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Research Article

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Relative Nielsen Numbers, Braids and Periodic Segments

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Abstract: The aim of this paper is to establish a connection between the method of period segments and the relative Nielsen fixed point theory. We prove that if W is a periodic segment over $[0, T]$ for the T-periodic semiprocess Φ, then the Poincaré map P has at least $N(m_W, \overline{W_0 \setminus W_0^{--}})$ fixed points with trajectories contained in W, where $N(m_W, \overline{W_0 \setminus W_0^{--}})$ is the relative Nielsen number defined by Zhao. It is also shown that if the sequence $N(\overline{m}^n)$ is bounded and $N^\infty(m) > 1$, then the Poincaré map has infinitely many periodic points. We prove that there exists a compact set $I \subset W_0$, invariant for the Poincaré map, such that the topological entropy $h(P|_I)$ is bounded from below by $\log N^\infty(m) - h(\overline{m})$. In particular, if $h(\overline{m}) = 0$, then $h(P|_I) \geq \log N^\infty(m)$. We adapt the result obtained by Jiang to get a concrete example of a braid-like periodic segment with $N^\infty(m) > 1$.

Keywords: Nielsen Number, Lefschetz Number, Periodic Segments, Braid Group, Topological Entropy

MSC 2010: Primary 54H20; secondary 37B40, 20F36

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1 Introduction

The notion of periodic segments introduced by Srzednicki proved to be a very useful tool for detecting periodic solutions and chaotic dynamics generated by periodic in time non-autonomous ODEs [17, 18]. If $v\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth time-dependent vector field on $\mathbb{R}^n$, then the ordinary differential equation

$$\dot{t} = 1, \quad \dot{x} = v(t, x),$$

generates a local flow on the extended phase space $\mathbb{R} \times \mathbb{R}^n$. If f is T-periodic with respect to time, then the Poincaré map P associated to the vector field v (defined on some open subset of $\mathbb{R}^n$) is given as follows: $P(x_0)$ is the value of the solution of the problem

$$\dot{x} = v(t, x), \quad x(0) = x_0,$$

at time T. The k-periodic points of the Poincaré map P correspond to the kT-periodic solutions of $\dot{x} = v(t, x)$.

The periodic segment W is a compact subset of $[0, T] \times \mathbb{R}^n$ with some special behavior of the vector field $(1, v)$ on the boundary of W. Namely, the set of points (called the exit set W^-) on the boundary of W at which the vector field $(1, v)$ is pointing out with respect to the segment is closed. Periodicity of the segment means that the time 0 and time T sections of W are equal, $W_0 = W_T$. Moreover, there exists a compact set $W^{--} \subset W^-$ (called *the essential exit set*) such that $W_0^{--} = W_T^{--}$, and (W, W^{--}) is a pair of trivial bundles over $[0, T]$ with the fiber (W_0, W_0^{--}). Intuitively, W consists of the left-hand side $\{0\} \times W_0$, the right-hand side $\{T\} \times W_T = \{T\} \times W_0$, and the main part located over the open interval $(0, T)$. Because of the specific behavior of the flow (it moves along the time-axis with speed 1), it is clear that the right-hand side of W must belong to the exit set (see Figure 1).

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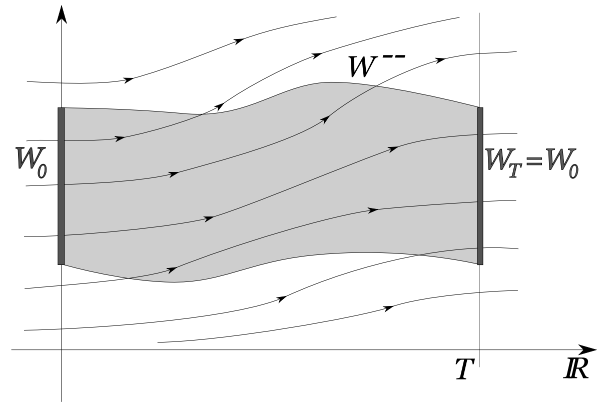

Figure 1. The periodic segment W over $[0, T]$ with the essential exit set W^{--}.

