Research Article

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Periodic Impact Motions at Resonance of a Particle Bouncing on Spheres and Cylinders

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Abstract: We investigate the existence of periodic trajectories of a particle, subject to a central force, which can hit a sphere or a cylinder. We will also provide a Landesman–Lazer-type condition in the case of a nonlinearity satisfying a *double* resonance condition. Afterwards, we will show how such a result can be adapted to obtain a new result for the impact oscillator at *double* resonance.

Keywords: Keplerian Problem, Periodic Solutions, Impact, Resonance, Landesman–Lazer Condition

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1 Introduction

In this paper, we are interested in periodic solutions of the differential equation

$$\mathbf{x}'' + f(t, |\mathbf{x}|)\mathbf{x} = 0, \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^d$, with $|\mathbf{x}| \geq R_0$, and $f : \mathbb{R} \times [R_0, +\infty) \to \mathbb{R}$ is a continuous function, T-periodic in the first variable, with a fixed positive constant R_0. We are going to study the existence of *bouncing* periodic solutions. In particular, we are looking for solutions $\mathbf{x} : \mathbb{R} \to \mathbb{R}^d$ solving (1.1) when $|\mathbf{x}| > R_0$ and satisfying a *perfect bounce condition* on the sphere $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = R_0\}$: the speed has the same value before and after the bounce but the sign of the radial component changes.

The main results can also be applied to a class of systems with a particle hitting a cylinder $\mathbb{S}^{d_1-1} \times \mathbb{R}^{d_2}$. In the case of a proper cylinder (i.e. for $d_1 = 2$ and $d_2 = 1$), it models, for example, a particle subject to a periodic central electric field and to an elastic force (see Figure 1). Similar situations can be seen in the case of a proper sphere, for $d = 3$. We will focus our attention, at first, to the case of spheres, postponing the treatment of the case of cylinders to Section 4.

By the radial symmetry of the equation, every solution of (1.1) is contained in a plane, so we can pass to polar coordinates and consider solutions to the following system:

$$\begin{cases} \rho'' - \dfrac{L^2}{\rho^3} + f(t,\rho) = 0, & \rho > R_0, \\ \rho^2 \vartheta' = L, \end{cases} \tag{1.2}$$

where $f(t,\rho) = f(t,\rho)\rho$ and $L \in \mathbb{R}$ is the angular momentum. The bounce condition could be easily written in the following way:

$$\rho'(t_0^+) = -\rho'(t_0^-) \quad \text{if} \quad \rho(t_0) = R_0.$$

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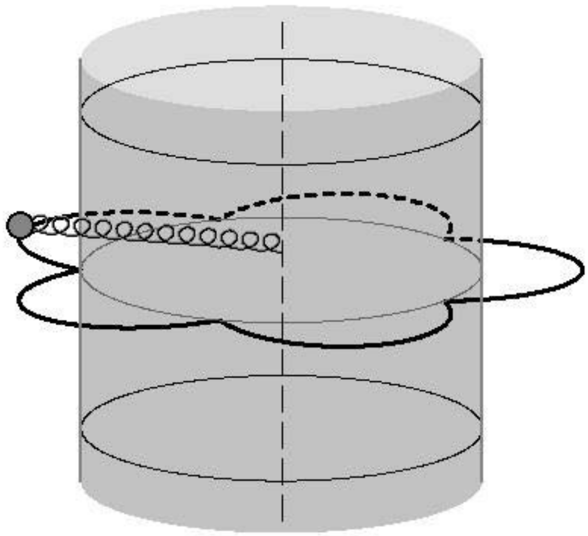

Figure 1. A bouncing particle periodically rotating around a cylinder.

We emphasize that the bounce does not affect the behavior of ϑ. In this paper, we are going to study the existence of rotating periodic solutions, performing a certain number ν of revolutions around the sphere in the time kT and T-periodic in the variable ρ, i.e. such that

$$\begin{cases} \rho(t+T) = \rho(t), \\ \vartheta(t+kT) = \vartheta(t) + 2\pi\nu. \end{cases} \tag{1.3}$$

The existence of periodic solutions of large period kT was previously studied by Fonda and Toader in [21] in the setting of a Keplerian-like system where the planet is viewed as a point (see also [18, 19, 22] for other situations).

Problems modeling the motion of a particle hitting some surfaces have been widely studied in literature in different situations; see, e.g., [2, 4, 5, 26–28, 34]. The simpler system with impacts is given by the so-called impact oscillator (see, e.g., [1, 3, 25, 33]) where a particle hits a wall attracted towards it by an elastic force. The existence of bouncing periodic solutions of such systems has been discussed, for example, in [3, 17, 28, 30–33, 37]. However, to the best of our knowledge, it seems that similar existence results on rotating periodic solutions with impact on spheres (or cylinders) of positive radius have not been presented yet.

Let us now explain in detail what we mean by the term “bouncing solution”, borrowing the definition given by Bonheure and Fabry in [3], which we recall for the reader’s convenience. In this definition, the constant w indicates the x-coordinate of the *wall* against which the solution bounces.

Definition 1.1. Consider a scalar second order differential equation

$$x'' + p(t, x) = 0,$$

where $p : \mathbb{R} \times [w, +\infty) \to \mathbb{R}$ is a continuous function. A *w-bouncing solution* is a continuous function $x(t)$, defined on a certain interval (a, b), such that $x(t) \geq w$ for every $t \in (a, b)$, satisfying the following properties:

(i) If $t_0 \in (a, b)$ is such that $x(t_0) > w$, then $x(t)$ is twice differentiable at $t = t_0$, and $x''(t_0) + p(t_0, x(t_0)) = 0$.

(ii) If $t_0 \in (a, b)$ is such that $x(t_0) = w$ and, in a neighborhood of t_0, there holds $x(t) > w$ for $t \neq t_0$, then $x'(t_0^-)$ and $x'(t_0^+)$ exist and $x'(t_0^-) = -x'(t_0^+)$.

(iii) If $t_0 \in (a, b)$ is such that $x(t_0) = w$ and either $x'(t_0^-)$ or $x'(t_0^+)$ exists and is different from 0, then, in a neighborhood of t_0, there holds $x(t) > w$ for $t \neq t_0$.

(iv) If $x(t) = w$ for all t in a non-trivial interval $I \subseteq (a, b)$, then $p(t, w) \geq 0$ for every $t \in I$.

Treating a Keplerian system like (1.2), we will say that (ρ, ϑ) is a w-bouncing solution if ρ is a w-bouncing solution of the first differential equation.

We are now ready to state one of the main results of this paper.

Theorem 1.2. *Assume*

$$\check{\mu} \le \liminf_{\rho\to+\infty} \frac{f(t,\rho)}{\rho} \le \limsup_{\rho\to+\infty} \frac{f(t,\rho)}{\rho} \le \hat{\mu} \tag{1.4}$$

with

$$\Big(\frac{N\pi}{T}\Big)^2 < \check{\mu} \le \hat{\mu} < \Big(\frac{(N+1)\pi}{T}\Big)^2 \tag{1.5}$$

for a suitable integer N. Then for every integer $\nu > 0$ there exists an integer $k_\nu > 0$ such that for every integer $k \ge k_\nu$ there exists at least one periodic R_0-bouncing solution (ρ, ϑ) of (1.2) *with period kT, which makes exactly ν revolutions around the origin in the period time kT, i.e. satisfying* (1.3).

