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Research Article

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Kirchhoff–Hardy Fractional Problems with Lack of Compactness

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Abstract: This paper deals with the existence and the asymptotic behavior of nontrivial solutions for some classes of stationary Kirchhoff problems driven by a fractional integro-differential operator and involving a Hardy potential and different critical nonlinearities. In particular, we cover the delicate *degenerate* case, that is, when the Kirchhoff function M is zero at zero. To overcome the difficulties due to the lack of compactness as well as the degeneracy of the models, we have to make use of different approaches.

Keywords: Stationary Kirchhoff–Dirichlet Problems, Nonlocal p-Laplacian Operators, Hardy Coefficients, Critical Nonlinearities, Variational Methods

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1 Introduction

Recently, great attention has been drawn to the study of fractional and nonlocal elliptic problems with lack of compactness. These models arise in a quite natural way in many different applications, and we refer to the recent monograph [27], to the extensive paper [10] and the references cited therein for further details.

This paper is devoted to the study of the existence of solutions of a series of Dirichlet problems in general open subsets Ω of $\mathbb{R}^N$, $N \geq 1$, possibly unbounded. The problems involve a Hardy coefficient and the so-called fractional p-Laplace operator $(-\Delta)_p^s$, with $0 < s < 1 < p < \infty$ and $ps < N$. Hence, $p_s^* = pN/(N - ps)$ is well defined and is *critical* in the sense of the fractional Sobolev theory. The operator $(-\Delta)_p^s$ (up to normalization factors) is defined for any $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ is the open ball of $\mathbb{R}^N$ centered at x and with radius $\varepsilon > 0$.

Let $0 \leq \alpha < ps$ and let $p_s^*(\alpha) = p(N - \alpha)/(N - ps) \leq p_s^*(0) = p_s^*$. The main results of the paper are based on the best fractional Hardy–Sobolev constant $H_\alpha = H(p, N, s, \alpha)$, given by

$$H_\alpha = \inf_{\substack{u \in Z(\Omega) \\ u \neq 0}} \frac{[u]_{s,p}^p}{\|u\|_{H_\alpha}^p}, \quad \|u\|_{H_\alpha}^{p_s^*(\alpha)} = \int_\Omega |u(x)|^{p_s^*(\alpha)} \frac{dx}{|x|^\alpha}, \tag{1.1}$$

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where $Z(\Omega)$ is the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$[\varphi]_{s,p} = \Big(\int_{\mathbb{R}^N} |D^s\varphi(x)|^p\,dx \Big)^{1/p}, \quad |D^s\varphi(x)|^p = \int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\,dy, \tag{1.2}$$

well defined along any test function $\varphi \in C_0^\infty(\Omega)$, extended to the entire $\mathbb{R}^N$ by putting $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$. We refer to Section 2 for details.

The constant H_α is well defined and strictly positive thanks to Lemma 2.1. However, the fractional Hardy embedding $Z(\Omega) \hookrightarrow L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha})$ is continuous but not compact. In order to handle the critical Hardy–Sobolev potential as well as the nonlocal term given by $(-\Delta)_p^s$, we first study the exact behavior of weakly convergent sequences of $Z(\Omega)$ in the space of measures. This behavior is described in the following theorem, where the assumption that Ω is bounded seems to play an essential role. If Ω is a bounded open subset of $\mathbb{R}^N$, then (1.1) reduces simply to the Poincaré theorem when $\alpha = ps$, that is, (1.1) holds for all $\alpha \in [0, ps]$, in other words for all $p_s^*(\alpha) \in [p, p_s^*]$.

Theorem 1.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and let $\alpha \in (0, ps]$. Let $(u_j)_j$ be a weakly convergent sequence in $Z(\Omega)$, with weak limit u. Then there exist two finite positive measures μ and ν in $\mathbb{R}^N$ such that*

$$|D^s u_j(x)|^p\,dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_j(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha} \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N). \tag{1.3}$$

Furthermore, there exist two nonnegative numbers μ_0, ν_0 such that

$$\nu = |u(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha} + \nu_0\delta_0 \tag{1.4}$$

and

$$\mu \ge |D^s u(x)|^p\,dx + \mu_0\delta_0, \quad 0 \le H_\alpha \nu_0^{p/p_s^*(\alpha)} \le \mu_0, \tag{1.5}$$

where H_α is the Hardy constant defined in (1.1).

For the case $\alpha = 0$ we refer to [29, Theorem 2.5]. As a direct consequence of Theorem 1.1 above and [29, Theorem 2.5], we prove that the functional

$$\mathcal{H}_{\gamma,\lambda}(u) = \frac{1}{p}\Big[\mathscr{M}([u]_{s,p}^p) - \frac{\gamma}{\theta}\|u\|_{H_\alpha}^{p\theta} - \lambda\|u\|_p^p\Big], \quad \mathscr{M}(t) = \int_0^t M(\tau)\,d\tau, \tag{1.6}$$

is weakly lower semi-continuous and coercive in $Z(\Omega)$, provided that α, θ, γ, and λ verify suitable restrictions, depending on the behavior of the Kirchhoff coefficient M, which is assumed to satisfy condition

($\mathcal{M}$) $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and nondecreasing. There exist numbers $c > 0$ and θ such that for all $t \in \mathbb{R}_0^+$,

$$\mathscr{M}(t) \ge ct^\theta, \quad \text{with } \theta \begin{cases} \in (1, p_s^*(\alpha)/p) \text{ and } \alpha \in [0, ps) & \text{if } M(0) = 0, \\ = 1 \text{ and } \alpha \in [0, ps] & \text{if } M(0) > 0. \end{cases} \tag{1.7}$$

A typical prototype for M, due to Kirchhoff, satisfying ($\mathcal{M}$) is given by

$$M(t) = a + b\vartheta t^{\vartheta-1}, \quad a, b \ge 0,\ a + b > 0, \quad \vartheta \in (1, p_s^*(\alpha)/p), \quad \alpha \in [0, ps), \tag{1.8}$$

with $c = a$ and $\theta = 1$ if $M(0) > 0$, while $c = b$ and $\theta = \vartheta$ if $M(0) = 0$, that is, $a = 0$. Indeed, if $M(0) = 0$ for (1.8), then $\vartheta > 1$ as a corollary of [7, Lemma 3.1].

The functional $\mathcal{H}_{\gamma,\lambda}$ is the basis of the elliptic part of some nonlinear Kirchhoff problems which are studied in Section 3. Theorem 1.1 is also applied in minimization arguments and critical point theorems to get existence and multiplicity results.

In general, when $M(t) > 0$ for all $t \in \mathbb{R}^+$, then the related Kirchhoff problem is said to be *non-degenerate* when $M(0) > 0$, while it is called *degenerate* if $M(0) = 0$.

In the second part of the paper, we treat Kirchhoff problems in general open sets Ω, with possibly $\Omega = \mathbb{R}^N$. The first problem is

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u - \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \sigma w(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.9}$$

where $0 < s < 1 < p < \infty$ and $0 \le \alpha < ps < N$, while σ is a real parameter. Naturally, the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ disappears when $\Omega = \mathbb{R}^N$. The exponent q satisfies $p\theta < q < p_s^*(\alpha) \le p_s^*$. The norm $[\,\cdot\,]_{s,p}$ is defined in (1.2).

Since $0 \le \alpha < ps < N$ in problem (1.9), we assume that the nonlocal Kirchhoff term satisfies assumption

($\widetilde{\mathcal{M}}$) $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function such that the following conditions hold:
- (M_1) there exists $\theta \in [1, p_s^*(\alpha)/p)$ such that $tM(t) \le \theta\mathscr{M}(t)$ for any $t \in \mathbb{R}_0^+$, where $\mathscr{M}$ is defined in (1.6); and either
- ($\widetilde{M_2}$) $\inf_{t\in\mathbb{R}_0^+} M(t) = a > 0$;
- or $M(0) = 0$ and M satisfies both properties
- (M_2) for any $\tau > 0$ there exists $m = m(\tau) > 0$ such that $M(t) \ge m$ for all $t \ge \tau$;
- (M_3) there exists a positive number $c > 0$ such that $M(t) \ge ct^{\theta-1}$ for all $t \in [0, 1]$.

Clearly, condition ($\widetilde{M_2}$) covers the so-called non-degenerate case and implies at once the validity of (M_2) and (M_3). Concerning the positive weight w, we assume

(w) $w \in L^{\wp}(\mathbb{R}^N)$, with $\wp = p_s^*/(p_s^* - q)$ and $1 < q < p_s^*$.

Condition (w) guarantees that the embedding $Z(\Omega) \hookrightarrow L^q(\Omega, w)$ is compact, even when Ω is the entire $\mathbb{R}^N$, as explained in Section 4. Indeed, the natural solution space for problem (1.9) is the fractional density space $Z(\Omega)$, that is, the closure of $C_0^\infty(\Omega)$ with respect to $[\,\cdot\,]_{s,p}$, given in (1.2). Thus,

$$\|u\|_{q,w} \le C_w[u]_{s,p} \quad \text{for all } u \in Z(\Omega), \tag{1.10}$$

with $C_w = H_0^{-1/p}\|w\|_{\wp}^{1/q} > 0$, as proved in Lemma 4.1.

In the next result, we partially answer the open question asked for the Kirchhoff–Hardy equation [7, (1.1)] since we cover for a slightly different equation also the degenerate case. Thanks to the variational nature of (1.9), (weak) solutions of (1.9) are exactly the critical points of the underlying functional $\mathcal{J}_\sigma$, which satisfies the geometry of the mountain pass lemma under the above structural assumptions. The critical points u_σ of $\mathcal{J}_\sigma$ in $Z(\Omega)$ are found at special mountain pass levels c_σ, and these solutions of (1.9) are simply called *mountain pass solutions*.

Theorem 1.2. *Assume that M and w satisfy* ($\widetilde{\mathcal{M}}$) *and* (w), *with $p\theta < q < p_s^*(\alpha) \le p_s^*$ and $0 \le \alpha < ps < N$. Then there exists $\sigma^* > 0$ such that for any $\sigma \ge \sigma^*$ problem* (1.9) *admits a nontrivial mountain pass solution u_σ in $Z(\Omega)$. Moreover,*

$$\lim_{\sigma\to\infty} [u_\sigma]_{s,p} = 0. \tag{1.11}$$

The degenerate nature of problem (1.9) does not allow us to apply Theorem 1.1 in the proof of Theorem 1.2. As is customary in elliptic problems, involving critical Hardy nonlinearities, the delicate point is the verification of the Palais–Smale condition. For this we exploit an asymptotic property of the mountain pass level c_σ, taking inspiration from the proof of [15, Theorem 1.3] and also for a somehow similar problem from the proof of [1, Theorem 1.1].

The case $\Omega = \mathbb{R}^N$ and $\alpha = 0$ of Theorem 1.2 was first treated in [7, Theorem 1.2]. Furthermore, Theorem 1.2 extends in several directions [1, Theorem 1.1 and Theorem 1.2 (i)], [12, Theorem 1.1], [22, Theorem 1.1], [24, Theorem 1.1], [26, Theorem 1.1], and [30, Theorem 1.1 (ii)].

Moreover, in the non-degenerate case, following [1, Theorem 1.2 (ii)], we have this nice addition.

Theorem 1.3. *Assume that M is continuous in $\mathbb{R}_0^+$, satisfying* ($\widetilde{M_2}$). *Suppose further that w verifies* (w), *with $p < q < p_s^*(\alpha) \le p_s^*$ and $0 \le \alpha < ps < N$, and that*

$$pM(0) < qa. \tag{1.12}$$

Then there exists $\sigma^* > 0$ such that for any $\sigma \geq \sigma^*$ problem (1.9) *admits a nontrivial mountain pass solution u_σ in $Z(\Omega)$, satisfying the asymptotic property* (1.11).

As explained in the introduction of [1], the request (1.12) is automatic whenever $M(0) = a$, with $p < q$. The case $M(0) = a$ occurs in the non-degenerate prototype case (1.8), and more generally, whenever M is monotone increasing in $\mathbb{R}_0^+$. As we shall see, Theorem 1.3 is proved via a truncation argument on M since the Kirchhoff function M could increase too quickly with respect to the other terms of problem (1.9). Theorem 1.3 extends in several directions [1, Theorem 1.2 (ii)], in particular to the case in which Ω could be possibly unbounded and also to the case $\Omega = \mathbb{R}^N$. Furthermore, Theorem 1.3 generalizes, e.g., [24, Theorem 1.2 (2) and (3)] and [12, Theorem 1.1].

In the last part of the work, we study the nonhomogeneous version of the Kirchhoff problem (1.9), considering M of the special type (1.8), with θ replacing ϑ for simplicity. Actually, we treat the general problem

$$\begin{cases} (a + b\theta[u]_{s,p}^{p(\theta-1)})(-\Delta)_p^s u - \gamma\|u\|_{H_\alpha}^{p\theta-p_s^*(\alpha)} \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} \\ \qquad = \sigma w(x)|u|^{q-2}u + \dfrac{|u|^{p_s^*(\beta)-2}u}{|x|^\beta} + g(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.13}$$

with $\alpha \in [0, ps)$ but $\beta \in [0, ps)$. Of course, the only interesting case occurs when $\theta > 1$, and we assume it without loss of generality, with possibly $b = 0$. Hence, the addition of a sufficiently small nontrivial perturbation allows us to balance both Hardy–Sobolev terms with the Kirchhoff coefficient M, possibly zero at zero. The latter Hardy–Sobolev term could coincide with the Sobolev critical nonlinearity when $\beta = 0$, a special interesting case.

Theorem 1.4. *Let $a, b \geq 0$ with $a + b > 0$. Assume that w satisfies* (w), *with $\theta > 1$, $0 \leq \alpha < ps < N$, $0 \leq \beta < ps$, $p\theta \leq p_s^*(\alpha)$, $1 < q < p_s^*(\beta)$, and $p\theta < p_s^*(\beta)$. Then for all $\gamma \in [0, c_{a,b}H_\alpha^\theta)$ with*

$$c_{a,b} = \begin{cases} \infty & \textit{if } a > 0, \\ b\theta & \textit{if } a = 0 \end{cases} \tag{1.14}$$

there exist a number $\delta > 0$ and $\sigma_* \in (0, \infty]$ such that for any perturbation $g \in L^\upsilon(\Omega)$ and any parameter σ, satisfying

$$(\|g\|_\upsilon, \sigma) \in \begin{cases} [0, \delta] \times (0, \sigma_*] & \textit{if either } 1 < q < p, \textit{ or } p \leq q < p\theta \textit{ and } a = 0, \\ (0, \delta] \times (-\infty, \sigma_*) & \textit{if either } p \leq q < p\theta \textit{ and } a > 0, \textit{ or } p\theta \leq q < p_s^*(\beta), \end{cases} \tag{1.15}$$

problem (1.13) *admits a nontrivial solution $u_{\gamma,\sigma,g}$ in $Z(\Omega)$ and*

$$\lim_{\sigma\to\infty} [u_{\gamma,\sigma,g}]_{s,p} = 0 \tag{1.16}$$

when either $p\theta < q < p_s^*(\beta)$, or $p < q \leq p\theta$ and $a > 0$.

The result of Theorem 1.4 can be summarized in Table 1.

The values $\delta > 0$ and $\sigma_* \in (0, \infty]$ are constructed in the technical Lemma 4.5. It is worth noting that $p\theta < p_s^*$ since $p\theta < p_s^*(\beta) \leq p_s^*$. Hence, $p\theta$ cannot be equal to $p_s^*(\alpha) = p_s^*(0) = p_s^*$ when $\alpha = 0$.

The strategy used in the proof of Theorem 1.2 seems not to work for (1.13). This is why we had to require that M is of the special canonical form since this allows us to prove that the weak limit of a minimizing sequence is actually a nontrivial local interior minimum point of the functional corresponding to (1.13), that is, a nontrivial solution of (1.13).

Theorem 1.4 extends in several directions [24, Theorems 1.3 and 1.4], and generalizes the existence part contained in the multiplicity results given in [10, Theorem 2.1.1], [14, Theorem 1.1], [21, Theorem 1.1], [26, Theorem 1.2], and [35, Theorem 1.1].

A natural appealing open problem is to prove the existence of nontrivial solutions for (1.9) and (1.13) when $\alpha = ps$, a case not covered in Theorems 1.2–1.4. When $\alpha = ps$, $\beta = 0$ and $g \equiv 0$, a problem somehow

q	a	$\|g\|_u$	σ
$1 < q < p$	≥ 0	$\in [0, \delta]$	$\in (0, \sigma_*]$
$q = p$	$= 0$	$\in [0, \delta]$	$\in (0, \sigma_*]$
$q = p$	> 0	$\in (0, \delta]$	$\in (-\infty, a/C_w)$
$p < q < p\theta$	$= 0$	$\in [0, \delta]$	$\in (0, \sigma_*]$
$p < q < p\theta$	> 0	$\in (0, \delta]$	$\in \mathbb{R}$
$q = p\theta$	$= 0$	$\in (0, \delta]$	$\in (-\infty, b - \gamma^+/\theta H_\alpha^\theta)$
$q = p\theta$	> 0	$\in (0, \delta]$	$\in \mathbb{R}$
$p\theta < q < p_s^*(\beta)$	≥ 0	$\in (0, \delta]$	$\in \mathbb{R}$

Table 1. This table summarizes the conclusions of Theorem 1.4.

related to (1.13) is treated in details in [7, Theorem 1.1], but only in the non-degenerate setting, that is, under condition $(\widetilde{\mathrm{M}_2})$ and under a suitable geometrical restriction on γ. Finally, the case $\alpha = ps$ and $\beta = 0$ is covered in [32, Theorem 1.3] for a problem similar to (1.13), but again in the non-degenerate setting, that is, when $a > 0$.

