} = \frac{\vec{v}}{|\vec{v}|} \in U_k \cap V_s$. Contradiction! Hence,

$$\cos(\vec{u}, \vec{v}) \leq K_0 < 1 \quad \text{for any nonzero vectors } \vec{u} \in U_k,\ \vec{v} \in V_s. \tag{2.1}$$

On the other hand, $\mathcal{A}(\mathcal{P}_n) \leq M_0 + 1$. If $0 \leq t_1 < t_2 \leq 1$, we have

$$\frac{m_j |q_j^{(n)}(t_2) - q_j^{(n)}(t_1)|^2}{2d(t_2 - t_1)} \leq \int_{t_1}^{t_2} \frac{m_j |\dot{q}_j(t)|^2}{2}\, dt \leq \mathcal{A}(\mathcal{P}_n) \leq M_0 + 1.$$

This implies that for any $1 \leq j \leq N$ and any t_1, t_2 satisfying $0 \leq t_1 < t_2 \leq 1$,

$$|q_j^{(n)}(t_2) - q_j^{(n)}(t_1)| \leq \sqrt{\frac{2d(t_2 - t_1)(M_0 + 1)}{m_j}}. \tag{2.2}$$

Let $m^* = \min\{m_1, m_2, \ldots, m_N\}$. Then for all $1 \le j \le N$,

$$|q_j^{(n)}(t_2) - q_j^{(n)}(t_1)| \le \sqrt{\frac{2d(M_0+1)}{m^*}}.$$

In each $\mathcal{P}_n$, its element

$$q^{(n)}(t) = \begin{bmatrix} q_1^{(n)} \\ \vdots \\ q_N^{(n)} \end{bmatrix}$$

can be rewritten as

$$\tilde{q}^{(n)}(t) = \begin{bmatrix} (q_1^{(n)})^T \\ \vdots \\ (q_N^{(n)})^T \end{bmatrix}.$$

Then for any $t \in [0, 1]$,

$$|\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(t)| \le N\sqrt{\frac{2d(M_0+1)}{m^*}}. \tag{2.3}$$

Note that by (2.1), the angle between any two nonzero vectors

$$\tilde{q}^{(n)}(0) \in \{\widetilde{Q_{\text{start}}} : (a_1, \ldots, a_k) \in \mathcal{S}_1\}, \quad \tilde{q}^{(n)}(1) \in \{\widetilde{Q_{\text{end}}} : (b_1, \ldots, b_s) \in \mathcal{S}_2\}$$

satisfies

$$\left\langle \frac{\tilde{q}^{(n)}(0)}{|\tilde{q}^{(n)}(0)|}, \frac{\tilde{q}^{(n)}(1)}{|\tilde{q}^{(n)}(1)|} \right\rangle = \cos(\tilde{q}^{(n)}(0), \tilde{q}^{(n)}(1)) \le K_0 < 1.$$

It follows that

$$\begin{aligned}
\frac{2dN^2(M_0+1)}{m^*} &\ge |\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(1)|^2 \\
&= |\tilde{q}^{(n)}(0)|^2 + |\tilde{q}^{(n)}(1)|^2 - 2|\tilde{q}^{(n)}(0)||\tilde{q}^{(n)}(1)| \cos(\tilde{q}^{(n)}(0), \tilde{q}^{(n)}(1)) \\
&\ge |\tilde{q}^{(n)}(0)|^2 + |\tilde{q}^{(n)}(1)|^2 - 2K_0|\tilde{q}^{(n)}(0)||\tilde{q}^{(n)}(1)| \\
&= [K_0|\tilde{q}^{(n)}(0)| - |\tilde{q}^{(n)}(1)|]^2 + (1-K_0^2)|\tilde{q}^{(n)}(0)|^2 \\
&\ge (1-K_0^2)|\tilde{q}^{(n)}(0)|^2.
\end{aligned}$$

Hence

$$|\tilde{q}^{(n)}(0)| \le \sqrt{\frac{2dN^2(M_0+1)}{m^*(1-K_0^2)}}. \tag{2.4}$$

By inequalities (2.3) and (2.4), it follows that for any $t \in [0, 1]$,

$$|\tilde{q}^{(n)}(t)| \le |\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(t)| + |\tilde{q}^{(n)}(0)| \le N\sqrt{\frac{2d(M_0+1)}{m^*}} + N\sqrt{\frac{2d(M_0+1)}{m^*(1-K_0^2)}},$$

which is a uniform bound for $|\tilde{q}^{(n)}(t)|$. Therefore, the path sequence $\mathcal{P}_n = \mathcal{P}_n(t)$ is uniformly bounded.

Next, we show the path sequence $\{\mathcal{P}_n = \mathcal{P}_n(t)\}$ is equicontinuous. In fact, by inequality (2.2),

$$|q_j^{(n)}(t_2) - q_j^{(n)}(t_1)| \le \sqrt{\frac{2d(M_0+1)}{m^*}}|t_2 - t_1|^{\frac{1}{2}}.$$

Then for any $\epsilon > 0$, let $\delta = \frac{\epsilon^2 m^*}{2d(M_0+1)}$. Whenever $|t_2 - t_1| \le \delta$, the following inequality holds:

$$|q_j^{(n)}(t_2) - q_j^{(n)}(t_1)| \le \sqrt{\frac{2d(M_0+1)}{m^*}}|t_2 - t_1|^{\frac{1}{2}} = \epsilon.$$

It implies that for each $j = 1, 2, \ldots, N$, $q_j^{(n)}(t)$ is equicontinuous. It follows that the path sequence $\mathcal{P}_n$ is equicontinuous.

By the Arzelà–Ascoli theorem, there exists a subsequence $\{\mathcal{P}_{n_l}\}$ which converges uniformly. The limit $\mathcal{P}_0 = \mathcal{P}_0(t)$ is in $H^1([0,1],\chi)$ and it satisfies

$$\lim_{n_l\to\infty} \mathcal{P}_{n_l}(t) = \mathcal{P}_0(t) \quad \text{for all } t \in [0,1].$$

In particular,

$$\lim_{n_l\to\infty} \mathcal{P}_{n_l}(0) = \mathcal{P}_0(0), \qquad \lim_{n_l\to\infty} \mathcal{P}_{n_l}(1) = \mathcal{P}_0(1).$$

It follows that

$$\lim_{n_l\to\infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l\to\infty} b_{j_{n_l}} = b_{j_0}, \quad i = 1,\dots,k,\ j = 1,\dots,s,$$

and $\mathcal{P}_0$ satisfies

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{q(0)=Q_{\text{start}},\, q(1)=Q_{\text{end}},\, q(t)\in H^1([0,1],\chi),\, a_i=a_{i_0},\, b_j=b_{j_0}\ (i=1,\dots,k,\, j=1,\dots,s)\}} \mathcal{A}.$$