The statement of the theorem requires that the nonlinearity f has a *nonresonant* asymptotically linear growth at infinity in the following sense: the constants $\mu_j = (\pi j/T)^2$ in (1.5) are the values of the vertical asymptotes $\mu = \mu_j$ of the j-th curve of the periodic Dancer–Fučík spectrum associated to the asymmetric oscillator $x'' + \mu x^+ - \nu x^- = 0$. The first equation in (1.2) presents a *wall* at $\rho = R_0$ against which the particle bounces. Such a wall can be approximatively modeled as a spring with a very large elasticity constant (see, e.g., [3, 4, 28, 37]). For this reason, it is natural to require that the nonlinearity f satisfies an asymptotic behavior at infinity as in (1.4). Conditions like (1.4) have already been introduced by treating scalar equations with a singularity; see, e.g., [6, 14, 20, 35]. Moreover, let us underline that if $f(s, R_0) < 0$ for some $s \in [0, T]$, then necessarily the periodic solutions satisfy $\rho(t) > R_0$ for some $t \in [0, T]$.

We will also see, in Theorem 3.1, how we can relax condition (1.5) by introducing a Landesman–Lazer-type condition, thus obtaining a similar existence result for nonlinearities *next to* resonance. In particular, we will introduce a double resonance condition for nonlinearities satisfying (1.4) with $\check{\mu} = \mu_N$ and $\hat{\mu} = \mu_{N+1}$. A double Landesman–Lazer-type condition has been treated in other situations; see, e.g., [8, 9, 13, 36].

Moreover, following the proof of Theorem 3.1, we will obtain Corollary 3.2 which extends to the resonant case a previous result obtained by Fonda and the author in [17] for impact oscillators.

The next section is devoted to the proof of Theorem 1.2. In Section 3, we will present the results *next to* resonance and we will show how to modify the proof of Theorem 1.2 in order to prove such a result. Then, in Section 4, we will present how to extend the applications to systems with a particle bouncing on a cylinder.

2 Nonresonant Case, Proof of Theorem 1.2

Let us consider the change of coordinate $\rho = r + R_0$. System (1.2) is equivalent to

$$\begin{cases} r'' - \dfrac{L^2}{(r+R_0)^3} + f(t, r+R_0) = 0, \quad r > 0, \\ \vartheta' = \dfrac{L}{(r+R_0)^2}. \end{cases} \tag{2.1}$$

Let us define the function $g : \mathbb{R}\times\mathbb{R}\times[0,+\infty) \to \mathbb{R}$ as

$$g(L, t, r) = -\frac{L^2}{(r+R_0)^3} + f(t, r+R_0),$$

so that the first differential equation in (2.1) becomes

$$r'' + g(L, t, r) = 0 \tag{2.2}$$

and it is easy to verify that

$$\check{\mu} \le \liminf_{r\to+\infty} \frac{g(L,t,r)}{r} \le \limsup_{r\to+\infty} \frac{g(L,t,r)}{r} \le \hat{\mu}$$

uniformly in $t \in [0, T]$ and L in a compact set. In what follows, we will assume that L varies in a compact set containing zero, but it is not restrictive to assume L to be non-negative. So, fixing $L_0 > 0$, in what follows, we will always assume $L \in [0, L_0]$.

We are looking for 0-bouncing solutions to (2.2) such that

$$r(t+T) = r(t),$$
$$\vartheta(t+kT) = \vartheta(t) + 2\pi\nu \quad \text{with } k, \nu \in \mathbb{Z}.$$

We define, for a fixed small $\delta > 0$, for every $n \in \mathbb{N}$ the functions $g_n : [0, L_0] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$g_n(L,t,x) = \begin{cases} g(L,t,x), & x \geq 1/n, \\ nx(g(L,t,x)+\delta) - \delta, & 0 < x < 1/n, \\ nx - \delta, & x \leq 0 \end{cases} \tag{2.3}$$

and we consider the differential equations

$$x'' + g_n(L,t,x) = 0, \tag{2.4}$$

where x varies in $\mathbb{R}$. Let us spend a few words to motivate the introduction of the small constant $\delta > 0$ which can appear unessential: it will be useful to simplify the proof of the validity of the fourth property in Definition 1.1 for the bouncing solution we are going to find (cf. [17]).

It is well known that such a differential equation has at least one T-periodic solution when n is large enough and L is fixed (cf. [7, 10, 16]). In the next section, we will prove the existence of a common a priori bound.

Let us introduce the set of C^1 functions which are T-periodic

$$C^1_P = \{x \in C^1([0,T]) : x(0) = x(T),\ x'(0) = x'(T)\}.$$

We are going to look for an open bounded set $\Omega \subset C^1_P$ containing 0 such that any periodic solution of (2.4), for n sufficiently large, belongs to Ω.

Then in Section 2.2 we will prove the existence of periodic solutions of (2.2) by a limit procedure: we will show that, for every integer $\nu > 0$ and for every integer k sufficiently large, there exists a sequence $(L^{k,\nu}_n)_n \subset [0, L_0]$ and a sequence of solutions $(x^{k,\nu}_n)_n$ of (2.4), with $L = L_n$, converging to $L^{k,\nu}$ and $x^{k,\nu}$, respectively, where $x^{k,\nu}$ is the desired bouncing solution. The proof of Theorem 1.2 will be concluded easily in Section 2.3.

2.1 The A Priori Bound

The proof makes use of some phase-plane techniques, so it will be useful to define the following subsets of $\mathbb{R}^2$:

$$\Pi^- = \{(x,y) \in \mathbb{R}^2 : x \leq 0\}, \quad \Pi^+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0\}.$$

Further, we define the open balls centered at the origin

$$B_s = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < s^2\}.$$

Let $x \in C^1_P$ be a solution of (2.4) such that there exist some instants $t_1, t_2, t_3 \in [0, T]$ such that $x(t_i) = 0$ and $(-1)^i x'(t_i) > 0$. Assume moreover that $x(t) < 0$ for every $t \in (t_1, t_2)$ and $x(t) > 0$ for every $t \in (t_2, t_3)$. Let us first consider the interval $[t_1, t_2]$. There exists a positive constant c such that the orbit $(x(t), x'(t))$ in the phase plane consists, in this interval, of a branch of the ellipse

$$y^2 + nx^2 - 2\delta x = c^2. \tag{2.5}$$

In particular, $x'(t_1) = -c$ and $x'(t_2) = c$, and for every $t \in [t_1, t_2]$ one has

$$x(t) \geq \frac{\delta - \sqrt{\delta^2 + c^2 n}}{n} > -\frac{c}{\sqrt{n}}.$$

A computation shows that

$$t_2 - t_1 = \frac{1}{\sqrt{n}}\Big[\pi - 2\arcsin\Big(\frac{\delta}{\sqrt{nc^2 + \delta^2}}\Big)\Big] < \frac{\pi}{\sqrt{n}}$$

for every $c > 0$.