There exists a relative homeomorphism $m_W: (W_0, W_0^{--}) \to (W_0, W_0^{--})$ associated to the periodic segment W, called the *monodromy map*. It was proved by Srzednicki [18] that there exists an isolated set of fixed points F^k of P^k contained in $W_0 \setminus W_0^{--}$ such that

$$\operatorname{ind}(P^k, F^k) = L(m_W^k) = L(m^k) - L(\overline{m}^k),$$

where $m := W_0 \ni x \to m_W(x) \in W_0$, $\overline{m} = m|_{W_0^{--}}$. In particular, if $L(m_W) \neq 0$, then the Poincaré map has a fixed point x whose trajectory (with respect to the local flow on the extended phase space) is contained in the segment W. The proof of Srzednicki's fixed point index formula is based on the Lefschetz fixed point theorem and the properties of the fixed point index.

The problem of using other topological invariants to get more information on the structure of periodic solutions inside the segment was proposed in [16]. In addition to the existence of fixed points of the Poincaré map P, we are interested in finding a lower bound of the cardinality of the set of fixed points $\operatorname{Fix}(P)$. From that point of view, it seems to be very natural to try to find a connection between the method of periodic segments and the theory of Nielsen numbers (see [20]). The Nielsen fixed point theory is concerned with the determination of the minimal number of fixed points for all maps in the homotopy class of a given self map $f: X \to X$ (X is a compact ENR). The Nielsen number $N(f)$ provides a homotopy invariant lower bound for the number of fixed points of f.

The classical Nielsen number $N(f)$ is rather poor lower bound for the number of fixed point of f if $f: (X, A) \to (X, A)$ is a relative map. For example, let X be a 2-dimensional disk and let A be the circle bounding it. If $f: (X, A) \to (X, A)$ is a continuous map such that $\overline{f} := f|_A$ has degree d, then f has at least $|d - 1|$ fixed points in A. On the other hand, $N(f) = 1$. In 1980, Jiang observed the following phenomena in the problem concerning fixed point sets on the pants. Let P be the pants, i.e., the disk with two holes removed, and let f be the homeomorphism obtained by reflection on an axis of symmetry which interchanges the boundaries of the two holes. Then $N(f) = 1$, but any homeomorphism isotopic to f will map the outer boundary of P onto itself in an orientation-reversing manner, and hence have at least two fixed point on this boundary circle. Thus, $N(f)$ cannot be realized by a homeomorphism in the isotopy class.

In 1986, an extension of Nielsen theory to the relative setting was introduced by Schirmer in [14] and has developed rapidly since then ([25] is a very interesting survey on the subject). In this paper we will use the relative Nielsen number of $f: (X, A) \to (X, A)$ on the closure of the complement $N(f; \overline{X \setminus A})$, defined by Zhao [23–25].

We show (Theorem 4.4) that if W is a periodic segment over $[0, T]$, then the Poincaré map P has at least $N(m, \overline{W_0 \setminus W_0^{--}})$ fixed points with trajectories contained in the segment W. We give an example showing that $N(m, \overline{W_0 \setminus W_0^{--}})$ cannot be replaced by the relative Nielsen number $N(m; W_0, W_0^{--})$, defined by Schirmer. We will also study the relation between the number of periodic points of the Poincaré map P and the asymptotic Nielsen number $N^\infty(m)$ defined and developed by Jiang (see [6, 7, 9]). We prove (Theorem 5.2) that if the sequence $N(\overline{m}^n)$ is bounded and $N^\infty(m) > 1$, then the Poincaré map has infinitely many periodic points. We also show (Theorem 5.3) that there exists a compact set $I \subset W_0$, invariant for the Poincaré map, such that the topological entropy $h(P|_I)$ is bounded from below by $\log N^\infty(m) - h(\overline{m})$. In particular, if $h(\overline{m}) = 0$, then