The paper is organized as follows: In Section 2, we prove Theorem 1.1 with a result from Appendix A. In Section 3, we provide some applications of Theorem 1.1. Section 4 contains the proofs of Theorems 1.2–1.4. Finally, in Section 5, we extend the previous results when $(-\Delta)_p^s$ is replaced by a nonlocal integro-differential operator $\mathcal{L}_K^p$, generated by a general singular kernel K, satisfying the natural assumptions described by Caffarelli, e.g., in [6]; see also [33, 34].

2 A Concentration-Compactness Result

This section is devoted to the proof of Theorem 1.1, which concerns the delicate study of the exact behavior of weakly convergent sequences of $Z(\Omega)$ in the space of measures.

Let us first introduce the fractional Hardy–Sobolev inequality which is basic for (1.1). By [25, Theorems 1 and 2], we know that

$$\begin{cases} \|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leq c_{N,p} \dfrac{s(1-s)}{(N-ps)^{p-1}} [u]^p_{s,p}, \\ \displaystyle\int_{\mathbb{R}^N} |u(x)|^p \frac{dx}{|x|^{ps}} \leq c_{N,p} \frac{s(1-s)}{(N-ps)^p} [u]^p_{s,p} \end{cases} \tag{2.1}$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where $c_{N,p}$ is a positive constant depending only on N and p and $D^{s,p}(\mathbb{R}^N)$ is the fractional Beppo–Levi space, that is, the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm $[\,\cdot\,]_{s,p}$ defined in (1.2).

Let Ω be any open set of $\mathbb{R}^N$ and let $Z(\Omega)$ be the completion of $C_0^\infty(\Omega)$, with respect to the norm $[\,\cdot\,]_{s,p}$ defined in (1.2). When Ω is bounded, $p = 2$ and $K(x) = |x|^{-N-2s}$, then $Z(\Omega)$ is equivalent to the Hilbert space defined in [13]. Even if $Z(\Omega)$ is not a real space of functions, but a density space, the choice of this solution space is an improvement with respect to the space

$$X_0(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

fairly popular in recent papers devoted to nonlocal variational problems. Indeed, the density result proved in [16, Theorem 6] does not hold true for $X_0(\Omega)$ without assuming more restrictive conditions on the open bounded set Ω and on its boundary $\partial\Omega$; see in particular [16, Remark 7]. In conclusion, if Ω is an open bounded subset of $\mathbb{R}^N$, then $Z(\Omega) \subset X_0(\Omega)$, with possibly $Z(\Omega) \neq X_0(\Omega)$.

It is worth noting that if Ω is any open subset of $\mathbb{R}^N$ and $\tilde{u}$ denotes the natural extension of any $u \in Z(\Omega)$, then $\tilde{u} \in D^{s,p}(\mathbb{R}^N)$ by (2.1). In other words,

$$Z(\Omega) \subset \{u \in L^{p_s^*}(\Omega) : \tilde{u} \in D^{s,p}(\mathbb{R}^N)\}, \tag{2.2}$$

and equality holds when either $\Omega = \mathbb{R}^N$ or $\partial\Omega$ is continuous by [20, Theorem 1.4.2.2]. In what follows, with abuse of notation, we continue to write u in place of $\tilde{u}$ since the context is clear.

Therefore, the main density function space $Z(\mathbb{R}^N)$ reduces to $D^{s,p}(\mathbb{R}^N)$, and so

$$Z(\mathbb{R}^N) = D^{s,p}(\mathbb{R}^N) = \{u \in L^{p_s^*}(\mathbb{R}^N) : |\varphi(x) - \varphi(y)| \cdot |x-y|^{-s-N/p} \in L^p(\mathbb{R}^{2N})\}.$$

Clearly, $Z(\mathbb{R}^N) = D^{s,p}(\mathbb{R}^N)$ is the suitable solution space for problems (1.9) and (1.13) when $\Omega = \mathbb{R}^N$.

Thus, by the interpolation and the Hölder inequalities we easily get the next fractional Hardy–Sobolev inequality, proved for $p = 2$ in [19, Lemma 2.1]. However, for the sake of completeness we give the proof when $0 < \alpha < ps < N$ since when either $\alpha = 0$ or $\alpha = ps$, the fractional Hardy–Sobolev inequality reduces exactly to (2.1).

Lemma 2.1. *Assume that $0 \le \alpha \le ps < N$. Then there exists a positive constant C, possibly depending only on N, p, s, α, such that*

$$\|u\|_{H_\alpha} \le C[u]_{s,p}$$

for all $u \in Z(\mathbb{R}^N)$.

Proof. By (2.1), it is enough to consider only the case $0 < \alpha < ps$, so that $p < p_s^*(\alpha) < p_s^*$. By (2.1) and the Hölder inequality, for all $u \in Z(\mathbb{R}^N)$,

$$\begin{aligned}
\|u\|_{H_\alpha}^{p_s^*(\alpha)} &= \int_{\mathbb{R}^N} \frac{|u(x)|^{\alpha/s}}{|x|^\alpha} |u(x)|^{p_s^*(\alpha)-\alpha/s}\,dx \\
&\le \Big(\int_{\mathbb{R}^N} |u(x)|^p \frac{dx}{|x|^{ps}}\Big)^{\alpha/ps} \Big(\int_{\mathbb{R}^N} |u(x)|^{(p_s^*(\alpha)-\frac{\alpha}{s})\frac{ps}{ps-\alpha}}\,dx\Big)^{(ps-\alpha)/ps} \\
&= \Big(\int_{\mathbb{R}^N} |u(x)|^p \frac{dx}{|x|^{ps}}\Big)^{\alpha/ps} \Big(\int_{\mathbb{R}^N} |u(x)|^{p_s^*}\,dx\Big)^{(ps-\alpha)/ps} \\
&\le c_1[u]_{s,p}^{\alpha/s} c_2[u]_{s,p}^{p_s^*(ps-\alpha)/ps} \\
&= c_1c_2[u]_{s,p}^{p_s^*(\alpha)},
\end{aligned}$$

as required. □

From Lemma 2.1 it is clear that the fractional Sobolev embedding $Z(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ and the fractional Hardy–Sobolev embedding $Z(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\alpha)}(\mathbb{R}^N, |x|^{-\alpha})$ are continuous, but not compact. However, we are able to introduce the best fractional Hardy–Sobolev constant $H_\alpha = H(p, N, s, \alpha)$, as stated in (1.1). Of course, the number H_α is strictly positive and it coincides with the best fractional Sobolev constant when $\alpha = 0$. Consequently, the embeddings $Z(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$ and $Z(\Omega) \hookrightarrow L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha})$ are continuous, but not compact.

The last part of this section is devoted to the proof of Theorem 1.1. Even if the proof of Theorem 1.1 is fairly similar to that of [15, Theorem 2.2] given in the case $p = 2$, here we do not use any longer [31, Lemmas 5 and 6], where the requirement that Ω is bounded seems to be crucial. However, the proof of Theorem 1.1 is based on the tightness of the sequence $(|D^s u_j|^p)_j$, and the tightness property is obtained as an application of [3, Theorem 8.6.2]. It is exactly at this step that we use that Ω is bounded.

Proof of Theorem 1.1. As noted above, the given sequence $(u_j)_j$ converges weakly to u also in $L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha})$. In particular, there exist two finite positive measures μ and ν in $\mathbb{R}^N$ such that (1.3) holds, with the measures

$$j \mapsto |D^s u_j(x)|^p\,dx, \quad j \mapsto |u_j(x)|^{p_s^*(\alpha)} \frac{dx}{|x|^\alpha}$$

in $\mathbb{R}^N$ being uniformly tight in j. Indeed, since Ω is bounded, we can find an open bounded set U of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U$. Hence, for a.a. $x \in \mathbb{R}^N \setminus U$ we have $u_j(x) = 0$, from which

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus U} |D^s u_j(x)|^p\,dx &= \int_{\mathbb{R}^N\setminus U} dx\Big(\int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x-y|^{N+ps}}\,dy\Big) \\
&= \int_{\mathbb{R}^N\setminus U} dx\Big(\int_{\mathbb{R}^N} \frac{|u_j(y)|^p}{|x-y|^{N+ps}}\,dy\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N\setminus U} dx\Big(\int_\Omega \frac{|u_j(y)|^p}{|x-y|^{N+ps}}\,dy\Big)\\
&\le \int_{\mathbb{R}^N\setminus U} \frac{dx}{\operatorname{dist}(x,\overline{\Omega})^{N+ps}} \int_\Omega |u_j(y)|^p\,dy\\
&\le \|u_j\|_p^p\Big(\int_{\mathbb{R}^N\setminus U} \frac{dx}{\operatorname{dist}(x,\overline{\Omega})^{N+ps}}\Big)\\
&\le \big(\sup_j \|u_j\|_p^p\big)\Big(\int_{\mathbb{R}^N\setminus U} \frac{dx}{\operatorname{dist}(x,\overline{\Omega})^{N+ps}}\Big),
\end{aligned}
\tag{2.3}
$$

and the last integral is finite since $\operatorname{dist}(\mathbb{R}^N\setminus U,\overline{\Omega})>0$ and $N+ps>N$.

Reasoning as above and considering that $u_j=0$ in $\mathbb{R}^N\setminus\Omega$, we get also the tightness of $(u_j/|x|^{\alpha/p_s^*(\alpha)})_j$.

Put $v_j=u_j-u$. Clearly, $v_j\rightharpoonup 0$ in $Z(\Omega)$ as $j\to\infty$. Repeating the above argument, we get the existence of two positive measures $\widehat{\mu}$ and $\widehat{\nu}$ on $\mathbb{R}^N$ such that

$$
|D^s v_j(x)|^p\,dx \overset{*}{\rightharpoonup} \widehat{\mu} \quad\text{and}\quad |v_j(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha} \overset{*}{\rightharpoonup} \widehat{\nu} \qquad \text{in } \mathcal{M}(\mathbb{R}^N). \tag{2.4}
$$

By [9, Corollary 7.2], with $0<\alpha\le ps$, the sequence $(u_j)_j$ strongly converges to u in $L^p(\Omega)$, with Ω being bounded, and so in $L^p(\mathbb{R}^N)$ by the trivial extension to the entire $\mathbb{R}^N$. Thus [4, Theorem 4.9] implies that up to a subsequence, still named $(u_j)_j$, there exists $h\in L^p(\Omega)$, with

$$
u_j \to u \text{ a.e. in } \Omega, \qquad |u_j|\le h \text{ a.e. in } \Omega \text{ and all } j. \tag{2.5}
$$

Hence, for any $\varphi\in C_0^\infty(\Omega)$,

$$
\begin{aligned}
\int_\Omega |\varphi(x)|^{p_s^*(\alpha)}d\nu - \|\varphi u\|_{H_\alpha}^{p_s^*(\alpha)} &= \lim_{j\to\infty}\|\varphi u_j\|_{H_\alpha}^{p_s^*(\alpha)} - \|\varphi u\|_{H_\alpha}^{p_s^*(\alpha)}\\
&= \lim_{j\to\infty}\|\varphi v_j\|_{H_\alpha}^{p_s^*(\alpha)} = \int_\Omega |\varphi(x)|^{p_s^*(\alpha)}d\widehat{\nu}
\end{aligned}
$$

by the Brezis–Lieb lemma; see [5]. This yields that

$$
\nu = \widehat{\nu} + |u(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha}
$$

since $\varphi\in C_0^\infty(\Omega)$ is arbitrary.

Let us first prove (1.4). To this end, fix $\varepsilon>0$ and $\varphi\in C_0^\infty(\Omega)$. Then there exists $C_\varepsilon>0$ such that $|\xi+\eta|^p\le(1+\varepsilon)|\xi|^p+C_\varepsilon|\eta|^p$ for all numbers $\xi,\eta\in\mathbb{R}$. Hence, the Leibniz formula gives for all j,

$$
\int_{\mathbb{R}^N}|D^s(v_j\varphi)(x)|^p\,dx \le (1+\varepsilon)\int_{\mathbb{R}^N}|D^s v_j(x)|^p|\varphi(x)|^p\,dx + C_\varepsilon\int_{\mathbb{R}^N}|D^s\varphi(x)|^p|v_j(x)|^p\,dx.
$$

Thus, the Hardy inequality (1.1) along the sequence $(\varphi v_j)_j$ of $Z(\Omega)$ yields

$$
H_\alpha\|\varphi v_j\|_{H_\alpha}^p \le [\varphi v_j]_{s,p}^p \le (1+\varepsilon)\int_{\mathbb{R}^N}|D^s v_j(x)|^p|\varphi(x)|^p\,dx + C_{\varepsilon,\varphi}\|v_j\|_p^p \tag{2.6}
$$

for an appropriate constant $C_{\varepsilon,\varphi}>0$ since

$$
|D^s\varphi(x)|^p = \int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\,dy \le 2^p\|\varphi\|_{C^1(\mathbb{R}^N)}^p\int_{\mathbb{R}^N}\frac{\min\{1,|x-y|^p\}}{|x-y|^{N+ps}}\,dy \le C_\varphi, \tag{2.7}
$$

where $C_\varphi>0$ depends also on N, p and s. By (2.4), (2.6) and the fact that $v_j=u_j-u\to 0$ in $L^p(\Omega)$ as $j\to\infty$, we obtain at once that

$$
\Big(\int_{\mathbb{R}^N}|\varphi(x)|^{p_s^*(\alpha)}d\widehat{\nu}\Big)^{p/p_s^*(\alpha)} \le \frac{1+\varepsilon}{H_\alpha}\int_{\mathbb{R}^N}|\varphi(x)|^p\,d\widehat{\mu},
$$

that is, $\widehat{\nu}$ is absolutely continuous with respect to $\widehat{\mu}$. Hence, by [23, Lemma 1.2] the measure $\widehat{\nu}$ is decomposed as a sum of Dirac masses.

It remains to show that $\widehat{\nu}$ is concentrated at 0. Here we assume that $0 \notin \mathrm{Supp}(\varphi)$, so that $|\varphi(x)|^{p_s^*(\alpha)}/|x|^\alpha$ is in $L^\infty(\mathrm{Supp}(\varphi))$. In turn, [9, Corollary 7.2] yields

$$\|\varphi v_j\|_{H_\alpha}^{p_s^*(\alpha)} = \int_{\mathrm{Supp}(\varphi)} \frac{|\varphi(x)|^{p_s^*(\alpha)}}{|x|^\alpha}|v_j(x)|^{p_s^*(\alpha)}\,dx \le C \int_{\mathrm{Supp}(\varphi)} |v_j(x)|^{p_s^*(\alpha)}\,dx \to 0$$

as $j \to \infty$ since $0 < \alpha \le ps$, so that $p \le p_s^*(\alpha) < p_s^*$. This, combined with (2.4), gives $\int_\Omega |\varphi(x)|^{p_s^*(\alpha)} d\widehat{\nu} = 0$. In other words, $\widehat{\nu}$ is a measure concentrated in 0. Hence $\widehat{\nu} = \nu_0\delta_0$, and (1.4) is proved.

In order to show (1.5), arguing as in (2.6) and replacing v_j by u_j, we have

$$H_\alpha\Big(\int_\Omega |\varphi(x)|^{p_s^*(\alpha)} d\nu\Big)^{p/p_s^*(\alpha)} \le (1+\varepsilon)\int_{\mathbb{R}^N} |\varphi(x)|^p\,d\mu + C_\varepsilon \int_{\mathbb{R}^N} |D^s\varphi(x)|^p|u(x)|^p\,dx \tag{2.8}$$

as $j \to \infty$ by (2.4) and (2.5).

Let now $\varphi \in C_0^\infty(\mathbb{R}^N)$, with $0 \le \varphi \le 1$, $\varphi(0) = 1$, $\mathrm{Supp}(\varphi) = B(0,1)$, and put $\varphi_{\tilde\varepsilon}(x) = \varphi(x/\tilde\varepsilon)$ for $\tilde\varepsilon > 0$ sufficiently small. Since $\nu \ge \nu_0\delta_0$, choosing $\varphi_{\tilde\varepsilon}$ as test function in (2.8), we obtain

$$0 \le H_\alpha \nu_0^{p/p_s^*(\alpha)} \le (1+\varepsilon)\mu(B(0,\tilde\varepsilon)) + C_\varepsilon \int_{\mathbb{R}^N} |u(x)|^p|D^s\varphi_{\tilde\varepsilon}(x)|^p\,dx. \tag{2.9}$$

Note that $\|\nabla\varphi_{\tilde\varepsilon}\|_\infty \le C/\tilde\varepsilon$ by construction. Hence

$$\begin{aligned}\iint_{U\times V} \frac{|u(x)|^p|\varphi_{\tilde\varepsilon}(x) - \varphi_{\tilde\varepsilon}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \le{}& \frac{C}{\tilde\varepsilon^p}\iint_{U\times(V\cap\{|x-y|\le\tilde\varepsilon\})} \frac{|u(x)|^p}{|x-y|^{N+ps-p}}\,dx\,dy\\ &+ C\iint_{U\times(V\cap\{|x-y|>\tilde\varepsilon\})} \frac{|u(x)|^p}{|x-y|^{N+ps}}\,dx\,dy,\end{aligned} \tag{2.10}$$

where U and V are two generic subsets of $\mathbb{R}^N$.

