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**Research Article**

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# Local and Global Existence of Strong Solutions to Large Cross Diffusion Systems

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**Abstract:** We study the solvability of a general class of cross diffusion systems and establish the local and global existence of their strong solutions under the weakest assumption that they are VMO. This work simplifies the setting in our previous work [15] and provides new extensions which are more verifiable in applications.

**Keywords:** Cross Diffusion Systems, Hölder Regularity, Global Existence

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## 1 Introduction

In this paper, for $T_0 > 0$ and a bounded domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, $n \geq 2$, we consider the following general parabolic system of $m$ equations ($m \geq 2$):

$$\begin{cases} u_t = \operatorname{div}(A(u)Du) + \hat{f}(u, Du), & (x,t) \in Q = \Omega \times (0, T_0), \\ u(x, 0) = U_0(x), & x \in \Omega, \\ u = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0), \end{cases} \tag{1.1}$$

where $A(u)$ is an $m \times m$ matrix in $u$, and $u\colon \Omega \to \mathbb{R}^m, \hat{f}\colon \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}^m$ are vector-valued functions. The initial data $U_0$ is given in $W^{1,p_0}(\Omega, \mathbb{R}^m)$ for some $p_0 > n$, where $n$ is the dimension of $\Omega$. As usual, $W^{1,p}(\Omega, \mathbb{R}^m)$, $p \geq 1$, will denote the standard Sobolev spaces whose elements are vector-valued functions $u\colon \Omega \to \mathbb{R}^m$ with finite norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.$$

We say that $u$ is a strong solution if $u$ is continuous on $\bar{Q}$ with $Du \in L^\infty_{\mathrm{loc}}(Q)$ and $D^2u \in L^2_{\mathrm{loc}}(Q)$.

The strongly coupled system (1.1) appears in many physical applications, for instance, in Maxwell–Stephan systems describing the diffusive transport of multicomponent mixtures, in models of reaction and diffusion in electrolysis, in flows in porous media, in diffusion of polymers and in population dynamics [7, 21, 23], among others. We refer the reader to the recent work [9] and the references therein for the models and the existence of their *weak* solutions.

The first fundamental problem in the study of (1.1) is the local and global existence of its solutions. One can decide to work with either weak or strong solutions. In the first case, the existence of a weak solution can be achieved via Galerkin, time discretization (see [9]) or variational methods [6], but its regularity (e.g., boundedness, Hölder continuity of the solution and its higher derivatives) is still an open issue. Several works have been done along this line to improve the early work [5] and establish *partial regularity* of *bounded* weak solutions to (1.1).

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On the other hand, if strong solutions are considered, then their existence can be established via semigroup theories as in the works of Amann [2, 3]. Using the interpolation theories of Sobolev's spaces, Amann established local and global existence of a strong solution $u$ of (1.1) under the assumption that one can control $\|u\|_{W^{1,p}(\Omega,\mathbb{R}^m)}$ for some $p > n$.

In both aforementioned approaches, the assumption on the boundedness of $u$ must be the starting point. For strongly coupled systems like (1.1), as invariant/maximum principles for cross diffusion systems are generally unavailable, the boundedness of the solutions is already a hard problem. One usually needs to use ad hoc techniques on a case by case basis to show that $u$ is bounded (see [10, 20]). Even for bounded weak solutions, we know that they are only Hölder continuous almost everywhere (see [5]). In addition, there exist counter examples for systems ($m > 1$) which exhibit solutions that start smoothly and remain bounded but develop singularities in higher norms in finite times (see [8]).

In our recent works [12–15], we chose a different approach, making use of fixed point theory and discuss the solvability of (1.1) under the weakest assumption that $u$ is VMO (see (1.3) below) and much more general structural conditions, compared to [2, 3], on the data of (1.1). The proof in [15] relies on fixed point theories, instead of the semigroup approach in [3], and weighted Gagliardo–Nirenberg inequalities involving BMO norms.

In particular, we assumed in [15] the following conditions:

(A) $A(u)$ is $C^1$ in $u$ and there exist constants $\lambda_0, C_* > 0$ and a scalar $C^1$ function $\lambda(u)$ such that for all $u \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^{mn}$, we have

$$\lambda(u) \ge \lambda_0, \quad \lambda(u)|\zeta|^2 \le \langle A(u)\zeta, \zeta\rangle \quad \text{and} \quad |A(u)| \le C_*\lambda(u). \tag{1.2}$$

In addition, $|A_u| \le C|\lambda_u|$ and the following number is finite:

$$\Lambda = \sup_{u\in\mathbb{R}^m} \frac{|\lambda_u(u)|}{\lambda(u)}.$$

With a slight abuse of notation, $A(u)\zeta$, $\langle A(u)\zeta, \zeta\rangle$ in (1.2) should be understood in the following way: For $A(u) = [a_{ij}(u)]$, $\zeta \in \mathbb{R}^{mn}$, we write $\zeta = [\zeta_i]_{i=1}^m$ with $\zeta_i = (\zeta_{i,1}, \dots, \zeta_{i,n})$ and

$$A(u)\zeta = \left[\sum_{j=1}^m a_{ij}\zeta_j\right]_{i=1}^m, \quad \langle A(u)\zeta, \zeta\rangle = \sum_{i,j=1}^m a_{ij}\langle \zeta_i, \zeta_j\rangle.$$

Also, here and throughout this paper, if $B$ is a $C^1$ function in $u \in \mathbb{R}^m$ then we abbreviate its derivative $\frac{\partial B}{\partial u}$ by $B_u$.

(F) There exist a constant $C$ and a differentiable function $f\colon \mathbb{R}^m \to \mathbb{R}^m$ such that for any differentiable vector-valued functions $u\colon \mathbb{R}^n \to \mathbb{R}^m$ and $p\colon \mathbb{R}^n \to \mathbb{R}^{mn}$, we have

$$\begin{aligned}
|\hat{f}(u,p)| &\le C\lambda^{1/2}(u)|p| + f(u),\\
|D\hat{f}(u,p)| &\le C\lambda^{1/2}(u)|Dp| + C\frac{|\lambda_u(u)|}{\lambda^{1/2}(u)}|Du||p| + |f_u(u)||Du|,\\
|f_u(u)| &\le C\lambda(u).
\end{aligned}$$

The local existence of a strong solution of (1.1) was proved in [15] under the key assumption that any strong solution $u$ of the system satisfies the following condition: For any given $\mu_0 > 0$, there exists $R_{\mu_0} > 0$ such that

$$\Lambda^2 \sup_{x_0\in\bar{\Omega},\, t\in(0,T_0)} \|u(\cdot\,,t)\|^2_{\mathrm{BMO}(B_{R_{\mu_0}}(x_0)\cap\Omega)} \le \mu_0. \tag{1.3}$$

This condition was referred to as condition (M') in [15].

Here and throughout this paper, $B_R(y)$ denotes a ball centered at $y$ with radius $R$, and a locally integrable function $U\colon \Omega \to \mathbb{R}^m$ is said to be in $\mathrm{BMO}(\Omega)$ if the following quantity is finite:

$$[U]_* := \sup_{B_R(y)\subset\Omega} \fint_{B_R(y)} |U - U_{B_R(y)}|\,dx.$$

We denote by $U_A$ the average of $U$ over a measurable set $A$: $U_A = \frac{1}{|A|}\int_A U(x)\,dx$.

The Banach space $\mathrm{BMO}(\Omega, \mathbb{R}^m)$ consists of functions with finite norm

$$\|U\|_{\mathrm{BMO}(\Omega,\mathbb{R}^m)} := [U]_* + \|U\|_{L^1(\Omega,\mathbb{R}^m)}.$$

We also say that $U$ is VMO in $\Omega$ if $\inf_{R>0,B_R\subset\Omega}\|U\|_{\mathrm{BMO}(B_R,\mathbb{R}^m)} = 0$.

In this paper, for simplicity of presentation and with models in applications in mind, we consider only the following special form of the reaction terms which are linear in $Du$, namely, $\hat{f}(u, Du) = B(u)Du + f(u)$, and study local and global existence of strong solutions. Thanks to this form of $\hat{f}$ the fixed point argument in [15] can be greatly simplified. Furthermore, we will provide conditions which are a bit stronger than (1.3) but verifiable in applications. In particular, we will show that a strong solution $u$ exists globally if the norm $\|u\|_{W^{1,n}(\Omega)}$ does not blow up in finite time. This relaxes Amann's conditions in [3] which required a control on $\|u\|_{W^{1,p}(\Omega)}$ for some $p > n$. Again, we are not assuming that $u$ is bounded and our structural conditions (A) and (F) are more general than those in [3, 16]. Our results also hold for general $\hat{f}(u, Du)$ with linear or quadratic growth in $Du$, see Remark 2.4.

We organize our paper as follows. In Section 2 we state our main results. In Section 3 we state another version of the local weighted Gagliardo–Nirenberg inequality [15, Lemma 2.4], which is one of the main ingredients of the proof in [15] and of our main theorem in this paper. Technical results and auxiliary lemmas needed for the proof of the main results for linear reaction terms are given in Section 4. We conclude the paper with an appendix presenting a full and simpler proof for the global and local weighted Gagliardo–Nirenberg inequalities in [15].