$$h(P|_I) \geq \log N^\infty(m).$$

The usefulness of our main results (Theorems 5.2 and 5.3) depends on the possibility of computation of $N^\infty(m)$. The setting in which a great work has been carried out is that of homeomorphisms of compact surfaces. The central area in the topological dynamics of surface homeomorphism is their classification up to isotopy, due to Nielsen and Thurston (see [8, 12, 19]). Given a homeomorphism f (relative to some given finite f-invariant subset A) of a compact surface X, perhaps with boundary ∂X, there exists a canonical Thurston representative g in the isotopy class of f that is one of the following three types: finite order (so $g^n = \mathrm{id}$ for some $n \geq 1$), pseudo-Anosov, or reducible. In the third case, the surface may be cut up into subsurfaces along a tubular neighborhood of a finite g-invariant set of mutually-disjoint curves, and the restriction of an appropriate iterate of g to each subsurfaces is either finite order or pseudo-Anosov. Given the action of f on the fundamental group $\pi_1(X \setminus A)$, one can effectively decide its Thurston type using an algorithm due to Bestvina and Handel [2]. If W_0 is a compact surface with boundary, then the Thurston–Nielsen canonical form decomposes m into periodic and pseudo-Anosov pieces where there is a stretching factor λ which describes each pseudo-Anosov piece. It was proved by Jiang (see [6, 7, 9]) that if the Euler–Poincaré characteristic W_0 is negative, then $N^\infty(m)$ equals the largest such stretching factor $\lambda > 1$ ($\lambda := 1$ if there is no pseudo-Anosov piece). In particular, if there is at least one pseudo-Anosov piece, then $N^\infty(m) > 1$.

As a natural possible area of applications, we will study braid-like periodic segments for periodic in time ODEs on the plane $\mathbb{R}^2$. Using the technique developed in [8, 9], we give an example of a periodic braid-like segment with $N^\infty(m) > 1$. It is based on result of Fadell and Husseini (see [4]), using the Fox's free differential calculus.

The study of geometric braids in the context of the existence of periodic solutions of periodic system of differential equations on the plane was started by Matsuoka in [10–13] (see also [1, 8]). Based on the braiding information of known solutions, he obtained lower bounds for the number of extra periodic solutions.

This approach has one failing from the point of view of the applications to dynamics generated by ordinary differential equations. It is assumed that every solution of the equation $\dot{x} = v(t, x)$ extends forever in both directions of time. In particular, the Poincaré map P has to be defined globally as a diffeomorphism $P\colon \mathbb{R}^2 \to \mathbb{R}^2$. Matsuoka's method applies also to the case of dissipative systems, when the Poincaré map P has a closed disk D such that $P(D) \subset D$. It is rather a rare phenomenon in the concrete examples of ODEs. The advantage of our approach lies in the fact that we consider the local flows, so the solutions can blow up to infinity in finite time and the Poincaré map does not have to be globally defined. Moreover, we allowed the periodic segments with the non-empty essential exit sets. It should be stressed that Matsuoka's approach and the method described in [6, 7, 9] cannot be directly applied to study the number of periodic points of the Poincaré map restricted to W_0 if $W_0^{--} \neq \emptyset$. Our viewpoint sheds some new light on the structure of periodic orbits in that more general case.

We emphasize that if W_0 is contractible, then the Nielsen numbers $N(m^n, \overline{W_0 \setminus W_0^{--}})$ do not give more information on the number of periodic points than the Lefschetz numbers $L(m_W^n)$. We show that in some examples of chaotic planar periodic ODEs, the number of k-periodic points of the Poincaré map is bounded from below by $|L((I - \mu_W)^k)|$, where μ_W is induced by m_W in homologies.

The paper is organized as follows. In Section 2 we have compiled some basic facts concerning Nielsen fixed point theory. In Section 3 we introduce the notion of a periodic segment. Section 4 establishes the relation between the method of periodic segments and Nielsen theory. In Section 5 will be concerned with the topological entropy of the Poincaré map. Section 6 is devoted to studying braid-like periodic segments. In Section 7 we give an example of a braid-like periodic segment that forces complicated dynamics. Section 8 deals with the case of contractible W_0. For the convenience of the reader, in Appendix A we give a brief survey of the used results in the Nielsen theory based on [8].

2 Relative Nielsen Numbers

In this section we summarize without proofs the relevant material on the Nielsen theory. Let X be a compact ENR and let $f\colon X \to X$ be a continuous map. The fixed point set of f,

$$\mathrm{Fix}(f) := \{x \in X : f(x) = x\},$$

splits into a disjoint union of fixed point classes – two fixed points are in the same class F if and only if they can be joined by a path which is homotopic (rel end-points) to its own f-image. More precisely, $x_0, x_1 \in \mathrm{Fix}(f)$ are in the Nielsen relation if there exists a continuous map $\alpha\colon [0,1] \to X$ such that $\alpha(i) = x_i$ for $i = 0, 1$ and

$$\alpha \simeq f \circ \alpha, \quad \mathrm{rel}\{0,1\}.$$

Then

$$\mathrm{Fix}(f) = F_1 \cup \cdots \cup F_k,$$

where F_i are Nielsen fixed point classes. Each Nielsen class is an isolated set of fixed points of f, so the fixed point index $\mathrm{ind}(f, F_i) \in \mathbb{Z}$ is defined. A fixed point class F is called *essential* if $\mathrm{ind}(f, F) \neq 0$. The *Nielsen number* $N(f)$ of f is defined as the number of essential fixed point classes. Every map homotopic to f has at least $N(f)$ fixed points.