The proof is complete. □

As an application of Theorem 2.1, we first check if the two configurations Q_{start} and Q_{end} defined in (1.1) satisfy the assumptions in Theorem 2.1. Recall that

$$Q_{\text{start}} = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \\ 0 & -c_1 \\ 0 & c_1 - a_1 - b_1 \end{bmatrix}, \quad Q_{\text{end}} = \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \\ -c_2 & -\frac{a_2+b_2}{2} \\ c_2 & -\frac{a_2+b_2}{2} \end{bmatrix} R(\theta),$$

where $\vec{a} = (a_1, b_1, c_1, a_2, b_2, c_2) \in \mathbb{R}^6$, and

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

For any given $\theta \in (0, \frac{\pi}{2})$, it is clear that

$$\{Q_{\text{start}} : \vec{a} \in \mathbb{R}^6\} \cap \{Q_{\text{end}} : \vec{a} \in \mathbb{R}^6\} = \{\vec{0}\}.$$

By Theorem 2.1, there exists an action minimizer $\mathcal{P}_0 \in H^1([0,1],\chi)$ and a vector $\vec{a}_0$ such that

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{\vec{a}\in\mathbb{R}^6\}} \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A} = \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}}),\, \vec{a}=\vec{a}_0\}} \mathcal{A}.$$

In order to find local action minimizers, one can fix the order of the four bodies on Q_{end}. We assume $a_2 \geq b_2$ and $c_2 \geq 0$, which means that at $t = 1$, on the configuration $Q_{\text{end}} \cdot R(-\theta)$, body 1 is above body 2 on the y-axis and body 3 is on the left of body 4. We can then set different orders of the four bodies at $t = 0$. Basically, the four bodies have 24 different orders on the y-axis at $t = 0$. For each given order of the four bodies at $t = 0$, we can define a subset Γ_i $(i = 1, 2, \dots, 24)$ of $\vec{a}$. It follows that we can define 24 different subsets of $\vec{a}$. For example, we define Γ_1 as follows:

$$\Gamma_1 = \{\vec{a} : a_1 \leq b_1 \leq -c_1 \leq c_1 - a_1 - b_1,\ a_2 \geq b_2,\ c_2 \geq 0\},$$

which implies that the y-components of the four bodies at $t = 0$ satisfy

$$q_{1y}(0) \leq q_{2y}(0) \leq q_{3y}(0) \leq q_{4y}(0).$$

The other 23 subsets Γ_i $(i = 2, 3, \dots, 24)$ can be defined similarly. By Theorem 2.1, it follows that for each Γ_i $(1 \leq i \leq 24)$, there exists a minimizer $\mathcal{P}_{\Gamma_i} \in H^1([0,1],\chi)$ such that

$$\mathcal{A}(\mathcal{P}_{\Gamma_i}) = \inf_{\{\vec{a}\in\Gamma_i\}} \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A}.$$

Remark. Note that the boundary points of each Γ_i $(1 \leq i \leq 24)$ are collisions. The possible collisions in the minimizer $\mathcal{P}_{\Gamma_i}$ could be eliminated by using the level estimate method in [4, 5] or local deformation arguments in [15, 16] .

3 Exclusion of Collisions in $\mathcal{P}_0$

In this section, we prove the following:

Theorem 3.1. *The minimizing path $\mathcal{P}_0$ has no collision.*

Note that the minimizing path $\mathcal{P}_0$ is defined for $t \in [0, 1]$. By the celebrated results of Marchal [12] and Chenciner [7], $\mathcal{P}_0$ has no collision in $(0, 1)$. So we only need to exclude the possible collisions on boundaries $q(0) = Q_{\text{start}}$ and $q(1) = Q_{\text{end}}$. To exclude possible collision singularities of Q_{start} and Q_{end} in the minimizer $\mathcal{P}_0$, we introduce the following theorem [1]:

Theorem 3.2 ([1, Theorem 4.3]). *Let $\mathcal{S} = \mathbb{R}^{k+s}$. If the intersection of the two configuration subsets is at origin, i.e.*

$$\{Q_{\text{start}} : (a_1, \dots, a_k) \in \mathbb{R}^k\} \cap \{Q_{\text{end}} : (b_1, \dots, b_s) \in \mathbb{R}^s\} = \{\vec{0}\},$$

the action minimizer $\mathcal{P}_0 \in H^1([0, 1], \chi)$ in Theorem 2.1 has no binary collision.

As an application, in our case $\vec{a} = (a_1, b_1, c_1, a_2, b_2, c_2) \in \mathbb{R}^6$, it follows that there is no binary collision in $\mathcal{P}_0$.

Note that Q_{start} is a collinear configuration. We can then apply Chen's result [3, 6] on collinear configurations to exclude possible collisions on Q_{start}:

Theorem 3.3 ([3, Theorem 2.1]). *In the minimizer $\mathcal{P}_0$, the collinear configuration Q_{start} has no collision singularity.*

Furthermore, it is known that the bodies involved in a partial collision or total collision will approach a set of central configurations. More information can be known if the solution under concern is an action minimizer:

Lemma 3.1 ([14, Theorem 4.1.18] or [7, Section 3.2.1]). *If a minimizer q of the fixed-ends problem on time interval $[\tau_1, \tau_2]$ has an isolated collision of $k \le N$ bodies, then there is a parabolic homothetic collision-ejection solution $\hat{q}$ of the k-body problem which is also a minimizer of the fixed-ends problem on $[\tau_1, \tau_2]$.*

Lemma 3.2 ([3, Proposition 5] or [9, Section 7]). *Let X be a proper linear subspace of $\mathbb{R}^d$. Suppose that a local minimizer x of $\mathcal{A}_{t_0,t_1}$ on $B_{t_0,t_1}(x(t_0), X) := \{x \in H^1([t_0, t_1], (\mathbb{R}^d)^N) : x(t_0) \text{ is fixed}, x_i(t_1) \in X,\ i = 1, 2, \dots, N\}$ has an isolated collision of $k \le N$ bodies at $t = t_1$. Then there is a homothetic parabolic solution $\bar{y}$ of the k-body problem with $\bar{y}(t_1) = 0$ such that $\bar{y}$ is a minimizer of $\mathcal{A}^*_{\tau,t_1}$ on $B_{\tau,t_1}(\bar{y}(\tau), X)$ for any $\tau < t_1$. Here $\mathcal{A}^*_{\tau,t_1}$ denotes the action of this k-body subsystem.*

By Theorem 3.3, we are left to exclude possible collisions on Q_{end}. The matrix form of Q_{end} in (1.1) implies that Q_{end} can be a double isosceles, a kite or a diamond. Hence it may have binary collisions, triple collisions and total collision. By Theorem 3.2, there is no binary collision on Q_{end}. Hence, we need to exclude possible triple collisions and total collision on Q_{end} of $\mathcal{P}_0$. We will use the level estimate method [4, 5] to exclude the total collision first.