## 2 Preliminaries and Main Results

In this section we state the main results of this paper. Our first main result concerns the local existence of strong solutions to (1.1) with $\hat{f}$ being linear in $Du$, i.e.,

$$\hat{f}(u, Du) = B(u)Du + f(u). \tag{2.1}$$

We imbed (1.1) in the following family of systems:

$$\begin{cases} u_t = \operatorname{div}(A(\sigma u)Du) + \hat{f}(\sigma u, \sigma Du), & (x, t) \in Q = \Omega \times (0, T_0), \sigma \in [0, 1], \\ u(x, 0) = U_0(x), & x \in \Omega, \\ u = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \tag{2.2}$$

In [15] we assumed the spectral gap condition, which requires that the eigenvalues of the matrix $A(u)$ are not too far apart. Namely, we need that $\frac{n-2}{n} < C_*^{-1}$, where $C_*$ is, in certain sense, the ratio of the largest and smallest eigenvalues of $A(u)$. One should note that there exist counterexamples for global existence of strong solutions if the eigenvalues of the matrix $A(u)$ are far apart [1, 17]. We then again assume that

$$n^* = \frac{2C_*}{C_* - 1} > n. \tag{2.3}$$

Our first main result is the following.

**Theorem 2.1.** *Assume that* (A), (F) *and* (2.3) *hold. Assume also that the following conditions hold for* any strong *solution u of* (2.2) *with* (2.1)*:*

(M1) *There exists $\mu_0 > 0$ sufficiently small in terms of the constants in* (A) *and* (F) *such that for some $R_{\mu_0} > 0$, which may depends on $T_0$, we have*

$$\Lambda^2 \sup_{x_0\in\bar{\Omega},t\in(0,T_0)} \|u(\cdot\,, t)\|^2_{\mathrm{BMO}(B_R(x_0)\cap\Omega)} \le \mu_0.$$

(M2) *The following quantity is finite:*

$$C_{T_0} := \iint_{\Omega\times(0,T_0)} |Du|^2 \, dx.$$

(L) *There exist constants* $L(T_0)$ *and* $r^* > \frac{n}{p^*-n}$, *with* $p^* = \min\{n^*, p_0\}$, *assuming* $n^*$ *satisfies* (2.3) *and* $p_0 > n$, *such that*

$$\sup_{t\in(0,T_0)} \|\lambda(u(\cdot,t))\|_{L^{r^*}(\Omega)} \le L(T_0).$$

*Then* (1.1) *has a unique strong solution* $u$ *on* $\Omega \times (0, T_0)$. *Moreover, if the above assumptions hold for all* $T_0 > 0$, *then* (1.1) *has a unique strong solution* $u$ *which exists globally on* $\Omega \times (0, \infty)$.

The next results are more applicable and improve those of Amann in [2, 3]. Basically, we need only to control the $W^{1,n}(\Omega)$ norm of strong solutions while [2, 3] required that their $W^{1,p}(\Omega)$ norms do not blow up in finite time for some $p > n$, and thus the boundedness of the solutions is needed in his results.

**Corollary 2.2.** *The conclusion of Theorem 2.1 holds if* (M1) *and* (M2) *are replaced by the following assumption:*
(D) *There exists a constant* $C_{T_0}$ *such that for any* $t \in (0, T_0)$,

$$\|u(\cdot,t)\|_{W^{1,n}(\Omega)} \le C_{T_0}.$$

*If this condition holds for all* $T_0 > 0$, *then* $u$ *exists globally.*

Finally, concerning the integrability condition of $\lambda(u)$ in (L), we can assume a weaker integrability of $\lambda(u)$ if it has a polynomial growth. We have the following result.

**Corollary 2.3.** *The conclusion of Corollary 2.2 holds if there exist constants* $\Lambda_1, \varepsilon_0 > 0$ *such that*

$$|\lambda_u(u)| \le \Lambda_1 \lambda^{1-\varepsilon_0}(u) \quad \text{for all } u \in \mathbb{R}^m, \tag{2.4}$$

*and* (L) *is replaced by the following weaker one:*
(L′) *There exist constants* $L(T_0)$, $s_0 > 0$ *such that*

$$\sup_{t\in(0,T_0)} \|\lambda^{s_0}(u(\cdot,t))\|_{L^1(\Omega)} \le L(T_0).$$

It is easy to see that condition (2.4) holds if $\lambda(u)$ has polynomial growth in $u$.

**Remark 2.4.** The results of this paper also hold for reaction terms with linear growth in $Du$. Namely, we can assume that

$$|\partial_\zeta \hat{f}(u,\zeta)| \le C\lambda^{1/2}(u), \quad |\partial_u \hat{f}(u,0)| \le C\lambda(u) \quad \text{for all } u \in \mathbb{R}^m \text{ and all } \zeta \in \mathbb{R}^{nm}. \tag{2.5}$$

In fact, it is possible to obtain the same results for $\hat{f}$ with quadratic growth in $Du$. That is, we can assume that $|\partial_\zeta \hat{f}(u,\zeta)| \le C\lambda^{1/2}(u)(|\zeta| + 1)$, which is clearly implied by (2.5). The proof is of course more involved and will be reported in our forthcoming work.

## 3 Technical Results

In this section we state another version of the local weighted Gagliardo–Nirenberg inequality [15, Lemma 2.4], which is one of the main ingredients of the proof in [15] and our main theorem in this paper. In order to state the assumption for this type of inequalities, we recall some well-known notions from Harmonic Analysis. For $\gamma \in (1, \infty)$, we say that a nonnegative locally integrable function $w$ is an $A_\gamma$ weight if the quantity

$$[w]_\gamma := \sup_{B_R(y)\subset\Omega} \Big( \fint_{B_R(y)} w\,dx \Big)\Big( \fint_{B_R(y)} w^{1-\gamma'}\,dx \Big)^{\gamma-1} \text{ is finite.} \tag{3.1}$$

Here, $\gamma' = \frac{\gamma}{\gamma-1}$. For more details on these classes, we refer the reader to [19, 22].

Throughout this paper, when there exists no ambiguity $C, C_i$ will denote universal constants that can change from line to line in our argument. If necessary, $C(\cdots)$ or $C_{(\dots)}$ are used to denote quantities which are bounded in terms of theirs parameters in $(\cdots)$. We will also write $a \sim b$ if there exist two generic positive constants $C_1, C_2$ such that $C_1 b \le a \le C_2 b$. Furthermore, we denote by $B_R(x_0)$ a ball with center $x_0 \in \bar{\Omega}$. In the sequel, if the center $x_0$ is already specified, then we simply write $B_R$ and $\Omega_R$ for $B_R(x_0)$ and $B_R(x_0) \cap \Omega$, respectively.

We have the following version of [15, Lemma 2.4].

**Lemma 3.1.** *Let $u, U\colon \Omega \to \mathbb{R}^m$ be vector-valued functions with $u \in C^1(\Omega)$, $U \in C^2(\Omega)$, and let $\Phi\colon \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ function and $\Phi(u)^\alpha$ be an $A_{\beta+1}$ weight for some $\alpha > \frac{2}{p+2}$ and $\beta < \frac{p}{p+2}$. Suppose that either $U$ or $\Phi^2(u)\frac{\partial U}{\partial \nu}$ vanish on the boundary $\partial\Omega$ of $\Omega$. For any ball $B_t(x_0)$ with center $x_0 \in \bar{\Omega}$, we set*

$$I_1(t) := \int_{\Omega_t} \Phi^2(u)|DU|^{2p+2}\,dx, \qquad \hat{I}_1(t) := \int_{\Omega_t} \Phi^2(u)|Du|^{2p+2}\,dx,$$

$$\bar{I}_1(t) := \int_{\Omega_t} |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2})\,dx, \quad I_2(t) := \int_{\Omega_t} \Phi^2(u)|DU|^{2p-2}|D^2U|^2\,dx.$$

*Consider any ball $B_s$ concentric with $B_t$, $0 < s < t$, and any nonnegative $C^1$ function $\psi$ such that $\psi = 1$ in $B_s$ and $\psi = 0$ outside $B_t$. Then, for any $\varepsilon > 0$, there exist positive constants $C_{\varepsilon,\Phi}$, which depend on $[\Phi^\alpha]_{\beta+1}$ and $C_\varepsilon$, such that*

$$I_1(s) \le \varepsilon[I_1(t) + \hat{I}_1(t)] + C_{\varepsilon,\Phi}\|U\|^2_{\mathrm{BMO}(\Omega_t)}[\bar{I}_1(t) + I_2(t)] + C_\varepsilon\|U\|^2_{\mathrm{BMO}(\Omega_t)} \sup_{x\in B_t}|D\psi(x)|^2 \int_{\Omega_t} \Phi^2(u)|DU|^{2p}\,dx. \tag{3.2}$$

The only differences between the two versions are that the factor $\|U\|^2_{\mathrm{BMO}(B_t)}$ in the last terms of (3.2) replaces the factor $\|U\|_{\mathrm{BMO}(B_t)}$ in (2.17) of [15, Lemma 2.4] and the condition on $[\Phi^\alpha]_{\beta+1}$ (both facts are not important in this paper and other applications). The two proofs differ only by the order of using Young's inequality in the argument. Since this inequality and its global version will be very useful for other purposes, we present their proof in Appendix A. Our proofs are somehow simpler than that in [15].