We claim that the last term on the right-hand side of (2.9) goes to 0 as $\tilde\varepsilon \to 0$. If $U = V = \mathbb{R}^N \setminus B(0,\tilde\varepsilon)$, all integrals in (2.10) are equal to 0, indeed. Now, if $U \times V = B(0,\tilde\varepsilon) \times \mathbb{R}^N$ and $U \times V = \mathbb{R}^N \times B(0,\tilde\varepsilon)$, by Lemma A.1 we have

$$\begin{cases} \displaystyle\lim_{\tilde\varepsilon\to 0} \frac{1}{\tilde\varepsilon^p}\iint_{U\times(V\cap\{|x-y|\le\tilde\varepsilon\})} \frac{|u(x)|^p}{|x-y|^{N+ps-p}}\,dx\,dy = 0,\\ \displaystyle\lim_{\tilde\varepsilon\to 0} \iint_{U\times(V\cap\{|x-y|>\tilde\varepsilon\})} \frac{|u(x)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0.\end{cases} \tag{2.11}$$

Thus, combining (2.10) with (2.11), we get

$$\lim_{\tilde\varepsilon\to 0} \int_{\mathbb{R}^N} |D^s\varphi_{\tilde\varepsilon}(x)|^p|u(x)|^p\,dx = 0,$$

as claimed.

Hence, letting $\tilde\varepsilon \to 0$ and $\varepsilon \to 0$ in (2.9), we have $0 \le H_\alpha\nu_0^{p/p_s^*(\alpha)} \le \mu_0$. By the Fatou lemma, $\mu \ge |D^s u(x)|^p\,dx$, and this concludes the proof of (1.5) since $|D^s u(x)|^p\,dx$ and $\mu_0\delta_0$ are orthogonal. □

An immediate consequence of Theorem 1.1 is the following result, where H_α is given in (1.1) and M verifies $(\mathcal{M})$. This assumption will be useful to get balance between the Kirchhoff term and the Hardy–Sobolev critical nonlinearity. For this we also use the variational characterization of the first eigenvalue of the fractional p-Laplacian given by

$$\lambda_1 = \min_{u\in Z(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |D^s u(x)|^p\,dx}{\int_\Omega |u(x)|^p\,dx}, \tag{2.12}$$

which is positive by [18, Theorem 4.1], with Ω being bounded. In passing we recall that $p^*(0) = p_s^*$.

Theorem 2.2. *Let M satisfy* $(\mathcal{M})$, *with $c > 0$ and α given as in* (1.7). *For all $\gamma \in [0, c\theta H_\alpha^\theta)$ and $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$, with*

$$m_{\gamma,\theta} = \begin{cases} \infty & \text{if } \theta > 1, \\ c - \gamma/H_\alpha & \text{if } \theta = 1, \end{cases} \tag{2.13}$$

the functional $\mathcal{H}_{\gamma,\lambda} : Z(\Omega) \to \mathbb{R}$, defined by (1.6), *is weakly lower semi-continuous and coercive in $Z(\Omega)$.*

Proof. Let $(u_j)_j$ be a sequence such that $u_j \rightharpoonup u$ in $Z(\Omega)$. Clearly, $u_j \to u$ in $L^p(\Omega)$ since Ω is bounded, and there exist two positive measures, verifying (1.3). Now, $\mathscr{M}$ is super-additive in $\mathbb{R}_0^+$ since $\mathscr{M}$ is convex in $\mathbb{R}_0^+$ and $\mathscr{M}(0) = 0$. Let us divide the proof into two parts.

Case $\alpha \in (0, ps]$: Since $\mathscr{M}$ is continuous in $\mathbb{R}_0^+$, $\theta p < p_s^*(\alpha)$ by (1.6) and $\gamma \ge 0$, Theorem 1.1 yields

$$\begin{aligned} \liminf_{j\to\infty} \mathcal{H}_{\gamma,\lambda}(u_j) &= \liminf_{j\to\infty} \frac{1}{p}\Big[\mathscr{M}([u_j]_{s,p}^p) - \frac{\gamma}{\theta}\|u_j\|_{H_\alpha}^{\theta p} - \lambda\|u_j\|_p^p\Big] \\ &\ge \frac{1}{p}\Big[\mathscr{M}([u]_{s,p}^p + \mu_0) - \frac{\gamma}{\theta}(\|u\|_{H_\alpha}^{p_s^*(\alpha)} + \nu_0)^{\theta p/p_s^*(\alpha)} - \lambda\|u\|_p^p\Big] \\ &\ge \frac{1}{p}\Big[\mathscr{M}([u]_{s,p}^p) + \mathscr{M}(\mu_0) - \frac{\gamma}{\theta}(\|u\|_{H_\alpha}^{\theta p} + \nu_0^{\theta p/p_s^*(\alpha)}) - \lambda\|u\|_p^p\Big] \\ &= \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\Big(\mathscr{M}(\mu_0) - \frac{\gamma}{\theta}\nu_0^{\theta p/p_s^*(\alpha)}\Big) \\ &\ge \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\Big(\mathscr{M}(\mu_0) - \frac{\gamma}{\theta H_\alpha^\theta}\mu_0^\theta\Big) \\ &\ge \mathcal{H}_{\gamma,\lambda}(u) + \frac{\mu_0^\theta}{p}\Big(c - \frac{\gamma}{\theta H_\alpha^\theta}\Big), \end{aligned} \tag{2.14}$$

where in the last step we have used (1.7).

Case $\alpha = 0$: In this case, [29, Theorem 2.5] gives the existence of an at most denumerable set of index Λ, $x_n \in \overline{\Omega}$, $\mu_n \ge 0$, $\nu_n \ge 0$, with $\mu_n + \nu_n > 0$ for all $n \in \Lambda$, such that

$$\nu = |u(x)|^{p_s^*}\,dx + \sum_{n\in\Lambda} \nu_n \delta_{x_n}, \quad \mu \ge |D^s u(x)|^p\,dx + \sum_{n\in\Lambda} \mu_n \delta_{x_n},$$

and $0 \le H_0 \nu_n^{p/p_s^*} \le \mu_n$ for all $n \in \Lambda$, where H_0 is the Sobolev constant defined in (1.1), with $\alpha = 0$. Since $\mathscr{M}$ is continuous in $\mathbb{R}_0^+$, $\theta p < p_s^*$ by (1.6) and $\gamma \ge 0$, then as before

$$\begin{aligned} \liminf_{j\to\infty} \mathcal{H}_{\gamma,\lambda}(u_j) &= \liminf_{j\to\infty} \frac{1}{p}\Big[\mathscr{M}([u_j]_{s,p}^p) - \frac{\gamma}{\theta}\|u_j\|_{p_s^*}^{\theta p} - \lambda\|u_j\|_p^p\Big] \\ &\ge \frac{1}{p}\Big[\mathscr{M}\Big([u]_{s,p}^p + \sum_{n\in\Lambda}\mu_n\Big) - \frac{\gamma}{\theta}\Big(\|u\|_{p_s^*}^{p_s^*} + \sum_{n\in\Lambda}\nu_n\Big)^{\theta p/p_s^*} - \lambda\|u\|_p^p\Big] \\ &\ge \frac{1}{p}\Big[\mathscr{M}([u]_{s,p}^p) + \sum_{n\in\Lambda}\mathscr{M}(\mu_n) - \frac{\gamma}{\theta}\Big(\|u\|_{p_s^*}^{\theta p} + \sum_{n\in\Lambda}\nu_n^{\theta p/p_s^*}\Big) - \lambda\|u\|_p^p\Big] \\ &= \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\sum_{n\in\Lambda}\Big(\mathscr{M}(\mu_n) - \frac{\gamma}{\theta}\nu_n^{\theta p/p_s^*}\Big) \\ &\ge \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\sum_{n\in\Lambda}\Big(\mathscr{M}(\mu_n) - \frac{\gamma}{\theta H_0^\theta}\mu_n^\theta\Big) \\ &\ge \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\Big(c - \frac{\gamma}{\theta H_0^\theta}\Big)\sum_{n\in\Lambda}\mu_n^\theta. \end{aligned} \tag{2.15}$$

In conclusion, the weak lower semi-continuity of $\mathcal{H}_{\gamma,\lambda}$ in $Z(\Omega)$ follows at once in both cases thanks to (2.14), (2.15) and the fact that $\gamma < c\theta H_\alpha^\theta$, where α and θ are related by (1.7).

Now, by (1.1), (1.7) and (2.12) we also get for all $u \in Z(\Omega)$,

$$\mathcal{H}_{\gamma,\lambda}(u) \ge \frac{1}{p}\Big(c - \frac{\gamma}{\theta H_\alpha^\theta}\Big)[u]_{s,p}^{p\theta} - \frac{\lambda^+}{p\lambda_1}[u]_{s,p}^p. \tag{2.16}$$

Consequently, $\mathcal{H}_{\gamma,\lambda}(u) \to \infty$ as $[u]_{s,p} \to \infty$, provided that $\gamma < c\theta H_\alpha^\theta$ and $\lambda < m_{\gamma,\theta}\lambda_1$, as required. □

The case $M \equiv 1$, $\alpha = 0$ and $p = 2$ of Theorem 2.2 was first treated in [28, Theorem 1]. Clearly, when $\theta > 1$, that is, $m_{\gamma,\theta} = \infty$, Theorem 2.2 holds for all $\lambda \in \mathbb{R}$. This standard convention is used also in what follows.

3 Some Applications on Bounded Domains

Following [15], we present some applications of Theorem 2.2. Hence, throughout the section we assume that Ω is a bounded open subset of $\mathbb{R}^N$, that $0 < s < 1 < p < \infty$ and $ps < N$.

Theorem 3.1 (Superlinear f). *Let M verify* $(\mathcal{M})$*, with $c > 0$ and α given as in* (1.7)*. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the conditions*

(f_1) $\sup\{|f(x,t)| : \text{a.e. } x \in \Omega,\ t \in [0, C]\} < \infty$ *for any* $C > 0$;

(f_2) $f(x,t) = o(|t|^{p_s^*-1})$ *as* $|t| \to \infty$ *uniformly a.e. in* $x \in \Omega$;

(f_3) *there exist a non-empty open set $A \subseteq \Omega$ and a set $B \subseteq A$ of positive Lebesgue measure such that*

$$\limsup_{t\to 0^+} \frac{\operatorname{ess\,inf}_{x\in B} F(x,t)}{t^p} = \infty \quad \text{and} \quad \liminf_{t\to 0^+} \frac{\operatorname{ess\,inf}_{x\in A} F(x,t)}{t^p} > -\infty,$$

where $F(x,t) = \int_0^t f(x,\tau)\,d\tau$.

Then for all $\gamma \in [0, c\theta H_\alpha^\theta)$ and $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$, where $m_{\gamma,\theta}$ is given in (2.13)*, there exists a positive constant $\overline{\sigma} = \overline{\sigma}(\lambda,\gamma)$ such that for any $\sigma \in (0,\overline{\sigma})$ the problem*

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u - \gamma \|u\|_{H_\alpha}^{p\theta - p_s^*(\alpha)} \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \lambda |u|^{p-2}u + \sigma f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.1}$$

has a nontrivial solution $u_{\gamma,\lambda,\sigma} \in Z(\Omega)$.

Moreover, if $\gamma \in [0, cH_\alpha^\theta)$ and either $\lambda \in \mathbb{R}_0^-$ when $\theta > 1$, or $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$ when $\theta = 1$, then

$$\lim_{\sigma\to 0^+} [u_{\gamma,\lambda,\sigma}]_{s,p} = 0. \tag{3.2}$$

Proof. Fix $\gamma \in [0, c\theta H_\alpha^\theta)$ and $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$. Problem (3.1) can be seen as the Euler–Lagrange equation of the functional $\mathcal{J}_{\gamma,\lambda,\sigma}$ defined by

$$\mathcal{J}_{\gamma,\lambda,\sigma}(u) = \mathcal{H}_{\gamma,\lambda}(u) - \sigma\Psi(u), \quad u \in Z(\Omega),$$

where $\mathcal{H}_{\gamma,\lambda}$ is the functional given in (1.6), while

$$\Psi(u) = \int_\Omega F(x,u(x))\,dx.$$

Clearly, the functionals $\mathcal{H}_{\gamma,\lambda}$ and Ψ are Fréchet differentiable in $Z(\Omega)$, and actually $\mathcal{J}_{\gamma,\lambda,\sigma}$ is of class $C^1(Z(\Omega))$.

Furthermore, by Theorem 2.2 we know that $\mathcal{H}_{\gamma,\lambda}$ is weakly lower semi-continuous and coercive in $Z(\Omega)$. From (f_1) and (f_2) for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \delta(\varepsilon) > 0$ such that

$$|F(x,t)| \le \varepsilon |t|^{p_s^*} + \delta_\varepsilon |t| \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Hence, the Vitali convergence theorem yields that Ψ is continuous in the weak topology of $Z(\Omega)$.

From this point, arguing essentially as in the proof of [15, Theorem 1.1] but working in the functional space $Z(\Omega) = (Z(\Omega), [\,\cdot\,]_{s,p})$, we prove the existence of a nontrivial solution $u_{\gamma,\lambda,\sigma}$ for any $\sigma \in (0,\overline{\sigma})$. Moreover, the family $\{[u_{\gamma,\lambda,\sigma}]_{s,p}\}_{\sigma\in(0,\overline{\sigma})}$ is uniformly bounded in σ.

It remains to show the asymptotic behavior (3.2). By (f_1) and (f_2), with $\varepsilon = 1$, and [9, Theorem 6.5], we have

$$\left|\int_\Omega f(x, u_{\gamma,\lambda,\sigma}(x)) u_{\gamma,\lambda,\sigma}(x)\,dx\right| \le H_0^{-p_s^*/p} [u_{\gamma,\lambda,\sigma}]_{s,p}^{p_s^*} + \delta_1 C_1 [u_{\gamma,\lambda,\sigma}]_{s,p} \le C_{\gamma,\lambda}, \tag{3.3}$$

with $C_{\gamma,\lambda}$ independent of σ since $\{[u_{\gamma,\lambda,\sigma}]_{s,p}\}_{\sigma\in(0,\overline{\sigma})}$ is uniformly bounded in σ.

Fix $\gamma \in [0, cH_\alpha^\theta)$ and λ as in the last part of the statement. Since $\langle \mathcal{J}'_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle_{Z(\Omega),Z'(\Omega)} = 0$ for any $\sigma \in (0, \overline{\sigma})$, we have

$$\begin{aligned} M([u_{\gamma,\lambda,\sigma}]_{s,p}^p)[u_{\gamma,\lambda,\sigma}]_{s,p}^p - \gamma\|u_{\gamma,\lambda,\sigma}\|_{H_\alpha}^{p\theta} - \lambda\|u_{\gamma,\lambda,\sigma}\|_p^p &= \langle \mathcal{H}'_{\gamma,\lambda}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle_{Z(\Omega),Z'(\Omega)} \\ &= \sigma \int_\Omega f(x, u_{\gamma,\lambda,\sigma}(x))u_{\gamma,\lambda,\sigma}(x)\,dx. \end{aligned}$$

This, (1.1), (2.12), (3.3), and the monotonicity of M, combined with (1.7), yield

$$\Big(c - \frac{\gamma}{H_\alpha^\theta}\Big)[u_{\gamma,\lambda,\sigma}]_{s,p}^{p\theta} - \frac{\lambda^+}{\lambda_1}[u_{\gamma,\lambda,\sigma}]_{s,p}^p \le \langle \mathcal{H}'_{\gamma,\lambda}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle_{Z(\Omega),Z'(\Omega)} \le \sigma C_{\gamma,\lambda}.$$

Letting $\sigma \to 0^+$, we get (3.2) by the choices of γ and λ. □

The case $M \equiv 1$, $\alpha = 0$ and $p = 2$ of Theorem 3.1 was first treated in [28, Theorem 4]. Furthermore, Theorem 3.1 extends in several directions the existence part contained in the multiplicity [26, Theorem 1.1].