Lemma 3.3. *In the minimizer $\mathcal{P}_0$, there is no total collision at $t = 1$.*

Proof. We assume that at $t = 1$, Q_{end} experiences a total collision. Note that $q \in \chi$. By Chen's binary decomposition method [2, 5], the action can be written into the following form:

$$\mathcal{A} = \int_0^1 \sum_{i=1}^4 \frac{1}{2}|\dot{q}_i|^2 + \sum_{1\le i<j\le 4} \frac{1}{|q_i - q_j|}\, dt = \frac{1}{4} \sum_{1\le i<j\le 4} \int_0^1 \frac{1}{2}|\dot{q}_i - \dot{q}_j|^2 + \frac{4}{|q_i - q_j|}\, dt.$$

By the estimates in [2, 5], if q_i and q_j $(1 \le i < j \le 4)$ has a collision when $t \in [0, 1]$, the following inequality holds:

$$\int_0^1 \frac{1}{2}|\dot{q}_i - \dot{q}_j|^2 + \frac{4}{|q_i - q_j|}\, dt \ge \frac{3}{2}(16\pi^2)^{\frac{1}{3}}.$$

It follows that the action $\mathcal{A}_{\text{total collision}}$ of a total collision path satisfies

$$\mathcal{A}_{\text{total collision}} \geq \frac{6}{4} \times \frac{3}{2}(16\pi^2)^{\frac{1}{3}} \geq 12.16.$$

For any given $\theta \in (0, \frac{\pi}{2})$, note that at $t = 1$, the double isosceles configuration can be degenerated to a diamond configuration. A testing path can be defined as follows:

$$\bar{q}_1 = \sqrt{2}e^{(\theta t+\frac{\pi}{2})i}, \quad \bar{q}_2 = -\bar{q}_1, \quad \bar{q}_3 = e^{[(\theta-\frac{\pi}{2})t+\frac{\pi}{2}]i}, \quad \bar{q}_4 = -\bar{q}_3.$$

The action $\overline{\mathcal{A}}$ of this testing path is

$$\begin{aligned}\overline{\mathcal{A}} &= \int_0^1 \left[2\theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 + \frac{1}{2\sqrt{2}} + \frac{1}{2}\right] dt + \int_0^1 \frac{2}{|\sqrt{2}e^{(\theta t+\frac{\pi}{2})i} + e^{[(\theta-\frac{\pi}{2})t+\frac{\pi}{2}]i}|} + \frac{2}{|\sqrt{2}e^{(\theta t+\frac{\pi}{2})i} - e^{[(\theta-\frac{\pi}{2})t+\frac{\pi}{2}]i}|}\, dt \\
&= 2\theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 + \frac{1}{2\sqrt{2}} + \frac{1}{2} + \int_0^1 \frac{2}{\sqrt{3 + 2\sqrt{2}\cos(t\frac{\pi}{2})}} + \frac{2}{\sqrt{3 - 2\sqrt{2}\cos(t\frac{\pi}{2})}}\, dt \\
&\leq 2\theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 + \frac{1}{2\sqrt{2}} + \frac{1}{2} + 3.3386.\end{aligned}$$

Note that $\theta \in (0, \frac{\pi}{2})$, it follows that

$$\overline{\mathcal{A}} \leq 2\theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 + \frac{1}{2\sqrt{2}} + \frac{1}{2} + 3.3386 < \frac{\pi^2}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2} + 3.3386 < 9.13 < 12.16.$$

Therefore, there is no total collision at $t = 1$ in the minimizer $\mathcal{P}_0$. The proof is complete. □

To exclude the possible triple collisions on Q_{end} of the minimizer $\mathcal{P}_0$, we will apply the blow-up results in Lemma 3.1 and Lemma 3.2. Possible central configurations are discussed case by case so that we can lower the action by perturbation in each case. A local deformation argument is introduced in Lemma 3.4.

Lemma 3.4. *The minimizer $\mathcal{P}_0$ has no triple collision at $t = 1$.*

Proof. The possible triple collisions in Q_{end} could be a collision involving bodies 1, 3 and 4, or a collision involving bodies 2, 3 and 4. Note that in either case, the three bodies form an isosceles triangle configuration. Without loss of generality, we assume that the collision bodies are $\{1, 3, 4\}$ and q_2 is away from them on Q_{end}. Note that $\theta \in (0, \frac{\pi}{2})$ is always fixed. For simplicity, we may shift the collision time from $t = 1$ to $t = 0$ and assume Q_{end} to be

$$\begin{bmatrix} 0 & a_2 \\ 0 & b_2 \\ -c_2 & -\frac{a_2+b_2}{2} \\ c_2 & -\frac{a_2+b_2}{2} \end{bmatrix}. \tag{3.1}$$

In the above configuration (3.1), the triple collision among bodies 1, 3 and 4 happens at $t = 0$ when $c_2 = 0$ and $a_2 = -\frac{a_2+b_2}{2} \neq b_2$.

By the analysis of the blow-up in Lemma 3.1 and Lemma 3.2, there exists a parabolic homothetic solution $q_i(t) = a_i t^{\frac{2}{3}}$ $(i = 1, 3, 4)$, which is also a minimizer of the three-body problem on $[0, \tau]$ for any $\tau > 0$. For convenience, we denote the minimizer by $q = (q_1, q_3, q_4)$. Furthermore, (a_1, a_3, a_4) forms a central configuration, and the three vectors a_1, a_3 and a_4 satisfy the energy constraint

$$\sum_{i=1,3,4} \frac{1}{2}\left|\frac{2}{3}a_i\right|^2 - \sum_{i<j,\, i,j\in\{1,3,4\}} \frac{1}{|a_i - a_j|} = 0.$$

Note that by the formula of Q_{end} in (3.1), q_1, q_3 and q_4 always form an isosceles triangle at $t = 0$. To exclude the triple collision of q_1, q_3 and q_4 at $t = 0$, we need to show that the action $\mathcal{A}^*$ of the subsystem ($q = (q_1, q_3, q_4)$ with $q_i = a_i t^{\frac{2}{3}}$ $(i = 1, 3, 4)$) can be lowered by deforming this parabolic homothetic solution under the boundary constraint at $t = 0$. In fact, under this boundary constraint, it is challenging to apply the averaging method [7, 9, 12]. For this reason, the analysis of local deformation is done case by case for every possible central configuration.

Fix $\epsilon > 0$. When the central configuration of (q_1, q_3, q_4) is an equilateral triangle, we use complex notations for simplicity. In fact, one can assume that $a_1 = \gamma e^{i\theta_0}$ with $\theta_0 \in [0, \pi]$ and $\gamma = (\frac{3\sqrt{3}}{2})^{\frac{1}{3}}$. The case when $\theta_0 \in [\pi, 2\pi)$ can be handled similarly. Since there is no constraint on the order of body 3 and body 4 on the horizontal line, we can further assume that $a_3 = a_1 e^{\frac{2}{3}\pi i}$, $a_4 = a_3 e^{\frac{2}{3}\pi i}$. Now consider

$$\tilde{q}_1 = r(t)e^{i\theta(t)}, \quad \tilde{q}_3 = \tilde{q}_1 e^{\frac{2}{3}\pi i}, \quad \tilde{q}_4 = \tilde{q}_3 e^{\frac{2}{3}\pi i},$$

where

$$r(t) = \begin{cases} \gamma(\frac{\epsilon}{N})^{\frac{2}{3}}, & 0 \le t \le \frac{\epsilon}{N}, \\ \gamma t^{\frac{2}{3}}, & \frac{\epsilon}{N} \le t \le \epsilon, \end{cases}$$