We now let $\Phi \equiv 1$ and $\psi$ be a cutoff function for $B_s, B_t$, i.e., $\psi = 1$ in $B_s$ and $\psi = 0$ outside $B_t$ and $|D\psi| \le \frac{1}{t-s}$. Then $\Phi$ is an $A_\gamma$ weight for all $\gamma > 1$ and $\Phi_u \equiv 0$. The following version of the above lemma with $u = U$ suffices for our purpose in this paper.

**Lemma 3.2.** *Let $U\colon \Omega \to \mathbb{R}^m$ be a vector-valued function in $C^2(\Omega)$. Suppose that either $U$ or $\frac{\partial U}{\partial \nu}$ vanish on the boundary $\partial\Omega$ of $\Omega$. Then, for any two concentric balls $B_s, B_t$, with $s < t$, and any $p \ge 1$, $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\int_{\Omega_s} |DU|^{2p+2}\,dx \le \varepsilon \int_{\Omega_t} |DU|^{2p+2}\,dx + C_\varepsilon\|U\|^2_{\mathrm{BMO}(\Omega_t)} \int_{\Omega_t} [|DU|^{2p-2}|D^2U|^2 + (t-s)^{-2}|DU|^{2p}]\,dx. \tag{3.3}$$

## 4 The Proof of the Main Results

In this section we present the proof of Theorem 2.1 and its corollaries. The proof relies on the Leray–Schauder theorem. We obtain the existence of a strong solution $u$ of (1.1) as a fixed point of a nonlinear map defined on an appropriate Banach space.

Let us consider the Banach space $\mathbf{X} = C(Q, \mathbb{R}^m)$, where $Q = \Omega \times (0, T_0)$. For any given $u \in \mathbf{X}$ and $\sigma \in [0, 1]$, we consider the following linear system:

$$\begin{cases} w_t = \operatorname{div}(A(\sigma u)Dw) + B(\sigma u)Dw + f(\sigma u), & (x,t) \in Q, \\ w(x,0) = U_0(x), & x \in \Omega, \\ w = 0 \quad \text{or} \quad \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \tag{4.1}$$

We then define $T_\sigma(u) = w$. It is clear that a fixed point of $T_\sigma$ solves (2.2). In order to apply the Leray–Schauder theorem, we need to establish the following steps:
Step 1. The map $T_\sigma \colon \mathbf{X} \to \mathbf{X}$ is well defined and compact.
Step 2. There exists a constant $M$ such that $\|u\|_{\mathbf{X}} \le M$ for any fixed points of $u = T_\sigma(u)$.

Step 1 is fairly standard thanks to the following lemma.

**Lemma 4.1.** *The map $T_\sigma \colon \mathbf{X} \to \mathbf{X}$ is well defined and compact.*

*Proof.* For each $u \in \mathbf{X}$, $A(\sigma u)$ satisfies the ellipticity condition (A), and the data of the linear system (4.1) are bounded and continuous. So, (4.1) satisfies the assumptions of [11, Chapter VII, Theorem 1.1], see also [4], which applies to the system

$$w_t = \operatorname{div}(\mathbf{a}Dw) + \mathbf{b}Dw + \mathbf{g},$$

under the assumption that $\mathbf{a}$, $\mathbf{b}$ and $\|\mathbf{g}\|_{q,r,Q}$ are bounded for sufficiently large $q, r$ such that $\frac{1}{r} + \frac{n}{2q} = 1$. Here, for any vector-valued function $F$,

$$\|F\|_{q,r,Q} := \bigg( \int_0^{T_0} \bigg( \int_\Omega |F(x,t)|^q \, dx \bigg)^{r/q} dt \bigg)^{1/r}.$$

Theorem 1.1 in [11, Chapter VII] shows that $w$ exists uniquely, and so $T_\sigma(u)$ is well defined. Moreover, as the initial condition $w(\cdot, 0) = U_0(x)$ belongs to $W^{1,p_0}(\Omega)$ and then $C_0^\beta(\Omega)$ for $\beta_0 = 1 - \frac{n}{p_0} > 0$, a combination of [11, Chapter VII, Theorems 2.1 and 3.1] shows that $w$ belongs to $C^{\alpha_0,\alpha_0/2}(\bar{Q}, \mathbb{R}^m)$ for some $\alpha_0 > 0$. In addition, the norm $\|w\|_{C^{\alpha_0,\alpha_0/2}(\bar{Q})}$ depends on $\beta_0$ and $\|A(\sigma u)\|_\infty$, $\|B(\sigma u)\|_\infty$, $\|f_\sigma(u)\|_{q,r,Q}$. Thus, if $u$ belongs to a bounded set $K$ of $\mathbf{X}$, then $\|u\|_{\mathbf{X}} \le M$ for some $M$, and there exists a constant $C$ such that

$$\|w\|_{C^{\alpha_0,\alpha_0/2}(\bar{Q})} \le C(M, \|U_0(\cdot, 0)\|_{C^{\beta_0}(\Omega)}).$$

Hence, $T_\sigma(K)$ is compact in $\mathbf{X}$ and $T_\sigma \colon \mathbf{X} \to \mathbf{X}$ is a compact map. □

We now turn to Step 2, the hardest part of the proof, and provide a uniform estimate for the fixed points of $T_\sigma$. Such a fixed point $u$ of $T_\sigma$ satisfies (4.1) and belongs to $\mathbf{X}$. Therefore, $u$ is a bounded weak solution and continuous, and so [5, Theorems 2.1 and 3.2] apply and yield that $Du$ is bounded in $\Omega \times (t_0, T_0)$ for all $t_0 > 0$. Thus, $Du$ is locally bounded in $\Omega \times (0, T_0)$. It is then well known that $D^2u$ exists in $L^2_{\mathrm{loc}}(\Omega \times (0, T_0)$ and that $u$ is a strong solution in $\Omega \times (0, T_0)$.

Thus, in the rest of this section, we consider a strong solution $u$ of (4.1). As the data of (4.1) satisfy the structural conditions (A), (F) with the same set of constants and assumptions (M1), (M2) and (L) are assumed to be uniform for all $\sigma \in [0, 1]$, we will only present the proof for $\sigma = 1$ in the sequel.

We should also emphasize that the estimates in the rest of this section do not require the special form of $\hat{f}$ in (2.1) but the growth condition in (F).

For any two concentric balls $B_s, B_t$ with $s < t$, we say that $\psi$ is a cutoff function for $B_s, B_t$ if $\psi$ is a $C^1$ function satisfying $\psi \equiv 1$ in $B_s$, and $\psi \equiv 0$ outside $B_t$ and $|D\psi| \le \frac{1}{t-s}$. Similarly, for $T_1 < T_2 < T_3$, we say that $\eta$ is a cutoff function for $(T_1, T_3)$, $(T_2, T_3)$ if $\eta$ is a $C^1$ function satisfying $\eta(t) \equiv 0$ for $t \le T_1$, and $\eta(t) \equiv 1$ if $t \ge T_3$ and $|\eta_t| \le \frac{1}{T_2 - T_1}$.

We begin with the following energy estimate for $Du$.

**Lemma 4.2.** *We assume that $A, \hat{f}$ satisfy* (A), (F). *Suppose that $u$ is a strong solution of* (1.1) *on $\Omega \times (0, T_0)$. Consider any given triple $t_0, T, T'$ satisfying $0 < t_0 < T < T' \le T_0$ and $p \in [1, \frac{n^*}{2})$, see the definition* (2.3) *of $n^*$. Then there exists a constant $C$, which depends only on the parameters in* (A) *and* (F), *such that for any two concentric balls $B_s, B_t$ with center $x_0 \in \bar{\Omega}$ and $s < t$, we have*

$$\sup_{t \in (T, T')} \int_{\Omega_s} \lambda^{-1}(u)|Du|^{2p} \, dx + \iint_{Q_{s,t_0}} |Du|^{2p-2}|D^2u|^2 \eta \, dx$$
$$\le C\Lambda^2 \iint_{Q_{t,t_0}} |Du|^{2p+2} \eta \, dx + C\bigg( (t-s)^{-2} \iint_{Q_{t,t_0}} |Du|^{2p} \, dx + t_0^{-1} \int_{T-t_0}^{T} \int_{\Omega_t} |Du|^{2p} \, dx \, ds \bigg). \tag{4.2}$$

*Here, $Q_{t,t_0} = \Omega_t \times (T - t_0, T')$ and $\eta$ is a cutoff function for $(T - t_0, T')$, $(T, T')$.*

The above lemma is a special case of the energy estimate for $Du$ in [15, Lemma 3.2] with $W = U = u$ and $\beta(u) = \lambda^{-1}(u)$. Roughly speaking, we differentiated the system in $x$ to obtain

$$(Du)_t = \operatorname{div}(A(u)D^2v + A_u(u)DuDu) + D\hat{f}(u, Du). \tag{4.3}$$

We then test the above with $\lambda^{-1}(u)|Du|^{2p-2}Du\psi^2(x)\eta(t)$, where $\psi$ is a cutoff function for $B_s$, $B_t$, and $\eta$ is a cutoff function for $(T - t_0, T')$, $(T, T')$. Because $2p < n^*$, from the definition (2.3) of $n^*$, it is clear that $\frac{2p-2}{2p} < C_*^{-1}$, and so the spectral gap condition needed in the proof of [15, Lemma 3.2] is available here. Some simple use of Hölder and Young's inequalities gives (4.2). Here, the last integral in (4.2) comes from the integration by parts in time, and we made use of the assumption that $\beta(u) = \lambda^{-1}(u)$ is bounded from above, $|\eta'| \le t_0^{-1}$ and that $|\eta'|$ is zero outside $[T - t_0, T]$.