Call Σ_c^n the open region delimited by the ellipse in (2.5). Besides, for every $c > 0$, it is possible to find $n_0 = n_0(c)$ large enough to have

$$\frac{1}{n}(\delta - \sqrt{\delta^2 + c^2 n}) > -R_0/2 \quad \text{for every } n \geq n_0(c). \tag{2.6}$$

For our purposes it is useful to define the set

$$\Xi_c^n = (\Pi^- \cap \Sigma_c^n) \cup (\Pi^+ \cap B_c).$$

In particular, we have $\Xi_c^{n+1} \subset \Xi_c^n$ and

$$\Xi_c^{n_0(c)} \subset (-R_0/2, c) \times (-c, c).$$

We define the following set of periodic functions:

$$\Omega_c^n = \{x \in C_P^1 : (x(t), x'(t)) \in \Xi_c^n \text{ for every } t \in [0, T]\}.$$

Let us now focus our attention on the second interval $[t_2, t_3]$. It is possible to verify that there exists $\bar{\chi} > 0$ sufficiently large (which can be chosen independently of L and n) with the following property: if x satisfies $(x'(t))^2 + (x(t))^2 > \bar{\chi}^2$ for every $t \in [t_2, t_3]$, then (cf. [15])

$$\frac{\pi}{\sqrt{\hat{\mu}}} \leq t_3 - t_2 \leq \frac{\pi}{\sqrt{\check{\mu}}}.$$

Hence, we can conclude that any solution to (2.4) such that $x^2 + x'^2 > \bar{\chi}^2$ when $x \geq 0$ must rotate in the phase plane, and it needs a time

$$\tau \in \Big[\frac{\pi}{\sqrt{\hat{\mu}}}, \frac{\pi}{\sqrt{\check{\mu}}} + \frac{\pi}{\sqrt{n}}\Big] \tag{2.7}$$

to complete a rotation. So, by (1.5), choosing n large enough, we can find that it performs more than N rotations and less than $N + 1$ rotations in the phase plane. In particular, it cannot perform an integer number of rotations.

Moreover, by assumption (1.4), if we consider a solution x to (2.4) satisfying $x^2 + x'^2 > \bar{\chi}^2$ when $x \geq 0$, introducing polar coordinates

$$\begin{cases} x(t) = \varrho(t)\cos(\theta(t)), \\ x'(t) = \varrho(t)\sin(\theta(t)), \end{cases}$$

we can find (enlarging $\bar{\chi}$, if necessary) the following uniform bound for the radial and angular velocity when x is positive:

$$-\theta'(t) = \frac{x'(t)^2 + x(t)g_n(L, t, x(t))}{x(t)^2 + x'(t)^2} > \theta_0 > 0,$$
$$|\varrho'(t)| = \frac{|x'(t)(x(t) - g_n(L, t, x(t)))|}{\sqrt{x(t)^2 + x'(t)^2}} < l_0\varrho(t).$$

Thus we find

$$\Big|\frac{d\varrho}{d(-\theta)}\Big| < \frac{l_0}{\theta_0}\varrho = K\varrho. \tag{2.8}$$

Let us now state a lemma which will also be useful in Section 3.

Lemma 2.1. *For every $\chi \geq \bar{\chi}$ there exists $R = R(\chi) > 0$ and an integer $n_1 = n_1(\chi)$ such that every solution x to* (2.4)*, with $n > n_1$, satisfying*

$$(x(t_0), x'(t_0)) \in \overline{B}_\chi^+ = \overline{B}_\chi \cap \Pi^+$$

at a certain time $t_0 \in [0, T]$ is such that $(x(t), x'(t)) \in \Xi_R^{n_1}$ for every $t \in [t_0, t_0 + T]$.

Proof. It is not restrictive to consider a solution such that $(x(t_0), x'(t_0)) \in \partial B_\chi \cap \Pi^+$ and $(x(s), x'(s)) \notin B_\chi \cap \Pi^+$ for every $s \in (t_0, t_0 + T]$. Such a solution rotates clockwise in the phase plane and will vanish for the first time at $t_1 > t_0$, thus having $(x(t_1), x'(t_1)) = (0, -y_1)$ such that $0 < y_1 < e^{K\pi}\chi$ by (2.8). The solution vanishes again at $t_2 > t_1$ such that $(x(t_2), x'(t_2)) = (0, y_1)$. Then the solution will perform a complete rotation in the interval $[t_2, t_3]$, thus obtaining $x'(t_3) < e^{K\pi}x'(t_2)$. In the time interval $[t_0, t_0 + T]$ the solution cannot perform more than $N + 1$ rotations, so choosing $R(\chi) = e^{(N+2)K\pi}\chi$ and $n_1(\chi) = n_0(R(\chi))$, as in (2.6), we conclude the proof of the lemma. □

With a similar reasoning, we can prove that if x' vanishes at a time τ_0 with $x(\tau_0) = \bar{x} > 0$, then the solution x will vanish for the first time at τ_1 with $-x'(\tau_1) < \bar{x}e^{K\pi/2}$. Then the solution will reach a negative minimum at τ_2 and again will vanish at τ_3 with $x'(\tau_3) = -x'(\tau_1)$. Recalling that, in Π^-, the orbit of x is contained in a certain ellipse of equation (2.5), we have $x(\tau_2) > -x'(\tau_3)/\sqrt{n}$. Hence, setting $C = e^{K\pi/2}$, we have immediately

$$\frac{1}{C}\|x\|_\infty \leq \|x'\|_\infty \leq C\|x\|_\infty \quad \text{and} \quad x(t) > -\frac{\|x'\|_\infty}{\sqrt{n}} \geq -\frac{C\|x\|_\infty}{\sqrt{n}}. \tag{2.9}$$

Such estimates will be useful in Section 3.

The set $\Xi = \Xi_R^{n_1}$ provided by the previous lemma is the a priori bound we were looking for. In fact, suppose to have a T-periodic solution such that $(x(\zeta), x'(\zeta)) \notin \Xi$ at a certain time $\zeta \in [0, T]$. If the solution remains outside $B_{\bar{\chi}}^+$, then it cannot perform an integer number of rotations around the origin in the period time T. Hence, the solution must enter the set at a certain time $\zeta' > \zeta$, and the previous lemma gives us a contradiction.

Summing up, in this section we have proved the following estimate.

Lemma 2.2. *There exist an open bounded set $\Xi \subset (-R_0/2, +\infty) \times \mathbb{R}$ and a positive integer $\bar{n}$ such that every T-periodic solution to* (2.4) *with $n > \bar{n}$ belongs to*

$$\Omega = \{x \in C_P^1 : (x(t), x'(t)) \in \Xi \text{ for every } t \in [0, T]\}.$$

2.2 Degree Theory

It is well known that the existence of periodic solutions of equation (2.4) is strictly related to the existence of a fixed point of a completely continuous operator $\Psi_{L,n} : C_P^1 \to C_P^1$ (see, e.g., [21]) defined as

$$\Psi_{L,n} = (\mathcal{L} - \sigma I)^{-1}(\mathcal{N}_{L,n} - \sigma I),$$

where $\mathcal{L} : D(\mathcal{L}) \to L^1(0, T)$ is defined in $D(\mathcal{L}) = \{x \in W^{2,1}(0, T) : x(0) = x(T),\ x'(0) = x'(T)\}$ as $\mathcal{L}x = x''$ and σ does not belong to its spectrum. The operator $(\mathcal{N}_{L,n}x)(t) = -g_n(L, t, x(t))$ is the so-called Nemytzkii operator and I is the identity operator.

By classical results (see, e.g., [7, 10]) one has

$$d_{LS}(I - \Psi_{L,n}, \Omega) \neq 0 \tag{2.10}$$

for every $n \geq n_0$, with Ω given by Lemma 2.2.