Let $f\colon (X, A) \to (X, A)$ be a continuous map of the pair (X, A) of compact ENRs. The restriction of f to A is written as $\bar{f} := f|_A\colon A \to A$.

Definition 2.1 (Common Fixed Point Class). A fixed point class F of f is said to be a *common* fixed point class if it contains an essential fixed point class of $\bar{f}$.

Example 2.2 (Non-Common Fixed Point Class). Let $f\colon (\mathbb{D}, \mathbb{S}^1) \to (\mathbb{D}, \mathbb{S}^1)$ be the identity map on the 2-disk $\mathbb{D}$. Then f has one fixed point class $F = \mathbb{B}^2$. It is an essential fixed point class because $\mathrm{ind}(f, F) = L(f) = 1$. The restricted map $\bar{f} = \mathrm{id}_{\mathbb{S}^1}$ has one fixed point class $\mathbb{S}^1$, and it is not essential because $\mathrm{ind}(\bar{f}, \mathbb{S}^1) = L(\mathrm{id}_{\mathbb{S}^1}) = 0$. It follows that F does not contain the essential class of $\bar{f}$, so F is not common.

Example 2.3 (Common Fixed Point Class Does Not Have to Be Essential). Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the identity and let $A = \{1\}$. Then f has one fixed point class $F = \mathbb{S}^1$ which is inessential because $\mathrm{ind}(\mathrm{id}_{\mathbb{S}^1}, \mathbb{S}^1) = L(\mathrm{id}_{\mathbb{S}^1}) = 0$. Since $A = \{1\}$ is an essential fixed point class of $\bar{f}$, we have that F is a common fixed point class.

By $N(f, \bar{f})$ we denote the number of common and essential fixed point classes of f. It follows that

$$N(f, \bar{f}) \leq N(f), \quad N(f, \bar{f}) \leq N(\bar{f}).$$

Definition 2.4 (Relative Nielsen Number). Let $f\colon (X, A) \to (X, A)$ be a relative map. The *relative Nielsen number* of f is defined by

$$N(f; X, A) := N(f) + N(\bar{f}) - N(f, \bar{f}).$$

Theorem 2.5 (Lower Bound). *Any relative map $f\colon (X, A) \to (X, A)$ has at least $N(f; X, A)$ fixed points.*

Definition 2.6. We say that a fixed point class F of $f\colon X \to X$ *assumes its index* in A if

$$\mathrm{ind}(f, F) = \mathrm{ind}(\bar{f}, F \cap A).$$

The number of fixed point classes of f which do not assume their indices in A is denoted by $N(f; \overline{X \setminus A})$ and is called the *relative Nielsen number of f on the closure of the complement*.

Proposition 2.7 ([25]). *Let $f\colon (X, A) \to (X, A)$ be a relative map. If there exists a neighborhood V of A in X such that $f(V) \subset A$, then for any fixed point class F of f, the set $F \cap (X \setminus A)$ is an isolated fixed point set of f with*

$$\mathrm{ind}(f, F \cap (X \setminus A)) = \mathrm{ind}(f, F) - \mathrm{ind}(\bar{f}, F \cap A).$$

Corollary 2.8. *Let $f\colon (X,A)\to(X,A)$ be a relative map and assume that there exists a neighborhood V of A in X such that $f(V)\subset A$. Then any fixed point class which does not assume its index in A contains a fixed point in $X\setminus A$. In particular, f has at least $N(f,\overline{X\setminus A})$ fixed points in $X\setminus A$.*

Theorem 2.9 ([25]). *Any relative map $f\colon (X,A)\to(X,A)$ has at least $N(f,\overline{X\setminus A})$ fixed points on the closure* $\mathrm{cl}(X\setminus A)$.