Next, we have the following technical result.

**Lemma 4.3.** *In addition to the assumptions of Lemma 4.2, we suppose that the quantity*

$$C_{t_0,T,T'} := \iint\limits_{\Omega\times(T-t_0,T')} |Du|^2 \, dx \text{ is finite}. \tag{4.4}$$

*There exists $\mu_0 > 0$ sufficiently small, in terms of the constants in* (A) *and* (F), *such that if for some positive $R_{\mu_0}$, which may depends on $t_0$, $T$, $T'$, such that*

$$\Lambda^2 \sup_{x_0\in\bar{\Omega},t\in(T-t_0,T')} \|u(\cdot\,,t)\|^2_{\mathrm{BMO}(\Omega_R(x_0))} \le \mu_0, \tag{4.5}$$

*then there exist $p > \frac{n}{2}$, an integer $k_0$ and a constant $C$ depending only on the parameters of* (A) *and* (F), *$C_{t_0,T,T'}$, $R_{\mu_0}$, and $t_0$, $T$, $T'$, such that*

$$\sup_{t\in(T,T')} \int\limits_{\Omega_R} \lambda^{-1}(u)|Du|^{2p} \, dx \le C \quad \text{for any } R < 2^{-k_0}R_{\mu_0}. \tag{4.6}$$

*Proof.* We follow the argument in the proof of [15, Proposition 3.1] with $W = U = u$. Suppose that the energy estimate (4.2) in Lemma 4.2 holds for some $p \ge 1$. We write it as

$$\mathcal{A}(s) + \mathcal{H}(s) \le C\Lambda^2\mathcal{B}(t) + C[(t-s)^{-2}\mathcal{C}(t) + t_0^{-1}\hat{\mathcal{C}}(t)], \quad 0 < s < t, \tag{4.7}$$

where the functions $\mathcal{A}$, $\mathcal{H}$, $\mathcal{B}$,$\mathcal{C}$ and $\hat{\mathcal{C}}$ are defined by

$$\mathcal{A}(s) := \sup_{t\in(T,T')} \int\limits_{\Omega_s} \lambda^{-1}(u)|Du|^{2p} \, dx, \quad \mathcal{H}(s) := \iint\limits_{Q_{s,t_0}} |Du|^{2p-2}|D^2u|^2\eta \, dx,$$

$$\mathcal{B}(s) := \iint\limits_{Q_{s,t_0}} |Du|^{2p+2}\eta \, dx, \quad \mathcal{C}(s) := \iint\limits_{Q_{s,t_0}} |Du|^{2p} \, dx, \quad \hat{\mathcal{C}}(s) := \int\limits_{T-t_0}^{T} \int\limits_{\Omega_s} |Du|^{2p} \, dx \, ds.$$

On the other hand, we apply Lemma 3.2 to estimate $\mathcal{B}(t)$, the integral of $|Du|^{2p+2}$, on the right-hand side of (4.7). Namely, we let $U = u$, multiply (3.3) by $\Lambda^2\eta$ and integrate the result over $(T - t_0, T')$ to get (recalling the definition of $\mu_0$ in (4.5))

$$\Lambda^2\mathcal{B}(s) \le \varepsilon\Lambda^2\mathcal{B}(t) + C(\varepsilon)\mu_0\mathcal{H}(t) + C(\varepsilon)\mu_0(t-s)^{-2}\mathcal{C}(t), \quad 0 < s < t \le R_{\mu_0}.$$

Let us define $F(t) := \Lambda^2\mathcal{B}(t)$, $G(t) := \mathcal{H}(t)$, $g(t) := \mathcal{C}(t)$ and $h(t) := t_0^{-1}\hat{\mathcal{C}}(t)$. Then the above yields

$$F(s) \le \varepsilon_0[F(t) + G(t)] + C(t-s)^{-2}g(t),$$

where $\varepsilon_0 = \Lambda^2\varepsilon + C(\varepsilon)\mu_0$. This obviously gives

$$F(s) \le \varepsilon_0[F(t) + G(t)] + C(t-s)^{-2}g(t) + Ch(t).$$

On the other hand, (4.7) implies

$$G(s) \le C[F(t) + (t-s)^{-2}g(t) + h(t)].$$

As $\varepsilon = \Lambda^2\varepsilon + C(\varepsilon)\mu_0$, it is clear that we can choose and fix some $\varepsilon$ sufficiently small, and then for $\mu_0$ small in terms of $C, \varepsilon$ so that $2C\varepsilon_0 < 1$. Thus, if $\mu_0$ is sufficiently small in terms of the constants in (A), (F), then we can apply a simple iteration argument, see [15, Lemma 3.11], to obtain

$$F(s) + G(s) \le C[(t-s)^{-2}g(t) + h(t)] \quad \text{for } 0 < s < t \le R_{\mu_0}.$$

Hence, for any $R < \frac{R_{\mu_0}}{2}$, we take $t = 2R$ and $s = R$ in the above to obtain

$$\iint\limits_{Q_{R,t_0}} (|Du|^{2p-2}|D^2u|^2 + |Du|^{2p+2})\,dx \le C_1\Bigg(R^{-2}\iint\limits_{Q_{2R,t_0}} |Du|^{2p}\,dx + t_0^{-1}\int\limits_{T-t_0}^{T}\int\limits_{\Omega_{2R}} |Du|^{2p}\,dx\,ds\Bigg). \tag{4.8}$$

The above argument shows that if there exist $p \ge 1$ and a constant $C(R, t_0)$ such that the energy estimate (4.2) holds for $p$ and

$$\iint\limits_{Q_{2R,t_0}} \lambda^{-1}(u)|Du|^{2p}\,dx \le C(R, t_0), \tag{4.9}$$

then this estimate also holds for $p$ being replaced by any $q \in (p, p+1]$, via (4.8) and Hölder's inequality. By assumption (4.4), (4.9) holds for $p = 1$. It is now clear that we can repeat the argument $k_0$ times to find a number $p > \frac{n}{2}$, as long as $2p < n^*$ (so that (4.2) holds by Lemma 4.2). We then see that (4.8) and (4.9) hold for such $p$, and therefore estimate (4.6) follows from the energy estimate for (4.2), with $t = 2R$, $s = R$. The lemma is proved. ☐

Lemma 4.3 made use of a cutoff function $\eta$ for the intervals $[T - t_0, T]$ and $[T, T']$ to avoid the dependence on the initial data at $t = 0$. This type of result is useful when one wants to discuss the long time dynamics and global attractors of the system.

In order to establish the local and global existence results, we have to provide bounds for $u$ in $\Omega \times [0, T_0)$ and allow $t_0 = 0$. The next lemma considers this case.

**Lemma 4.4.** *Let the assumptions in Lemma 4.3 with $T = t_0 = 0$ hold. That is,*

$$\iint\limits_{\Omega\times(0,T')} |Du|^2\,dx \text{ is finite} \quad \text{and} \quad \Lambda^2 \sup_{x_0\in\bar\Omega, t\in(0,T')} \|u(\cdot, t)\|^2_{\mathrm{BMO}(\Omega_R(x_0))} \le \mu_0 \tag{4.10}$$

*for some positive $\mu_0$, $R_{\mu_0}$ sufficiently small, in terms of the constants in* (A), (F) *and $T'$. In addition, for some $T_1 \in (0, T_0)$ and $p \ge 1$, assume that*

$$u \in C([0, T_1), L^{2p}(\Omega)), \tag{4.11}$$

$$\sup_{t\in[0,T_1)} \|Du(\cdot, t)\|_{L^{2p}(\Omega)} \text{ is finite}. \tag{4.12}$$

*If* (4.5) *holds, then for the same constant C, the conclusion* (4.6) *now reads*

$$\sup_{t\in(0,T')} \int\limits_{\Omega_R} \lambda^{-1}(u)|Du|^{2p}\,dx \le C + C\|Du(\cdot, 0)\|^{2p}_{L^{2p}(\Omega)} \quad \text{for any } R < 2^{-k_0}R_{\mu_0}. \tag{4.13}$$

*Proof.* Thanks to assumption (4.12), we can let $T, t_0 \to 0$ in (4.8) to see that if $\|Du\|_{L^{2p}(Q_{2R,0})}$ is finite for $Q_{R,0} = \Omega_R \times (0, T')$, then

$$\iint\limits_{Q_{R,0}} (|Du|^{2p-2}|D^2u|^2 + |Du|^{2p+2})\,dx \text{ is finite}. \tag{4.14}$$

Using the difference quotient operator $\delta_h$ instead of $D$ in (4.3) in the proof of Lemma 4.2, we obtain

$$(\delta_h u)_t = \operatorname{div}(A(u)D(\delta_h u) + \delta_h(A(u))Du) + \delta_h\hat f(u, Du).$$