and $\theta(t) = (\theta_0 - \frac{\pi}{2})\frac{t}{\epsilon} + \frac{\pi}{2}$. At $t = 0$, $\tilde{q}_1(0) = r(0)e^{i\theta(0)}$, $\tilde{q}_3(0) = \tilde{q}_1(0)e^{\frac{2}{3}\pi i}$ and $\tilde{q}_4(0) = \tilde{q}_3(0)e^{\frac{2}{3}\pi i}$, which satisfies the configuration Q_{end} in (3.1). At $t = \epsilon$, $\tilde{q}_i(\epsilon) = q_i(\epsilon)$ $(i = 1, 3, 4)$. Let $\mathcal{A}^*$ be the action of the parabolic homothetic ejection solution $q_i(t) = a_i t^{\frac{2}{3}}$ $(i = 1, 2, 4)$. Let $\tilde{\mathcal{A}}^*$ be the action of the perturbed path $\tilde{q} = (\tilde{q}_1, \tilde{q}_3, \tilde{q}_4)$. It follows that

$$\tilde{\mathcal{A}}^* - \mathcal{A}^* = \bigg(\Big(\frac{9\gamma^2}{14} + \frac{6\gamma^2}{7N^{\frac{7}{3}}}\Big)\Big(\theta_0 - \frac{\pi}{2}\Big)^2 - \frac{2\gamma^2 + \frac{2\sqrt{3}}{\gamma}}{N^{\frac{1}{3}}}\bigg)\epsilon^{\frac{1}{3}}.$$

By taking $N = 5$, we have $\tilde{\mathcal{A}}^* - \mathcal{A}^* < 0$ for every $\theta_0 \in [0, \pi]$.

When the central configuration of (q_1, q_3, q_4) is an Euler collinear configuration, there must be one body staying at the origin. We discuss it in several cases. If $a_1 = (0, 0)$, $a_3 = (\xi_3, \eta_3)$ and $a_4 = -a_3$, then we define a perturbed path as follows:

$$\tilde{q}_1 = q_1, \quad \tilde{q}_3 = (\xi_3 \epsilon^{\frac{2}{3}}, \eta_3 t^{\frac{2}{3}}), \quad \tilde{q}_4 = -(\xi_3 \epsilon^{\frac{2}{3}}, \eta_3 t^{\frac{2}{3}}),$$

where $\tilde{q}_1(0) = (0, 0)$, $\tilde{q}_3(0) = (\xi_3 \epsilon^{\frac{2}{3}}, 0)$ and $\tilde{q}_4(0) = -(\xi_3 \epsilon^{\frac{2}{3}}, 0)$ and at $t = \epsilon$, $\tilde{q}_i(\epsilon) = q_i(\epsilon)$ $(i = 1, 3, 4)$. It is clear that $\tilde{\mathcal{A}}^* < \mathcal{A}^*$ in this case.

If $a_1 \neq (0, 0)$, then one of the other two bodies must stay at the origin. Without loss of generality, we can assume $a_3 = (0, 0)$. Let $a_1 = (\xi_1, \eta_1)$. If $\eta_1 \neq 0$, we define

$$\tilde{q}_1 = (\xi_1 t^{\frac{2}{3}}, \eta_1 \epsilon^{\frac{2}{3}}), \quad \tilde{q}_3 = \Big(0, -\frac{\eta_1}{2}(\epsilon^{\frac{2}{3}} - t^{\frac{2}{3}})\Big), \quad \tilde{q}_4 = \Big(-\xi_1 t^{\frac{2}{3}}, -\frac{\eta_1}{2}(\epsilon^{\frac{2}{3}} + t^{\frac{2}{3}})\Big),$$

where $\tilde{q}_1(0) = (0, \eta_1 \epsilon^{\frac{2}{3}})$, $\tilde{q}_3(0) = (0, -\frac{\eta_1}{2}\epsilon^{\frac{2}{3}})$ and $\tilde{q}_4(0) = (0, -\frac{\eta_1}{2}\epsilon^{\frac{2}{3}})$ and at $t = \epsilon$, $\tilde{q}_i(\epsilon) = q_i(\epsilon)$, $(i = 1, 3, 4)$. It follows that the kinetic energy is decreased, and

$$|\tilde{q}_3 - \tilde{q}_4| = |q_3 - q_4|, \quad |\tilde{q}_1 - \tilde{q}_4| \ge |q_1 - q_4|.$$

For $t \in [0, \epsilon]$,

$$|\tilde{q}_1 - \tilde{q}_3| = \sqrt{\Big(\xi_1^2 + \frac{\eta_1^2}{4}\Big)t^{\frac{4}{3}} - \frac{3}{2}\eta_1^2 \epsilon^{\frac{2}{3}} t^{\frac{2}{3}} + \frac{9}{4}\eta_1^2 \epsilon^{\frac{4}{3}}} \ge \sqrt{\Big(\xi_1^2 + \frac{\eta_1^2}{4}\Big)t^{\frac{4}{3}} + \frac{3}{4}\eta_1^2 \epsilon^{\frac{4}{3}}}.$$

It implies that

$$\int_0^\epsilon \frac{dt}{|\tilde{q}_1 - \tilde{q}_3|} < \int_0^\epsilon \frac{dt}{|q_1 - q_3|}$$

and

$$\int_0^\epsilon \frac{dt}{|\tilde{q}_1 - \tilde{q}_4|} < \int_0^\epsilon \frac{dt}{|q_1 - q_4|}.$$

Hence, $\tilde{\mathcal{A}}^* < \mathcal{A}^*$.

If $\eta_1 = 0$ and $\xi_1 \neq 0$, then

$$q_1 = (\xi_1 t^{\frac{2}{3}}, 0), \quad q_3 = (0, 0), \quad q_4 = (-\xi_1 t^{\frac{2}{3}}, 0).$$

In this case we define

$$\tilde{q}_1 = \Big(\xi_1 t^{\frac{2}{3}}, 2\delta\Big(1 - \frac{t^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}}\Big)\Big), \quad \tilde{q}_3 = \Big(0, -\delta\Big(1 - \frac{t^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}}\Big)\Big), \quad \tilde{q}_4 = \Big(-\xi_1 t^{\frac{2}{3}}, -\delta\Big(1 - \frac{t^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}}\Big)\Big),$$

where $\tilde{q}_1(0) = (0, 2\delta)$, $\tilde{q}_3(0) = (0, -\delta)$ and $\tilde{q}_4(0) = (0, -\delta)$ and at $t = \epsilon$, $\tilde{q}_i(\epsilon) = q_i(\epsilon)$ $(i = 1, 3, 4)$. We set

$$\delta = \frac{\epsilon^{\frac{2}{3}}}{\sqrt{N}}.$$

Let K^* be the kinetic energy of the subsystem $q = (q_1, q_3, q_4)$. Let $\tilde{K}^*$ be the kinetic energy of the perturbed path $\tilde{q} = (\tilde{q}_1, \tilde{q}_3, \tilde{q}_4)$. It follows that

$$\triangle K^* = \int_0^\epsilon (\tilde{K}^* - K^*)\, dt = \frac{4}{N}\epsilon^{\frac{1}{3}}.$$

