Research Article

Serena Dipierro, Aram Karakhanyan and Enrico Valdinoci*

New Trends in Free Boundary Problems

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Abstract: We present a series of recent results on some new classes of free boundary problems. Differently from the classical literature, the problems considered have either a “nonlocal” feature (e.g., the interaction or/and the interfacial energy may depend on global quantities) or a “nonlinear” flavor (namely, the total energy is the nonlinear superposition of energy components, thus losing the standard additivity and scale invariances of the problem). The complete proofs and the full details of the results presented here are given in [17, 26, 28, 31, 35, 39].

Keywords: Free Boundary Problems, Nonlocal Equations, Regularity Theory, Free Boundary Conditions, Scaling Properties, Instability

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1 Introduction

In this survey, we would like to present some recent research directions in the study of variational problems whose minimizers naturally exhibit the formation of free boundaries. Differently than the cases considered in most of the existing literature, the problems that we present here are either *nonlinear* (in the sense that the energy functional is the nonlinear superposition of classical energy contributions) or *nonlocal* (in the sense that some of the energy contributions involve objects that depend on the global geometry of the system).

In these settings, the problems typically show new features and additional difficulties with respect to the classical cases. In particular, as we will discuss in further details: the regularity theory is more complicated, there is a lack of scale invariance for some problems, the natural scaling properties of the energy may not be compatible with the optimal regularity, the condition at the free boundary may be of nonlocal or nonlinear type and involve the global behavior of the solution itself, and some problems may exhibit a variational instability (e.g., minimizers in large domains and in small domains may dramatically differ the ones from the others).

We will also discuss how the classical free boundary problems in [3, 4, 6] are recovered either as limit problems or after a blow-up, under appropriate structural conditions on the energy functional.

We recall the classical free boundary problems of [3, 4, 6] in Section 2.

Serena Dipierro: School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia, e-mail: s.dipierro@unimelb.edu.au
Aram Karakhanyan: Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom, e-mail: aram.karakhanyan@ed.ac.uk
***Corresponding author: Enrico Valdinoci:** School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia; and Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, Via Ferrata 1, 27100 Pavia, Italy; and Weierstraß Institut für angewandte Analysis und Stochastik, Mohrenstr. 39, 10117 Berlin, Germany; and Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy, e-mail: enrico@mat.uniroma3.it

The results concerning nonlocal free boundary problems will be presented in Section 3, while the case of nonlinear energy superposition is discussed in Section 4.

2 Two Classical Free Boundary Problems

A classical problem in fluid dynamics is the description of a two-dimensional ideal fluid in terms of its stream function, i.e., of a function whose level sets describe the trajectories of the fluid. For this, we consider an incompressible, irrotational and inviscid fluid which occupies a given planar region $\Omega \subset \mathbb{R}^2$. If $V : \Omega \to \mathbb{R}^2$ represents the velocity of the particles of the fluid, the incompressibility condition implies that the flow of the fluid through any portions of Ω is zero (the amount of fluid coming in is exactly the same as the one going out), that is, for any $\Omega_o \Subset \Omega$, and denoting by ν the exterior normal vector,

$$0 = \int_{\partial\Omega_o} \nabla V \cdot \nu = \int_{\Omega_o} \operatorname{div} V.$$

Since this is valid for any subdomain of Ω, we thus infer that

$$\operatorname{div} V = 0 \quad \text{in } \Omega. \tag{2.1}$$

Now, we use that the fluid is irrotational to write equation (2.1) as a second order PDE. To this aim, let us analyze what a “vortex” is. Roughly speaking, a vortex is given by a close trajectory, say $\gamma : S^1 \to \Omega$, along which the fluid particles move. In this way, the velocity field V is always parallel to the tangent direction γ', and therefore

$$0 \neq \int_{S^1} V(\gamma(t)) \cdot \gamma'(t)\, dt = \oint_{\gamma} V, \tag{2.2}$$

where the standard notation for the circulation line integral is used. That is, if we denote by S the region inside Ω enclosed by the curve γ (hence, $\gamma = \partial S$), we infer by (2.2) and Stokes' theorem that

$$0 \neq \int_S \operatorname{curl} V \cdot e_3, \tag{2.3}$$

where, as usual, we write $\{e_1, e_2, e_3\}$ to denote the standard basis of $\mathbb{R}^3$, we identify the vector $V = (V_1, V_2)$ with its three-dimensional image $V = (V_1, V_2, 0)$, and

$$\operatorname{curl} V(x) := \nabla \times V(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ V_1(x_1, x_2) & V_2(x_1, x_2) & 0 \end{pmatrix} = (\partial_{x_1} V_2(x) - \partial_{x_2} V_1(x)) e_3.$$

In this setting, the fact that the fluid is irrotational is translated, in mathematical language, into the fact that the opposite of (2.3) holds true, namely,

$$0 = \int_S \operatorname{curl} V \cdot e_3$$

for any $S \subset \Omega$ (say, with smooth boundary). Since this is valid for any arbitrary region S, we thus can translate the irrotational property of the fluid into the condition $\operatorname{curl} V = 0$ in Ω, that is,

$$\partial_{x_1} V_2 - \partial_{x_2} V_1 = 0 \quad \text{in } \Omega. \tag{2.4}$$

Now, we consider the 1-form

$$\omega := V_2\, dx_1 - V_1\, dx_2, \tag{2.5}$$

and, thanks to (2.1), we have that

$$d\omega = -\partial_{x_2} V_2\, dx_1 \wedge dx_2 - \partial_{x_1} V_1\, dx_1 \wedge dx_2 = -\operatorname{div} V\, dx_1 \wedge dx_2 = 0.$$