Using the continuation principle, we have for every $n \geq n_0$ that there exists a continuum $\mathcal{C}_n$ in $[0, L_0] \times \Omega$, connecting $\{0\} \times \Omega$ to $\{L_0\} \times \Omega$, whose elements (L_o, x_o) are such that x_o is a solution of $x_o'' + g_n(L_o, t, x_o) = 0$ (see [21] for a similar approach). The function $\Theta : [0, L_0] \times \Omega \to \mathbb{R}$ defined as

$$\Theta(L, x) = \int_0^T \frac{L}{(R_0 + x(t))^2}\, dt$$

is well defined and continuous since $x(t) > -R_0/2$ holds by Lemma 2.2. In particular, one has $\Theta(0, x) = 0$ and

$$\frac{T}{(R_0 + R)^2} L < \Theta(L, x) < \frac{4T}{R_0^2} L, \tag{2.11}$$

where R is the constant provided by Lemma 2.1. Hence, for every integer $\nu > 0$, there exists k_ν with the following property: for every $k \geq k_\nu$ and for every $n \geq n_0$ there exists $(L_n^{k,\nu}, x_n^{k,\nu}) \in \mathcal{C}_n$ with $\Theta(L_n^{k,\nu}, x_n^{k,\nu}) = 2\pi\nu/k$.

Fix now ν and $k \geq k_\nu$ and consider the sequences $(L_n^{k,\nu})_n$ and $(x_n^{k,\nu})_n$. Let us simply denote them by $(L_n)_n$ and $(x_n)_n$. Both the sequences are contained in a compact set of $\mathbb{R}$ and C_P^1, respectively, so there exist, up to subsequences, $\overline{L} > 0$ – the estimate in (2.11) gives us that $\overline{L}$ is positive – and $\overline{x}$ such that $L_n \to \overline{L}$ and $x_n \to \overline{x}$ uniformly. Moreover, by the continuity of Θ, we have $\Theta(\overline{L}, \overline{x}) = 2\pi\nu/k$. We have to prove that $\overline{x}$ is a bouncing solution of the differential equation (2.2), where $L = \overline{L}$. The proof follows the same procedure as the one in [17, pp. 185–188] considering the sequence of approximating differential equations

$$x'' + f_n(t, x) = 0 \quad \text{with } f_n(t, x) = g_n(L_n, t, x).$$

For briefness we refer to that paper for the proof of this part.

2.3 Conclusion

In the previous section we have found, for every integer ν and for every integer k sufficiently large, a solution x of (2.2) and $L \in (0, L_0]$ such that $\Theta(L, x) = 2\pi\nu/k$. So, defining

$$\theta(t) = \theta_0 + \int_0^t \frac{L}{(R_0 + x(t))^2}\, dt$$

for a certain $\theta_0 \in [0, 2\pi)$, we have that (x, θ) is a bouncing periodic solution of (2.1), and using the change of coordinate $\rho = r + R_0$, we find the bouncing solution of (1.2) satisfying (1.3). The proof of Theorem 1.2 is thus completed.

3 Nonlinearities Next to Resonance, a Double Landesman–Lazer-Type Condition

In this section, we will see how we can relax the hypotheses of Theorem 1.2 in order to treat the situation of a nonlinearity which has an asymptotically linear growth next to resonance. We are going to provide a Landesman–Lazer-type condition for the case when the nonlinearity f satisfies (1.4) with $\check{\mu} = \mu_N$ and $\hat{\mu} = \mu_{N+1}$, where $\mu = \mu_j$ is the vertical asymptote of the j-th curve of the Dancer–Fučík spectrum. Such a situation has often been called *double resonance* (see, e.g., [8, 9, 13]). A *one-side* Landesman–Lazer condition for the scalar differential equation with singularity has been provided by Fonda and Garrione in [14]. This section has been inspired by this paper and some steps of the proof of Lemma 3.3 will appear similar. The main novelty occurs in the estimate in (3.12) and its proof, permitting us to treat a double resonance situation. In particular, the validity of (3.12) permits us to obtain a Landesman–Lazer condition involving the function

$$\psi_j(t) = \sin(\sqrt{\mu_j} t) \quad \text{with } t \in [0, \tfrac{T}{j}] \tag{3.1}$$

extended by periodicity to the whole real line, while, in [14], the Landesman–Lazer condition is weaker, being related to a function of the type

$$\tilde{\psi}_j(t) = \begin{cases} \sin(\sqrt{\mu_j} t), & t \in [0, \frac{T}{j}], \\ 0, & t \in [\frac{T}{j}, T] \end{cases} \tag{3.2}$$

extended by periodicity.

In this section, we will prove the following result.

Theorem 3.1. *Assume that there exists a constant $\hat{\eta}$ such that, for $N > 0$, there holds*

$$\mu_N x - \hat{\eta} \leq f(t,x) \leq \mu_{N+1} x + \hat{\eta}$$

for every $t \in [0,T]$ and every $x > R_0$. Moreover, for every $\tau \in [0,T]$ there holds

$$\int_0^T \limsup_{x\to+\infty} (f(t, x+R_0) - \mu_{N+1}x)\psi_{N+1}(t+\tau)\,dt < 0 \tag{3.3}$$

and

$$\int_0^T \liminf_{x\to+\infty} (f(t, x+R_0) - \mu_N x)\psi_N(t+\tau)\,dt > 0. \tag{3.4}$$

Then for every integer $\nu > 0$ there exists an integer $k_\nu > 0$ such that for every integer $k \geq k_\nu$ there exists at least one periodic R_0-bouncing solution (ρ, ϑ) of (1.2) *with period kT, which makes exactly ν revolutions around the origin in the period time kT, i.e. satisfying* (1.3).

The case $N = 0$ will be briefly treated at the end of this section.

First of all, we need to introduce a sequence of approximating equations with nonresonant nonlinearities. Then we will look for a common a priori bound.

It is possible to find, when n is chosen large enough, a constant $\kappa_n > 1$ such that

$$\mu_N < n\frac{(\sqrt{\kappa_n}-1)^2}{\kappa_n} < \mu_{N+1}.$$

Notice that $\lim_{n\to\infty}\kappa_n = 1$. In this way, we have

$$\begin{aligned}\frac{T}{(N+1)\pi} = \frac{1}{\sqrt{\mu_{N+1}}} &< \frac{1}{\sqrt{\kappa_n\mu_{N+1}}} + \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{\kappa_n\mu_N}} + \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{\mu_N}} = \frac{T}{N\pi}.\end{aligned} \tag{3.5}$$

We define the function g_n, similarly as in (2.3), as

$$g_n(L,t,x) = \begin{cases} \tilde{g}_n(L,t,x), & x \geq 1/n, \\ nx(\tilde{g}_n(L,t,x)+\delta) - \delta, & 0 < x < 1/n, \\ nx - \delta, & x \leq 0, \end{cases}$$

where

$$\tilde{g}_n(L,t,x) = -\frac{L^2}{(x+R_0)^3} + \kappa_n f(t, x+R_0).$$

We will consider the differential equation

$$x'' + g_n(L,t,x) = 0. \tag{3.6}$$

Notice that

$$\kappa_n\mu_N \leq \liminf_{\xi\to+\infty}\frac{g_n(L,t,\xi)}{\xi} \leq \limsup_{\xi\to+\infty}\frac{g_n(L,t,\xi)}{\xi} \leq \kappa_n\mu_{N+1},$$

thus giving us, by the previous computation in (3.5), that the nonlinearities g_n are nonresonant. So, if we find a common a priori bound Ω as in Lemma 2.2, uniform in L and n for every T-periodic solution of (3.6), then (2.10) holds, thus permitting us to end the proof of the theorem as in Sections 2.2 and 2.3. The Landesman–Lazer-type conditions introduced in (3.3) and (3.4) are needed in order to find the common a priori bound for every n sufficiently large.