Theorem 3.2 (Sublinear f). *Let M satisfy $(\mathcal{M})$, with $c > 0$ and α given as in* (1.7). *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:*

(f_4) *There exist $q \in (1, \theta p)$ and $a \in L^{p_s^*/(p_s^*-q)}(\Omega)$ such that*

$$|f(x, t)| \le a(x)(1 + |t|^{q-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

(f_5) *There exist $\tilde{q} \in (1, p)$, $\delta > 0$, $a_0 > 0$, and a nonempty open subset ω of Ω such that*

$$F(x, t) \ge a_0 t^{\tilde{q}} \quad \text{for all } (x, t) \in \omega \times (0, \delta).$$

Then for all $\gamma \in [0, c\theta H_\alpha^\theta)$, $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$, where $m_{\gamma,\theta}$ is given in (2.13), *and $\sigma > 0$, problem* (3.1) *has a nontrivial solution $u_{\gamma,\lambda,\sigma} \in Z(\Omega)$.*

Moreover, if $\gamma \in [0, cH_\alpha^\theta)$ and either $\lambda \in \mathbb{R}_0^-$ when $\theta > 1$, or $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$ when $\theta = 1$, then (3.2) *holds.*

Proof. Fix $\gamma \in [0, c\theta H_\alpha^\theta)$, $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$ and $\sigma > 0$. Using the notation of the proof of Theorem 3.1, by (2.16), (f_4) and the Hölder inequality, for all $u \in Z(\Omega)$ we have

$$\begin{aligned} \mathcal{J}_{\gamma,\lambda,\sigma}(u) &\ge \frac{1}{p}\Big(c - \frac{\gamma}{\theta H_\alpha^\theta}\Big)[u]_{s,p}^{p\theta} - \frac{\lambda^+}{p\lambda_1}[u]_{s,p}^p - \sigma\int_\Omega a(x)|u|^q\,dx - \sigma\|a\|_{(p_s^*)'}\|u\|_{p_s^*} \\ &\ge \frac{1}{p}\Big(c - \frac{\gamma}{\theta H_\alpha^\theta}\Big)[u]_{s,p}^{p\theta} - \frac{\lambda^+}{p\lambda_1}[u]_{s,p}^p - \sigma\|a\|_{p_s^*/(p_s^*-q)}\|u\|_{p_s^*}^q - \sigma\|a\|_{(p_s^*)'}\|u\|_{p_s^*} \\ &\ge \frac{1}{p}\Big(c - \frac{\gamma}{\theta H_\alpha^\theta}\Big)[u]_{s,p}^{p\theta} - \frac{\lambda^+}{p\lambda_1}[u]_{s,p}^p - \sigma H_0^{-q/p}\|a\|_{p_s^*/(p_s^*-q)}[u]_{s,p}^q - \sigma\|a\|_{(p_s^*)'}\|u\|_{p_s^*} \end{aligned} \tag{3.4}$$

since $(p_s^*)' < p_s^*/(p_s^* - q)$ and Ω is bounded. Hence $\mathcal{J}_{\gamma,\lambda,\sigma}$ is coercive and bounded below on $Z(\Omega)$. Furthermore, $\mathcal{H}_{\gamma,\lambda}$ is weakly lower semi-continuous in $Z(\Omega)$ by Theorem 2.2. Moreover, Ψ is weakly continuous in $Z(\Omega)$ by (f_4). Thus, $\mathcal{J}_{\gamma,\lambda,\sigma} = \mathcal{H}_{\gamma,\lambda} - \sigma\Psi$ is weakly lower semi-continuous in $Z(\Omega)$. Then there exists $u_{\gamma,\lambda,\sigma} \in Z(\Omega)$ such that

$$\mathcal{J}_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}) = \inf\{\mathcal{J}_{\gamma,\lambda,\sigma}(u) : u \in Z(\Omega)\}.$$

We claim that $u_{\gamma,\lambda,\sigma} \not\equiv 0$. Let $x_0 \in \omega$ and let $r > 0$ such that $B_r(x_0) \subset \omega$. Fix $\varphi \in C_0^\infty(B_r(x_0))$ with $0 \le \varphi \le 1$, $[\varphi]_{s,p} \le C_r$ and $\|\varphi\|_{L^q(B_r(x_0))} > 0$. Then, by $(\mathcal{M})$ and (f_5), for all $t \in (0, \delta)$,

$$\mathcal{J}_{\gamma,\lambda,\sigma}(t\varphi) \le \frac{1}{p}\Big(M((\delta C_r)^p)\delta^p[\varphi]_{s,p}^p - \frac{\gamma}{\theta}t^{p\theta}\|\varphi\|_{H_\alpha}^{p\theta} - \lambda t^p\|\varphi\|_p^p\Big) - \sigma t^{\tilde{q}}a_0\|\varphi\|_{L^{\tilde{q}}(B_r(x_0))} < 0,$$

by choosing $t > 0$ sufficiently small, since $1 < \tilde{q} < p$. Thus, the claim is proved. In other words, the nontrivial critical point $u_{\gamma,\lambda,\sigma}$ of $\mathcal{J}_{\gamma,\lambda,\sigma}$ in $Z(\Omega)$ is a nontrivial solution of (3.1).

To prove (3.2) fix $\gamma \in [0, c\theta H_\alpha^\theta)$ and $\lambda \in (-\infty, m_{\gamma,\theta}\lambda_1)$ and note that the family of nontrivial critical points $\{u_{\gamma,\lambda,\sigma}\}_{\sigma\in(0,1]}$, constructed above, is clearly uniformly bounded in $Z(\Omega)$ thanks to (3.4). Therefore, for any $\gamma \in [0, cH_\alpha^\theta)$ and either $\lambda \in \mathbb{R}_0^-$ when $\theta > 1$, or $\lambda \in (-\infty, m^*\lambda_1)$ when $\theta = 1$, with $m^* = c - \gamma/H_\alpha$ if $\theta = 1$, we can proceed exactly as in the last part of the proof of Theorem 3.1 and get (3.2). □

When Ω is the open unit ball B of center 0 and radius 1 of $\mathbb{R}^N$, a typical example of f, verifying (f_4) and (f_5), is given by $f(x,t) = a(x)(|t|^{\tilde{q}-2} + |t|^{q-2})t$, with $1 < \tilde{q} < p$, $1 < q < \theta p$, $a(x) = -\log|x|$, $\delta = 1$, and $\omega = \{x \in B : |x| > 1/2\}$.

Theorem 3.2 extends in several directions the existence result contained in the multiplicity [26, Theorem 1.3].

4 Critical Problems in General Open Sets Ω

As said in Sections 1 and 2, the best solution space for problems (1.9) and (1.13) is the fractional space $Z(\Omega) = (Z(\Omega), [\,\cdot\,]_{s,p})$, where Ω is any open subset of $\mathbb{R}^N$, possibly the entire $\mathbb{R}^N$ itself, and so $Z(\mathbb{R}^N) = D^{s,p}(\mathbb{R}^N)$. In any case, $Z(\Omega)$ is a uniformly convex Banach space when $0 < s < 1 < p < \infty$. Throughout this section, we assume that $ps < N$, $\alpha \in [0, ps)$ and that $\theta \in [1, p_s^*(\alpha)/p)$, except for (1.13) where $\alpha \in (0, ps)$. Moreover, M and w satisfy conditions ($\widetilde{\mathcal{M}}$) and (w), and q is any Lebesgue exponent, with $p\theta < q < p_s^*(\alpha)$.

By [2, Proposition A.6], the space $L^q(\Omega, w) = (L^q(\Omega, w), \|\cdot\|_{q,w})$ is a uniformly convex Banach space, endowed with the norm

$$\|u\|_{q,w} = \Big(\int_\Omega w(x)|u(x)|^q\,dx\Big)^{1/q}.$$

Essentially, as proved in [7, Lemma 2.1], the following result holds also in our context.

Lemma 4.1. *Let* (w) *hold with* $1 < q < p_s^*$. *Then the embedding* $Z(\Omega) \hookrightarrow L^q(\Omega, w)$ *is compact and* (1.10) *is valid with* $C_w = H_0^{-1/p}\|w\|_\wp^{1/q} > 0$.

Proof. By (w), (2.2), the Hölder inequality, and (1.1), for all $u \in Z(\Omega)$,

$$\|u\|_{q,w} \le \Big(\int_\Omega w(x)^\wp\,dx\Big)^{1/\wp q} \cdot \Big(\int_\Omega |u|^{p_s^*}\,dx\Big)^{1/p_s^*} \le H_0^{-1/p}\|w\|_\wp^{1/q}[u]_{s,p},$$

so that the embedding $Z(\Omega) \hookrightarrow L^q(\Omega, w)$ is continuous and (1.10) holds.

To complete the proof, it remains to show that if $u_j \rightharpoonup u$ in $Z(\Omega)$, then $u_j \to u$ in $L^q(\Omega, w)$ as $j \to \infty$. As noted in (2.2), the natural extensions of u_j and u, denoted by $\tilde{u}_j$ and $\tilde{u}$, have the property that $\tilde{u}_j \rightharpoonup \tilde{u}$ in $D^{s,p}(\mathbb{R}^N)$. Let $\tilde{w}$ be the natural extension of the weight w to $\mathbb{R}^N$. Hence, by the Hölder inequality,

$$\int_{\mathbb{R}^N\setminus B_R} \tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q\,dx \le L\Big(\int_{\mathbb{R}^N\setminus B_R} \tilde{w}(x)^\wp\,dx\Big)^{1/\wp} = o(1) \tag{4.1}$$

as $R \to \infty$, with $\tilde{w} \in L^\wp(\mathbb{R}^N)$ and $\sup_j\|\tilde{u}_j - \tilde{u}\|_{p_s^*}^q = L < \infty$ by (1.1). Moreover, for all $R > 0$ the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(B_R)$ is continuous, and so the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(B_R)$ is compact for all $\nu \in [1, p_s^*)$ by [9, Corollary 7.2]. Indeed, by (1.1) and the Hölder inequality,

$$\|\tilde{u}\|_{W^{s,p}(B_R)}^p \le C_R\|\tilde{u}\|_{p_s^*}^p + [\tilde{u}]_{s,p}^p \le (C_R/H_0 + 1)[\tilde{u}]_{s,p}^p$$

for all $\tilde{u} \in D^{s,p}(\mathbb{R}^N)$, where $C_R = (\omega_N/N)^{ps/N}R^{ps}$ and ω_N is the measure of the unit sphere

$$S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$$

of $\mathbb{R}^N$.

Fix $\varepsilon > 0$. There exists $R_\varepsilon > 0$ so large that $\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} \tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q\,dx < \varepsilon$ by (4.1). Take a subsequence $(\tilde{u}_{j_k})_k \subset (\tilde{u}_j)_j$. Since $\tilde{u}_{j_k} \to \tilde{u}$ in $L^\nu(B_{R_\varepsilon})$ for all $\nu \in [1, p_s^*)$, up to a further subsequence, still denoted by $(\tilde{u}_{j_k})_k$, we have that $\tilde{u}_{j_k} \to \tilde{u}$ a.e. in B_{R_ε}. Thus $\tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q \to 0$ a.e. in B_{R_ε}. Furthermore, for each measurable subset $E \subset B_{R_\varepsilon}$, by the Hölder inequality we have

$$\int_E \tilde{w}(x)|\tilde{u}_{j_k} - \tilde{u}|^q\,dx \le L\Big(\int_E \tilde{w}(x)^\wp\,dx\Big)^{1/\wp}.$$

Hence, $(\tilde{w}(x)|\tilde{u}_{j_k} - \tilde{u}|^q)_k$ is equi-integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$ since $\tilde{w} \in L^\wp(\mathbb{R}^N)$ by (w). Then the Vitali convergence theorem implies

$$\lim_{k\to\infty} \int_{B_{R_\varepsilon}} \tilde{w}(x)|\tilde{u}_{j_k} - \tilde{u}|^q\, dx = 0,$$

and so $\tilde{u}_j \to \tilde{u}$ in $L^q(B_{R_\varepsilon}, \tilde{w})$ since the sequence $(\tilde{u}_{j_k})_k$ is arbitrary.

Consequently, $\int_{B_{R_\varepsilon}} \tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q\, dx = o(1)$ as $j \to \infty$. In conclusion, as $j \to \infty$,

$$\|\tilde{u}_j - \tilde{u}\|^q_{q,\tilde{w}} = \int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} \tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q\, dx + \int_{B_{R_\varepsilon}} \tilde{w}(x)|\tilde{u}_j - \tilde{u}|^q\, dx \le \varepsilon + o(1),$$

that is, $\tilde{u}_j \to \tilde{u}$ in $L^q(\mathbb{R}^N, \tilde{w})$ as $j \to \infty$, with $\varepsilon > 0$ being arbitrary. In particular, $u_j \to u$ in $L^q(\Omega, w)$ as $j \to \infty$, and this completes the proof. □

We now turn back to problem (1.9). According to the variational nature, (weak) solutions of (1.9) correspond to critical points of the associated Euler–Lagrange functional $\mathfrak{J}_\sigma : Z(\Omega) \to \mathbb{R}$ defined by

$$\mathfrak{J}_\sigma(u) = \frac{1}{p}\mathscr{M}([u]^p_{s,p}) - \frac{1}{p^*_s(\alpha)}\|u\|^{p^*_s(\alpha)}_{H_\alpha} - \frac{\sigma}{q}\|u\|^q_{q,w}$$

for all $u \in Z(\Omega)$. Note that $\mathfrak{J}_\sigma$ is a $C^1(Z(\Omega))$ functional and for any $u, \varphi \in Z(\Omega)$,

$$\langle \mathfrak{J}'_\sigma(u), \varphi\rangle_{Z(\Omega),Z'(\Omega)} = M([u]^p_{s,p})\langle u, \varphi\rangle_{s,p} - \langle u, \varphi\rangle_{H_\alpha} - \sigma\langle u, \varphi\rangle_{q,w}, \tag{4.2}$$

where

$$\langle u, \varphi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)]}{|x - y|^{N+sp}}\, dx\, dy,$$

$$\langle u, \varphi\rangle_{H_\alpha} = \int_\Omega |u(x)|^{p^*_s(\alpha)-2}u(x)\varphi(x)\frac{dx}{|x|^\alpha},$$

$$\langle u, \varphi\rangle_{q,w} = \int_\Omega w(x)|u(x)|^{q-2}u(x)\varphi(x)\, dx.$$

In order to find the critical points of $\mathfrak{J}_\sigma$, we intend to apply the mountain pass theorem by checking that $\mathfrak{J}_\sigma$ possesses a suitable geometrical structure and that it satisfies the Palais–Smale compactness condition. In particular, to handle the Kirchhoff coefficient on a degenerate setting we need appropriate lower and upper bounds for M, given by (M_1) and (M_2), which first appear in [8].

Indeed, condition (M_2) implies that $M(t) > 0$ for any $t > 0$, and consequently by (M_1) for all $t \in (0, 1]$ we have $M(t)/\mathscr{M}(t) \le \theta/t$. Thus, integrating on $[t, 1]$, with $0 < t < 1$, we get

$$\mathscr{M}(t) \ge \mathscr{M}(1)t^\theta, \tag{4.3}$$

and (4.3) holds for all $t \in [0, 1]$ by continuity. Hence, (M_3) is a stronger request. Furthermore (4.3) is compatible with (M_3) since integrating (M_3), we have $\mathscr{M}(t) \ge ct^\theta/\theta$ for any $t \in [0, 1]$, from which $\mathscr{M}(1) \ge c/\theta$.

Similarly, for any $\varepsilon > 0$ there exists $r_\varepsilon = \mathscr{M}(\varepsilon)/\varepsilon^\theta > 0$ such that

$$\mathscr{M}(t) \le r_\varepsilon t^\theta \quad \text{for any } t \ge \varepsilon. \tag{4.4}$$

We point out that also when M satisfies (M_1) and $(\widetilde{\mathrm{M}_2})$, that is, we work on a non-degenerate setting, (4.3) and (4.4) immediately hold true. Finally, we recall that $p\theta < q < p^*_s(\alpha)$ and $0 \le \alpha < ps$.

Lemma 4.2. *For any $\sigma \in \mathbb{R}$ there exists a function $e \in Z(\Omega)$ with $[e]_{s,p} \ge 2$ and $\mathfrak{J}_\sigma(e) < 0$. Further, there exist $\rho \in (0, 1]$ and $\jmath > 0$ such that $\mathfrak{J}_\sigma(u) \ge \jmath$ for any $u \in Z(\Omega)$ with $[u]_{s,p} = \rho$.*

Proof. Fix $\sigma \in \mathbb{R}$. Now take $v \in Z(\Omega)$ such that $[v]_{s,p} = 1$. By (4.4), with $\varepsilon = 1$, we get, as $t \to \infty$,

$$\mathcal{J}_\sigma(tv) \leq \mathscr{M}(1)t^{p\theta} - \frac{\|v\|_{H_\alpha}^{p_s^*(\alpha)}}{p_s^*(\alpha)} t^{p_s^*(\alpha)} - \sigma \frac{\|v\|_{q,w}^q}{q} t^q \to -\infty \tag{4.5}$$

since $p\theta < q < p_s^*(\alpha)$. Hence, taking $e = t_* v$ with $t_* > 0$ large enough, we obtain that $[e]_{s,p} \geq 2$ and $\mathcal{J}_\sigma(e) < 0$.