We test this with $\lambda^{-1}(u)|\delta_h u|^{2p-2}\delta_h u\psi^2(x)$, where $\psi$ is a cutoff function for $B_s$, $B_t$. We easily see that the energy estimate in Lemma 4.2 holds with the operator $D$ being replaced by $\delta_h$. Since $u \in C([0, T'), L^{2p}(\Omega))$, by assumption (4.11), we can let $T$, $t_0 \to 0$ and obtain

$$\begin{aligned}\sup_{t\in(0,T')}\int_{\Omega_s}\lambda^{-1}(u)|\delta_h u|^{2p}\,dx + \iint_{Q_{s,0}}|\delta_h u|^{2p-2}|D\delta_h u|^2\,dx\\ \le C\Lambda^2\iint_{Q_{t,0}}|Du|^2|\delta_h u|^{2p}\,dx + (t-s)^{-2}\iint_{Q_{t,0}}|\delta_h u|^{2p}]\,dx + C\int_{\Omega_t}|\delta_h u(x,0)|^{2p}\,dx.\end{aligned}$$

As we now see, the integral in (4.14) is finite, and so we can let $h$ tend to 0 and obtain a similar energy estimate (4.2) for $Du$ with $t_0 = 0$ and $\eta \equiv 1$. Namely,

$$\begin{aligned}\sup_{t\in(0,T')}\int_{\Omega_s}\lambda^{-1}(u)|Du|^{2p}\,dx + \iint_{Q_{s,0}}|Du|^{2p-2}|D^2 u|^2\,dx\\ \le C\Lambda^2\iint_{Q_{t,0}}|Du|^{2p+2}\,dx + (t-s)^{-2}\iint_{Q_{t,0}}|Du|^{2p}\,dx + C\int_{\Omega_t}|Du(x,0)|^{2p}\,dx.\end{aligned}$$

Again, given the second assumption in (4.10), we can argue as in Lemma 4.3 to treat $\mathcal{B}(t)$, the integral of $|Du|^{2p+2}$, on the right-hand side and redefine $h(t) := \|Du(\cdot,0)\|^{2p}_{L^{2p}(\Omega)}$, a constant in $t$. The same argument then yields a version of (4.8) with $t_0 = 0$. In particular, we obtain

$$\iint_{Q_{R,0}}|Du|^{2p+2}\,dx \le C_1 R^{-2}\iint_{Q_{2R,0}}|Du|^{2p}\,dx + C_1\|Du(\cdot,0)\|^{2p}_{L^{2p}(\Omega)}.$$

With assumption (4.10) the iteration argument after (4.8) in the proof of Lemma 4.3 on the power $p$ then gives (4.13). This completes the proof. ☐

We are now ready to provide the proof of the main theorem.

*Proof of Theorem 2.1.* By Lemma 4.1, the map $T_\sigma\colon \mathbf{X} \to \mathbf{X}$ defined by (4.1) is compact. In order to apply the Leray–Schauder theorem and show that there exists a fixed point $u$ for $\sigma = 1$, which is the solution of (1.1), we need only to provide a uniform bound for the fixed points of $T_\sigma$ and conclude the proof. To this end, for any $\sigma \in [0, 1]$, we consider a fixed point $u$ of $T_\sigma$.

Since $u \in \mathbf{X}$, $u$ is a bounded weak solution and continuous, and so [5, Theorems 2.1 and 3.1] apply and yield that $Du$ is locally bounded in $Q = \Omega \times (0, T_0)$. It is then well known that $D^2 u \in L^2_{\mathrm{loc}}(Q)$, and thus $u$ is a strong solution in $Q$.

We will apply Lemma 4.4 here to provide a uniform bound. First of all, the continuity assumption (4.11) of the lemma is clear because $u \in \mathbf{X}$. Next, for any $q = 2p \in (n, p_0)$, we show that $\|u(\cdot,t)\|_{W^{1,q}(\Omega)}$ is bounded in $[0, T_0)$ to verify (4.12). For any $h > 0$ and any function $w$, we denote by $w_{(h)} = \phi_h * w$ the mollifier/regularizer of $w$. For any $f \in L^{q'}(\Omega)$, we have

$$\begin{aligned}\int_\Omega D(u(x,t)_{(h)})f(x)\,dx &= \int_\Omega (Du(x,t))_{(h)}f(x)\,dx = \int_\Omega Du(x,t)f_{(h)}(x)\,dx\\ &= \int_\Omega u(x,t)Df_{(h)}(x)\,dx \to \int_\Omega u(x,0)Df_{(h)}(x)\,dx \quad \text{as } t \to 0,\end{aligned}$$

because $u \in \mathbf{X}$. The last term in the above is bounded by $\|u(\cdot,0)\|_{W^{1,p}(\Omega)}\|f_h\|_{L^{q'}(\Omega)}$ and $u(\cdot,0) = U_0(\cdot)$. By the uniform boundedness principle, noting that $Du(\cdot,t) \in L^q(\Omega)$ for each $t > 0$, we see that $\|Du_{(h)}(\cdot,t)\|_{L^q(\Omega)}$ is uniformly bounded with respect to $h$ for all $h > 0$ and $t \in [0, T_0)$. By letting $h \to 0$, we derive that $\sup_{t\in[0,T_0)}\|Du(\cdot,t)\|_{L^{2p}(\Omega)}$ is finite. Thus, for each fixed point $u$ of $T_\sigma$, condition (4.12) holds.

Hence, from assumptions (M1) and (M2), the assumption (4.10) of Lemma 4.4 holds, and so the lemma can apply here to provide uniform constants $C^*$, $R_1$, depending only on the parameters of (A) and (F), $C_{T_0}$, $R_{\mu_0}$ and $\|DU_0\|_{L^{p_0}(\Omega)}$, such that if $p < p^* = \frac{1}{2}\min\{n^*, p_0\}$, then

$$\sup_{t\in(0,T_0)}\int_{\Omega_{R_1}}\lambda^{-1}(u)|Du|^{2p}\,dx \le C^*. \tag{4.15}$$

From the definition of $r^*$ it is clear that we can choose $p, p_1$ such that $n < p_1 < p < p^*$ and $r^* = \frac{p_1}{p-p_1}$. As $r^* = \frac{p_1}{p}(\frac{p}{p_1})'$, by Hölder's inequality, we have

$$\int_{\Omega_{R_1}} |Du|^{2p_1}\,dx \le \|\lambda(u)\|_{L^{r^*}(\Omega_{R_1})}\Bigl(\int_{\Omega_{R_1}} \lambda^{-1}(u)|Du|^{2p}\,dx\Bigr)^{p_1/p}. \tag{4.16}$$

From assumption (L) on $\lambda(u)$ and (4.15), the right-hand side of (4.16) will be bounded uniformly for all $\sigma \in [0,1]$.

We then have a uniform bound for $\|u\|_{W^{1,q}(\Omega)}$. As $q = 2p_1 > n$, by Sobolev's embedding theorem, we see that $\|u\|_{\mathbf{X}} \le M$ for some constant $M$ and all $\sigma \in [0,1]$. The Leray–Schauder theory then applies to provide a fixed point $u = T_1(u)$. This fixed point is the unique strong solution of system (1.1). □

*Proof of Corollary 2.2.* We just need to show that assumption (D) implies (M1) and (M2). It is clear that (D) yields (M2). To verify (M1), we argue by contradiction. If this is not the case, then there exist sequences $\{x_n\} \subset \bar{\Omega}$, $\{\sigma_n\} \subset [0,1]$, $\{t_n\} \subset (0,T_0)$, $\{r_n\}$, $r_n \to 0$, and a sequence of strong solutions $\{u_{\sigma_n}\}$ such that for $U_n(\cdot) = u_{\sigma_n}(\cdot, t_n)$,

$$\|U_n\|_{\mathrm{BMO}(B_{r_n}(x_n)\cap\Omega)} > \varepsilon_0 \quad \text{for some } \varepsilon_0 > 0.$$

By (D), we see that the sequence $\{U_n\}$ is bounded in $W^{1,n}(\Omega)$. We can then assume that $U_n$ converges weakly to some $U$ in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. We then have $\|U_n\|_{\mathrm{BMO}(B_R\cap\Omega)} \to \|U\|_{\mathrm{BMO}(B_R\cap\Omega)}$ for any given ball $B_R$. It is easy to see that $U \in W^{1,n}(\Omega)$ and, by Poincaré's inequality, $U$ is VMO and $\|U\|_{\mathrm{BMO}(B_R\cap\Omega)} < \frac{\varepsilon_0}{2}$ if $R$ is sufficiently small. The number $R$ is independent of $\lambda_0 \ge 1$ because $\|U\|_{W^{1,n}(\Omega)}$ is independent of $\lambda_0$. Furthermore, we can assume also that $x_n$ converges to some $x \in \bar{\Omega}$. Thus, for large $n$, we have $r_n < \frac{R}{2}$ and $x_n \in B_{R/2}(x)$. Then, for large $n$, $B_{r_n}(x_n) \subset B_R(x)$ and

$$\|U_n\|_{\mathrm{BMO}(B_{r_n}(x_n)\cap\Omega)} \le \|U_n\|_{\mathrm{BMO}(B_R(x)\cap\Omega)} \le \|U\|_{\mathrm{BMO}(B_R(x)\cap\Omega)} + \frac{\varepsilon_0}{2} < \varepsilon_0.$$

We obtain a contradiction. Thus, (M1) holds and the proof is complete. □

*Proof of Corollary 2.3.* We need only to show that (D) and (L′) together imply (L). Let $u$ be any strong solution of (2.2) and $\lambda(u)$ satisfy (2.4). There exist $s_0, C_0 > 0$ such that

$$\|\lambda^{s_0}(u)\|_{L^1(\Omega)} \le C_0(T_0). \tag{4.17}$$

We will show that for any $r > 1$, there exists a constant $C$, depending on $C_0$, $s_0$, $r$, $|\Omega|$, $T_0$ and $\|u\|_{W^{1,n}(\Omega)}$, such that $\|\lambda(u)\|_{L^r(\Omega)} \le C$.