A direct calculation implies that

$$|\tilde{q}_1 - \tilde{q}_3| \geq \sqrt{\frac{9\xi_1^2}{N\xi_1^2 + 9}}\,\epsilon^{\frac{2}{3}},$$
$$|\tilde{q}_1 - \tilde{q}_4| \geq \sqrt{\frac{36\xi_1^2}{4N\xi_1^2 + 9}}\,\epsilon^{\frac{2}{3}},$$
$$|\tilde{q}_3 - \tilde{q}_4| = |\xi_1| t^{\frac{2}{3}}.$$

It follows that

$$\tilde{\mathcal{A}}^* - \mathcal{A}^* \leq \left[\frac{4}{N} + \left(\frac{9\xi_1^2}{N\xi_1^2 + 9}\right)^{-\frac{1}{2}} + \left(\frac{36\xi_1^2}{4N\xi_1^2 + 9}\right)^{-\frac{1}{2}} - \frac{9}{2|\xi_1|}\right]\epsilon^{\frac{1}{3}}. \tag{3.2}$$

Note that in the Euler central configuration, $\xi_1 = (\frac{45}{8})^{\frac{1}{3}}$. By taking $N = 6$ in (3.2), we have $\tilde{\mathcal{A}}^* - \mathcal{A}^* < 0$.

Therefore, for all possible central configurations of $q = (q_1, q_3, q_4)$, there always exists some perturbed path $\tilde{q}$, such that

$$\tilde{\mathcal{A}}^* - \mathcal{A}^* < 0.$$

Contradiction! It implies that there is no triple collision among q_1, q_3 and q_4 in the minimizer $\mathcal{P}_0$. Similarly, we can show that there is no triple collision among q_2, q_3 and q_4 in $\mathcal{P}_0$. The proof is complete. □

The proof of Theorem 3.1 follows by Theorem 3.2, Theorem 3.3, Lemma 3.3 and Lemma 3.4. Hence, the minimizer $\mathcal{P}_0 = \mathcal{P}_0$ $(t \in [0, 1])$ is a classical solution of the N-body problem. However, it is hard to eliminate possible collisions in each $\mathcal{P}_{\Gamma_i}$ $(i = 1, 2, \dots, 24)$. We will check them numerically.

4 Extension of the Minimizer $\mathcal{P}_0$

By the previous section, we know that $\mathcal{P}_0$ $(t \in [0, 1])$ is a classical solution of the N-body problem. In this section, we show that the minimizer $\mathcal{P}_0$ $(t \in [0, 1])$ can be extended to a periodic or quasi-periodic orbit. First variation formulas are applied to the variables a_1, b_1, c_1 in Q_{start} and a_2, b_2, c_2 in Q_{end} of the minimizer $\mathcal{P}_0$, which imply several identities of the velocities on both boundaries. An extension formula of $\mathcal{P}_0$ $(t \in [0, 1])$ can then be defined by (4.2) in Theorem 4.1.

Proposition 4.1. *In the minimizer $\mathcal{P}_0$, the velocities $\dot{q}_i(t)$ $(i = 1, 2, 3, 4)$ satisfy*

$$\dot{q}_{1y}(0) = \dot{q}_{2y}(0) = \dot{q}_{3y}(0) = \dot{q}_{4y}(0) = 0,$$

and

$$\dot{q}_1(1) = -\dot{q}_1(1)BR(2\theta), \quad \dot{q}_3(1) = -\dot{q}_4(1)BR(2\theta),$$
$$\dot{q}_2(1) = -\dot{q}_2(1)BR(2\theta), \quad \dot{q}_4(1) = -\dot{q}_3(1)BR(2\theta),$$

where

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad R(2\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}.$$

Proof. Note that there exists some $\vec{a}_0 \in \mathbb{R}^6$ such that

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}}),\ \vec{a}=\vec{a}_0\}} \mathcal{A}.$$

Let $q = q(t)$ be the position matrix path of $\mathcal{P}_0$. Consider an admissible variation $\xi(t) \in P(Q_{\text{start}}, Q_{\text{end}})$ satisfying $\xi(0) \in \{Q_{\text{start}} : (a_1, \dots, a_k) \in \mathcal{S}_1\}$ and $\xi(1) \in \{Q_{\text{end}} : (b_1, \dots, b_s) \in \mathcal{S}_2\}$; then the first variation $\delta_\xi \mathcal{A}(q)$ satisfies

$$\begin{aligned}
0 &= \delta_\xi \mathcal{A}(q) \\
&= \lim_{\tau\to 0} \frac{\mathcal{A}(q+\tau\xi) - \mathcal{A}(q)}{\tau} \\
&= \int_0^1 \frac{1}{2}\sum_{i=1}^4 \lim_{\tau\to 0} m_i \frac{\|\dot{q}_i + \tau\dot{\xi}_i\|^2 - \|\dot{q}_i\|^2}{\tau} + \lim_{\tau\to 0}\frac{U(q+\tau\xi) - U(q)}{\tau}\, dt \\
&= \int_0^1 \bigg(\sum_{i=1}^4 m_i \langle \dot{q}_i, \dot{\xi}_i\rangle + \sum_{i=1}^4 \Big\langle \frac{\partial}{\partial q_i} U(q(t)), \xi_i \Big\rangle \bigg)\, dt \\
&= \sum_{i=1}^4 \bigg(m_i \langle \dot{q}_i, \xi_i\rangle\Big|_{t=0}^{t=1} + \int_0^1 \Big\langle -m_i \ddot{q}_i + \frac{\partial}{\partial q_i} U(q(t)), \xi_i \Big\rangle\, dt \bigg) \\
&= \sum_{i=1}^4 m_i \langle \dot{q}_i, \xi_i\rangle\Big|_{t=0}^{t=1}.
\end{aligned} \tag{4.1}$$

Note that $m_1 = m_2 = m_3 = m_4 = 1$. Let $\xi(t) \in P(Q_{\text{start}}, Q_{\text{end}})$ such that

$$\xi(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \xi(1) = 0.$$

By identity (4.1), it follows that

$$\dot{q}_{1y}(0) - \dot{q}_{4y}(0) = 0.$$

Similarly, by taking different values of $\xi(0)$, we have

$$\dot{q}_{2y}(0) - \dot{q}_{4y}(0) = 0, \quad -\dot{q}_{3y}(0) + \dot{q}_{4y}(0) = 0.$$

Note that

$$\sum_{i=1}^4 \dot{q}_{iy}(0) = 0.$$

It follows that

$$\dot{q}_{1y}(0) = \dot{q}_{2y}(0) = \dot{q}_{3y}(0) = \dot{q}_{4y}(0) = 0.$$

Similar arguments on a_2, b_2, c_2 imply that

$$\begin{aligned}
\dot{q}_1(1) &= -\dot{q}_1(1)BR(2\theta), & \dot{q}_3(1) &= -\dot{q}_4(1)BR(2\theta), \\
\dot{q}_2(1) &= -\dot{q}_2(1)BR(2\theta), & \dot{q}_4(1) &= -\dot{q}_3(1)BR(2\theta),
\end{aligned}$$

where

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R(2\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}.$$

The proof is complete. □

The extension of the path $\mathcal{P}_0(t \in [0,1])$ then follows by Proposition 4.1 and the uniqueness of solutions of an ODE system.