Namely, ω is closed, and thus exact (by the Poincaré lemma, at least if Ω is star-shaped). This says that there exists a function u such that

$$\omega = du = \partial_{x_1} u\, dx_1 + \partial_{x_2} u\, dx_2.$$

By comparing this and (2.5), we conclude that

$$\partial_{x_1} u = V_2 \quad \text{and} \quad \partial_{x_2} u = -V_1. \tag{2.6}$$

We observe that u is a stream function for the fluid, namely, the fluid particles move along the level sets of u. Indeed, if $x(t)$ is the position of the fluid particle at time t, we have that $\dot{x}(t) = V(x(t))$ is the velocity of the fluid, and

$$\begin{aligned}\frac{d}{dt} u(x(t)) &= \partial_{x_1} u(x(t))\dot{x}_1(t) + \partial_{x_2} u(x(t))\dot{x}_2(t)\\ &= \partial_{x_1} u(x(t)) V_1(x(t)) + \partial_{x_2} u(x(t)) V_2(x(t))\\ &= V_2(x(t)) V_1(x(t)) - V_1(x(t)) V_2(x(t))\\ &= 0,\end{aligned}$$

in view of (2.6).

The stream function u also satisfies a natural overdetermined problem. First of all, since $\partial\Omega$ represents the boundary of the fluid, and the fluid motion occurs on the level sets of u, up to constants, we may assume that $u = 0$ along $\partial\Omega$. In addition, along $\partial\Omega$ Bernoulli's law prescribes that the velocity is balanced by the pressure (which we take here to be $p = p(x)$). That is, up to dimensional constants, we can write that, along $\partial\Omega$,

$$p = |V|^2 = |\nabla u|^2,$$

where we used again (2.6) in the last identity.

Also, (2.4) and (2.6) give that

$$\Delta u = \partial_{x_1} V_2 - \partial_{x_2} V_1 = 0 \quad \text{in } \Omega,$$

that is, summarizing,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ |\nabla u|^2 = p & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Notice that these types of overdetermined problems are, in general, not solvable, namely, only "very special" domains allow a solution of such overdetermined problem to exist (see, e.g., [45]). In this spirit, determining such domain Ω is part of the problem itself, and the boundary of Ω is, in this sense, a "free boundary" to be determined together with the solution u. These kinds of free boundary problems have a natural formulation, which was widely studied in [3, 4]. The idea is to consider an energy functional which is the superposition of a Dirichlet part and a volume term. By an appropriate domain variation, one sees that minimizers (or, more generally, critical points) of this functional correspond (at least in a weak sense) to solutions of (2.7) (compare, for instance, system (2.7) with [3, Lemma 2.4 and Theorem 2.5]). Needless to say, in this framework, the analysis of the minimizers of this energy functional and of their level sets becomes a crucial topic of research.

In [6] a different energy functional is taken into account, in which the volume term is substituted by a perimeter term. This modification provides a natural change in the free boundary condition (in this setting, the pressure of Bernoulli's law is replaced by the curvature of the level set, see [6, formula (6.1)]).

In the following sections we will discuss what happens when:

- we interpolate the volume term of the energy functional of [3, 4] and the perimeter term of the energy functional of [6] with a fractional perimeter term, which recovers the volume and the classical perimeter in the limit;
- we consider a nonlinear energy superposition, in which the total energy depends on the volume, or on the (possibly fractional) perimeter, in a nonlinear fashion.

3 Nonlocal Free Boundary Problems

A classical motivation for free boundary problems comes from the superposition of a "Dirichlet-type energy" $\mathfrak{D}$ and an "interfacial energy" $\mathfrak{I}$. Roughly speaking, one may consider the minimization problem of an energy functional

$$\mathfrak{E} := \mathfrak{D} + \mathfrak{I}, \tag{3.1}$$

which takes into account the following two tendencies of the energy contributions:

- the term $\mathfrak{D}$ tries to reduce the oscillations of the minimizers,
- the term $\mathfrak{I}$ penalizes the formation of interfaces.

Two classical approaches appear in the literature to measure these interfaces, taking into account the "volume" of the phases or the "perimeter" of the phase separations. The first approach, based on a "bulk" energy contribution, was widely studied in [3]. In this setting, the energy superposition in (3.1) (with respect to a reference domain $\Omega \subset \mathbb{R}^n$) takes the form

$$\begin{cases} \mathfrak{D} = \mathfrak{D}(u) := \displaystyle\int_\Omega |\nabla u(x)|^2 \, dx, \\ \mathfrak{I} = \mathfrak{I}(u) := \displaystyle\int_\Omega \chi_{\{u>0\}}(x) \, dx = \mathscr{L}^n(\Omega \cap \{u > 0\}), \end{cases} \tag{3.2}$$

where $\mathscr{L}^n$ denotes, as customary, the n-dimensional Lebesgue measure. The case of two phase contributions (namely, the one which takes into account the bulk energy of both $\{u > 0\}$ and $\{u < 0\}$) was also considered in [4].