Take any $u \in Z(\Omega)$ with $[u]_{s,p} \leq 1$. By (1.1), (w) and (4.3),

$$\begin{aligned}\mathcal{J}_\sigma(u) &\geq \frac{\mathscr{M}(1)}{p}[u]_{s,p}^{p\theta} - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{\sigma}{q}\|u\|_{q,w}^q \\ &\geq \frac{\mathscr{M}(1)}{p}[u]_{s,p}^{p\theta} - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)}[u]_{s,p}^{p_s^*(\alpha)} - \frac{C_w \sigma^+}{q}[u]_{s,p}^q.\end{aligned}$$

Thus, setting

$$\eta_\sigma(t) = \frac{\mathscr{M}(1)}{p} t^{p\theta} - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)} t^{p_s^*(\alpha)} - \frac{C_w \sigma^+}{q} t^q,$$

we find some $\rho \in (0, 1)$ so small that $\max_{t\in[0,1]} \eta_\sigma(t) = \eta_\sigma(\rho)$ since $p\theta < q < p_s^*(\alpha)$. Consequently,

$$\mathcal{J}_\sigma(u) \geq \jmath = \eta_\sigma(\rho) > 0$$

for any $u \in Z(\Omega)$ with $[u]_{s,p} = \rho$. □

We discuss now the compactness property for the functional $\mathcal{J}_\sigma$, given by the Palais–Smale condition at a suitable mountain pass level c_σ. For this, we fix $\sigma \in \mathbb{R}$ and set

$$c_\sigma = \inf_{\xi\in\Gamma} \max_{t\in[0,1]} \mathcal{J}_\sigma(\xi(t)),$$

where

$$\Gamma = \{\xi \in C([0, 1], Z(\Omega)) : \xi(0) = 0,\ \xi(1) = e\}.$$

Clearly, $c_\sigma > 0$ by Lemma 4.2. We recall that $(u_j)_j \subset Z(\Omega)$ is a *Palais–Smale sequence for $\mathcal{J}_\sigma$ at level $c_\sigma \in \mathbb{R}$* if

$$\mathcal{J}_\sigma(u_j) \to c_\sigma \quad \text{and} \quad \mathcal{J}_\sigma'(u_j) \to 0 \qquad \text{as } j \to \infty. \tag{4.6}$$

We say that $\mathcal{J}_\sigma$ *satisfies the Palais–Smale condition at level c_σ* if any Palais–Smale sequence $(u_j)_j$ at level c_σ admits a convergent subsequence in $Z(\Omega)$.

Before proving the relative compactness of the Palais–Smale sequences, we introduce an asymptotic property for the level c_σ. This result is similar to [17, Lemma 6] and will be crucial not only to get (1.11), but above all to overcome the lack of compactness due to the presence of a Hardy term, which reduces to the standard critical nonlinearity when $\alpha = 0$.

Lemma 4.3. *There holds*

$$\lim_{\sigma\to\infty} c_\sigma = 0.$$

Proof. Let $e \in Z(\Omega)$ be the function obtained by Lemma 4.2 and corresponding to $\sigma = 0$. Hence $\mathcal{J}_\sigma$ satisfies the mountain pass geometry at 0 and e for all $\sigma \geq 0$. Thus there exists $t_\sigma > 0$ verifying $\mathcal{J}_\sigma(t_\sigma e) = \max_{t\geq 0} \mathcal{J}_\sigma(te)$. Hence, $\langle \mathcal{J}_\sigma'(t_\sigma e), e\rangle_{Z(\Omega),Z'(\Omega)} = 0$ and by (4.2),

$$t_\sigma^{p-1}[e]_{s,p}^p M(t_\sigma^p [e]_{s,p}^p) = \sigma t_\sigma^{q-1}\|e\|_{q,w}^q + t_\sigma^{p_s^*(\alpha)-1}\|e\|_{H_\alpha}^{p_s^*(\alpha)} \geq t_\sigma^{p_s^*(\alpha)-1}\|e\|_{H_\alpha}^{p_s^*(\alpha)}. \tag{4.7}$$

We claim that $\{t_\sigma\}_{\sigma\geq 0}$ is bounded in $\mathbb{R}^+$. Indeed, putting $\Sigma = \{\sigma \geq 0 : t_\sigma [e]_{s,p} \geq 1\}$, we see that

$$t_\sigma^p [e]_{s,p}^p M(t_\sigma^p [e]_{s,p}^p) \leq \theta \mathscr{M}(t_\sigma^p [e]_{s,p}^p) \leq \theta \mathscr{M}(1) t_\sigma^{p\theta} [e]_{s,p}^{p\theta} \quad \text{for any } \sigma \in \Sigma \tag{4.8}$$

by (M_1) and (4.4). Hence, from (4.7) and (4.8) there follows

$$t_\sigma^{p_s^*(\alpha)-p\theta} \leq \frac{\theta \mathscr{M}(1)[e]_{s,p}^{p\theta}}{\|e\|_{H_\alpha}^{p_s^*(\alpha)}} \quad \text{for any } \sigma \in \Sigma,$$

which implies that $\{t_\sigma\}_{\sigma\in\Sigma}$ is bounded in $\mathbb{R}^+$, and so, in turn, $\{t_\sigma\}_{\sigma\geq 0}$ is also bounded, concluding the proof of the claim.

Now we assert that

$$\lim_{\sigma\to\infty} t_\sigma = 0. \tag{4.9}$$

Indeed, assume by contradiction that $\limsup_{\sigma\to\infty} t_\sigma = \tau > 0$. Hence there is a sequence, say $j \mapsto \sigma_j \uparrow \infty$ such that

$$\lim_{j\to\infty} t_{\sigma_j} = \tau.$$

Clearly, $(t_{\sigma_j})_j$ is bounded. Thus, the continuity of M and (4.7) give at once

$$\infty > \frac{[e]_{s,p}^p}{\|e\|_{q,w}^q} \limsup_{j\to\infty} M(t_{\sigma_j}^p [e]_{s,p}^p) \geq \lim_{j\to\infty} \sigma_j t_{\sigma_j}^{q-p} = \infty,$$

which is the required contradiction since $p \leq p\theta < q$. This proves the assertion.

Consider now the path $\xi(t) = te$, $t \in [0, 1]$, belonging to Γ. By Lemma 4.2,

$$0 < c_\sigma \leq \max_{t\in[0,1]} \mathcal{J}_\sigma(te) \leq \mathcal{J}_\sigma(t_\sigma e) \leq \frac{1}{p}\mathscr{M}(t_\sigma^p [e]_{s,p}^p),$$

where by continuity $\mathscr{M}(t_\sigma^p [e]_{s,p}^p) \to 0$ as $\sigma \to \infty$ by (4.9). □

Now, we are ready to show the validity of the Palais–Smale condition.

Lemma 4.4. *There exists $\sigma^* > 0$ such that for any $\sigma \geq \sigma^*$ the functional $\mathcal{J}_\sigma$ satisfies the Palais–Smale condition at level c_σ.*

Proof. Take $\sigma > 0$ and let $(u_j)_j \subset Z(\Omega)$ be a Palais–Smale sequence for $\mathcal{J}_\sigma$ at level c_σ. Since by $(\widetilde{\mathcal{M}})$ our Kirchhoff term M could be possibly degenerate, we split the proof in two steps.

Step 1: Let the Kirchhoff function M verify $M(0) = 0$, (M_1), (M_2), and (M_3). Due to the degenerate nature of (1.9), two situations must be considered: either $\inf_{j\in\mathbb{N}}[u_j]_{s,p} = d_\sigma > 0$ or $\inf_{j\in\mathbb{N}}[u_j]_{s,p} = 0$. Hence, we divide the proof of the current step into two cases.

Case $\inf_{j\in\mathbb{N}}[u_j]_{s,p} = d_\sigma > 0$: First we prove that $(u_j)_j$ is bounded in $Z(\Omega)$. By (M_2), with $\tau = d_\sigma^p$, there exists $m_\sigma > 0$ such that

$$M([u_j]_{s,p}^p) \geq m_\sigma \quad \text{for any } j \in \mathbb{N}. \tag{4.10}$$

Furthermore, from (M_1) it follows that

$$\begin{aligned}\mathcal{J}_\sigma(u_j) - \frac{1}{q}\langle \mathcal{J}_\sigma'(u_j), u_j\rangle_{Z(\Omega),Z'(\Omega)} &\geq \frac{1}{p}\mathscr{M}([u_j]_{s,p}^p) - \frac{1}{q}M([u_j]_{s,p}^p)[u_j]_{s,p}^p + \Big(\frac{1}{q} - \frac{1}{p_s^*(\alpha)}\Big)\|u_j\|_{H_\alpha}^{p_s^*(\alpha)} \\ &\geq \Big(\frac{1}{p\theta} - \frac{1}{q}\Big)M([u_j]_{s,p}^p)[u_j]_{s,p}^p + \Big(\frac{1}{q} - \frac{1}{p_s^*(\alpha)}\Big)\|u_j\|_{H_\alpha}^{p_s^*(\alpha)},\end{aligned} \tag{4.11}$$

with $p\theta < q < p_s^*(\alpha)$. Hence, by (4.6), (4.10) and (4.11) there exists a β_σ such that, as $j \to \infty$,

$$\begin{cases} c_\sigma + \beta_\sigma[u_j]_{s,p} + o(1) \geq \Big(\dfrac{1}{p\theta} - \dfrac{1}{q}\Big)M([u_j]_{s,p}^p)[u_j]_{s,p}^p \geq \mu_\sigma[u_j]_{s,p}^p, \\ \mu_\sigma = \Big(\dfrac{1}{p\theta} - \dfrac{1}{q}\Big)m_\sigma > 0. \end{cases} \tag{4.12}$$

Therefore, $(u_j)_j$ is bounded in $Z(\Omega)$.

Now we can prove the validity of the *Palais–Smale* condition. Since $(u_j)_j$ is bounded in $Z(\Omega)$, Lemma 4.1 and [4, Theorem 4.9] give the existence of $u_\sigma \in Z(\Omega)$ such that, up to a subsequence still relabeled $(u_j)_j$, it follows that

$$\begin{cases} u_j \rightharpoonup u_\sigma \text{ in } Z(\Omega), & [u_j]_{s,p} \to \kappa_\sigma, \\ u_j \rightharpoonup u_\sigma \text{ in } L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha}), & \|u_j - u_\sigma\|_{H_\alpha} \to \iota_\sigma, \\ u_j \to u_\sigma \text{ in } L^q(\Omega, w), & u_j \to u_\sigma \text{ a.e. in } \Omega, \end{cases} \tag{4.13}$$

since $p\theta < q < p_s^*(\alpha) \le p_s^*$. Clearly, $\kappa_\sigma > 0$ since we have $d_\sigma > 0$. Therefore, $M([u_j]_{s,p}^p) \to M(\kappa_\sigma^p) > 0$ as $j \to \infty$ by continuity and the fact that 0 is the unique zero of M by (M_2).

In particular, by (4.6) and (4.11) we also have

$$c_\sigma + o(1) \ge \mu_\sigma [u_j]_{s,p}^p + \Big(\frac{1}{q} - \frac{1}{p_s^*(\alpha)}\Big)\|u_j\|_{H_\alpha}^{p_s^*(\alpha)}. \tag{4.14}$$

First, we assert that

$$\lim_{\sigma\to\infty} \kappa_\sigma = 0. \tag{4.15}$$

Otherwise, $\limsup_{\sigma\to\infty} \kappa_\sigma = \kappa > 0$. Hence there is a sequence, say $n \to \sigma_n \uparrow \infty$, such that $\kappa_{\sigma_n} \to \kappa$ as $n \to \infty$. Thus, letting $n \to \infty$ in (4.12), we get from Lemma 4.3 that

$$0 \ge \Big(\frac{1}{p\theta} - \frac{1}{q}\Big) M(\kappa^p)\kappa^p > 0$$

by (M_2), which is the desired contradiction and proves the assertion (4.15).

Now, $[u_\sigma]_{s,p} \le \lim_{j\to\infty} [u_j]_{s,p} = \kappa_\sigma$ since $u_j \rightharpoonup u_\sigma$ in $Z(\Omega)$, so that (1.1) and (4.15) imply at once

$$\lim_{\sigma\to\infty} \|u_\sigma\|_{H_\alpha} = \lim_{\sigma\to\infty} [u_\sigma]_{s,p} = 0. \tag{4.16}$$

By (4.6) we have, as $j \to \infty$,

$$o(1) = M([u_j]_{s,p}^p)\langle u_j, \varphi\rangle_{s,p} - \langle u_j, \varphi\rangle_{H_\alpha} - \sigma\langle u_j, \varphi\rangle_{q,w}$$

for any $\varphi \in Z(\Omega)$. As shown in the proof of [7, Lemma 2.4], by (4.13) the sequence $(\mathcal{U}_j)_j$, defined in $\mathbb{R}^{2N} \setminus \operatorname{Diag}\mathbb{R}^{2N}$ by

$$(x, y) \mapsto \mathcal{U}_j(x, y) = \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))}{|x - y|^{(N+ps)/p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$, as well as $\mathcal{U}_j \to \mathcal{U}_\sigma$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}_\sigma(x, y) = \frac{|u_\sigma(x) - u_\sigma(y)|^{p-2}(u_\sigma(x) - u_\sigma(y))}{|x - y|^{(N+ps)/p'}}.$$

Thus, up to a subsequence, we get $\mathcal{U}_j \rightharpoonup \mathcal{U}_\sigma$ in $L^{p'}(\mathbb{R}^{2N})$, and so $\langle u_j, \varphi\rangle_{s,p} \to \langle u_\sigma, \varphi\rangle_{s,p}$ since

$$|\varphi(x) - \varphi(y)| \cdot |x - y|^{-(N+ps)/p} \in L^p(\mathbb{R}^{2N}).$$

Then, using (4.13) and the facts that $|u_j|^{q-2}u_j \rightharpoonup |u_\sigma|^{q-2}u_\sigma$ in $L^{q'}(\Omega, w)$ and $|u_j|^{p_s^*(\alpha)-2}u_j \rightharpoonup |u_\sigma|^{p_s^*(\alpha)-2}u_\sigma$ in $L^{p_s^*(\alpha)'}(\Omega, |x|^{-\alpha})$ by [2, Proposition A.8], we obtain

$$M(\sigma_\sigma^p)\langle u_\sigma, \varphi\rangle_{s,p} - \langle u_\sigma, \varphi\rangle_{H_\alpha} = \sigma\langle u_\sigma, \varphi\rangle_{q,w}$$

for all $\varphi \in Z(\Omega)$.

Hence, u_σ is a critical point of the $C^1(Z(\Omega))$ functional

$$\mathcal{J}_{\kappa_\sigma}(u) = \frac{1}{p}M(\kappa_\sigma^p)[u]_{s,p}^p - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{\sigma}{q}\|u\|_{q,w}^q. \tag{4.17}$$

In particular, (4.6) and (4.13) imply that, as $j \to \infty$,

$$\begin{aligned} o(1) &= \langle \mathcal{J}_\sigma'(u_j) - \mathcal{J}_{\kappa_\sigma}'(u_\sigma), u_j - u_\sigma\rangle_{Z(\Omega),Z'(\Omega)} \\ &= M([u_j]_{s,p}^p)[u_j]_{s,p}^p + M(\kappa_\sigma^p)[u_\sigma]_{s,p}^p - \langle u_j, u_\sigma\rangle_{s,p}[M([u_j]_{s,p}^p) + M(\kappa_\sigma^p)] \\ &\quad - \int_\Omega (|u_j|^{p_s^*(\alpha)-2}u_j - |u_\sigma|^{p_s^*(\alpha)-2}u_\sigma)(u_j - u_\sigma)\frac{dx}{|x|^\alpha} - \sigma\int_\Omega w(x)(|u_j|^{q-2}u_j - |u_\sigma|^{q-2}u_\sigma)(u_j - u_\sigma)\,dx \\ &= M(\kappa_\sigma^p)(\kappa_\sigma^p - [u_\sigma]_{s,p}^p) - \|u_j\|_{H_\alpha}^{p_s^*(\alpha)} + \|u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} + o(1) \\ &= M(\kappa_\sigma^p)[u_j - u_\sigma]_{s,p}^p - \|u_j - u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} + o(1). \end{aligned}$$

Indeed, by (4.13),

$$\lim_{j\to\infty}\int_\Omega w(x)(|u_j|^{q-2}u_j - |u_\sigma|^{q-2}u_\sigma)(u_j - u_\sigma)\,dx = 0.$$

Moreover, again by (4.13) and the celebrated Brezis–Lieb lemma, see [5], as $j \to \infty$,

$$[u_j]_{s,p}^p = [u_j - u_\sigma]_{s,p}^p + [u_\sigma]_{s,p}^p + o(1), \quad \|u_j\|_{H_\alpha}^{p_s^*(\alpha)} = \|u_j - u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} + \|u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} + o(1).$$

Finally, we have used the fact that $[u_j]_{s,p} \to \kappa_\sigma$ by (4.13). Therefore, we have proved the crucial formula

$$M(\kappa_\sigma^p)\lim_{j\to\infty}[u_j - u_\sigma]_{s,p}^p = M(\kappa_\sigma^p)(\kappa_\sigma^p - [u_\sigma]_{s,p}^p) = \lim_{j\to\infty}\|u_j - u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} = \iota_\sigma^{p_s^*(\alpha)}. \tag{4.18}$$

By (1.1) and the notation in (4.13), for all $\sigma > 0$ we have

$$\iota_\sigma^{p_s^*(\alpha)} \geq H_\alpha M(\kappa_\sigma^p)\iota_\sigma^p. \tag{4.19}$$

We assert that there exists a $\sigma^* > 0$ such that $\iota_\sigma = 0$ for all $\sigma \geq \sigma^*$. Otherwise, there exists a sequence $n \mapsto \sigma_n \uparrow \infty$ such that $\iota_{\sigma_n} = \iota_n > 0$. Noting that (4.18) implies in particular that

$$M(\kappa_\sigma^p)(\kappa_\sigma^p - [u_\sigma]_{s,p}^p) = \iota_\sigma^{p_s^*(\alpha)},$$

we get along this sequence, using (4.19) and denoting $\kappa_{\sigma_n} = \kappa_n$, $u_{\sigma_n} = u_n$, that

$$\iota_n^{p_s^*(\alpha)-p} = (\iota_n^{p_s^*(\alpha)})^{ps/(N-\alpha)} = M(\kappa_n^p)^{ps/(N-\alpha)}(\kappa_n^p - [u_n]_{s,p}^p)^{ps/(N-\alpha)} \geq H_\alpha M(\kappa_n^p).$$

Hence, we obtain for all n sufficiently large by (M_3) and (4.15),

$$\kappa_n^{p\cdot\frac{ps}{N-\alpha}} \geq (\kappa_n^p - [u_n]_{s,p}^p)^{\frac{ps}{N-\alpha}} \geq H_\alpha M(\kappa_n^p)^{1-ps/(N-\alpha)} \geq C\kappa_n^{p(\theta-1)[1-ps/(N-\alpha)]},$$

where $C = H_\alpha c^{1-ps/(N-\alpha)} > 0$. Therefore, with $\kappa_n > 0$ for all n, it follows that for all n sufficiently large

$$\kappa_n^{p[ps-(\theta-1)(N-\alpha)+(\theta-1)ps]/(N-\alpha)} = \kappa_n^{p[\theta ps-(\theta-1)(N-\alpha)]/(N-\alpha)} \geq C,$$

which is impossible by (4.15) since

$$ps < N < \theta' ps + \alpha.$$

Indeed, $M(0) = 0$ implies that $\theta > 1$ by [7, Lemma 3.1]. The restriction

$$\frac{N-\alpha}{p\theta'} < s$$

follows directly from the fact that $1 < \theta < p_s^*(\alpha)/p = (N-\alpha)/(N-ps)$, so that

$$\theta' > \Big(\frac{N-\alpha}{N-ps}\Big)' = \frac{N-\alpha}{ps-\alpha}.$$

Therefore,

$$\frac{N-\alpha}{p\theta'} < \frac{ps-\alpha}{p} \leq s,$$

with $\alpha \geq 0$. In conclusion, the assertion is proved.