Theorem 4.1. *Let $\theta \in (0, \frac{\pi}{2})$. If $\frac{\theta}{\pi}$ is rational, the minimizer $\mathcal{P}_0$ $(t \in [0,1])$ can be extended to a periodic orbit. Otherwise, $\mathcal{P}_0$ $(t \in [0,1])$ can be extended to a quasi-periodic orbit.*

Proof. Let

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix}, \quad t \in [0,1],$$

be the position matrix path of the minimizer $\mathcal{P}_0$ $(t \in [0,1])$, where each $q_i(t) = (q_{ix}(t), q_{iy}(t))$ $(i = 1,2,3,4)$ is a 2-dimensional row vector path. Its extension is defined as follows:

$$q(t) = \begin{cases} (q_1^T(t), q_2^T(t), q_3^T(t), q_4^T(t))^T, & t \in [0,1], \\ (q_1^T(2-t), q_2^T(2-t), q_4^T(2-t), q_3^T(2-t))^T BR(2\theta), & t \in [1,2], \\ (q_1^T(t-2), q_2^T(t-2), q_4^T(t-2), q_3^T(t-2))^T R(2\theta), & t \in [2,4], \\ q(t-4k)R(4k\theta), & t \in [4k, 4k+4], \end{cases} \tag{4.2}$$

where

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $k \in \mathbb{Z}$. Recall that

$$Q_{\text{start}} = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \\ 0 & -c_1 \\ 0 & c_1 - a_1 - b_1 \end{bmatrix}, \quad Q_{\text{end}} = \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \\ -c_2 & -\frac{a_2+b_2}{2} \\ c_2 & -\frac{a_2+b_2}{2} \end{bmatrix} R(\theta).$$

By Proposition 4.1, it follows that at $t = 1$, $q(t)$ is actually C^1. Since $q(t)$ satisfies the Newtonian equation, it follows that $q(t)$ is smooth at $t = 1$. And at $t = 2$, the four bodies are on a line which is a counterclockwise 2θ rotation of the y-axis. Each velocity $\dot{q}_i$ $(i = 1,2,3,4)$ at $t = 2$ is perpendicular to the line of their configuration. Similarly, we can show that the orbit $q(t)$ is smooth at $t = 2$. Note that when $t = 4$, the position matrix q and the velocity matrix $\dot{q}$ satisfy

$$q(4) = q(0)R(4\theta), \quad \dot{q}(4) = \dot{q}(0)R(4\theta).$$

By the uniqueness of solutions of an ODE system, the extension of $q(t)$ $(t \in [0,4])$ is

$$q(t) = q(t-4k)R(4k\theta), \quad t \in [4k, 4k+4],$$

where $k \in \mathbb{Z}$. Therefore, the definition of $q(t)$ in (4.2) is smooth for all t. If $\frac{\theta}{\pi} = \frac{k_1}{l_1}$ is rational (k_1, l_1 are integers), then the minimizer $\mathcal{P}_0$ $(t \in [0,1])$ can be extended to a periodic orbit with a period $T = 4l_1$. If $\frac{\theta}{\pi}$ is irrational, then the extension of $\mathcal{P}_0$ $(t \in [0,1])$ in (4.2) is quasi-periodic. The proof is complete. □

Remark. If the local action minimizers $\mathcal{P}_{\Gamma_i}$ $(t \in [0,1])$ $(i = 1,2,\dots,24)$ are collision-free, similar arguments can be applied to show that each $\mathcal{P}_{\Gamma_i}$ $(i = 1,2,\dots,24)$ can be extended to a periodic or quasi-periodic orbit. And their extensions satisfy the same formula of $q(t)$ in (4.2).

5 Equivalence of Action Minimizers

In the last section, we discuss the equivalent class of all the 24 local action minimizers $\mathcal{P}_{\Gamma_i}$ $(i = 1,2,\dots,24)$. By assuming that $\frac{\theta}{\pi}$ is rational and each minimizer $\mathcal{P}_{\Gamma_i}$ $(i = 1,2,\dots,24)$ is a classical solution, we show that these 24 action minimizers can actually generate sixteen different periodic orbits.

Recall that each local action minimizer $\mathcal{P}_{\Gamma_i}$ $(i = 1, 2, \dots, 24)$ satisfies

$$\mathcal{A}(\mathcal{P}_{\Gamma_i}) = \inf_{\{\vec{a}\in\Gamma_i\}} \inf_{\{q(t)\in P(Q_{\text{start}}, Q_{\text{end}})\}} \mathcal{A},$$

where Γ_i is defined as follows. In every Γ_i $(i = 1, 2, \dots, 24)$, we assume that the variables on Q_{end} always satisfy $a_2 \geq b_2$ and $c_2 \geq 0$. Then in the collinear configuration Q_{start}, the 24 different orders $\Delta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the four bodies at $t = 0$ have a one-to-one correspondence to the 24 action minimizers $\mathcal{P}_{\Gamma_i}$ $(i = 1, 2, \dots, 24)$. For example, we define Γ_1 as

$$\Gamma_1 = \{\vec{a} : a_1 \leq b_1 \leq -c_1 \leq -a_1 - b_1 + c_1,\ a_2 \geq b_2,\ c_2 \geq 0\},$$

which has the following order at $t = 0$:

$$q_{1y}(0) \leq q_{2y}(0) \leq q_{3y}(0) \leq q_{4y}(0).$$

We denote it by $(1, 2, 3, 4)$. That is, the minimizer $\mathcal{P}_{\Gamma_1}$ corresponds to an order $(1, 2, 3, 4)$ in Q_{start}. The following result shows that some of the orders in Q_{start} correspond to the same solution of the four-body problem.

Lemma 5.1. *Given $a_2 \geq b_2$, $c_2 \geq 0$, the orders Δ and $\sigma_1\sigma_2\Delta$ in Q_{start} correspond to the same periodic solution, where*

$$\sigma_1 = (12)(34), \quad \sigma_2 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

Proof. Note that $q(t)$ and $-q(t)$ are considered as the same solution in the N-body problem. Assume that $q(0)$ has an order $\Delta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in Q_{start}; then $-q(0)$ has an order

$$(\alpha_4, \alpha_3, \alpha_2, \alpha_1) = \sigma_2\Delta,$$

where

$$\sigma_2 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

But the corresponding Q_{end} in $-q(t)$ has $a_2 \leq b_2$ and $c_2 \leq 0$. Let $\sigma_1 = (12)(34)$. Then in the orbit corresponding to $\sigma_1\sigma_2\Delta$, Q_{end} satisfies $a_2 \geq b_2$ and $c_2 \geq 0$. It implies that Δ and $\sigma_1\sigma_2\Delta = \sigma_2\sigma_1\Delta$ correspond to the same orbit. The proof is complete. □

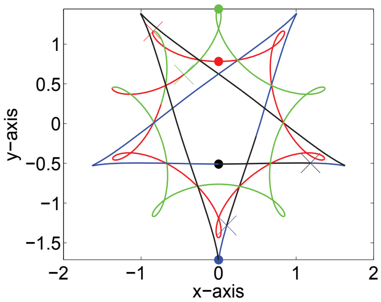
Theorem 5.1. *Assume that $\theta \in (0, \frac{\pi}{2})$ and $\frac{\theta}{\pi}$ is rational. If all the minimizers $\mathcal{P}_{\Gamma_i}$ $(i = 1, 2, \dots, 24)$ are classical solutions of the N-body problem, then there are sixteen different periodic orbits connecting Q_{start} and Q_{end} defined in* (1.1).