The second approach, based on a "surface tension" energy contribution, was introduced in [6]. In this setting, the energy superposition in (3.1) takes the form

$$\begin{cases} \mathfrak{D} = \mathfrak{D}(u) := \displaystyle\int_\Omega |\nabla u(x)|^2 \, dx, \\ \mathfrak{I} = \mathfrak{I}(u) := \operatorname{Per}(\{u > 0\}, \Omega), \end{cases} \tag{3.3}$$

where the notation

$$\operatorname{Per}(E, \Omega) := \int_\Omega |D\chi_E(x)| \, dx = [\chi_E]_{BV(\Omega)}$$

represents the perimeter of the set E in Ω. Hence, if E has smooth boundary, then

$$\operatorname{Per}(E, \Omega) = \mathscr{H}^{n-1}((\partial E) \cap \Omega),$$

with $\mathscr{H}^{n-1}$ being the $(n-1)$-dimensional Hausdorff measure.

As pointed out in [17], the two free boundary problems (3.2) and (3.3) can be settled into a unified framework, and in fact they may be seen as "extremal" problems of a family of energy functionals indexed by a continuous parameter $\sigma \in (0, 1)$.

To this aim, given two measurable sets $E, F \subset \mathbb{R}^n$, with $\mathscr{L}^n(E \cap F) = 0$, one considers the σ-interaction of E and F, as given by the double integral

$$\mathscr{S}_\sigma(E, F) := \sigma(1-\sigma) \iint_{E \times F} \frac{dx\, dy}{|x-y|^{n+\sigma}}.$$

In [15], the notion of σ-minimal surfaces has been introduced by considering minimizers of the σ-perimeter induced by such interaction. Namely, one defines the σ-perimeter of E in Ω as the contribution relative to Ω of the σ-interaction of E and its complement (which we denote by $E^c := \mathbb{R}^n \setminus E$), that is,

$$\operatorname{Per}_\sigma(E, \Omega) := \mathscr{S}_\sigma(E, E^c \cap \Omega) + \mathscr{S}_\sigma(E \cap \Omega, E^c \cap \Omega^c). \tag{3.4}$$

After [15], an intense activity has been performed to investigate the regularity and the geometric properties of σ-minimal surfaces; see, in particular, [7, 19, 37, 43, 44] for interior regularity results, and [11, 13, 32, 33] for the rather special behavior of σ-minimal surfaces near the boundary of the domain. See also [36] for a recent survey on σ-minimal surfaces.

The analysis of the asymptotics of the σ-perimeter as $\sigma \nearrow 1$ has been studied under several perspectives in [5, 9, 10, 18–20, 42]. Roughly speaking, up to normalization constants, we may say that Per_σ approaches the classical perimeter as $\sigma \nearrow 1$. On the other hand, as $\sigma \searrow 0$, we have that Per_σ approaches the Lebesgue measure (again, up to normalization constants, see [24, 25, 41] for precise statements and examples).

In virtue of these considerations, we have that the free boundary problem introduced in [17], which takes into account the energy superposition (3.1), of the form

$$\begin{cases} \mathfrak{D} = \mathfrak{D}(u) := \int\limits_{\Omega} |\nabla u(x)|^2\,dx, \\ \mathfrak{I} = \mathfrak{I}(u) := \mathrm{Per}_\sigma(\{u > 0\}, \Omega), \end{cases} \tag{3.5}$$

may be seen as an interpolation of the problems stated in (3.2) and (3.3) (that is, at least at a formal level, the energy functional in (3.5) reduces to that in (3.2) as $\sigma \searrow 0$ and to that in (3.3) as $\sigma \nearrow 1$).

A nonlocal variation of the classical Dirichlet energy has been also considered in [16, 22]. In this setting, the classical H^1-seminorm in Ω of a function u is replaced by a Gagliardo H^s-seminorm of the form

$$s(1-s) \iint\limits_{Q_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \tag{3.6}$$

where

$$Q_\Omega := (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega) \tag{3.7}$$

and $s \in (0, 1)$. More precisely, in [16, 22] a superposition of the Gagliardo seminorm and the Lebesgue measure of the positivity set is taken into account.

It is worth pointing out that the domain Q_Ω in (3.6) comprises all the interactions of points $(x, y) \in \mathbb{R}^{2n}$ which involve the domain Ω, since $Q_\Omega = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$.

In this sense, the integration over $Q_\Omega \subset \mathbb{R}^{2n}$ is the natural counterpart of the classical integration over Ω of the standard Dirichlet energy, since $\Omega = \mathbb{R}^n \setminus \Omega^c$.

Also, the double integral in (3.6) recovers the classical Dirichlet energy, see, e.g., [9, 23].

In this spirit, in [31, 35] a fully nonlocal counterpart of the free boundary problems (3.2) and (3.3) has been introduced, by studying energy superpositions of Gagliardo norms and fractional perimeters (see also [29] for the superpositions of Gagliardo norms and classical perimeters). More precisely, the energy superposition (3.1), considered in [31, 35], takes the form

$$\mathfrak{D} = \mathfrak{D}(u) := s(1-s) \iint\limits_{Q_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$
$$\mathfrak{I} = \mathfrak{I}(u) := \mathrm{Per}_\sigma(\{u > 0\}, \Omega),$$

where $s, \sigma \in (0, 1)$.

We summarize here a series of results recently obtained in [17, 31, 35] for these nonlocal free boundary problems (some of these results also rely on a notion of fractional harmonic replacement analyzed in [34]). First of all, we have that minimizers[1] of free boundary problems with fractional perimeter interfaces are continuous, possess suitable density estimates and have smooth free boundaries up to sets of codimension 3.

1 Here, for simplicity, we omit the fact that, in this setting, the minimization is performed not only on a function, but on a couple given by the function and its positivity set. See [17, Section 2] for a rigorous discussion on this important detail.