We choose and fix $s > 0$ and $p \in (1,n)$ such that $sp_* = s_0$, where $p_* = \frac{np}{n-p}$. Then (4.17) implies

$$\|\lambda^s(u)\|_{L^{p_*}(\Omega)} \le C_0^{1/p_*}(T_0). \tag{4.18}$$

We define $g(\cdot) = \lambda^{s+\varepsilon_0}(u(\cdot,t))$. The definition of $\Lambda_1$ in (2.4) gives

$$|Dg| \le C(s)\frac{|\lambda_u|}{\lambda^{1-\varepsilon_0}(u)}\lambda^s(u)|Du| \le C(s)\Lambda_1\lambda^s(u)|Du|.$$

Hence, by Hölder's inequality, $\|Dg\|_{L^p(\Omega)} \le C\|\lambda^s(u)\|_{L^{p_*}(\Omega)}\|Du\|_{L^n(\Omega)}$. This and (4.18) and (D) provide some $C(T_0)$ such that $\|Dg\|_{L^p(\Omega)} \le C(T_0)$. Using Hölder's inequality, we have

$$\|g\|_{L^1(\Omega)} \le C\|\lambda^s(u)\|_{L^{p_*}(\Omega)}^{(s+\varepsilon_0)/s} \le CC_0^{1+\varepsilon_0/(p_*s)}(T_0).$$

Hence,

$$\|g\|_{W^{1,p}(\Omega)} \le C(T_0) + CC_0^{1+\varepsilon_0/(p_*s)}(T_0).$$

By Sobolev's embedding theorem, $\|g\|_{L^{p_*}(\Omega)}$ is bounded. From the definition of $g$, we can find a constant $\bar{C}(T_0)$ such that $\|\lambda^{s+\varepsilon_0}(u)\|_{L^{p_*}(\Omega)} \le \bar{C}(T_0)$. Thus, there exists a constant $C_1(T_0)$ such that

$$\|\lambda^{s_0+p_*\varepsilon_0}(u)\|_{L^1(\Omega)} \le C_1(T_0).$$

This shows that if (4.17) holds for some $s_0$, then it also holds for $s_0$ being $s_0 + p_*\varepsilon_0$ and a new constant $C_1(T_0)$. It is then clear that we can repeat this argument to see that $\|\lambda^{s_0+kp_*\varepsilon_0}(u)\|_{L^1(\Omega)} \le C_k(T_0)$ for all integers $k$ and some $C_k(T_0)$. This fact and a simple use of Hölder's inequality show that (L) holds. The proof is complete. □

**Remark 4.5.** By (4.16), $u$ is Hölder in $x$. We can show that $u$ is also Hölder continuous in $x, t$. Indeed, (4.8) with $p = 1$ shows that $|D^2u|^2$, $|Du|^4$ are in $L^1(Q)$. From the system of $u$ and a simple use of Hölder's inequality, we obtain that

$$\|u_t\|_{L^1(Q)} \le \|A(u)\|_{L^2(Q)}\|D^2u\|_{L^2(Q)} + \|A_u(u)\|_{L^2(Q)}\||Du|^2\|_{L^2(Q)} + \|\hat{f}\|_{L^1(Q)}.$$

Since $|A(u)|, |A_u(u)| \le \lambda(u)$ and $\hat{f}$ has linear growth in $Du$, the right-hand side is finite and bounded by a constant independent of $\lambda_0$ (using (4.8) with $p = 1$ and then (4.16) to see that $\|D^2u\|_{L^2(Q)} \le C\lambda_0^{-1}$). Thus, $u_t$ belongs to $L^1(Q)$. It is well known that if $u$ is Hölder continuous in $x$, and $u_t$ is in $L^1(Q)$, then $u$ is Hölder in $x, t$ (see [18, Lemma 4]).

# A Appendix

In this section we provide the details of the key global and local weighted Gagliardo–Nirenberg interpolation inequalities, which allow us to control the $L^p$ norm of the derivatives of the solutions in the proof of our main theorems. The proof somehow simplifies that in [15], as we will not use the Muckenhoupt's inequality for the uncentered maximal operator but simple Hölder's inequality.

Again, we write $B_R(x)$ for a ball centered at $x$ with radius $R$ and will omit $x$ if no ambiguity arises. We use $C, C_1, \dots$ to denote various constants which can change from line to line but depend only on the parameters of the hypotheses in an obvious way. We will write $C(a, b, \dots)$ when the dependence of a constant $C$ on its parameters $a, b, \dots$ is needed to emphasize that $C$ is bounded in terms of its parameters.

For any measurable subset $A$ of $\Omega$ and any locally integrable function $U\colon \Omega \to \mathbb{R}^m$, we denote by $|A|$ the Lebesgue measure of $A$ and by $U_A$ the average of $U$ over $A$. That is,

$$U_A = \fint_A U(x)\,dx = \frac{1}{|A|}\int_A U(x)\,dx.$$

From the definition (3.1) of $A_\gamma$ weights, we clearly have

$$\Big(\fint_{B_R(y)} w\,dx\Big)\Big(\fint_{B_R(y)} w^{-1/\mu}\,dx\Big)^{\mu} \le [w]_{\mu+1} \quad \text{for all } \mu > 0. \tag{A.1}$$

A simple use of Hölder's inequality also gives

$$[w^\delta]_\gamma \le [w]_\gamma^\delta \quad \text{for all } \delta \in (0, 1). \tag{A.2}$$

We first have the following global weighted Gagliardo–Nirenberg inequality.

**Lemma A.1.** *Let $u, U\colon \Omega \to \mathbb{R}^m$ be vector-valued functions with $u \in C^1(\Omega)$, $U \in C^2(\Omega)$, and let $\Phi\colon \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ function. Suppose that either $U$ or $\Phi^2(u)\frac{\partial U}{\partial \nu}$ vanish on the boundary $\partial\Omega$ of $\Omega$. We set*

$$I_1 := \int_\Omega \Phi^2(u)|DU|^{2p+2}\,dx, \quad \hat{I}_1 := \int_\Omega \Phi^2(u)|Du|^{2p+2}\,dx, \tag{A.3}$$

$$\bar{I}_1 := \int_\Omega |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2})\,dx, \tag{A.4}$$

$$I_2 := \int_\Omega \Phi^2(u)|DU|^{2p-2}|D^2U|^2\,dx. \tag{A.5}$$

*Suppose that the following holds:*
(GN) *$\Phi^\alpha(u)$ belongs to the $A_{\beta+1}$ class for some $\alpha > \frac{2}{p+2}$ and $\beta < \frac{p}{p+2}$.*
*Then, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,\Phi}$ depending on $\varepsilon$ and $[\Phi^\alpha(u)]_{\beta+1}$ such that*

$$I_1 \le \varepsilon\hat{I}_1 + C_{\varepsilon,\Phi}\|U\|^2_{\mathrm{BMO}(\Omega)}[\bar{I}_1 + I_2]. \tag{A.6}$$

In the proof of this lemma we will make use of the following well-known facts from Harmonic Analysis. We first recall the definition of the *centered* Hardy–Littlewood maximal operator acting on function $F \in L^1_{\mathrm{loc}}(\Omega)$:

$$M((F)(y) = \sup_{\varepsilon}\Big\{ \int_{B_\varepsilon(y)} F(x)\,dx : \varepsilon > 0 \text{ and } B_\varepsilon(y) \subset \Omega \Big\}.$$

We also recall the Hardy–Littlewood theorem: For any $F \in L^q(\Omega)$, we have

$$\int_\Omega M(F)^q\,dx \le C(q) \int_\Omega F^q\,dx, \quad q > 1. \tag{A.7}$$

We also make use of the Hardy space $\mathcal{H}^1$. For any $y \in \Omega$ and $\varepsilon > 0$, let $\phi$ be a function in $C_0^\infty(B_1(y))$ with $|D\phi| \le C_1$. Let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ (then $|D\phi_\varepsilon| \le C_1\varepsilon^{-1-n}$). From [22], a function $g$ is in $\mathcal{H}^1(\Omega)$ if

$$\sup_{\varepsilon>0} g * \phi_\varepsilon \in L^1(\Omega) \quad \text{and} \quad \|g\|_{\mathcal{H}^1} = \|g\|_{L^1(\Omega)} + \Big\|\sup_{\varepsilon>0} g * \phi_\varepsilon\Big\|_{L^1(\Omega)}.$$

We are now ready to give the proof of Lemma A.1.