Proof. For convenience, we use $\Delta_{\alpha_1\alpha_2\alpha_3\alpha_4}$ to represent the corresponding action minimizer. For example, $\mathcal{P}_{\Gamma_1}$ corresponds to Δ_{1234}. By Lemma 5.1, we know that the 24 action minimizers can be classified into the following sixteen sets:

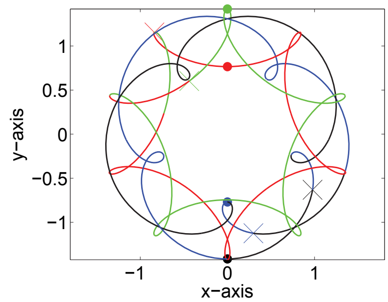
$$\begin{gathered}\{\Delta_{1234}, \Delta_{3412}\},\ \{\Delta_{1243}, \Delta_{4312}\},\ \{\Delta_{1324}, \Delta_{3142}\},\ \{\Delta_{1423}, \Delta_{4132}\},\\ \{\Delta_{2134}, \Delta_{3421}\},\ \{\Delta_{2143}, \Delta_{4321}\},\ \{\Delta_{2314}, \Delta_{3241}\},\ \{\Delta_{2413}, \Delta_{4231}\},\\ \{\Delta_{1342}\},\ \{\Delta_{1432}\},\ \{\Delta_{2341}\},\ \{\Delta_{2431}\},\ \{\Delta_{3124}\},\ \{\Delta_{3214}\},\ \{\Delta_{4123}\},\ \{\Delta_{4213}\}.\end{gathered}$$

Therefore, there are sixteen different periodic orbits connecting Q_{start} and Q_{end} defined in (1.1). The proof is complete. □

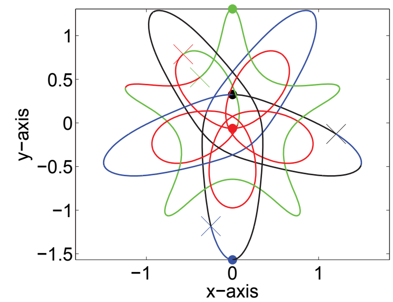
In the end, numerical results are presented for some rotation angle θ. Note that by the definitions of Q_{start} and Q_{end} in (1.1), it is easy to check that $\mathcal{P}_{\Gamma_i}$ $(i = 1, 2, \dots, 24)$ cannot be a relative equilibrium in the four-body problem. In other words, these sixteen periodic orbits are nontrivial. Numerically, we set $\theta = \frac{\pi}{5}$ and calculate the sixteen action minimizers by Matlab. The motions of the sixteen periodic orbits can then be drawn as in Figure 2. It is worth mentioning that some of the orbits are very complicated. It will be interesting if one can analytically prove their existences. For other values of $\theta \in (0, \frac{\pi}{2})$, similar multiplicity results of periodic orbits are expected.