Theorem 3.1 ([17, Theorems 1.1 and 1.2]). *Let $u_\star$ be a minimizer of*

$$\mathfrak{E}(u) := \int_{B_1} |\nabla u(x)|^2\,dx + \mathrm{Per}_\sigma(\{u > 0\}, B_1), \tag{3.8}$$

with $\sigma \in (0,1)$ and $0 \in \partial\{u_\star > 0\}$. Then $u_\star$ is locally $C^{1-\frac{\sigma}{2}}$ and, for any $r \in (0, \frac{1}{2})$,

$$\min\{\mathscr{L}^n(B_r \cap \{u_\star \geqslant 0\}), \mathscr{L}^n(B_r \cap \{u_\star \leqslant 0\})\} \geqslant cr^n$$

for some $c > 0$. Moreover, the free boundary is a C^∞-hypersurface possibly outside a small singular set of Haussdorff dimension $n-3$.

We remark that the Hölder exponent $1 - \frac{\sigma}{2}$ is consistent with the natural scaling of the problem (namely, $u_r(x) := r^{\frac{\sigma}{2}-1} u_\star(rx)$ is still a minimizer). Such type of regularity approaches the optimal exponent in [3, 4] as $\sigma \searrow 0$. Nevertheless, as $\sigma \nearrow 1$, minimizers in [6] are known to be Lipschitz continuous, therefore we think that it is a very interesting open problem to investigate the optimal regularity of $u_\star$ in Theorem 3.1 (we stress that this optimal regularity may well approach the Lipschitz regularity and so “beat the natural scaling of the problem”).

Also, we think it is very interesting to obtain optimal bounds on the dimension of the singular set in Theorem 3.1.

It is also worth observing that the minimizers in Theorem 3.1 satisfy a nonlocal free boundary condition. Namely, the normal jump $J_\star := |\nabla u_\star^+|^2 - |\nabla u_\star^-|^2$ along the smooth points of the free boundary coincides (up to normalizing constants) with the nonlocal mean curvature of the free boundary, which is defined by

$$\mathscr{K}^\sigma(x) := \int_{\mathbb{R}^n} \frac{\chi_{\{u\leqslant 0\}}(y) - \chi_{\{u>0\}}(y)}{|x-y|^{n+\sigma}}\,dy \tag{3.9}$$

for $x \in \partial\{u > 0\}$.

This free boundary condition has been presented in [17, formula (1.6)]. Since $\mathscr{K}^\sigma$ approaches the classical mean curvature as $\sigma \nearrow 1$ and a constant as $\sigma \searrow 0$ (see, e.g., [1] and [36, Appendix B]), we remark that this nonlocal free boundary condition recovers the classical ones in [6] and in [3, 4] as $\sigma \nearrow 1$, and as $\sigma \searrow 0$, respectively.

In [31, 35], we consider the fully nonlocal case in which both the energy components become of nonlocal type, namely we replace (3.8) with the energy functional

$$\mathfrak{E}(u) := s(1-s) \iint_{Q_{B_1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}_\sigma(\{u > 0\}, B_1), \tag{3.10}$$

with $s, \sigma \in (0,1)$, where the notation in (3.7) has been also used.

In this setting, we have the following theorem.

Theorem 3.2 ([35, Theorem 1.1]). *Let $u_\star$ be a minimizer of (3.10) with $0 \in \partial\{u_\star > 0\}$. Assume that $u_\star \geqslant 0$ in B_1^c and that*

$$\int_{\mathbb{R}^n} \frac{|u_\star(x)|}{1+|x|^{n+2s}}\,dx < +\infty.$$

Then, $u_\star$ is locally $C^{s-\frac{\sigma}{2}}$ and, for any $r \in (0, \frac{1}{2})$,

$$\min\{\mathscr{L}^n(B_r \cap \{u_\star \geqslant 0\}), \mathscr{L}^n(B_r \cap \{u_\star = 0\})\} \geqslant cr^n$$

for some $c > 0$.

We observe that the Hölder exponent in Theorem 3.2 recovers that of Theorem 3.1 as $s \nearrow 1$. Once again, we think that it would be very interesting to investigate the optimal regularity of the minimizers in Theorem 3.2. Also, Theorem 3.2 has been established in the “one-phase” case, i.e., under the assumption that the minimizer has a sign. It would be very interesting to establish similar results in the “two-phase” case in which minimizers can change sign. It is worth remarking that the case in which minimizers change sign is conceptually harder in the nonlocal setting than in the local one, since the two phases interact between each other, thus producing additional energy contributions which need to be carefully taken into account.

4 Nonlinear Free Boundary Problems

In [26, 28] a new class of free boundary problems has been considered, by taking into account "nonlinear energy superpositions". Namely, differently than in (3.1), the total energy functional considered in [26, 28] is of the form

$$\mathfrak{E} := \mathfrak{D} + \Phi_0(\mathfrak{I}), \tag{4.1}$$

for a suitable function Φ_0. When Φ_0 is linear, the energy functional in (4.1) boils down to its "linear counterpart" given in (3.1), but for a nonlinear function Φ_0 the minimizers of the energy functional in (4.1) may exhibit[2] interesting differences with respect to the classical case.