*Proof of Lemma A.1.* We can assume that $m = 1$, because the proof for the vectorial case is similar. Integrating by parts, we have

$$I_1 = \int_\Omega \Phi^2(u)|DU|^{2p+2}\,dx = -\int_\Omega U \operatorname{div}(\Phi^2(u)|DU|^{2p}DU)\,dx. \tag{A.8}$$

We will show that $g = \operatorname{div}(\Phi^2(u)|DU|^{2p}DU)$ belongs to the Hardy space $\mathcal{H}^1$ by showing that there exists a constant $C$ such that

$$\int_\Omega \sup_\varepsilon |g * \phi_\varepsilon|\,dx \le C[\bar{I}_1^{1/2} + I_2^{1/2}]I_1^{1/2} + C([\Phi^\alpha(u)]_{\beta+1})[\bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}], \tag{A.9}$$

$$\|g\|_{L^1(\Omega)} \le C[\bar{I}_1^{1/2} + I_2^{1/2}]I_1^{1/2} + C[\bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}]. \tag{A.10}$$

Once this is established, (A.8) and the Fefferman–Stein theorem on the duality of the BMO and Hardy spaces yield $I_1 \le \|U\|_{\mathrm{BMO}}\|g\|_{\mathcal{H}^1}$ (see [22]), and so

$$I_1 \le C([\Phi^\alpha(u)]_{\beta+1})\|U\|_{\mathrm{BMO}}[I_1^{1/2}(\bar{I}_1^{1/2} + I_2^{1/2}) + \bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}].$$

A simple use of Young's inequality to the right-hand side then gives (A.6).

Therefore, in the rest of the proof we need only to establish (A.9). We then write $g = g_1 + g_2$ with $g_i = \operatorname{div} V_i$, where

$$V_1 = \Phi(u)|DU|^{p+1}\Big(\Phi(u)|DU|^{p-1}DU - \fint_{B_\varepsilon} \Phi(u)|DU|^{p-1}DU\,dx\Big)$$

and

$$V_2 = \Phi(u)|DU|^{p+1} \fint_{B_\varepsilon} \Phi(u)|DU|^{p-1}DU\,dx.$$

Let us consider $g_1$ first. We define $h = \Phi(u)|DU|^{p-1}DU$. For any $y \in \Omega$ and $B_\varepsilon = B_\varepsilon(y) \subset \Omega$, we use integration by parts, the property of $\phi_\varepsilon$ and then Hölder's inequality for any $s > 1$ to get

$$\begin{aligned}
|g_1 * \phi_\varepsilon(y)| &= \Big|\int_{B_\varepsilon(y)} D\phi\Big(\frac{x-y}{\varepsilon}\Big)(h - h_{B_\varepsilon(y)})\Phi(u)|DU|^{p+1}\,dx\Big| \\
&\le \frac{C_1}{\varepsilon}\Big|\fint_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}|\Phi(u)|DU|^{p+1}\,dx\Big| \\
&\le \frac{C_1}{\varepsilon}\Big(\fint_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}|^s\,dx\Big)^{1/s}\Big(\fint_{B_\varepsilon(y)} \Phi^{s'}(u)|DU|^{(p+1)s'}\,dx\Big)^{1/s'}.
\end{aligned} \tag{A.11}$$

There exists a constant $C$ such that $|Dh| \le |\Phi_u(u)||Du||DU|^p + p\Phi|DU|^{p-1}|D^2U|$. The Poincaré–Sobolev inequality, with $s_* = n\frac{s}{n+s}$, then gives

$$\begin{aligned}\frac{C_1}{\varepsilon}\Big(\fint_{B_\varepsilon}|h - h_{B_\varepsilon}|^s\,dx\Big)^{1/s} &\le C\Big(\fint_{B_\varepsilon}|Dh|^{s_*}\,dx\Big)^{1/s_*}\\ &\le C\Big[\fint_{B_\varepsilon}|\Phi_u(u)|^{s_*}|Du|^{s_*}|DU|^{ps_*}\,dx + \fint_{B_\varepsilon}\Phi^{s_*}|DU|^{(p-1)s_*}|D^2U|^{s_*}\,dx\Big]^{1/s_*}.\end{aligned}$$

Using the above estimate in (A.11), we get

$$\sup_{\phi_\varepsilon}\Big|\int_\Omega \phi_\varepsilon g_1\,dx\Big| \le C[\Psi_1(y) + \Psi_2(y)]\Psi_3(y), \tag{A.12}$$

where $\Psi_i(y) = (M(F_i^{q_i}(y)))^{1/q_i}$ with $q_1 = q_2 = s_*$, $q_3 = s'$ and

$$F_1 = \Phi_u(u)|Du||DU|^p, \quad F_2 = \Phi(u)|DU|^{p-1}|D^2U|, \quad F_3 = \Phi(u)|DU|^{p+1}.$$

Take $s = \frac{2n}{n-1}$, then $s_* = s' = \frac{2n}{n+1}$. We see that $q_i < 2$. Hence, by (A.7),

$$\Big(\int_\Omega \Psi_i^2\,dx\Big)^{1/2} = \Big(\int_\Omega M(F_i^{q_*})^{2/q_*}\,dx\Big)^{1/2} \le \Big(\int_\Omega F_i^2\,dx\Big)^{1/2}.$$

Therefore, by Holder's inequality, the above estimates and the notations (A.3) and (A.5), we get

$$\int_\Omega \sup_\varepsilon|g_1 * \phi_\varepsilon|\,dx \le C[I_1^{1/2}\bar{I}_1^{1/2} + I_1^{1/2}I_2^{1/2}].$$

We consider $g_2$ and note that $|\mathrm{div}\,V_2| \le C(J_1 + J_2)$ for some constant $C$ and

$$J_1 := |\Phi_u(u)||Du||DU|^{p+1}J_3, \quad J_2 := \Phi(u)|Du|^p|D^2U|J_3,$$

with

$$J_3(y) := \Big|\fint_{B_\varepsilon(y)}\Phi(u)|DU|^p\,dx\Big|.$$

In the sequel, for any $r > \frac{1}{p+1}$, we define $r^* = 1 - \frac{1}{r(p+1)}$, $f = \Phi(u)|DU|^{p+1}$ and $\hat{f} = \Phi(u)|Du|^{p+1}$. We consider $J_3$. If $r_1 > \frac{1}{p+1}$, we use Hölder's inequality to have

$$\Big|\fint_{B_\varepsilon}\Phi^{1/(p+1)}\Phi^{p/(p+1)}|DU|^p\,dx\Big| \le \Big(\fint_{B_\varepsilon}(\Phi^{1/(p+1)})^{1/r_1^*}\,dx\Big)^{r_1^*}\Big(\fint_{B_\varepsilon}f^{pr_1}\,dx\Big)^{1/[r_1(p+1)]}.$$

This gives the following estimate for $J_3$:

$$J_3 \le \Big(\fint_{B_\varepsilon}\Phi(u)^{1/[r_1^*(p+1)]}\,dx\Big)^{r_1^*}\Big(\fint_{B_\varepsilon}f^{pr_1}\,dx\Big)^{1/[r_1(p+1)]}. \tag{A.13}$$

For $J_1$, we write $J_1 = KLJ_3$, with $K = |\Phi_u(u)||DU|^{p+1}|$ and $L = |Du|$. We have

$$|\phi_\varepsilon * J_1| \le \Big(\fint_{B_\varepsilon}K^{s'}\,dx\Big)^{1/s'}\Big(\fint_{B_\varepsilon}L^s\,dx\Big)^{1/s}J_3.$$

We write $L^s = \Phi^{-s/(p+1)}\Phi^{s/(p+1)}|Du|^s$ and use Hölder's inequality to have, for any $r > \frac{1}{p+1}$,

$$\Big(\fint_{B_\varepsilon}L^s\,dx\Big)^{1/s} \le \Big(\fint_{B_\varepsilon}\Phi^{-s/[r^*(p+1)]}\,dx\Big)^{r^*/s}\Big(\fint_{B_\varepsilon}\hat{f}^{sr}\,dx\Big)^{1/[rs(p+1)]}.$$

Combining these estimates with (A.13), we have

$$\sup_{\varepsilon}|\phi_\varepsilon * J_1| \le C_1 M(K^{s'})^{1/s'} M(\hat{f}^{sr})^{1/[rs(p+1)]} M(f^{pr_1})^{1/[r_1(p+1)]}, \tag{A.14}$$

where

$$C_1 = \Big(\fint_{B_\varepsilon} \Phi^{1/[r_1^*(p+1)]}\,dx\Big)^{r_1^*}\Big(\fint_{B_\varepsilon} \Phi^{-s/[r^*(p+1)]}\,dx\Big)^{r^*/s}.$$

We rewrite

$$C_1 = \Big[\sup_{\varepsilon}\Big(\fint_{B_\varepsilon} \Phi^{1/[r_1^*(p+1)]}\,dx\Big)\Big(\fint_{B_\varepsilon} \Phi^{-s/[r^*(p+1)]}\,dx\Big)^{r^*/(r_1^* s)}\Big]^{r_1^*}.$$

We now choose $s, r, r_1$ such that $s' = sr = pr_1$ and $sr < 2$. In this case if $r < 1$ and $s = \frac{r+1}{r}$, then $s' = r + 1$ and $r_1 = \frac{r+1}{p} > \frac{1}{p+1}$. Let

$$\alpha(r) = \frac{1}{r_1^*(p+1)} \quad \text{and} \quad \beta(r) = \frac{r^*}{r_1^* s}.$$