Let $f\colon (X,A)\to(X,A)$ be a relative map. Assume that $A_1,\dots,A_k$ are all components of A such that $f(A_i)\subset A_i$ ($i=1,\dots,k$). Then another relative number $N'(f,\overline{X\setminus A})$ is defined (see [25]) as the sum of the number of essential fixed point classes of all $f_{A_i}\colon A_i\to A_i$ with $\mathrm{int}(A_i)=\emptyset$ and the number of the fixed point classes which do not assume their indices in A and do not contain any essential fixed point classes of $f_{A_i}\colon A_i\to A_i$ with $\mathrm{int}(A_i)=\emptyset$. One can prove (see [25]) that $N'(f,\overline{X\setminus A})\geq N(f,\overline{X\setminus A})$.

We define the following auxiliary numbers:

- $N(f,A)$ is the number of essential fixed point classes of f which assume their index in A. Obviously, $0\leq N(f,A)\leq N(f,\bar f)$.
- $n(f;X,A)$ is the number of fixed point classes of f which do not assume their index in A and are a common point class of f and $\bar f$. Obviously, $n(f;X,A)\leq N(f,\overline{X\setminus A})$.

It follows that

$$N(f,\overline{X\setminus A})\geq N(f)-N(f,A)\geq N(f)-N(f,\bar f)=N(f,X,A)-N(\bar f).$$

By [15, Theorem 3.4], we have

$$N(f,\overline{X\setminus A})=n(f;X,A)+N(f)-N(f,\bar f),\quad N(f,\overline{X\setminus A})=N(f;X,A)+n(f;X,A)-N(\bar f).$$

3 Blocks and Periodic Segments

In this section we recall the notion of a Ważewski set and a periodic segment. We begin with the definitions of the basic concepts of the theory of continuous-time dynamical systems.

Let X be a topological space. A *local semiflow* on X is a continuous map $\phi\colon D\to X$, where D is an open subset of $X\times[0,\infty)$, such that for every $x\in X$, the set $\{t\in[0,\infty):(x,t)\in D\}$ is equal to an interval $[0,\omega_x)$ for some $0<\omega_x\leq\infty$. If $t\in[0,\omega_x)$, then $\omega_{\phi_t(x)}=\omega_x-t$ and the following equations hold:

$$\phi(x,0)=x,\quad \phi(x,s+t)=\phi(\phi(x,s),t).$$

Let ϕ be a local semiflow on X. For $B\subset X$ we define its *exit set* by

$$B^-=\{x\in B:\phi(x,[0,t])\not\subset B\text{ for all }t\in(0,\omega_x)\}.$$

Let another subset of B be defined as

$$B^*=\{x\in B:\text{there exists }t\in(0,\omega_x)\text{ such that }\phi_t(x)\notin B\}.$$

We call B a *Ważewski set* for ϕ if B and B^- are closed. A compact Ważewski set B is called a *block*.

Lemma 3.1. *If B is a Ważewski set, then the mapping*

$$\sigma\colon B^*\ni x\to\sup\{t\in[0,\omega_x):\phi(x,[0,t])\subset B\}\in[0,+\infty)$$

is continuous.

We will use the following notation: $\pi_2:\mathbb{R}\times X\to X$ and $\pi_1:\mathbb{R}\times X\to\mathbb{R}$ are projections. For $Z\subset\mathbb{R}\times X$ and $t\in\mathbb{R}$, we define the t-section of Z by

$$Z_t=\{x\in X:(t,x)\in Z\}.$$

By a *local semi-process* on a topological space X we mean a continuous map $\Phi\colon D\to X$, where D is open subset of $\mathbb{R}\times X\times[0,\infty)$ such that the map

$$\phi\colon D\in((\sigma,x),t)\to(\sigma+t,\Phi(\sigma,x,t))\in\mathbb{R}\times X$$

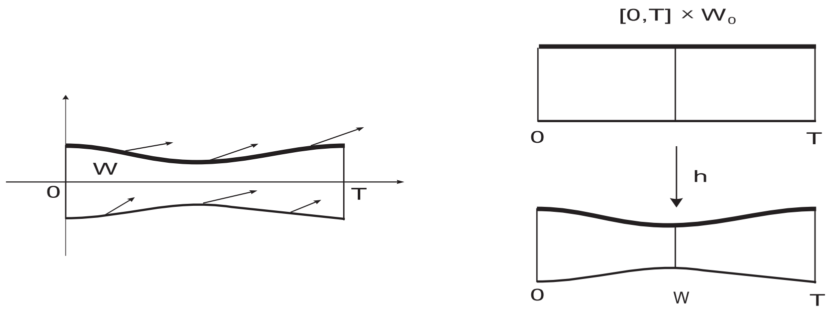

Figure 2. The periodic segment W and the homeomorphism h preserving t-sections.