A detailed analysis of free boundary problems as in (4.1) is given in [26, 28]. Here, we summarize some of the results obtained (we give here simpler statements, referring to [26, 28] for more general results). We take $\Phi_0 : [0, +\infty) \to [0, +\infty)$ to be monotone, nondecreasing, lower semicontinuous and coercive – in the sense that

$$\lim_{t \nearrow +\infty} \Phi_0(t) = +\infty.$$

We will also use the notation of writing Per_σ for every $\sigma \in [0, 1]$, with the convention that

- if $\sigma \in (0, 1)$, then Per_σ is the nonlocal perimeter defined in (3.4),
- if $\sigma = 1$, then Per_σ is the classical perimeter,
- if $\sigma = 0$, then $\mathrm{Per}_\sigma(E; \Omega) = \mathscr{L}^n(E \cap \Omega)$.

Then, in the spirit of (4.1), we consider energy functionals of the form

$$\mathfrak{E}(u) := \int_\Omega |\nabla u(x)|^2 \, dx + \Phi_0(\mathrm{Per}_\sigma(\{u > 0\}, \Omega)). \tag{4.2}$$

Notice that, for $\sigma \in (0, 1)$ and $\Phi_0(t) = t$, the energy in (4.2) reduces to that in (3.8). Similarly, for $\sigma = 0$ and $\sigma = 1$, the energy in (4.2) boils down to those in [3] and [6], respectively.

When $\sigma = 0$, a particularly interesting case of nonlinearity is given by $\Phi_0(t) = t^{\frac{n-1}{n}}$. Indeed, in this case, the interfacial energy depends on the n-dimensional Lebesgue measure, but it scales like an $(n-1)$-dimensional surface measure (also, by the isoperimetric inequality, the energy levels of the functional in [6] are above those in (4.2)).

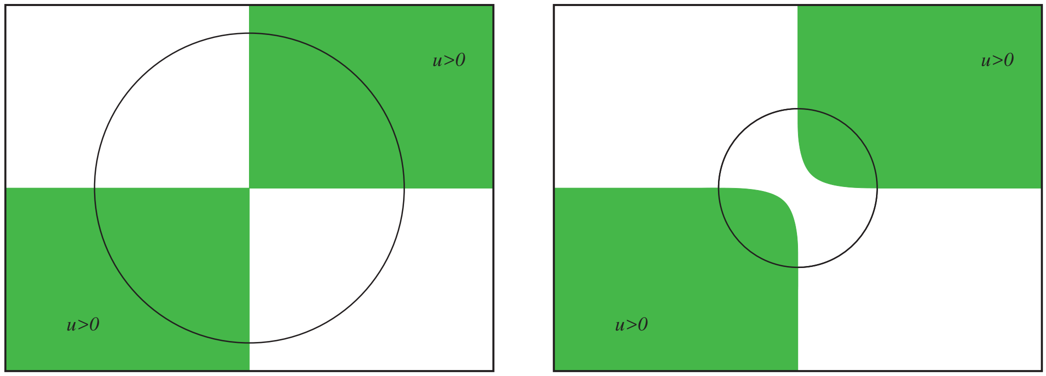
We point out that the free boundary problems (4.2) develop a sort of natural instability, in the sense that minimizers in a large ball, when restricted to smaller balls, may lose their minimizing properties. In fact, minimizers in large and small balls may be rather different from each other.

Theorem 4.1 ([26, Theorem 1.1]). *There exist a nonlinearity Φ_0 and radii $R_0 > r_0 > 0$ such that a minimizer for (4.2) in $\Omega := B_{R_0}$ is not a minimizer for (4.2) in $\Omega := B_{r_0}$.*

The counterexample in Theorem 4.1, which clearly shows the lack[3] of scaling invariance of the problem, is constructed by taking advantage of the different rates of scaling produced by a suitable nonlinear function Φ_0, chosen to be constant on an interval. Namely, the "saddle function" in the plane $u_0(x_1, x_2) = x_1 x_2$ is harmonic and therefore minimizes the Dirichlet energy of (4.2). In large balls, the interface of u_0 (as well

2 Let us give a brief motivation for the nonlinear interface case. The classical energy functionals in [3, 4, 6] may be also considered in view of models arising in population dynamics. Namely, one can consider the regions $\{u > 0\}$ and $\{u \leqslant 0\}$ as areas occupied by two different populations, that have reciprocal "hostile" feelings. Then, the diffusive behavior of the populations (which is encoded by the Dirichlet term of the energy) is influenced by the fact that the two populations will have the tendency to minimize the contact between themselves, and so to reduce an interfacial energy as much as possible.
In this setting, it is natural to consider the case in which the reaction of the populations to the mutual contact occurs in a nonlinear way. For instance, the case in which additional irrationally motivated hostile feelings arise from further contacts between the populations is naturally modeled by a superlinear interfacial energy, while the case in which the interactions between the populations favor the possibility of compromises and cultural exchanges is naturally modeled by a sublinear interfacial energy.

3 Just to recall the importance of scaling invariances in the classical free boundary problems, let us quote [4, p. 114]: "for (small) balls B_r [. . .] let us assume (see 3.1) $B_r = B_1(0)$".



(a) For a large ball B_{R_0}. **(b)** For a small ball B_{r_0}.

Figure 1. The minimizer in Theorem 4.1.

as the ones of its competitors) produces a contribution that lies in the constant part of Φ_0, thus reducing the minimization problem of (4.2) to the one coming from the Dirichlet contribution, and so favoring u_0 itself. Viceversa, in small balls, the interface of u_0 produces more (possibly fractional) perimeter than the one of the competitors whose positivity sets do not come to the origin, and this fact implies that u_0 is not a minimizer in small balls.