Hence, for all $\sigma \geq \sigma^*$,

$$\lim_{j\to\infty}\|u_j - u_\sigma\|_{H_\alpha}^{p_s^*(\alpha)} = 0.$$

Thus, (4.18) yields $u_j \to u_\sigma$ in $Z(\Omega)$ as $j \to \infty$ for all $\sigma \geq \sigma^*$ since $M(\kappa_\sigma^p) > 0$ by (M_2) and the fact that $d_\sigma > 0$. This completes the proof of the first case.

Case $\inf_{j\in\mathbb{N}}[u_j]_{s,p} = 0$: Here, either 0 is an accumulation point for the real sequence $([u_j]_{s,p})_j$ and so there is a subsequence of $(u_j)_j$ strongly converging to $u = 0$, or 0 is an isolated point of $([u_j]_{s,p})_j$. The first case can not occur since it implies that the trivial solution is a critical point at level c_σ. This is impossible since $0 = \mathcal{J}_\sigma(0) = c_\sigma > 0$. Hence only the latter case can occur, so that there is a subsequence, denoted by $([u_{j_k}]_{s,p})_k$, such that $\inf_{k\in\mathbb{N}}[u_{j_k}]_{s,p} = d_\sigma > 0$ and we can proceed as before. This completes the proof of the second case and of this step.

Step 2: Let the Kirchhoff function M satisfy (M_1) *and* $(\widetilde{\mathrm{M}_2})$. In this case, the proof of Step 1 simplifies, but we repeat the main argument where necessary. Hence, (M_1), $(\widetilde{\mathrm{M}_2})$ and (4.6) yield now that, as $j \to \infty$,

$$\begin{aligned} c_\sigma + \beta_\sigma[u_j]_{s,p} + o(1) &\geq \Big(\frac{1}{p\theta} - \frac{1}{q}\Big) M([u_j]_{s,p}^p)[u_j]_{s,p}^p + \Big(\frac{1}{q} - \frac{1}{p_s^*(\alpha)}\Big)[u_j]_{H_\alpha}^{p_s^*(\alpha)} \\ &\geq \Big(\frac{1}{p\theta} - \frac{1}{q}\Big) a[u_j]_{s,p}^p, \quad \text{with } \Big(\frac{1}{p\theta} - \frac{1}{q}\Big) a > 0. \end{aligned} \tag{4.20}$$

Therefore, $(u_j)_j$ is bounded in $Z(\Omega)$, and proceeding exactly as in the proof of Step 1, we get the main formulas (4.13)–(4.19).

As above, we assert that there exists a $\sigma^* > 0$ such that $\imath_\sigma = 0$ for all $\sigma \geq \sigma^*$. Otherwise, there exists a sequence $n \mapsto \sigma_n \uparrow \infty$ such that $\imath_{\sigma_n} = \imath_n > 0$. By (4.18) and (4.19), denoting $\kappa_{\sigma_n} = \kappa_n$ and $u_{\sigma_n} = u_n$, we still get

$$\imath_n^{p_s^*(\alpha)-p} = (\imath_n^{p_s^*(\alpha)})^{ps/(N-\alpha)} = M(\kappa_n^p)^{ps/(N-\alpha)}(\kappa_n^p - [u_n]_{s,p}^p)^{ps/(N-\alpha)} \geq H_\alpha M(\kappa_n^p).$$

Hence, by $(\widetilde{\mathrm{M}_2})$ and (4.15),

$$\kappa_n^{p \cdot ps/(N-\alpha)} \geq (\kappa_n^p - [u_n]_{s,p}^p)^{ps/(N-\alpha)} \geq H_\alpha M(\kappa_n^p)^{1-ps/(N-\alpha)} \geq C,$$

where $C = H_\alpha a^{1-ps/(N-\alpha)} > 0$. This fact immediately contradicts (4.15).

From this point we can conclude exactly as in Step 1. □

Proof of Theorem 1.2. Lemmas 4.2 and 4.4 guarantee that for any $\sigma \geq \sigma^*$ the functional $\mathcal{J}_\sigma$ satisfies all assumptions of the mountain pass theorem. Hence, for any $\sigma \geq \sigma^*$ there exists a critical point $u_\sigma \in Z(\Omega)$ for $\mathcal{J}_\sigma$ at level c_σ. Since $\mathcal{J}_\sigma(u_\sigma) = c_\sigma > 0 = \mathcal{J}_\sigma(0)$, we have that $u_\sigma \not\equiv 0$. Moreover, the asymptotic behavior (1.11) holds thanks to (4.16). □

We now turn to the setting stated in Theorem 1.3. Since (M_1) is no longer in charge, in order to control the growth of the elliptic part of (1.9), we use a truncation argument, as in [1] and in other previous works.

Proof of Theorem 1.3. Take $m \in \mathbb{R}$ with $0 < a \leq M(0) < m < aq/p$, which is possible since $pM(0) < aq$ by assumption. Put for all $t \in \mathbb{R}_0^+$,

$$M_m(t) = \begin{cases} M(t) & \text{if } M(t) \leq m, \\ m & \text{if } M(t) > m, \end{cases}$$

so that

$$M_m(0) = M(0), \quad \min_{t \in \mathbb{R}_0^+} M_m(t) = a,$$

and denote by $\mathscr{M}_m$ its primitive. Let us consider the auxiliary problem

$$\begin{cases} M_m([u]_{s,p}^p)(-\Delta)_p^s u - \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \sigma w(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.21}$$

We are going to solve (4.21), using a mountain pass argument as done in Step 2 of the proof of Theorem 1.2, but replacing the Kirchhoff function M with M_m.

Clearly, (4.21) can be thought as the Euler–Lagrange equation of the C^1 functional

$$\mathcal{J}_{m,\sigma}(u) = \frac{1}{p}\mathscr{M}_m([u]_{s,p}^p) - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{\sigma}{q}\|u\|_{q,w}^q$$

for all $u \in Z(\Omega)$. First let us observe that for the functional $\mathcal{J}_{m,\sigma}$ Lemmas 4.2 and 4.3 continue to hold. Indeed, for Lemma 4.2 it is enough to observe that (4.5) is now replaced by

$$\mathcal{J}_{m,\sigma}(tv) \leq mt^p - \frac{\|v\|_{H_\alpha}^{p_s^*(\alpha)}}{p_s^*(\alpha)} t^{p_s^*(\alpha)} - \sigma \frac{\|v\|_{q,w}^q}{q} t^q \to -\infty,$$

as $t \to \infty$, since $p < q < p_s^*(\alpha)$. Similarly, also Lemma 4.3 can be proved in a simpler way, by observing that now, since $t_\sigma > 0$ for all $\sigma > 0$, inequality (4.7) becomes

$$mt_\sigma^p[e]_{s,p}^p \geq t_\sigma^p[e]_{s,p}^p M_m(t_\sigma^p[e]_{s,p}^p) \geq t_\sigma^{p_s^*(\alpha)}\|e\|_{H_\alpha}^{p_s^*(\alpha)}$$

for any $\sigma \in \mathbb{R}^+$. This implies at once that $\{t_\sigma\}_{\sigma\in\mathbb{R}^+}$ is bounded in $\mathbb{R}$. The rest of the proof is unchanged. Hence Lemmas 4.2 and 4.3 are valid for $\mathcal{J}_{m,\sigma}$, and it remains to prove for $\mathcal{J}_{m,\sigma}$ the main Lemma 4.4.

Proceeding as in Step 2 of the proof of Theorem 1.2, by $(\widetilde{\mathrm{M}_2})$ now (4.20) becomes

$$c_\sigma + \beta_\sigma[u_j]_{s,p} + o(1) \geq \Big(\frac{a}{p} - \frac{m}{q}\Big)[u_j]_{s,p}^p + \Big(\frac{1}{q} - \frac{1}{p_s^*(\alpha)}\Big)\|u_j\|_{H_\alpha}^{p_s^*(\alpha)}, \quad \text{with } \frac{a}{p} - \frac{m}{q} > 0, \tag{4.22}$$

since $m < aq/p$. The other key formulas hold true with no relevant modifications. Thus, arguing as before, we find that for all $m \in (M(0), aq/p)$ there exists a suitable $\sigma_0 = \sigma_0(m) > 0$ such that problem (4.21) admits a nontrivial solution $u_\sigma \in Z(\Omega)$ with $\mathcal{J}_{m,\sigma}(u_\sigma) = c_\sigma$. Hence, (4.22) implies that for all $\sigma \geq \sigma_0$,

$$c_\sigma \geq \Big(\frac{a}{p} - \frac{m}{q}\Big)[u_\sigma]_{s,p}^p, \quad \text{with } \frac{a}{p} - \frac{m}{q} > 0,$$

so that (1.11) follows at once by Lemma 4.3.

Fix $m \in (M(0), aq/p)$. By (1.11),

$$a \leq M(0) = M_m(0) = \lim_{\substack{\sigma\to\infty\\ \sigma\geq\sigma_0}} M_m([u_\sigma]_{s,p}^p).$$

Therefore, there exists $\sigma^* = \sigma^*(m) \geq \sigma_0$ such that

$$a \leq M_m([u_\sigma]_{s,p}^p) < m \quad \text{for all } \sigma \geq \sigma^*.$$

In conclusion, for all $m \in (M(0), aq/p)$ there exists a threshold $\sigma^* = \sigma^*(m) > 0$ such that for all $\sigma \geq \sigma^*$ the mountain pass solution u_σ of (4.21) is also a solution of problem (1.9). □

We conclude the section with the proof of Theorem 1.4, and recall that for (1.13) the Kirchhoff function M is of the type (1.8), but possibly $M(0) = 0$, that is, $a = 0$. Hence in this part of the section we assume, without further mentioning, that $a, b \in \mathbb{R}_0^+$ with $a + b > 0$, that w satisfies (w) with $\theta > 1$, $0 \leq \alpha < ps < N$, $0 \leq \beta < ps$, $p\theta \leq p_s^*(\alpha) \leq p_s^*$, $1 < q < p_s^*(\beta) \leq p_s^*$, and $p\theta < p_s^*(\beta)$. Let us finally recall that $c_{a,b}$ is introduced in (1.14). Problem (1.13) is the Euler–Lagrange equation of the C^1 functional $\mathcal{I}_{\gamma,\sigma,g}$ defined by

$$\mathcal{I}_{\gamma,\sigma,g}(u) = \frac{1}{p}\Big(a[u]_{s,p}^p + b[u]_{s,p}^{p\theta} - \frac{\gamma}{\theta}\|u\|_{H_\alpha}^{p\theta}\Big) - \frac{\sigma}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*(\beta)}\|u\|_{H_\beta}^{p_s^*(\beta)} - \int_\Omega g(x)u(x)\,dx$$

for any $u \in Z(\Omega)$.

Lemma 4.5. *Fix $\gamma < c_{a,b}H_\alpha^\theta$. Then every function of the parametric family $\{\eta_{\gamma,\varepsilon}\}_{\varepsilon\geq 0}$, defined for all $t \in [0, 1]$ by*

$$\eta_{\gamma,\varepsilon}(t) = \frac{1}{p}\Big[at^{p-1} + \Big(b - \frac{\gamma^+}{\theta H_\alpha^\theta}\Big)t^{p\theta-1}\Big] - \frac{1}{H_\beta^{p_s^*(\beta)/p}\,p_s^*(\beta)}t^{p_s^*(\beta)-1} - \frac{C_w}{q}\varepsilon t^{q-1}, \tag{4.23}$$

with

$$\varepsilon = \begin{cases} 0 & \text{if either } 1 < q < p, \text{ or } p \leq q < p\theta \text{ and } a = 0,\\ \sigma^+ & \text{if either } p\theta \leq q < p_s^*(\beta), \text{ or } p \leq q < p\theta \text{ and } a > 0,\end{cases}$$

admits maximum value $\eta_{\gamma,\varepsilon}(\rho) > 0$ at a point $\rho \in (0, 1)$ for all $\varepsilon \geq 0$ if either $1 < q < p$, or $p \leq q < p\theta$ and $a = 0$, or $p < q \leq p\theta$ and $a > 0$, or $p\theta < q < p_s^*(\beta)$, and for all $\varepsilon \in [0, \varepsilon_*)$, with $\varepsilon_* > 0$ given by

$$\varepsilon_* = \begin{cases} a/C_w & \text{if } q = p \text{ and } a > 0,\\ b - \gamma^+/\theta H_\alpha^\theta & \text{if } q = p\theta \text{ and } a = 0,\end{cases}$$

whenever either $q = p$ and $a > 0$, or $q = p\theta$ and $a = 0$. Furthermore, putting $\delta = H_0^{1/p}\eta_{\gamma,\varepsilon}(\rho)/3$ and

$$\sigma_* = \begin{cases} q\eta_{\gamma,0}(\rho)/3C_w & \text{if either } 1 < q < p, \text{ or } p \le q < p\theta \text{ and } a = 0, \\ \infty & \text{if either } p < q \le p\theta \text{ and } a > 0, \text{ or } p\theta < q < p_s^*(\beta), \\ \varepsilon_* & \text{if either } q = p \text{ and } a > 0, \text{ or } q = p\theta \text{ and } a = 0, \end{cases}$$

we have $\mathcal{I}_{\gamma,\sigma,g}(u) \ge \jmath = \rho\eta_{\gamma,\varepsilon}(\rho)/3 > 0$ for all $u \in Z(\Omega)$ with $[u]_{s,p} = \rho$, and for all $g \in L^{\upsilon}(\Omega)$ and σ with $\|g\|_{\upsilon} \le \delta$ and

$$\sigma \in \begin{cases} (-\infty, \sigma_*] & \text{if either } 1 < q < p, \text{ or } p \le q < p\theta \text{ and } a = 0, \\ (-\infty, \sigma_*) & \text{if either } p \le q < p\theta \text{ and } a > 0, \text{ or } p\theta \le q < p_s^*(\beta). \end{cases} \tag{4.24}$$

Proof. Fix $\gamma \in (-\infty, c_{a,b}H_\alpha^\theta)$ and $\sigma \in \mathbb{R}$. By (1.1) and Lemma 4.1,

$$\begin{aligned} \mathcal{I}_{\gamma,\sigma,g}(u) &\ge \frac{a}{p}[u]_{s,p}^p + \frac{b}{p}[u]_{s,p}^{p\theta} - \frac{\gamma}{p\theta}\|u\|_{H_\alpha}^{p\theta} - \frac{\sigma}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*(\beta)}\|u\|_{H_\beta}^{p_s^*(\beta)} - \|g\|_{\upsilon}\|u\|_{p_s^*} \\ &\ge \frac{a}{p}[u]_{s,p}^p + \Big(\frac{b}{p} - \frac{\gamma^+}{p\theta H_\alpha^\theta}\Big)[u]_{s,p}^{p\theta} - \frac{1}{H_\beta^{p_s^*(\beta)/p}p_s^*(\beta)}[u]_{s,p}^{p_s^*(\beta)} - \frac{C_w\sigma^+}{q}[u]_{s,p}^q - \frac{1}{H_0^{1/p}}\|g\|_{\upsilon}[u]_{s,p} \\ &= [u]_{s,p}\eta_{\gamma,\varepsilon}([u]_{s,p}) - \frac{C_w(\sigma^+ - \varepsilon)}{q}[u]_{s,p}^q - \frac{\|g\|_{\upsilon}}{H_0^{1/p}}[u]_{s,p} \\ &\ge [u]_{s,p}\Big(\eta_{\gamma,\varepsilon}([u]_{s,p}) - \frac{C_w(\sigma^+ - \varepsilon)}{q} - \frac{\|g\|_{\upsilon}}{H_0^{1/p}}\Big) \end{aligned} \tag{4.25}$$

for all $u \in Z(\Omega)$ with $[u]_{s,p} \le 1$ since $q > 1$. It remains to show that $\mathcal{I}_{\gamma,\sigma,g}(u) \ge \jmath$ for any $u \in Z(\Omega)$ with $[u]_{s,p} = \rho$, where $\rho \in (0, 1)$ is the maximum point of $\eta_{\gamma,\varepsilon}$ in $[0, 1]$. To this end, take δ and σ_* as in the statement, so that for any $u \in Z(\Omega)$ with $[u]_{s,p} = \rho$, and for any $g \in L^{\upsilon}(\Omega)$ with $\|g\|_{\upsilon} \le \delta$, and for any σ as in (4.24), we have

$$\begin{aligned} \mathcal{I}_{\gamma,\sigma,g}(u) &\ge \rho \cdot \begin{cases} \Big(\eta_{\gamma,0}(\rho) - \frac{C_w\sigma_*}{q} - \frac{\delta}{H_0^{1/p}}\Big) & \text{if either } 1 < q < p, \text{ or } p \le q < p\theta \text{ and } a = 0, \\ \Big(\eta_{\gamma,\sigma^+}(\rho) - \frac{\delta}{H_0^{1/p}}\Big) & \text{if either } p\theta \le q < p_s^*(\beta), \text{ or } p \le q < p\theta \text{ and } a > 0, \end{cases} \\ &\ge \frac{\rho\eta_{\gamma,\varepsilon}(\rho)}{3} = \jmath, \end{aligned}$$

as stated. □

Proof of Theorem 1.4. Fix $\gamma \in [0, c_{a,b}H_\alpha^\theta)$. Take g and σ as in (1.15) with upper bounds δ and σ_* given in Lemma 4.5.