The reader will notice that the next result comes out free by the proof of the previous theorem (setting $L = 0$ everywhere). Such a corollary extends a previous result provided by Fonda and the author in [17].

Corollary 3.2. *Assume that there exists a constant $\hat{\eta}$ such that*

$$\mu_N x - \hat{\eta} \le g(t,x) \le \mu_{N+1} x + \hat{\eta}$$

for every $t \in [0,T]$ and every $x > 0$. Moreover, for every $\tau \in [0,T]$ there holds

$$\int_0^T \limsup_{x\to+\infty} (g(t,x) - \mu_{N+1}x)\psi_{N+1}(t+\tau)\,dt < 0$$

and

$$\int_0^T \liminf_{x\to+\infty} (g(t,x) - \mu_N x)\psi_N(t+\tau)\,dt > 0.$$

Then there exists a 0-bouncing solution for the equation $x'' + g(t,x) = 0$.

Repeating the reasoning explained in Section 2.1, which provides us the estimate in (2.7), we can find for every $\varepsilon > 0$ a value $\chi_\varepsilon > 0$ such that every solution x of (3.6) satisfying $x^2 + x'^2 > \chi_\varepsilon^2$ when $x \ge 0$ must rotate in the phase plane spending a time $\tau = (t_3 - t_2) + (t_2 - t_1)$ with

$$t_3 - t_2 \in \Big(\frac{\pi}{\sqrt{\mu_{N+1}}} - \varepsilon, \frac{\pi}{\sqrt{\mu_N}} + \varepsilon\Big). \tag{3.7}$$

When ε is chosen sufficiently small, one has that a T-periodic solution x of (3.6) such that $x^2 + x'^2 > \chi_\varepsilon^2$ when $x \ge 0$ must perform exactly N or $N+1$ rotations around the origin. The previous reasoning holds uniformly for every n and every L, so we can fix such a suitable ε and find the constant $\bar{\chi} = \chi_\varepsilon$.

We underline that Lemma 2.1 still holds under the hypotheses of Theorem 3.1, too. We underline that (2.9) remains valid.

The needed a priori bound is given by the following lemma.

Lemma 3.3. *There exist $\bar{R} \ge R(\bar{\chi})$ (given by Lemma 2.1) and $\bar{n} \ge n_0(\bar{R})$, as in (2.6), such that, for every $n > \bar{n}$ and every $L \in [0, L_0]$, any T-periodic solution x of (3.6) is such that $(x(t), x'(t)) \in B^+_{\bar{R}}$ when $x(t) \ge 0$. In particular, we immediately have $x \in \Omega^{\bar{n}}_{\bar{R}}$.*

Proof. In order to get a contradiction, suppose that there exist an increasing sequence $(R_m)_m$ with $R_m > R(\bar{\chi})$ and $\lim_m R_m = +\infty$, an increasing sequence $(n_m)_m$ of integers $n_m > n_0(R_m)$, a sequence $(L_m)_m \subset [0, L_0]$, a sequence $(x_m)_m$ of solutions to

$$x_m'' + g_{n_m}(L_m, t, x_m) = 0, \tag{3.8}$$

and a sequence of times $(t_m^1)_m \subset [0,T]$ such that $(x(t_m^1), x'(t_m^1)) \notin B_{R_m}$ and $x(t_m^1) \ge 0$.

Since $R_m > R(\bar{\chi})$, thanks to Lemma 2.1, the solutions cannot enter $B^+_{\bar{\chi}}$, so that they must perform exactly N or $N+1$ rotations around the origin.

We define the sequence of functions

$$v_m = \frac{x_m}{\|x_m\|_\infty}$$

which are solutions of

$$v_m(t)'' + \frac{g_{n_m}(L_m, t, x_m(t))}{\|x_m\|_\infty} = 0. \tag{3.9}$$

By (2.9) we have

$$-\frac{1}{\sqrt{n_m}} \le v_m(t) \le 1 \quad \text{for every } t \in [0,T] \text{ and } \frac{1}{C_0} < \|v_m'\|_\infty < C_0$$

for a suitable $C_0 > 0$. We have immediately that $(v_m)_m$ is bounded in $H^1(0,T)$, so that, up to subsequences, we have $v_m \to v$ weakly in H^1 and uniformly; moreover, we can assume that $L_m \to L_\dagger$. In particular, $v \ne 0$ since $\|v\|_\infty = 1$, and it is non-negative and T-periodic.

We assume that, up to subsequences, all solutions make exactly $N+1$ rotations. We will discuss the other situation later on.

We are now going to prove that v solves $v'' + \mu_{N+1} v = 0$ for almost every t. Denote

$$\mathcal{I}^+ = \{t \in \mathbb{R} : v(t) > 0\},$$

which is an at most countable union of open intervals. Consider a function ϕ with compact support $K_\phi \subset \mathcal{I}^+$. Multiplying (3.9) by ϕ and integrating in K_ϕ, we obtain

$$\int_{K_\phi} v_m'(t)\phi'(t)\,dt = \int_{K_\phi} \frac{g_{n_m}(L_m, t, x_m(t))}{x_m(t)} v_m(t)\phi(t)\,dt. \tag{3.10}$$

By compactness we have $\min_{K_\phi} v > \hat{\delta}$ for a suitable $\hat{\delta} > 0$, thus giving us that $\lim_m x_m(t) = +\infty$ uniformly for $t \in K_\phi$. Assuming, up to subsequences, $x_m(t) > 1$ for every $t \in K_\phi$, we can find for every $j > 0$ an index m_j such that

$$\mu_N - \frac{1}{j} < \frac{g_{n_{m_j}}(L_{m_j}, t, x_{m_j}(t))}{x_{m_j}(t)} < \mu_{N+1} + \frac{1}{j}.$$

Hence, the subsequence

$$\left(\frac{g_{n_{m_j}}(L_{m_j}, t, x_{m_j}(t))}{x_{m_j}(t)}\right)_j$$

is bounded in $L^2(K_\phi)$, so up to a subsequence it converges weakly to a certain function $p(t)$ such that $\mu_N \le p(t) \le \mu_{N+1}$ for almost every $t \in K_\phi$ almost everywhere. Hence, passing to the limit in (3.10), we obtain

$$\int_{K_\phi} v'(t)\phi'(t)\,dt = \int_{K_\phi} p(t)v(t)\phi(t)\,dt.$$

It is possible to extend the function p to the whole set $\mathcal{I}^+$, so that

$$\int_{\mathcal{I}^+} v'(t)\phi'(t)\,dt = \int_{\mathcal{I}^+} p(t)v(t)\phi(t)\,dt,$$

thus giving us that v is a *weak* solution of $v'' + p(t)v = 0$ in $\mathcal{I}^+$. In particular, $v \in H^2_{\text{loc}}(\mathcal{I}^+)$ and $v \in C^1(\mathcal{I}^+)$. We must show that $p(t) = \mu_{N+1}$ for almost every $t \in \mathcal{I}^+$.