This argument, which is depicted in Figure 1, is rigorously explained in [26, Section 3].

In spite of this instability and of the lack of self-similar properties of the energy functional, some regularity results for minimizers of (4.2) still hold true, under appropriate assumptions on the nonlinearity Φ_0 (notice that, for Φ_0 with a constant part Theorem 4.1 would produce the minimizer u_0 whose free boundary is a singular cone, see Figure 1 (a), hence any regularity result on the free boundary has to rule out this possibility by a suitable assumption on Φ_0). In this sense, we have the following results.

Theorem 4.2 ([26, Corollary 1.4 and Theorems 1.5 and 1.6]). *Let $\sigma \in (0,1]$, and let Φ_0 be Lipschitz continuous and strictly increasing. Let $u_\star$ be a minimizer of (4.2) in $\Omega := B_R$, with $0 \leqslant u_\star \leqslant M$ on ∂B_R for some $M > 0$. Then, $u_\star \in C^{1-\frac{\sigma}{2}}(B_{R/4})$. Also, for any $r \in (0, \frac{R}{4})$,*

$$\min\{\mathscr{L}^n(B_r \cap \{u_\star \geqslant 0\}),\ \mathscr{L}^n(B_r \cap \{u_\star = 0\})\} \geqslant cr^n$$

for some $c > 0$.

For $\sigma = 0$, a result similar to that in Theorem 4.2 holds true, in the sense that $u_\star$ is Lipschitz, see [28, Theorems 1.3, 8.1 and 9.2]. Moreover, in this case one obtains additional results, such as the nondegeneracy of the minimizers, the partial regularity of the free boundary and the full regularity in the plane.

Theorem 4.3 ([28, Theorems 1.4, 1.6 and 1.7]). *Let $\sigma = 0$, and let Φ_0 be Lipschitz continuous and strictly increasing. Let $u_\star$ be a minimizer of (4.2) in Ω, with $0 \in \partial\{u_\star > 0\}$ (in the measure theoretic sense). Then, for any $D \Subset \Omega$, there exists $c > 0$ such that for any $r > 0$ for which $B_r \Subset D$, we have that*

$$\int_{B_r \cap \{u_\star > 0\}} u_\star^2(x)\,dx \geqslant cr^{n+2}.$$

Also, $\nabla u_\star$ is locally BMO, in the sense that

$$\sup_{B_r \Subset D} \fint_{B_r} |\nabla u_\star(x) - \langle \nabla u_\star \rangle_r|\,dx \leqslant C$$

for some $C > 0$, where

$$\langle \nabla u_\star \rangle_r := \fint_{B_r} \nabla u_\star(x)\,dx.$$

In addition, $\mathcal{H}^{n-1}(B_r \cap (\partial\{u_\star > 0\})) < +\infty$. Finally, if $n = 2$, then $B_r \cap (\partial\{u_\star > 0\})$ is made of continuously differentiable curves.

The BMO-type regularity and the partial regularity of the free boundary in Theorem 4.3 rely in turn on some techniques developed in [27].

It is also interesting to remark that the case $\sigma = 0$ recovers the classical problems in [3] after a blow-up.

Theorem 4.4 ([28, Theorem 1.5 and Proposition 10.1]). *Let $\sigma = 0$, and let Φ_0 be Lipschitz continuous and strictly increasing. Let $u_\star$ be a minimizer of (4.2) in Ω, with $0 \in \Omega$. For any $r > 0$, let $u_r(x) := \frac{u_\star(rx)}{r}$. Then, there exists the blow-up limit*

$$u_0(x) := \lim_{r \searrow 0} u_r(x).$$

Also, u_0 is continuous and with linear growth, and it is a minimizer of the functional

$$\mathfrak{E}_0(u) := \int_{B_\rho} |\nabla u(x)|^2\, dx + \lambda_0 \mathcal{L}^n(B_\rho \cap \{u > 0\}), \tag{4.3}$$

where

$$\lambda_0 := \Phi_0'(\mathcal{L}^n(\Omega \cap \{u_\star > 0\})). \tag{4.4}$$

We stress that the energy functional in (4.3) coincides with that analyzed in the classical paper [3]. Nevertheless, the "scaling constant" λ_0 in (4.3) depends on the original minimizer $u_\star$, as prescribed by (4.4) (only in the case of a linear Φ_0, we have that λ_0 is a structural constant independent of $u_\star$).

The fact that geometric and physical quantities arising in this type of problems are not universal constants but depend on the minimizer itself is, in our opinion, an intriguing feature of such problems. In this sense, we recall that in [3] the free boundary condition coincides with the classical Bernoulli's law, namely, the normal jump $J_\star := |\nabla u_\star^+|^2 - |\nabla u_\star^-|^2$ along the smooth points of the free boundary is constant (in [6] it coincides with the mean curvature of the free boundary). Differently from the classical cases, in our nonlinear setting, the free boundary condition depends on the minimizer itself. Indeed, in our case the normal jump $J_\star$ coincides with

$$\mathscr{K}^\sigma \Phi_0'(\mathrm{Per}_\sigma(\{u_\star > 0\}, \Omega)), \tag{4.5}$$

where $\mathscr{K}^\sigma$ is the nonlocal mean curvature of the free boundary, as defined in (3.9) (see [26, formula (1.12)] and [28, formulas (1.13) and (1.14)]).