We see that

$$\frac{\alpha(r)}{\beta(r)} = \frac{s}{r^*(p+1)}.$$

By the definition of weights in (3.1), $w = \Phi^{\alpha(r)}$ and (A.1) with $\mu = \beta(r)$, it is clear that

$$C_1 \le C[\Phi^{\alpha(r)}]_{\beta(r)+1}^{r_1^*}.$$

We see that

$$\alpha(r) = \frac{r+1}{rp+r+1} \quad \text{and} \quad \beta(r) = \frac{r(p+1)-1}{r(p+1)+1}.$$

Clearly, $\alpha(r)$ decreases to $\frac{2}{p+2}$ and $\beta(r)$ increases to $\frac{p}{p+2}$ as $r \to 1^-$. Thus, if $\alpha > \frac{2}{p+2}$ and $\beta < \frac{p}{p+2}$, then for $r$ close to 1, we have $\alpha(r) < \alpha$ and $\beta(r) > \beta$, and so $[\Phi^{\alpha(r)}]_{\beta(r)+1} \le C([\Phi^\alpha]_{\beta+1})$, see (A.2). Hence, for $r < 1$ and $r$ being near 1, from (A.14) and the above estimates, we obtain

$$\sup_{\varepsilon}|\phi_\varepsilon * J_1| \le C([\Phi^\alpha]_{\beta+1}) M(K^{sr})^{1/(rs)} M(\hat{f}^{sr})^{1/[rs(p+1)]} M(f^{sr})^{p/[rs(p+1)]}.$$

Integrating the above over $\Omega$ and applying Hölder's inequality and then (A.7) (because $rs < 2$) to the right-hand side, we obtain

$$\int_{\Omega} \sup_{\varepsilon}|\phi_\varepsilon * J_1|\,dx \le C([\Phi^\alpha]_{\beta+1})\|K\|_{L^2(\Omega)}\|\hat{f}\|_{L^{2(p+1)}(\Omega)}\|f\|_2^{p/(p+1)}.$$

Using Young's inequality for the last two factors and the notations (A.3) and (A.4), we obtain

$$\int_{\Omega} \sup_{\varepsilon}|\phi_\varepsilon * J_1|\,dx \le C([\Phi^\alpha]_{\beta+1})\bar{I}_1^{1/2}(\hat{I}_1^{1/2} + I_1^{1/2}).$$

Next, we write $J_2 = \Phi|DU|^{p-1}|D^2U||DU|J_3 = KLJ_3$ with $K = \Phi|DU|^{p-1}|D^2U|$ and $L = |DU|$.
We repeat the same argument, and estimate (A.14) now reads

$$\sup_{\varepsilon}|\phi_\varepsilon * J_2| \le C_2 M(K^{s'})^{1/s'} M(f^{sr})^{1/[rs(p+1)]} M(f^{pr_1})^{1/[r_1(p+1)]},$$

where $C_2$ also satisfies $C_2 \le C[\Phi^\alpha]_{\beta+1}^{r_1^*}$. We then use the previous arguments, starting from (A.14), to have

$$\int_{\Omega} \sup_{\varepsilon}|\phi_\varepsilon * J_2|\,dx \le C([\Phi^\alpha]_{\beta+1}) I_2^{1/2} I_1^{1/2}.$$

Combining the estimates, we obtain

$$\int_{\Omega} \sup_{\varepsilon}|g * \phi_\varepsilon|\,dx \le C[\bar{I}_1^{1/2} + I_2^{1/2}]I_1^{1/2} + C([\Phi^\alpha(u)]_{\beta+1})[\bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}].$$

This proves (A.10). It is not difficult to establish (A.10) estimating $\|g\|_{L^1(\Omega)}$. We simply use Hölder inequality in a similar way in treating $J_1, J_2$, and replacing $J_3$ by $\Phi(u)|DU|^p$. We leave the details to the readers. The proof is then complete. ☐

We now give the proof of Lemma 3.1. Consider any ball $B_s$ concentric with $B_t$, $0 < s < t$, and any nonnegative $C^1$ function $\psi$ such that $\psi = 1$ in $B_s$ and $\psi = 0$ outside $B_t$. Recall the following notations $\Omega_t = \Omega \cap B_t$ and

$$I_1(t) := \int_{\Omega_t} \Phi^2(u)|DU|^{2p+2}\,dx, \qquad \hat{I}_1(t) := \int_{\Omega_t} \Phi^2(u)|Du|^{2p+2}\,dx,$$

$$\bar{I}_1(t) := \int_{\Omega_t} |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2})\,dx, \quad I_2(t) := \int_{\Omega_t} \Phi^2(u)|DU|^{2p-2}|D^2U|^2\,dx.$$

*Proof of Lemma 3.1.* We revisit the proof of Lemma A.1. By integrating by parts and noting that $\psi = 0$ on $\partial\Omega$, we have

$$\int_{\Omega} \Phi^2(u)\psi^2|DU|^{2p+2}\,dx = -\int_{\Omega} U\,\mathrm{div}(\Phi^2(u)\psi^2|DU|^{2p}DU)\,dx.$$

Again, we will show that $g = \mathrm{div}(\Phi^2\psi^2|DU|^{2p}DU)$ belongs to the Hardy space $\mathcal{H}^1$. We write $g = g_1 + g_2$ with $g_i = \mathrm{div}\,V_i$, where

$$V_1 = \Phi(u)\psi|DU|^{p+1}\Big(\Phi(u)\psi|DU|^{p-1}DU - \fint_{\Omega_\varepsilon} \Phi(u)\psi|DU|^{p-1}DU\,dx\Big),$$

$$V_2 = \Phi(u)\psi|DU|^{p+1} \fint_{\Omega_\varepsilon} \Phi(u)\psi|DU|^{p-1}DU\,dx.$$

In estimating $V_1$, we follow the proof of Lemma A.1 and replace $\Phi(u)$ by $\Phi(u)\psi(x)$. There will be some extra terms in the proof in computing $D(\Phi(u)\psi)$. In particular, in estimating $Dh$ in the right-hand side of (A.11), we have an extra term, which can be estimated as follows:

$$\Big(\fint_{\Omega_\varepsilon} \Phi^{s_*}(u)|D\psi|^{s_*}|DU|^{ps_*}\,dx\Big)^{1/s_*} \le \sup_{x\in B_t}|D\psi|\Big(\fint_{\Omega_\varepsilon} \Phi^{s_*}(u)|DU|^{ps_*}\,dx\Big)^{1/s_*}.$$

Accordingly, in the right-hand side of (A.12), we have the following term:

$$\sup|D\psi|\Psi_3 M(\Phi^{s_*}(u)|DU|^{ps_*})^{1/s_*}.$$

Using Hölder's inequality, we have

$$\int_{\Omega_t} \Psi_3 M(\Phi^{s_*}(u)|DU|^{ps_*})^{1/s_*}\,dx \le I_1^{1/2}\Big(\int_{\Omega_t} M(\Phi^{s_*}(u)|DU|^{ps_*})^{2/s_*}\,dx\Big)^{1/2}.$$

The last integral can be bounded via (A.7) by

$$I_* := \int_{\Omega_t} \Phi^2(u)|DU|^{2p}\,dx. \tag{A.15}$$

Using the fact that $|\psi| \le 1$ and $\Omega = B_t$, the proof can continue and give

$$\int_{\Omega_t} \sup_{\varepsilon}|g_1 * \phi_\varepsilon|\,dx \le C\Big[I_1^{1/2}\bar{I}_1^{1/2} + I_1^{1/2}I_2^{1/2} + \sup_{B_t}|D\psi|I_1^{1/2}I_*^{1/2}\Big]. \tag{A.16}$$

Similarly, in considering $g_2 = \mathrm{div}V_2$, we will have the following extra term in the definition of $J_1$:

$$|D\psi|\Phi(u)|DU|^{p+1}\fint_{\Omega_\varepsilon} \Phi(u)|DU|^p\,dx,$$

which can be estimated by

$$\sup_{B_t}|D\psi|M(\Phi(u)|DU|^{p+1})M(\Phi(u)|DU|^p).$$

Again, using Hölder's inequality and the Hardy–Littlewood inequality (A.7), the integral over $B_t$ of this quantity is bounded by $\sup_{B_t}|D\psi|I_1^{1/2}I_*^{1/2}$.

Therefore, (A.16) holds true with $g_1$ being replaced by $g_2$. Combining these estimates for $g_1, g_2$, we get

$$\int_{\Omega_t} \sup_{\varepsilon}|g * \phi_\varepsilon|\,dx \le \sup_{B_t}|D\psi|I_1^{1/2}I_*^{1/2} + C_\Phi[\bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}].$$

The above gives an estimate for the $\mathcal{H}^1$ norm of $g$. By the Fefferman–Stein duality theorem again, we obtain

$$\begin{aligned}\int_{\Omega_t} \Phi^2(u)\psi^2|DU|^{2p+2}\,dx &\le \|U\|_{\mathrm{BMO}(\Omega_t)}\|g\|_{\mathcal{H}^1}\\ &\le \|U\|_{\mathrm{BMO}(\Omega_t)}\Big(\sup_{B_t}|D\psi|I_1^{1/2}I_*^{1/2} + C_\Phi[\bar{I}_1^{1/2}(I_1^{1/2} + \hat{I}_1^{1/2}) + I_1^{1/2}I_2^{1/2}]\Big).\end{aligned}$$

A simple use Young's inequality, the definition of $I_*$ given in (A.15) and then the fact that $\psi = 1$ in $B_s$ give (3.2). The proof is complete. ☐

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