When $\|g\|_{\upsilon} \ne 0$, since $g \in L^{\upsilon}(\Omega)$, there exists $\psi \in C_0^\infty(\Omega)$ such that $\int_\Omega g(x)\psi(x)\,dx > 0$. Indeed, there exists a sequence $(g_j)_j$ in $C_0^\infty(\Omega)$ such that $g_j \to g$ strongly in $L^{p_s^*}(\Omega)$ since $C_0^\infty(\Omega)$ is dense in $L^{p_s^*}(\Omega)$. Hence, there exists $j_0 \in \mathbb{N}$ so large that

$$\|g_{j_0} - g\|_{p_s^*} \le \frac{1}{2}\|g\|_{\upsilon}^{\upsilon-1}.$$

Thus, by the Hölder inequality, we have

$$\int_\Omega g_{j_0}(x)g(x)\,dx \ge -\|g_{j_0} - g\|_{p_s^*}\|g\|_{\upsilon} + \|g\|_{\upsilon}^{\upsilon} > 0$$

since $\upsilon = (p_s^*)'$. Taking $\psi = g_{j_0}$, we obtain the claim.

Hence, for $t \in (0, 1)$ small enough,

$$\begin{aligned} \mathcal{I}_{\gamma,\sigma,g}(t\psi) &\le \frac{t^p}{p}a[\psi]_{s,p}^p + \frac{t^{p\theta}}{p}b[\psi]_{s,p}^{p\theta} - \frac{t^{p\theta}}{p\theta}\gamma\|\psi\|_{H_\alpha}^{p\theta} - \frac{t^q}{q}\sigma\|\psi\|_{q,w}^q - \frac{t^{p_s^*(\beta)}}{p_s^*(\beta)}\|\psi\|_{H_\beta}^{p_s^*(\beta)} - t\int_\Omega g(x)\psi(x)\,dx \\ &< 0 \end{aligned} \tag{4.26}$$

since $1 < p < p\theta < p_s^*(\beta)$ and $1 < q$.

It remains to consider the case $\|g\|_\nu = 0$ and $\sigma > 0$, when either $1 < q < p$, or $p \le q < p\theta$ and $a = 0$. Hence, for a fixed $v \in Z(\Omega)$ with $\|v\|_{q,w}^q = 1$, for $t \in (0,1)$ small enough we still have

$$\mathcal{J}_{\gamma,\sigma,g}(tv) \le \frac{t^p}{p} a[v]_{s,p}^p + \frac{t^{p\theta}}{p} b[v]_{s,p}^{p\theta} - \frac{t^{p\theta}}{p\theta}\gamma\|v\|_{H_\alpha}^{p\theta} - \frac{t^q}{q}\sigma - \frac{t^{p_s^*(\beta)}}{p_s^*(\beta)}\|v\|_{H_\beta}^{p_s^*(\beta)}\,dx < 0 \tag{4.27}$$

since $\sigma > 0$ and either $1 < q < p$, or $p \le q < p\theta$ and $a = 0$.

Thus, using the notation of Lemma 4.5, by (4.26) and (4.27),

$$c_0 = \inf\{\mathcal{J}_{\gamma,\sigma,g}(u) : u \in \overline{B}_\rho\} < 0,$$

and $\mathcal{J}_{\gamma,\sigma,g}$ is bounded below in $\overline{B}_\rho$ thanks to (4.25) with $\rho \in (0,1)$. By the *Ekeland* variational principle in [11] and by Lemma 4.5, there exists a sequence $(u_j)_j \subset B_\rho$ such that

$$c_0 \le \mathcal{J}_{\gamma,\sigma,g}(u_j) \le c_0 + \frac{1}{j} \quad \text{and} \quad \mathcal{J}_{\gamma,\sigma,g}(v) \ge \mathcal{J}_{\gamma,\sigma,g}(u_j) - \frac{1}{j}[v-u_j]_{s,p} \tag{4.28}$$

for all $v \in \overline{B}_{\rho_0}$. For a fixed $j \in \mathbb{N}$, for all $z \in \partial B_1$ and for all $\varepsilon > 0$ so small that $u_j + \varepsilon z \in \overline{B}_\rho$, we have

$$\mathcal{J}_{\gamma,\sigma,g}(u_j + \varepsilon z) - \mathcal{J}_{\gamma,\sigma,g}(u_j) \ge -\frac{\varepsilon}{j}$$

by (4.28). Since $\mathcal{J}_{\gamma,\sigma,g}$ is Gâteaux differentiable in $Z(\Omega)$, we have

$$\langle \mathcal{J}'_{\gamma,\sigma,g}(u_j), z\rangle_{Z(\Omega),Z'(\Omega)} = \lim_{\varepsilon\to 0}\frac{\mathcal{J}_{\gamma,\sigma,g}(u_j+\varepsilon z) - \mathcal{J}_{\gamma,\sigma,g}(u_j)}{\varepsilon} \ge -\frac{1}{j}$$

for all $z \in \partial B_1$. Hence $|\langle \mathcal{J}'_{\gamma,\sigma,g}(u_j), z\rangle_{Z(\Omega),Z'(\Omega)}| \le 1/j$ since $z \in \partial B_1$ is arbitrary. Consequently, $\mathcal{J}'_{\gamma,\sigma,g}(u_j) \to 0$ in $Z'(\Omega)$ as $j \to \infty$.

Furthermore, since $(u_j)_j$ is bounded in B_ρ, Lemma 4.1 and [4, Theorem 4.9] give the existence of $u_{\gamma,\sigma,g} \in \overline{B}_\rho$ such that, up to a subsequence still relabeled $(u_j)_j$, it follows that

$$\begin{cases} u_j \rightharpoonup u_{\gamma,\sigma,g} \text{ in } Z(\Omega), & [u_j]_{s,p} \to d_{\gamma,\sigma,g}, \\ u_j \rightharpoonup u_{\gamma,\sigma,g} \text{ in } L^{p_s^*(\alpha)}(\mathbb{R}^N, |x|^{-\alpha}), & u_j \rightharpoonup u_{\gamma,\sigma,g} \text{ in } L^{p_s^*(\beta)}(\mathbb{R}^N, |x|^{-\beta}) \\ \|u_j\|_{H_\alpha} \to \ell_{\gamma,\sigma,g}, & u_j \to u_{\gamma,\sigma,g} \text{ in } L^q(\Omega, w), \quad u_j \to u_{\gamma,\sigma,g} \text{ a.e. in } \Omega, \end{cases} \tag{4.29}$$

since $q < p_s^*(\beta) \le p_s^*$. Hence, as $j \to \infty$, we easily get

$$\begin{aligned} 0 &= \langle \mathcal{J}'_{\gamma,\sigma,g}(u_j), u_{\gamma,\sigma,g}\rangle_{Z(\Omega),Z'(\Omega)} + o(1) = (a + b\theta[u_j]_{s,p}^{p(\theta-1)})\langle u_j, u_{\gamma,\sigma,g}\rangle_{s,p} - \gamma\|u_j\|_{H_\alpha}^{p\theta - p_s^*(\alpha)}\langle u_j, u_{\gamma,\sigma,g}\rangle_{H_\alpha} \\ &\quad - \sigma\langle u_j, u_{\gamma,\sigma,g}\rangle_{q,w} - \langle u_j, u_{\gamma,\sigma,g}\rangle_{H_\beta} - \int_\Omega g(x)u_j(x)\,dx \\ &= (a + b\theta d_{\gamma,\sigma,g}^{p(\theta-1)})[u_{\gamma,\sigma,g}]_{s,p}^p - \gamma\ell_{\gamma,\sigma,g}^{p\theta-p_s^*(\alpha)}\|u_{\gamma,\sigma,g}\|_{H_\alpha}^{p_s^*(\alpha)} - \sigma\|u_{\gamma,\sigma,g}\|_{q,w}^q \\ &\quad - \|u_{\gamma,\sigma,g}\|_{H_\beta}^{p_s^*(\beta)} - \int_\Omega g(x)u_{\gamma,\sigma,g}(x)\,dx. \end{aligned} \tag{4.30}$$

Since $u_{\gamma,\sigma,g} \in \overline{B}_{\rho_0}$, we have $\mathcal{J}_{\gamma,\sigma,g}(u_{\gamma,\sigma,g}) \ge c_0$. Multiplying the expression in (4.30) by $1/p\theta$ and subtracting below, by (4.29), the weakly lower semi-continuity of the norms and the facts that $\gamma \ge 0$ and $p\theta \le p_s^*(\alpha) \le p_s^*$, as $j \to \infty$ we have

$$\begin{aligned} c_0 \le \mathcal{J}_{\gamma,\sigma,g}(u_{\gamma,\sigma,g}) &\le \frac{a + b d_{\gamma,\sigma,g}^{p(\theta-1)}}{p}[u_{\gamma,\sigma,g}]_{s,p}^p - \frac{\gamma\ell_{\gamma,\sigma,g}^{p\theta-p_s^*(\alpha)}}{p\theta}\|u_{\gamma,\sigma,g}\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{\sigma}{q}\|u_{\gamma,\sigma,g}\|_{q,w}^q \\ &\quad - \frac{1}{p_s^*(\beta)}\|u_{\gamma,\sigma,g}\|_{H_\beta}^{p_s^*(\beta)} - \int_\Omega g(x)u_{\gamma,\sigma,g}(x)\,dx \\ &= \frac{a}{p}\Big(1 - \frac{1}{\theta}\Big)[u_{\gamma,\sigma,g}]_{s,p}^p - \sigma\Big(\frac{1}{q} - \frac{1}{p\theta}\Big)\|u_{\gamma,\sigma,g}\|_{q,w}^q - \Big(\frac{1}{p_s^*(\beta)} - \frac{1}{p\theta}\Big)\|u_{\gamma,\sigma,g}\|_{H_\beta}^{p_s^*(\beta)} \\ &\quad - \Big(1 - \frac{1}{p\theta}\Big)\int_\Omega g(x)u_{\gamma,\sigma,g}(x)\,dx \end{aligned}$$

$$\leq \frac{a}{p}\Big(1-\frac{1}{\theta}\Big)[u_j]_{s,p}^p - \sigma\Big(\frac{1}{q}-\frac{1}{p\theta}\Big)\|u_j\|_{q,w}^q + \Big(\frac{1}{p\theta}-\frac{1}{p_s^*(\beta)}\Big)\|u_j\|_{H_\beta}^{p_s^*(\beta)} - \Big(1-\frac{1}{p\theta}\Big)\int_\Omega g(x)u_j(x)\,dx$$

$$= \mathcal{J}_{\gamma,\sigma,g}(u_j) - \frac{1}{p\theta}\langle \mathcal{J}'_{\gamma,\sigma,g}(u_j), u_j\rangle_{Z(\Omega),Z'(\Omega)} + o(1) = c_0$$

since $p\theta < p_s^*(\beta)$. Thus, $u_{\gamma,\sigma,g}$ is a minimizer of $\mathcal{J}_{\gamma,\sigma,g}$ in $\overline{B}_{\rho_0}$ and $\mathcal{J}_{\gamma,\sigma,g}(u_{\gamma,\sigma,g}) = c_0 < 0 < \jmath \leq \mathcal{J}_{\gamma,\sigma,g}(u)$ for all $u \in \partial\overline{B}_\rho$ by Lemma 4.5. Hence $u_{\gamma,\sigma,g} \in B_\rho$, so that $\mathcal{J}'_{\gamma,\sigma,g}(u_{\gamma,\sigma,g}) = 0$. In other words, $u_{\gamma,\sigma,g}$ is a nontrivial solution of (1.13).

It remains to show the asymptotic behavior (1.16). From the proof of Lemma 4.5 it is clear that

$$0 < [u_{\gamma,\sigma,g}]_{s,p} < \rho = \rho(\gamma,\sigma),$$

where by (4.23), when either $p < q \leq p\theta$ and $a > 0$, or $p\theta < q < p_s^*(\beta)$, the positive function $\rho(\gamma,\sigma)$ verifies the identity

$$\frac{a}{p'} + \frac{p\theta-1}{p}\Big(b - \frac{\gamma}{\theta H_\alpha^\theta}\Big)\rho(\gamma,\sigma)^{p\theta-p} = \sigma^+\frac{C_w}{q'}\rho(\gamma,\sigma)^{q-p} + \frac{1}{H^{p_s^*(\beta)/p}(p_s^*(\beta))'}\rho(\gamma,\sigma)^{p_s^*(\beta)-p}.$$

This implies at once that

$$\lim_{\sigma\to\infty}\rho(\gamma,\sigma) = 0$$

since either $p < p\theta < q < p_s^*(\beta)$, or $p < q \leq p\theta < p_s^*(\beta)$ and $a > 0$. The proof of (1.16) is now completed. □

5 General Nonlocal Operators

In this section, we show that Theorems 1.2–1.4 continue to hold when $(-\Delta)_p^s$ in (1.9) and (1.13) is replaced by a more general nonlocal integro-differential operator $\mathcal{L}_K^p$, defined for any $x \in \mathbb{R}^N$ as

$$\mathcal{L}_K^p\varphi(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)\,dy$$

along any function $\varphi \in C_0^\infty(\mathbb{R}^N)$, where the singular kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function satisfying the following conditions:

(K_1) $mK \in L^1(\mathbb{R}^N)$ with $m(x) = \min\{1, |x|^p\}$.

(K_2) There exists $K_0 > 0$ such that $K(x) \geq K_0|x|^{-(N+ps)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

Obviously, the operator $-\mathcal{L}_K^p$ reduces to the fractional p-Laplacian $(-\Delta)_p^s$ when $K(x) = |x|^{-N-ps}$.

Here we denote by $Z_K(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to

$$[\varphi]_{s,p,K} = \Big(\int_{\mathbb{R}^N}|D_K^s\varphi(x)|^p\,dx\Big)^{1/p}, \quad \text{where } |D_K^s\varphi(x)|^p = \int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^pK(x-y)\,dy,$$

which is well defined by (K_1) along all $\varphi \in C_0^\infty(\Omega)$. Clearly, the embedding $Z_K(\Omega) \hookrightarrow Z(\Omega)$ is continuous since

$$[u]_{s,p} \leq K_0^{-1/p}[u]_{s,p,K} \quad \text{for any } u \in Z_K(\Omega) \tag{5.1}$$

by (K_2). Hence, also by Lemma 4.1 the embedding $Z_K(\Omega) \hookrightarrow L^q(\Omega, w)$ is compact under condition (w) since $1 < q < p_s^*$.

A weak solution of the problem

$$\begin{cases} -M([u]_{s,p,K}^p)\mathcal{L}_K^p u - \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \sigma w(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.2}$$

is a function $u \in Z_K(\Omega)$ such that

$$M([u]_{s,p,K}^p)\langle u,\varphi\rangle_{s,p} - \langle u,\varphi\rangle_{H_\alpha} = \sigma\langle u,\varphi\rangle_{q,w} \quad \text{for any } \varphi \in Z_K(\Omega),$$
$$\langle u,\varphi\rangle_{s,p,K} = \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p-2}[u(x)-u(y)]\cdot[\varphi(x)-\varphi(y)]K(x-y)\,dx\,dy.$$

It is worth noting that, as in [1], it is not restrictive to assume K even, since the odd part of K does not give contribution in the integral above. Indeed, write $K = K_e + K_o$, where for all $x \in \mathbb{R}^N \setminus \{0\}$,

$$K_e(x) = \frac{K(x)+K(-x)}{2} \quad \text{and} \quad K_o(x) = \frac{K(x)-K(-x)}{2}.$$

Then it is apparent that for all u and $\varphi \in Z_K(\Omega)$,

$$\langle u,\varphi\rangle_{s,p,K} = \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p-2}[u(x)-u(y)]\cdot[\varphi(x)-\varphi(y)]K_e(x-y)\,dx\,dy.$$

Actually, the solutions of problem (5.2) correspond to critical points of the functional $\mathcal{J}_{\sigma,K} : Z_K(\Omega) \to \mathbb{R}$, defined for all $u \in Z_K(\Omega)$ by

$$\mathcal{J}_{\sigma,K}(u) = \frac{1}{p}\mathscr{M}([u]_{s,p,K}^p) - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{\sigma}{q}\|u\|_{q,w}^q.$$

Now, by using (5.1), Lemmas 4.2–4.4 continue to hold, with obvious changes in their proofs. Thus we have proved the following two results.