We point out that (4.5) recovers the classical cases in [3, 6] when $\sigma \in \{0, 1\}$ and Φ_0 is linear. On the other hand, when Φ_0 is not linear, the free boundary condition in (4.5) takes into account the global behavior of the free boundary and the (possibly fractional) perimeter of the minimizer in the whole of the domain. In this sense, this type of condition is "self-driven", since it is influenced by the minimizer itself and not only by the environmental conditions and the structural constants.

5 Regularity of Stationary Points of the Alt–Caffarelli Functional

In this section we discuss some recent results on the further connections between the Alt–Caffarelli problem and the minimal surfaces. More specifically, we consider the stationary points (in particular, minimizers) of the functional

$$\mathfrak{E}_{\mathrm{AC}}[u] = \int_\Omega |\nabla u|^2 + \lambda^2 \chi_{\{u>0\}}, \tag{5.1}$$

and the capillarity surfaces in the sphere of radius λ (notice that the critical points of the functional in (5.1) are related to system (2.7), see [3, Theorem 2.5] for details on this).

The starting point of our analysis is to study the classical capillary drop problem.

5.1 Capillarity Drop Problem

We first revisit the sessile drop problem and its higher dimensional analogue.

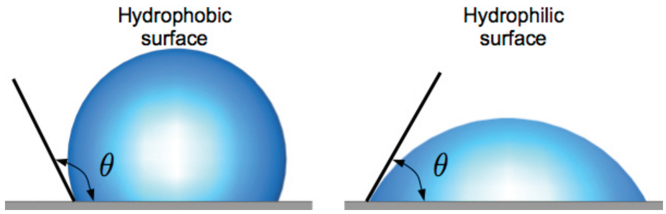


Figure 2. Two regimes of drop-surface interactions.

We consider the functional

$$\mathfrak{J}(E) := \int_{\Omega_0} |D\chi_E| + g \int_{\Omega_0} x_{n+1}\chi_E \, dx - \int_{\partial\Omega_0} \lambda(x')\chi_E \, dx', \tag{5.2}$$

where $\Omega_0 := \{x = (x', x_{n+1}), x_{n+1} > 0\}$, $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$, $g > 0$ is a given constant, $|\lambda(x')| < 1$, and χ_E is the characteristic function of $E \in \mathcal{A}$, where

$$\mathcal{A} := \{E \subset \overline{\Omega_0} : E \text{ has finite perimeter and } \mathcal{H}^{n+1}(E) = V\}.$$

Here, the parameter $V > 0$ is the volume fraction of the droplet.

For $n = 2$, the functional in (5.2) is related to the sessile drop problem, i.e., the problem of a (three-dimensional) capillarity drop occupying the set E and sitting in the halfspace $\{x_{n+1} > 0\}$.

We observe that the first term in $\mathfrak{J}(E)$ is the energy due to the surface tension, the second term is the gravitational energy and the last term is the wetting energy which produces a contact angle $\theta(x')$ such that $\cos\theta(x') = \lambda(x')$ (see Figure 2).

By a Taylor expansion, we see that

$$\sqrt{1 + |\nabla u|^2} = 1 + \frac{1}{2}|\nabla u|^2 + \cdots .$$

Hence, if ∂E is the graph of a (smooth) function $u \geqslant 0$ (with small gradient), we obtain the approximation

$$\int_{\Omega_0} |D\chi_E| = \mathcal{H}^{n-1}((\partial E) \cap \Omega_0) = \int_{(\partial\Omega_0)\cap\{u>0\}} \sqrt{1 + |\nabla u|^2} = \int_{\partial\Omega_0} \chi_{\{u>0\}} + \frac{1}{2}\int_{\partial\Omega_0} |\nabla u|^2 + \cdots .$$

In other words, the functional $\mathfrak{E}_{AC}$ in (5.1) is the linearization of the sessile drop problem described by the functional $\mathfrak{J}$ in (5.2), with no gravity and constant wetting energy density. This suggests that there must be a strong link between the regularity of the minimizers of $\mathfrak{E}_{AC}$ and the minimal surfaces. We will now discuss[4] in which sense this link rigorously occurs.

5.2 Homogeneity of Blow-Ups and the Support Function

The first of such direct links was established in [21], where it is showed that the singular axisymmetric critical point of $\mathfrak{E}_{AC}$ is an energy minimizer in dimension 7. This singular energy minimizer of the Alt–Caffarelli problem can be seen as the analog of the Simons cone

$$S = \left\{ x \in \mathbb{R}^8 \; : \; \sum_{i=1}^{4} x_i^2 = \sum_{i=5}^{8} x_i^2 \right\},$$

which is an example of a singular hypersurface of least perimeter in dimension 8. The minimality of the Simons cone was first proved by Bombieri, De Giorgi and Giusti in [8].

4 For completeness, we recall that a nonlocal capillarity theory has been recently developed in [30, 40].

The cones with non-negative mean curvature arise naturally in the blow-up procedure of the minimizer u at a free boundary point. By Weiss' monotonicity formula (see [46]), any blow-up limit u_0 of an energy minimizer of $\mathfrak{E}_{AC}$ is defined on $\mathbb{R}^n$ and must be a homogeneous function of degree one.