We recall that the functions x_m perform in the phase plane exactly $N + 1$ rotations around the origin, so there exist

$$\alpha_1^m < \beta_1^m < \alpha_2^m < \beta_2^m < \cdots < \alpha_{N+1}^m < \beta_{N+1}^m < \alpha_{N+2}^m = \alpha_1^m + T$$

such that, for every $r \in \{1, \ldots, N + 1\}$, there holds

$$\begin{aligned} x_m(t) > 0 \quad &\text{for every } t \in (\alpha_r^m, \beta_r^m), \\ x_m(t) < 0 \quad &\text{for every } t \in (\beta_r^m, \alpha_{r+1}^m). \end{aligned}$$

Up to subsequences, we can assume that $\alpha_r^m \to \check{\xi}_r$ and $\beta_r^m \to \hat{\xi}_r$ such that

$$\check{\xi}_1 \le \hat{\xi}_1 \le \check{\xi}_2 \le \hat{\xi}_2 \le \cdots \le \check{\xi}_{N+1} \le \hat{\xi}_{N+1} \le \check{\xi}_{N+2} = \check{\xi}_1 + T.$$

Since $\alpha_{r+1}^m - \beta_r^m < \pi/\sqrt{n_m}$, there holds $\check{\xi}_{r+1} = \hat{\xi}_r$. Moreover, using the estimate in (3.7), from

$$\beta_r^m - \alpha_r^m > \pi/\sqrt{\mu_{N+1}} - \varepsilon = \frac{T}{(N+1)} - \varepsilon$$

we have

$$\hat{\xi}_r - \hat{\xi}_{r-1} = \frac{T}{(N+1)}.$$

From $v_n(\alpha_r^m) = v_m(\beta_r^m) = 0$ we have $v(\xi_r) = 0$, where we denote $\xi_r = \hat{\xi}_r = \check{\xi}_{r+1}$.

We now consider an interval $[\alpha, \beta]$ with $v(\alpha) = v(\beta) = 0$ and $v(t) > 0$ in (α, β). Writing in polar coordinates

$$\begin{cases} v(t) = \hat{\rho}(t)\cos(\hat{\vartheta}(t)), \\ v'(t) = \hat{\rho}(t)\sin(\hat{\vartheta}(t)), \end{cases}$$

we obtain the expression of the angular velocity of the solution in the phase plane

$$-\hat{\vartheta}'(t) = \frac{p(t)v(t)^2 + v'(t)^2}{v(t)^2 + v'(t)^2}.$$

Thus obtaining

$$\frac{-\hat{\vartheta}'(t)}{\mu_{N+1}\cos^2(\hat{\vartheta}(t)) + \sin^2(\hat{\vartheta}(t))} \le 1 \le \frac{-\hat{\vartheta}'(t)}{\mu_N\cos^2(\hat{\vartheta}(t)) + \sin^2(\hat{\vartheta}(t))},$$

so that

$$\frac{T}{N+1} \le \beta - \alpha \le \frac{T}{N}.$$

So the only reasonable conclusion is that $[\alpha, \beta] = [\xi_r, \xi_{r+1}]$ for a certain index r. Passing to modified polar coordinates

$$\begin{cases} v(t) = \dfrac{1}{\sqrt{\mu_{N+1}}}\tilde{\rho}(t)\cos(\tilde{\vartheta}(t)), \\ v'(t) = \tilde{\rho}(t)\sin(\tilde{\vartheta}(t)), \end{cases}$$

and integrating $-\tilde{\vartheta}'$ on $[\alpha, \beta]$, we obtain

$$\pi = \sqrt{\mu_{N+1}}\int_\alpha^\beta \frac{p(t)v(t)^2 + v'(t)^2}{\mu_{N+1}v(t)^2 + v'(t)^2}\,dt \le \sqrt{\mu_{N+1}}\frac{T}{N+1} = \pi,$$

thus giving us $p(t) = \mu_{N+1}$ for almost every $t \in [\alpha, \beta]$. In particular, in every interval $[\xi_r, \xi_{r+1}]$ we have

$$v(t) = c_r \sin\big(\sqrt{\mu_{N+1}}(t - \xi_r)\big) \tag{3.11}$$

with $c_r \in [0, 1]$ and at least one of them is equal to 1 since $\|v\|_\infty = 1$.

We now prove that

$$c_r = 1 \quad \text{for every } r \in \{1, \dots, N+1\}. \tag{3.12}$$

The functions v_m solve equation (3.9), which we rewrite in the simpler form

$$v_m'' + h_m(t, v_m) = 0 \quad \text{with } h_m(t, v) = \frac{g_{n_m}(L_m, t, \|x_m\|_\infty v)}{\|x_m\|_\infty},$$

where, for every m, we have

$$|h_m(t, v)| \le d(v + 1) \quad \text{for every } t \in [0, T] \text{ and } v \ge 0 \tag{3.13}$$

for a suitable constant $d > 0$.

We show that if v is positive in $[a, b] \subset [0, T]$, then v_m C^1-converges to v in this interval. We have already seen that $(v_m)_m$ is bounded in C^1, and by (3.13) we get $|v_m''(t)| \le |h_m(t, v_m)| \le 2d$ for every $t \in [a, b]$ as an immediate consequence. So, since v_m is bounded in C^2 in such an interval, by the Ascoli–Arzelà theorem, we have that v_m C^1-converges to v in $[a, b]$.

The C^1-convergence and the estimate in (3.13) are the ingredients we need to prove that the solution v has only isolated zeros ξ_r.

We start by proving that if the left derivative $-v'(\xi_r^-) = \eta > 0$ for a certain index r, then $-v_m'(\beta_r^m) > \eta/2$ for m large enough. For every $\epsilon_0 > 0$ we can find $0 < s_1 < s_2$ sufficiently small to have

$$\frac{1}{2}\epsilon_0 < v(\xi_r - s) < \frac{3}{2}\epsilon_0 \quad \text{and} \quad |v'(\xi_r - s) + \eta| < \epsilon_0$$

for every $s \in (s_1, s_2)$. Since v_m is C^1-convergent to v in $[\xi_r - s_2, \xi_r - s_1]$, for m large enough, we have

$$\frac{1}{2}\epsilon_0 < v_m(\xi_r - s) < \frac{3}{2}\epsilon_0 \quad \text{and} \quad |v_m'(\xi_r - s) + \eta| < 2\epsilon_0$$

for every $s \in (s_1, s_2)$. Since $|v_m''| \le 2d$ in this interval, we find

$$v_m'^2(\beta_r^m) \ge (\eta - 2\epsilon_0)^2 - 6d\epsilon_0 > \eta^2/4,$$

choosing ϵ_0 sufficiently small.

We prove now that if $-v'(\xi_r^-) = \eta > 0$ for a certain index r, then ξ_r is an isolated zero of v. Suppose by contradiction that there exists $\varepsilon_0 \in (0, \eta/8d)$, with d as in (3.13), such that $v(\xi_r + \varepsilon_0) = 0$. For every m large enough we have $|\alpha_{r+1}^m - \xi_r| < \varepsilon_0/4$ and by the previous computation $v_m'(\alpha_{r+1}^m) = -v_m'(\beta_r^m) > \eta/2$.