Theorem 5.1. *Let K verify* (K_1) *and* (K_2)*. Assume that M and w satisfy* ($\widetilde{\mathcal{M}}$) *and* (w)*, with $p\theta < q < p_s^*(\alpha) \le p_s^*$ and $0 \le \alpha < ps < N$. Then there exists $\sigma^* > 0$ such that for any $\sigma \ge \sigma^*$ problem* (5.2) *admits a nontrivial mountain pass solution u_σ in $Z_K(\Omega)$. Moreover,*

$$\lim_{\sigma\to\infty}[u_\sigma]_{s,p,K} = 0. \tag{5.3}$$

Theorem 5.2. *Let K verify* (K_1) *and* (K_2)*. Assume that M is continuous in $\mathbb{R}_0^+$, satisfying* ($\widetilde{\mathrm{M}_2}$)*. Suppose that w verifies* (w)*, with $p < q < p_s^*(\alpha) \le p_s^*$ and $0 \le \alpha < ps < N$, and that* (1.12) *holds. Then there exists $\sigma^* > 0$ such that for any $\sigma \ge \sigma^*$ problem* (5.2) *admits a nontrivial mountain pass solution u_σ in $Z_K(\Omega)$, satisfying the asymptotic property* (5.3).

We can generalize also the study of problem (1.13), that is,

$$\begin{cases} -(a+b\theta[u]_{s,p,K}^{p(\theta-1)})\mathcal{L}_K^p u - \gamma\|u\|_{H_\alpha}^{p\theta-p_s^*(\alpha)}\dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \sigma w(x)|u|^{q-2}u + \dfrac{|u|^{p_s^*(\beta)-2}u}{|x|^\beta} + g(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.4}$$

In this case, by (5.1), Lemma 4.5 continues to hold, provided that $\gamma \in [0, c_{a,b}H_\alpha^\theta K_0^\theta)$ and σ satisfies (4.24) with a suitable new σ_*. Thus we have proved the following result.

Theorem 5.3. *Let K verify* (K_1) *and* (K_2) *and let $a, b \ge 0$ with $a+b>0$. Assume that w satisfies* (w)*, with $\theta > 1$, $0 \le \alpha < ps < N$, $0 \le \beta < ps$, $p\theta \le p_s^*(\alpha)$, $1 < q < p_s^*(\beta)$, and $p\theta < p_s^*(\beta)$. Then for all $\gamma \in [0, c_{a,b}H_\alpha^\theta K_0^\theta)$, with $c_{a,b}$ given in* (1.14)*, there exist a number $\delta > 0$ and $\sigma_* \in (0,\infty]$ such that for any perturbation $g \in L^\upsilon(\Omega)$ and any parameter σ satisfying* (1.15)*, problem* (5.4) *admits a nontrivial solution $u_{\gamma,\sigma,g}$ in $Z_K(\Omega)$ and*

$$\lim_{\sigma\to\infty}[u_{\gamma,\sigma,g}]_{s,p,K} = 0$$

when either $p\theta < q < p_s^*(\beta)$, or $p < q \le p\theta$ and $a > 0$.

Also Theorem 1.1 and all results of Section 3 derivable from it can be easily proved for general operators $\mathcal{L}_K^p$ provided that the assumption (K_2) is strengthened and replaced by condition

$(\widetilde{\mathrm{K}_2})$ there exists $K_0 > 0$ such that $K_0|x|^{-(N+ps)} \le K(x) \le |x|^{-(N+ps)}/K_0$ for any x in $\mathbb{R}^N \setminus \{0\}$.

Let us state the results.

Theorem 5.4. *Assume that K verifies (K_1) and $(\widetilde{\mathrm{K}_2})$, that Ω is an open bounded subset of $\mathbb{R}^N$ and that $\alpha \in (0, ps]$. Let $(u_j)_j$ be a weakly convergent sequence in $Z_K(\Omega)$ with weak limit u. Then there exist two finite positive measures μ and ν in $\mathbb{R}^N$ such that*

$$|D_K^s u_j(x)|^p\,dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_j(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha} \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

Furthermore, there exist two nonnegative numbers μ_0, ν_0 such that

$$\nu = |u(x)|^{p_s^*(\alpha)}\frac{dx}{|x|^\alpha} + \nu_0\delta_0$$

and

$$\mu \ge |D_K^s u(x)|^p\,dx + \mu_0\delta_0, \quad 0 \le H_\alpha K_0 \nu_0^{p/p_s^*(\alpha)} \le \mu_0,$$

where H_α is the Hardy constant defined in (1.1).

The proof is almost exactly as that of Theorem 1.1. The new assumption $(\widetilde{\mathrm{K}_2})$ is used only to derive (2.3), (2.6), (2.10), and so their consequences. It is worth noting that (2.7) comes directly from (K_1). In conclusion, we have the following results.

Theorem 5.5 (Superlinear f). *Assume that K verifies (K_1) and $(\widetilde{\mathrm{K}_2})$, that Ω is an open bounded subset of $\mathbb{R}^N$ and that M satisfies $(\mathcal{M})$, with $c > 0$ and α given as in* (1.7). *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions* (f_1)–(f_3) *of Theorem* 3.1. *Then for all $\gamma \in [0, c\theta H_\alpha^\theta K_0^\theta)$ and $\lambda \in (-\infty, m_{\gamma,\theta,K_0}\lambda_1 K_0)$, where m_{γ,θ,K_0} is given as*

$$m_{\gamma,\theta,K_0} = \begin{cases} \infty & \text{if } \theta > 1, \\ c - \gamma^+/H_\alpha K_0 & \text{if } \theta = 1, \end{cases} \tag{5.5}$$

there exists a positive constant $\overline{\sigma} = \overline{\sigma}(\lambda, \gamma)$ such that for any $\sigma \in (0, \overline{\sigma})$ the problem

$$\begin{cases} -M([u]_{s,p,K}^p)\mathcal{L}_K^p u - \gamma\|u\|_{H_\alpha}^{p\theta - p_s^*(\alpha)}\dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \lambda|u|^{p-2}u + \sigma f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$K_{\gamma,\lambda,\sigma}$}$$

has a nontrivial solution $u_{\gamma,\lambda,\sigma} \in Z_K(\Omega)$.

Moreover, if $\gamma \in [0, cH_\alpha^\theta K_0^\theta)$ and either $\lambda \in \mathbb{R}_0^-$ when $\theta > 1$, or $\lambda \in (-\infty, m_{\gamma,\theta,K_0}\lambda_1 K_0)$ when $\theta = 1$, then

$$\lim_{\sigma \to 0^+}[u_{\gamma,\lambda,\sigma}]_{s,p,K} = 0. \tag{5.6}$$

Clearly, when $\theta > 1$, that is, $m_{\gamma,\theta,K_0} = \infty$, then the existence part of Theorem 5.5 holds for all $\lambda \in \mathbb{R}$.

Theorem 5.6 (Sublinear f). *Assume that K verifies (K_1) and $(\widetilde{\mathrm{K}_2})$, that Ω is an open bounded subset of $\mathbb{R}^N$ and that M satisfies $(\mathcal{M})$, with $c > 0$ and α given as in* (1.7). *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions* (f_4) *and* (f_5) *of Theorem* 3.2. *For all $\gamma \in [0, c\theta H_\alpha^\theta K_0^\theta)$, $\lambda \in (-\infty, m_{\gamma,\theta,K_0}\lambda_1 K_0)$, where m_{γ,θ,K_0} is given in* (5.5), *and $\sigma > 0$, problem $(K_{\gamma,\lambda,\sigma})$ has a nontrivial solution $u_{\gamma,\lambda,\sigma} \in Z_K(\Omega)$.*

Moreover, if $\gamma \in [0, cH_\alpha^\theta K_0^\theta)$ and either $\lambda \in \mathbb{R}_0^-$ when $\theta > 1$, or $\lambda \in (-\infty, m_{\gamma,\theta,K_0}\lambda_1 K_0)$ when $\theta = 1$, then (5.6) *holds.*

A Lemma A.1 and its proof

This last section is devoted to the proof of Lemma A.1. This technical lemma plays a crucial role in the study of concentration and compactness results since it allows us to handle the nonlocal nature of the operator

$u \mapsto |D^s u|^p$. The proof of Lemma A.1 is fairly similar to that of [17, Proposition 7], stated in the case $p = 2$. For the sake of completeness, we give it here.

Lemma A.1. *Let $w \in \mathbb{R}^N$ and $u \in L^{p_s^*}(\mathbb{R}^N)$. Let $\varepsilon > 0$ and let either $U \times V = B_\varepsilon(w) \times \mathbb{R}^N$ or $U \times V = \mathbb{R}^N \times B_\varepsilon(w)$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{-p} \iint\limits_{U \times (V \cap \{|x-y| \le \varepsilon\})} |u(x)|^p |x-y|^{p-N-ps} \, dx \, dy = 0 \tag{A.1}$$

and

$$\lim_{\varepsilon \to 0} \iint\limits_{U \times (V \cap \{|x-y| > \varepsilon\})} |u(x)|^p |x-y|^{-N-ps} \, dx \, dy = 0. \tag{A.2}$$

Proof. Let $w \in \mathbb{R}^N$, $u \in L^{p_s^*}(\mathbb{R}^N)$ and $\varepsilon > 0$ be fixed. Set

$$\xi_\varepsilon = \Big(\int\limits_{B_\varepsilon(w)} |u(x)|^{p_s^*} \, dx \Big)^{p/p_s^*}.$$

Clearly,

$$\lim_{\varepsilon \to 0} \xi_\varepsilon = 0. \tag{A.3}$$

By the Hölder inequality,

$$\int\limits_{B_\varepsilon(w)} |u(x)|^p \, dx \le \Big(\int\limits_{B_\varepsilon(w)} |u(x)|^{p_s^*} \, dx \Big)^{p/p_s^*} \Big(\int\limits_{B_\varepsilon(w)} 1 \, dx \Big)^{ps/N} \le C \xi_\varepsilon \varepsilon^{ps} \tag{A.4}$$

for some $C > 0$ independent of ε (in what follows, we will possibly change C from line to line). We claim that

$$(U \times V) \cap \{|x-y| \le \varepsilon\} \subseteq B_{2\varepsilon}(w) \times B_{2\varepsilon}(w). \tag{A.5}$$

Indeed, if $(x, y) \in U \times V = B_\varepsilon(w) \times \mathbb{R}^N$, with $|x-y| \le \varepsilon$, we have

$$|w-y| \le |w-x| + |x-y| \le \varepsilon + \varepsilon,$$

and so the validity of (A.5). On the other hand, if $(x, y) \in U \times V = \mathbb{R}^N \times B_\varepsilon(w)$, with $|x-y| \le \varepsilon$, then

$$|w-x| \le |w-y| + |y-x| \le \varepsilon + \varepsilon.$$

This completes the proof of (A.5).

By (A.5) and the change of variables $z = x - y$, we have

$$\begin{aligned} \iint\limits_{U \times (V \cap \{|x-y| \le \varepsilon\})} |u(x)|^p |x-y|^{p-N-ps} \, dx \, dy &\le \int\limits_{B_{2\varepsilon}(w)} |u(x)|^p \, dx \int\limits_{B_{2\varepsilon}(w) \cap \{|x-y| \le \varepsilon\}} |x-y|^{p-N-ps} \, dy \\ &\le \int\limits_{B_{2\varepsilon}(w)} |u(x)|^p \, dx \int\limits_{B_\varepsilon} |z|^{p-N-ps} \, dz \\ &\le C \varepsilon^{p-ps} \int\limits_{B_{2\varepsilon}(w)} |u(x)|^p \, dx. \end{aligned}$$

Using this and (A.4), we obtain

$$\varepsilon^{-p} \iint\limits_{U \times (V \cap \{|x-y| \le \varepsilon\})} |u(x)|^p |x-y|^{p-N-ps} \, dx \, dy \le C \varepsilon^{-ps} \int\limits_{B_{2\varepsilon}(w)} |u(x)|^p \, dx \le C \xi_\varepsilon.$$

This and (A.3) imply (A.1).

Let us now prove (A.2). For this aim, fix an auxiliary parameter $K > 2$, which will be taken arbitrarily large at the end after sending $\varepsilon \to 0$. We claim that

$$U \times V \subseteq (B_{K\varepsilon}(w) \times \mathbb{R}^N) \cup ((\mathbb{R}^N \setminus B_{K\varepsilon}(w)) \times B_\varepsilon(w)). \tag{A.6}$$

Indeed, if $U \times V = B_\varepsilon(w) \times \mathbb{R}^N$, then of course $U \times V \subseteq B_{K\varepsilon}(w) \times \mathbb{R}^N$, hence (A.6) is obvious. If instead $(x, y) \in U \times V = \mathbb{R}^N \times B_\varepsilon(w)$, we distinguish two cases: if $x \in B_{K\varepsilon}(w)$, then

$$(x, y) \in B_{K\varepsilon}(w) \times \mathbb{R}^N;$$

if $x \in \mathbb{R}^N \setminus B_{K\varepsilon}(w)$, then

$$(x, y) \in (\mathbb{R}^N \setminus B_{K\varepsilon}(w)) \times V = (\mathbb{R}^N \setminus B_{K\varepsilon}(w)) \times B_\varepsilon(w).$$

This completes the proof of (A.6).

By (A.4),

$$\begin{aligned}\iint\limits_{B_{K\varepsilon}(w)\times(\mathbb{R}^N\cap\{|x-y|>\varepsilon\})} |u(x)|^p |x-y|^{-N-ps}\,dx\,dy &= \int\limits_{B_{K\varepsilon}(w)} |u(x)|^p\,dx \int\limits_{\mathbb{R}^N\setminus B_\varepsilon} |z|^{-N-ps}\,dz \\ &= C\varepsilon^{-ps} \int\limits_{B_{K\varepsilon}(w)} |u(x)|^p\,dx \\ &\le C\xi_{K\varepsilon}.\end{aligned} \tag{A.7}$$

If $x \in \mathbb{R}^N \setminus B_{K\varepsilon}(w)$ and $y \in B_\varepsilon(w)$, then

$$|x-y| \ge |x-w| - |y-w| = \frac{|x-w|}{2} + \frac{|x-w|}{2} - |y-w| \ge \frac{|x-w|}{2} + \frac{K\varepsilon}{2} - \varepsilon \ge \frac{|x-w|}{2}.$$

Hence, the Hölder inequality gives

$$\begin{aligned}&\iint\limits_{(\mathbb{R}^N\setminus B_{K\varepsilon}(w))\times B_\varepsilon(w)} |u(x)|^p |x-y|^{-N-ps}\,dx\,dy \\ &\quad\le 2^{N+ps} \int\limits_{\mathbb{R}^N\setminus B_{K\varepsilon}(w)} |u(x)|^p |x-w|^{-N-ps}\,dx \int\limits_{B_\varepsilon(w)} dy \\ &\quad\le C\varepsilon^N \Big(\int\limits_{\mathbb{R}^N\setminus B_{K\varepsilon}(w)} |u(x)|^{p_s^*}\,dx\Big)^{p/p_s^*} \Big(\int\limits_{\mathbb{R}^N\setminus B_{K\varepsilon}(w)} |x-w|^{-(N+ps)N/ps}\,dx\Big)^{ps/N} \\ &\quad\le C\varepsilon^N \|u\|_{p_s^*}^p \Big(\int_{K\varepsilon}^{\infty} r^{-((N+ps)N/ps)+(N-1)}\,dr\Big)^{ps/N} \\ &\quad= CK^{-N}\|u\|_{p_s^*}^p.\end{aligned} \tag{A.8}$$

Combining (A.6), (A.7) and (A.8), we obtain

$$\begin{aligned}&\iint\limits_{U\times(V\cap\{|x-y|>\varepsilon\})} |u(x)|^p |x-y|^{-N-ps}\,dx\,dy \\ &\le \int\limits_{B_{K\varepsilon}(w)} |u(x)|^p\,dx \int\limits_{\mathbb{R}^N\cap\{|x-y|>\varepsilon\}} |x-y|^{-N-ps}\,dy + \int\limits_{\mathbb{R}^N\setminus B_{K\varepsilon}(w)} |u(x)|^p\,dx \int\limits_{B_\varepsilon(w)} |x-y|^{-N-ps}\,dy \\ &\le C\xi_{K\varepsilon} + CK^{-N}\|u\|_{p_s^*}^p.\end{aligned}$$

Sending first $\varepsilon \to 0$ and then $K \to \infty$, we readily obtain (A.2), thanks to (A.3). □

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