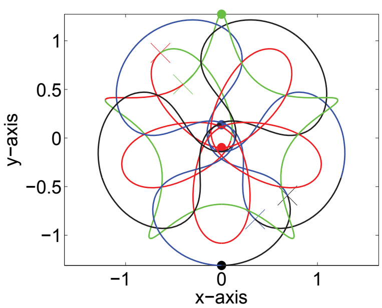
(a) $\Delta_{1234}, \Delta_{3412}$



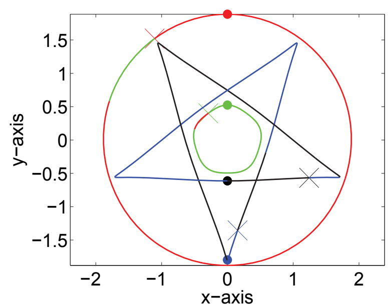
(b) $\Delta_{1243}, \Delta_{4312}$



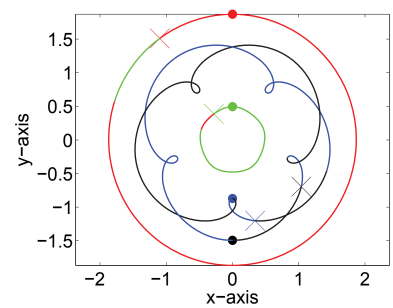
(c) $\Delta_{1324}, \Delta_{3142}$



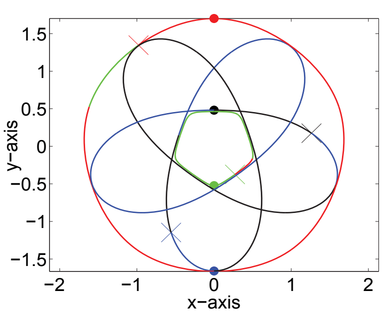
(d) $\Delta_{1423}, \Delta_{4132}$



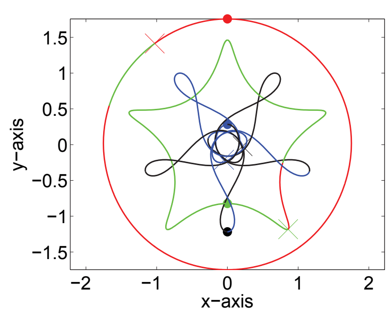
(e) $\Delta_{2134}, \Delta_{3421}$



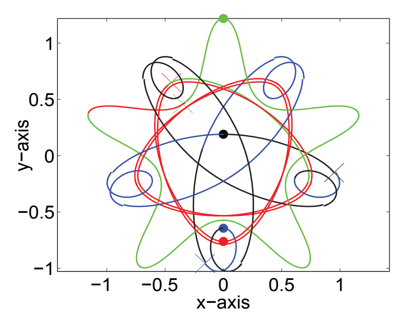
(f) $\Delta_{2143}, \Delta_{4321}$



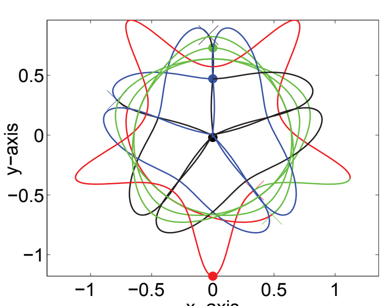
(g) $\Delta_{2314}, \Delta_{3241}$



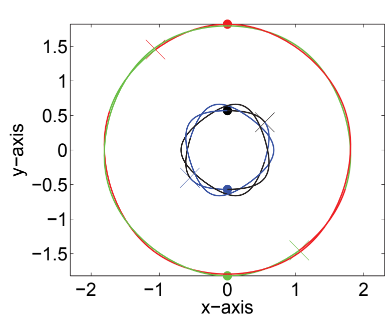
(h) $\Delta_{2413}, \Delta_{4231}$



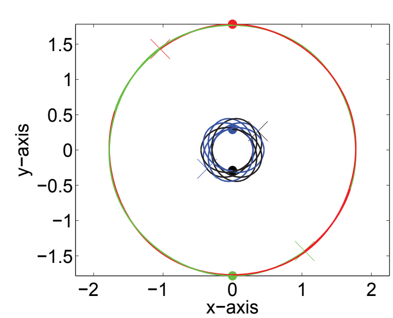
(i) Δ_{1342}



(j) Δ_{1432}

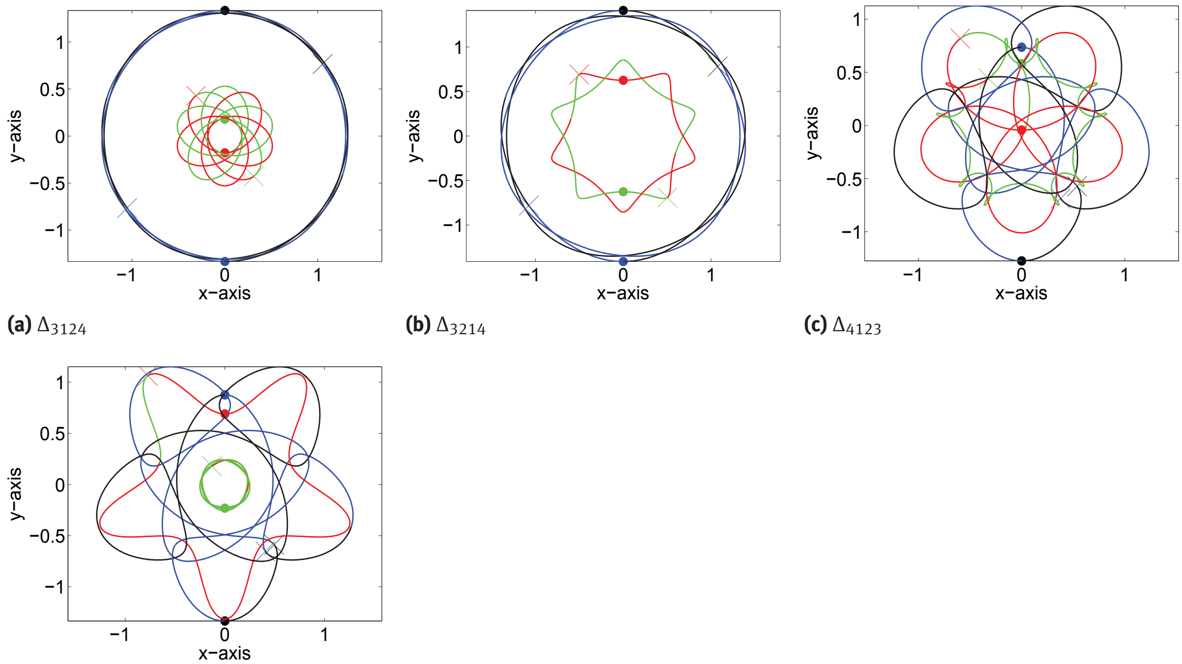


(k) Δ_{2341}



(l) Δ_{2431}

Figure 2. Sixteen different periodic orbits connecting Q_{start} and Q_{end} in (1.1) with $\theta = \frac{\pi}{5}$ (continued on next page).



(a) Δ_{3124} **(b)** Δ_{3214} **(c)** Δ_{4123} **(d)** Δ_{4213}

Figure 2 (continued). Sixteen different periodic orbits connecting Q_{start} and Q_{end} in (1.1) with $\theta = \frac{\pi}{5}$.

Acknowledgment: We thank the referees for carefully checking our manuscript, and for their many constructive suggestions. Part of this work was done while Duokui Yan was visiting Brigham Young University; he sincerely thanks the department of mathematics there for its help and support.

Funding: The research of Duokui Yan is supported in part by NSFC (No. 11432001) and the China Scholarship Council.

References

[1] X. Chang, T. Ouyang and D. Yan, Linear stability of the criss-cross orbit in the equal-mass three-body problem, *Discrete Contin. Dyn. Syst.* **36** (2016), no. 11, 5971–5991.

[2] K.-C. Chen, Binary decompositions for planar *N*-body problems and symmetric periodic solutions, *Arch. Ration. Mech. Anal.* **170** (2003), no. 3, 247–276.

[3] K.-C. Chen, Removing collision singularities from action minimizers for the *N*-body problem with free boundaries, *Arch. Ration. Mech. Anal.* **181** (2006), no. 2, 311–331.

[4] K.-C. Chen, Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses, *Ann. of Math. (2)* **167** (2008), no. 2, 325–348.

[5] K.-C. Chen and Y.-C. Lin, On action-minimizing retrograde and prograde orbits of the three-body problem, *Comm. Math. Phys.* **291** (2009), no. 2, 403–441.

[6] K.-C. Chen, T. Ouyang and Z. Xia, Action-minimizing periodic and quasi-periodic solutions in the *n*-body problem, *Math. Res. Lett.* **19** (2012), no. 2, 483–497.

[7] A. Chenciner, Action minimizing solutions of the Newtonian *n*-body problem: From homology to symmetry, in: *Proceedings of the International Congress of Mathematicians. Vol. III* (Beijing 2002), Higher Education Press, Beijing (2002), 279–294.

[8] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math. (2)* **152** (2000), no. 3, 881–901.

[9] D. L. Ferrario and S. Terracini, On the existence of collisionless equivariant minimizers for the classical *n*-body problem, *Invent. Math.* **155** (2004), no. 2, 305–362.

[10] G. Fusco, G. F. Gronchi and P. Negrini, Platonic polyhedra, topological constraints and periodic solutions of the classical *N*-body problem, *Invent. Math.* **185** (2011), no. 2, 283–332.

[11] Y. M. Long, Multiple solutions of perturbed superquadratic second order Hamiltonian systems, *Trans. Amer. Math. Soc.* **311** (1989), no. 2, 749–780.

[12] C. Marchal, How the method of minimization of action avoids singularities, *Celestial Mech. Dynam. Astronom.* **83** (2002), no. 1–4, 325–353.

[13] T. Ouyang and Z. Xie, Star pentagon and many stable choreographic solutions of the Newtonian 4-body problem, *Phys. D* **307** (2015), 61–76.

[14] A. Venturelli, *Application de la minimisation de l'action au problème des N corps dans le plan et dans l'espace*, Thesis, Université de Paris 7, 2002.

[15] G. Yu, Spatial double choreographies of the Newtonian 2*n*-body problem, preprint (2016), https://arxiv.org/abs/1608.07956.

[16] G. Yu, Simple choreographies of the planar Newtonian N-body problem, *Arch. Ration. Mech. Anal.* **225** (2017), no. 2, 901–935.

[17] S. Zhang, Multiple geometrically distinct closed noncollision orbits of fixed energy for *N*-body type problems with strong force potentials, *Proc. Amer. Math. Soc.* **124** (1996), no. 10, 3039–3046.

[18] S. Zhang, Multiple closed orbits of fixed energy for *N*-body-type problems with gravitational potentials, *J. Math. Anal. Appl.* **208** (1997), no. 2, 462–475.

[19] S. Zhang, Multiple closed orbits for *N*-body-type problems, *Bull. Austral. Math. Soc.* **58** (1998), no. 1, 1–13.