The property that $v_m'(\alpha_{r+1}^m) = -v_m'(\beta_r^m)$, which follows directly by the fact that the nonlinearities g_n do not depend on t when $x < 0$, leads us to treat a Landesman–Lazer condition involving function ψ_j as in (3.1) rather than as in (3.2). Here lies one of the main differences between our result and the one obtained by Fonda and Garrione in [14] (see in particular [14, Remark 2.5]).

Since $|v_m''| \le 2d$ when v_m is positive, we can show that if $s < \eta/4d$, then $v_m(\alpha_{r+1}^m + s) > s\eta/4$. By construction $\xi_r + \varepsilon_0 = \alpha_{r+1}^m + s_0$ for a certain $s_0 \in (\varepsilon_0/2, \eta/4d)$, so we obtain $v_m(\xi_r + \varepsilon_0) = v_m(\alpha_{r+1}^m + s_0) > \eta\varepsilon_0/8$ for every m large enough, thus contradicting $v_m \to v$.

With v as in (3.11), and $c_r = 1$ for at least one value $r \in \{1, \dots, N+1\}$, for such an index $v'(\xi_{r+1}^-) < 0$ holds. The previous reasoning gives us that $c_{r+1} > 0$. Iterating the procedure, we can prove that $c_r > 0$ for every index $r \in \{1, \dots, N+1\}$.

We prove now that for every $r \in \{1, \dots, N+1\}$ the left and right derivatives satisfy $-v'(\xi_r^-) = v'(\xi_r^+)$, thus we can conclude that $c_r = 1$ for every $r \in \{1, \dots, N+1\}$.

Suppose by contradiction that there exists $r \in \{1, \dots, N+1\}$ such that

$$v'(\xi_r^+) + v'(\xi_r^-) \ne 0.$$

Without loss of generality we suppose this value to be positive. The other case follows similarly. So, assume

$$v'(\xi_r^+) + v'(\xi_r^-) > p_0 > 0 \quad \text{and} \quad 0 < q_0 \le \min\{-v'(\xi_r^-), v'(\xi_r^+)\}.$$

Arguing as above, for every $\epsilon_0 > 0$ we can find $0 < s_1 < s_2$ sufficiently small to have

$$\frac{1}{2}\epsilon_0 < v(\xi_r \pm s) < \frac{3}{2}\epsilon_0 \quad \text{and} \quad |v'(\xi_r \pm s) - v'(\xi_r^\pm)| < \epsilon_0$$

for every $s \in (s_1, s_2)$. Since v_m is C^1-convergent to v in $[\xi_r - s_2, \xi_r - s_1]$ and in $[\xi_r + s_1, \xi_r + s_2]$, for m large enough, we have

$$\frac{1}{2}\epsilon_0 < v_m(\xi_r \pm s) < \frac{3}{2}\epsilon_0 \quad \text{and} \quad |v_m'(\xi_r \pm s) - v'(\xi_r^\pm)| < 2\epsilon_0$$

for every $s \in (s_1, s_2)$. Since $|v_m''| \le 2d$, we find

$$[v'(\xi_r^-) - 2\epsilon_0]^2 + 4\epsilon_0 d \ge v_m'^2(\beta_r^m) = v_m'^2(\alpha_{r+1}^m) \ge [v'(\xi_r^+) - 2\epsilon_0]^2 - 4\epsilon_0 d.$$

Hence

$$0 < p_0 < v'(\xi_r^+) + v'(\xi_r^-) \le \frac{4\epsilon_0 d}{q_0} + 4\epsilon_0,$$

thus giving us a contradiction for ϵ_0 sufficiently small.

We have proved that v, in every interval $[\alpha, \beta] = [\xi_r, \xi_{r+1}]$, satisfies

$$v(t) = \sin(\sqrt{\mu_{N+1}}(t - \alpha)).$$

So, v is a solution of the following Dirichlet problem:

$$\begin{cases} v'' + \mu_{N+1} v = 0, \\ v(\alpha) = 0, \quad v(\beta) = 0. \end{cases}$$

Let us consider the orthonormal basis $(\phi_k)_k$ of $L^2(\alpha,\beta)$ made of the eigenfunctions solving the Dirichlet problem

$$\begin{cases} \phi_k'' + \mu_k\phi_k = 0, \\ \qquad \phi_k(\alpha) = 0, \quad \phi_k(\beta) = 0, \end{cases}$$

where $\mu_k = (k\pi/T)^2$ is the k-th eigenvalue. Denoting by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ the scalar product and the norm in $L^2(\alpha,\beta)$, respectively, we can write the Fourier series of all functions x_n and split it as follows:

$$x_m = \sum_{k=1}^{\infty}\langle x_m,\phi_k\rangle\phi_k = \underbrace{\langle x_m,\phi_{N+1}\rangle\phi_{N+1}}_{x_m^0} + \underbrace{\sum_{k\neq N+1}\langle x_m,\phi_k\rangle\phi_k}_{x_m^\perp}$$

with the property

$$(x_m'')^0 = (x_m^0)'' \quad\text{and}\quad (x_m'')^\perp = (x_m^\perp)''.$$

Moreover, one has $v_m = v_m^0 + v_m^\perp$ with

$$v_m^0 = \frac{x_m^0}{\|x_m\|_\infty} \quad\text{and}\quad v_m^\perp = \frac{x_m^\perp}{\|x_m\|_\infty}.$$

Since $v = \|v\|_{L^2(\alpha,\beta)}\phi_{N+1}$, we have $v_m^0 \to v$ uniformly in $[\alpha,\beta]$. Moreover, $v_m^0 \geq 0$ for m sufficiently large.

Multiplying equation (3.8) by v_m^0 and integrating in the interval $[\alpha,\beta]$, we obtain

$$\begin{aligned}\int_\alpha^\beta g_{n_m}(L_m,t,x_m(t))v_m^0(t)\,dt &= -\int_\alpha^\beta (x_m^0)''(t)v_m^0(t)\,dt \\ &= -\int_\alpha^\beta x_m^0(t)(v_m^0)''(t)\,dt \\ &= \int_\alpha^\beta \mu_{N+1}x_m^0(t)v_m^0(t)\,dt \\ &= \int_\alpha^\beta \mu_{N+1}x_m(t)v_m^0(t)\,dt.\end{aligned}$$

Defining $r_m(t,x) = g_{n_m}(L_m,t,x) - \mu_{N+1}x$, we have

$$\int_\alpha^\beta r_m(t,x_m(t))v_m^0(t)\,dt = 0,$$

and applying Fatou's lemma, we have

$$\int_\alpha^\beta \limsup_{m\to\infty} r_m(t,x_m(t))v_m^0(t)\,dt \geq 0.$$

It is easy to see that for every $s_0 \in (\alpha,\beta)$ it is possible to find $m(s_0)$ such that $x_m(s_0) > 1$ for every $m > m(s_0)$. Hence, since $v_m^0 \to v$ and $L_m \to L_\dagger$, we have

$$\int_\alpha^\beta \limsup_{x\to+\infty}[f(t,x+R_0) - \mu_{N+1}x]v(t)\,dt \geq 0. \tag{3.14}$$

The previous estimate can be obtained for every interval (ξ_r,ξ_{r+1}), thus contradicting (3.3) by setting $\tau = \xi_1$.

We finished the proof of the case in which the sequence $(x_m)_m$ consists of solutions performing $N+1$ rotations in the phase plane around the origin for an infinite number of index m. We now treat the case in which the solutions perform N rotations around the origin.