Let us write

$$u_0(x) = rg(\sigma), \tag{5.3}$$

where $\sigma \in \mathbb{S}^{n-1}$ ($\mathbb{S}^{n-1}$ being, as usual, the unit sphere in $\mathbb{R}^n$). Since u_0 is also a global minimizer of $\mathfrak{E}_{AC}$, it follows that $\Delta u_0 = 0$ in $\Omega^+ = \{u_0 > 0\} \cap \mathbb{R}^n$. Rewriting the equation $\Delta u_0 = 0$ in polar coordinates, we infer that g is a solution of the equation

$$\Delta_{\mathbb{S}^{n-1}} g + ng = 0. \tag{5.4}$$

Here $\Delta_{\mathbb{S}^{n-1}} g$ is the Laplace–Beltrami operator on the sphere. We observe that $u_0 > 0$ if and only if $g > 0$ and $g = 0$ on the free boundary of u_0, which is a cone due to the homogeneity of u_0.

Equation (5.4) can be rewritten as

$$\text{Trace}[g_{ij} + \delta_{ij} g] = 0. \tag{5.5}$$

It is well known (see [2]) that

(C) the eigenvalues of the matrix $g_{ij} + \delta_{ij} g$ are the principal radii of curvature of the surface S determined by the parametrization

$$\mathbb{S}^{n-1} \ni \sigma \mapsto X(\sigma) := \sigma g(\sigma) + \nabla_\sigma g(\sigma).$$

In addition, we have that S and the sphere $\lambda \mathbb{S}^{n-1}$ are perpendicular at the contact points, see [39].

In this sense, one can interpret g as the Minkowski support function of the surface S. In other words, $X(\sigma) \cdot \sigma = g(\sigma)$ and it is the distance of the tangent plane with normal σ from the origin.

5.3 The Mean Radius Equation

The previous discussion tells us that the sum of the principal radii of the surface S is zero. Indeed, let $\kappa_i = \frac{1}{R_i}$, $i = 1, 2, \dots, n-1$ be the ith principal curvature of S and R_i the corresponding principal radius. Then, in view of (C), we have that the matrix $g_{ij} + \delta_{ij} g$ has eigenvalues $\frac{1}{\kappa_1}, \dots, \frac{1}{\kappa_{n-1}}$, and so its trace is equal to $\sum_{i=1}^{n-1} \frac{1}{\kappa_i}$. From this and (5.5), we thus obtain that

$$\sum_{i=1}^{n-1} R_i = \sum_{i=1}^{n-1} \frac{1}{\kappa_i} = 0 \quad \text{in } \{g > 0\}. \tag{5.6}$$

This is called the mean radius equation. Recalling (5.3), the free boundary condition given in [3] (and corresponding to the constancy of $|\nabla u_0|$) along $\{g = 0\}$ now becomes

$$|\nabla_\sigma g| = \lambda.$$

This means that the surface S is contained in the sphere of radius λ.

We point out that in dimension $n = 2$, formula (5.6) reduces to

$$\frac{\kappa_1 + \kappa_2}{\kappa_1 \kappa_2} = 0,$$

and therefore the mean curvature vanishes whenever the Gauss curvature is nonzero (i.e., $\kappa_1 \kappa_2 \neq 0$). If $n \geqslant 3$, then such simple interpretation is not possible.

In terms of the classification of global cones for the Alt–Caffarelli problem, we recall that the open question is whether for $n = 5, 6$ the stable solution of the mean radius equation in $\lambda \mathbb{S}^{n-1}$ such that $|\nabla_\sigma g| = \lambda$ on $g = 0$ is the disk passing through the origin (when $n = 2$, this question is settled in [3], the case $n = 3$ was addressed in [14], and $n = 4$ was proved in [38]).

An open and challenging question is to classify the stationary solutions g and the corresponding zero mean radius surfaces of given topological type.

5.4 Flame Models

A closely related problem is the behavior of the solutions to the singular perturbation problem

$$\begin{cases} \Delta u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \leqslant 1 & \text{in } B_1, \end{cases} \tag{5.7}$$

where $\varepsilon > 0$ is small and

$$\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta\Big(\frac{t}{\varepsilon}\Big), \quad \beta(t) \geqslant 0, \quad \operatorname{supp}\beta \subset [0,1], \quad \int_0^1 \beta(t)\,dt = M > 0,$$

is an approximation of the Dirac measure. It is well known that (5.7) models propagation of equidiffusional premixed flames with high activation of energy, see [12].

The limit $u_0 := \lim_{\varepsilon_j \to 0} u_{\varepsilon_j}$ (for a suitable sequence $\varepsilon_j \to 0$) solves a Bernoulli type free boundary problem with the following free boundary condition:

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M.$$

In fact, we have that u_0 is a stationary point of the Alt–Caffarelli functional in (5.1).

If we choose $\{u_\varepsilon\}$ to be a family of minimizers of the functional

$$\mathfrak{E}_\varepsilon[u_\varepsilon] := \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}\Big(\frac{u_\varepsilon}{\varepsilon}\Big), \quad \text{where } \mathcal{B}(t) = \int_0^t \beta(s)\,ds,$$

then u_ε inherits the generic features of the Alt–Caffarelli minimizers as described in [3, 4] (e.g., nondegeneracy, rectifiability of $\partial\{u > 0\}$, etc.). Consequently, by sending $\varepsilon \to 0$, one can see that the limit u exists and it is a minimizer of the Alt–Caffarelli functional

$$\int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}}.$$

As it was mentioned above, the singular set of minimizers is empty in dimensions 2, 3 and 4. However, if u_ε is not a minimizer, then not much is known about the classification of the blow-ups of the limit function u. An interesting question is to classify these stationary points of given topological type or Morse index. One recent result in this direction is that if the associated surface S that we constructed via the support function is of ring type, then S is a piece of catenoid, see [39].

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