In this case we have to prove that the limit function v solves $v'' + \mu_N v = 0$ for almost every t. With the same procedure we can show that there exists an L^2-function $q(t)$, satisfying $\mu_N \leq q(t) \leq \mu_{N+1}$ almost everywhere, such that v is a weak solution of $v'' + q(t)v = 0$. Then, with a similar procedure, it is possible to introduce some instants α_r^m, β_r^m, when v_m vanishes, converging to some values ξ_r (with $r \in \{1, \dots, N\}$) satisfying $\xi_{r+1} - \xi_r = T/N$. Now, with a similar procedure, we can prove that $q(t) = \mu_N$, thus giving us that $v(t) = c_r \sin(\sqrt{\mu_N}(t - \xi_r))$ in every interval $[\xi_r, \xi_{r+1}]$. Also in this case, in the same way, it is possible to conclude that $c_r = 1$ for every $r \in \{1, \dots, N\}$. In fact, we have that v_m C^1-converges to v when v is positive, and arguing as above, we can prove that whenever $v'(\xi_r^-) < 0$ for a certain index $r \in \{1, \dots, N\}$, then ξ_r is an isolated zero and $v'(\xi_r^+) = -v'(\xi_r^-) > 0$. Then we can consider an interval $[\alpha, \beta]$ with $v > 0$ in (α, β), and with a similar reasoning we can obtain a *liminf* estimate similar to the one obtained in (3.14), thus gaining a contradiction to assumption (3.4). The lemma is thus proved. □

Let us spend a few words about the possibility of extending Theorem 3.1 to the case $N = 0$, where $\mu_0 = 0$ and $\mu_1 = (\pi/T)^2$. There is extensive literature (cf. [11, 12, 23, 24, 29]) treating nonlinearities *lying* under the first curve of the Dancer–Fučík spectrum. The Landesman–Lazer condition (3.4) in this case reduces to a sign condition on the nonlinearity f. Unfortunately, it is not possible to obtain a proof with the same procedure; cf. the estimate in (2.8). For briefness we do not enter in such details in this paper. However, let us state the following weaker result for a nonlinearity with a *one-side* resonance condition, the proof of which works similarly to the one of Theorem 3.1.

Theorem 3.4. *Assume that there exists a constant $\bar{\varepsilon}$ such that*

$$\liminf_{x\to\infty} \frac{f(t,x)}{x} \geq \bar{\varepsilon} > 0$$

uniformly for every $t \in [0, T]$ and that there exists a constant $\hat{\eta}$ such that, for $N > 0$, there holds

$$f(t,x) \leq \mu_1 x + \hat{\eta}$$

for every $t \in [0, T]$ and every $x > R_0$. Moreover,

$$\int_0^T \liminf_{x\to+\infty} (f(t, x + R_0) - \mu_1 x)\psi_1(t+\tau)\, dt > 0$$

for every $\tau \in [0, T]$. Then for every integer $\nu > 0$ there exists an integer $k_\nu > 0$ such that for every integer $k \geq k_\nu$ there exists at least one periodic R_0-bouncing solution (ρ, ϑ) of (1.2) *with period kT, which makes exactly ν revolutions around the origin in the period time kT, i.e. satisfying* (1.3).

4 Systems on Cylinders

In this section, we briefly explain how the previous results could be applied to a class of systems defined in $\mathbb{R}^{d_1+d_2}$ which model a particle hitting a cylinder $\mathbb{S}^{d_1-1} \times \mathbb{R}^{d_2}$. The case $d_1 = 2$ and $d_2 = 1$ models bounces on a proper cylinder. For briefness we will present the result for *nonresonant* nonlinearities. We consider the differential equations

$$\begin{cases} \mathbf{x}'' + f_1(t, |\mathbf{x}|)\mathbf{x} + \mathbf{b}_1(t, \mathbf{x}, \mathbf{y}) = 0, \\ \mathbf{y}'' + \mathbf{f}_2(t, \mathbf{y}) + \mathbf{b}_2(t, \mathbf{x}, \mathbf{y}) = 0, \end{cases} \tag{4.1}$$

where $\mathbf{x} \in \mathbb{S}^{d_1-1}$ and $\mathbf{y} \in \mathbb{R}^{d_2}$. For the sake of simplicity we assume all functions to be continuous. We suppose that $f = f_1(t,\rho)\rho$ satisfies the assumptions of Theorem 1.2. The function $\mathbf{b}_1 : \mathbb{R}^{1+d_1+d_2} \to \mathbb{R}^{d_1}$ satisfies $\mathbf{b}_1(t, \mathbf{x}, \mathbf{y}) = b_1(t, \mathbf{x}, \mathbf{y})\mathbf{x}$ with

$$\lim_{|\mathbf{x}|\to\infty} \frac{b_1(t, \mathbf{x}, \mathbf{y})}{|\mathbf{x}|} = 0$$

uniformly in t and $\mathbf{y}$. Assume that the second equation in (4.1) can be viewed, in every component, as

$$y_i'' + f_{2,i}(t, y_i) + b_{2,i}(t, \mathbf{x}, \mathbf{y}) = 0,$$

where

$$\lim_{|y_i|\to\infty} \frac{b_{2,i}(t, \mathbf{x}, \mathbf{y})}{y_i} = 0$$

uniformly in all other variables. We can assume, as an example of application,

$$\check{\mu}_i \le \liminf_{y_i\to+\infty} \frac{f_{2,i}(t, y_i)}{y_i} \le \limsup_{y_i\to+\infty} \frac{f_{2,i}(t, y_i)}{y_i} \le \hat{\mu}_i,$$
$$\check{\nu}_i \le \liminf_{y_i\to-\infty} \frac{f_{2,i}(t, y_i)}{y_i} \le \limsup_{y_i\to-\infty} \frac{f_{2,i}(t, y_i)}{y_i} \le \hat{\nu}_i$$

uniformly in t, with

$$\frac{T}{(N_i+1)\pi} < \frac{1}{\sqrt{\hat{\mu}_i}} + \frac{1}{\sqrt{\hat{\nu}_i}} \le \frac{1}{\sqrt{\check{\mu}_i}} + \frac{1}{\sqrt{\check{\nu}_i}} < \frac{T}{N_i\pi}$$

for some positive constants $\check{\mu}_i, \check{\nu}_i, \hat{\mu}_i, \hat{\nu}_i$, and an integer $N_i > 0$.

Theorem 4.1. *Under the previous assumptions, for every integer $\ell > 0$ there exists an integer $k_\ell > 0$ such that for every integer $k \ge k_\ell$ there exists at least one periodic solution $(\mathbf{x}, \mathbf{y})$ of (4.1) such that $\mathbf{x}$ can be parametrized in polar coordinates (ρ, ϑ) and ρ is a R_0-bouncing solution. Such solutions satisfy the following periodicity conditions:*

$$\rho(t+T) = \rho(t),$$
$$\vartheta(t+kT) = \vartheta(t) + 2\pi\nu,$$
$$\mathbf{y}(t+T) = \mathbf{y}(t).$$

The proof of such a result can be obtained by gluing together the results contained in this paper (for the $\mathbf{x}$ coordinate) and classical results (for the $\mathbf{y}$ coordinate). The key tool is the fact that the equations are *weakly* coupled.

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