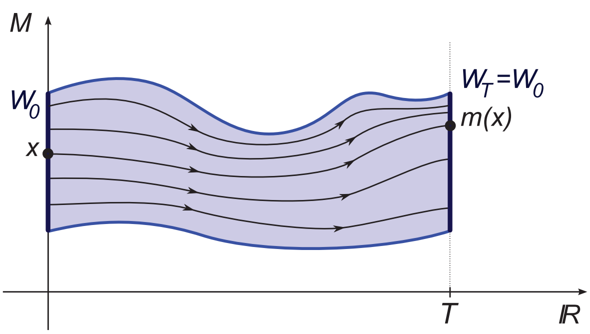

Figure 3. The monodromy map $m\colon W_0 \to W_0$ for some periodic segment W over $[0, T]$ induced by some homeomorphism $h\colon [0, T] \times W_0 \to W$.

is a local semiflow on $\mathbb{R} \times X$. In the sequel, we write $\Phi_{(\sigma,t)}(x)$ instead of $\Phi(\sigma, x, t)$. We say that Φ is T-periodic if $\Phi_{(\sigma,t)} = \Phi_{(\sigma+T,t)}$ for each σ and t.

Let Φ be a local T-periodic semi-process on X and let ϕ be the corresponding local semiflow on $\mathbb{R} \times X$. A set $W \subset [0, T] \times X$ is called a *periodic segment* over $[0, T]$ if it is a block with respect to ϕ such that the following conditions hold:

- There exists a compact subset W^{--} of W^- (called the *essential exit set*) such that
$$W^- = W^{--} \cup (\{T\} \times W_T), \quad W^- \cap ([0, T) \times X) \subset W^{--}.$$
- $W_0 = W_T$, $W_0^{--} = W_T^{--}$, where W_0 and W_0^{--} are compact ENRs.
- There exists a homeomorphism $h\colon [0, T] \times W_0 \to W$ such that $\pi_1 \circ h = \pi_1$ and
$$h([0, T] \times W_0^{--}) = W^{--}.$$

For the periodic segment W over $[0, T]$ one can define the corresponding *monodromy map* (see Figures 2 and 3)
$$m_W : (W_0, W_0^{--}) \to (W_T, W_T^{--}) = (W_0, W_0^{--}), \quad m_W(x) = \pi_2 h(T, \pi_2 h^{-1}(0, x)).$$
The monodromy map is actually a homeomorphism. We will use the following notation:
$$m\colon W_0 \ni x \to m_W(x) \in W_0, \quad \overline{m} = m|_{W_0^{--}} : W_0^{--} \to W_0^{--}.$$

For $s \in [0, T]$, we define two auxiliary functions by
$$m^s\colon (W_0, W_0^{--}) \to (W_s, W_s^{--}), \qquad m^s(x) = \pi_2 h(s, \pi_2 h^{-1}(0, x)),$$
$$m_s\colon (W_s, W_s^{--}) \to (W_T, W_T^{--}) = (W_0, W_0^{--}), \quad m_s(x) = \pi_2 h(T, \pi_2 h^{-1}(s, x)).$$

It follows that
$$m^T = m_W, \quad m^0 = \mathrm{id}_{(W_0, W_0^{--})}, \qquad m_T = \mathrm{id}_{(W_0, W_0^{--})}, \quad m_0 = m_W.$$

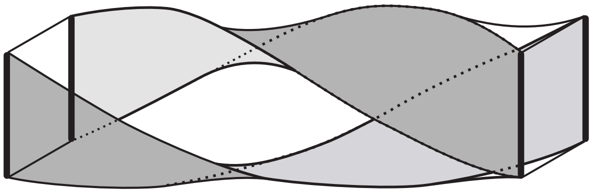

Figure 4. The periodic segment over $[0, 2\pi]$ for the 2π-periodic equation $\dot{z} = \bar{z}e^{it}$, $z \in \mathbb{C}$. The essential exit set W^{--} is shaded. The monodromy map is a rotation by π.

Lemma 3.2. *A different choice of the homeomorphism h in the definition of periodic segment provides the monodromy map homotopic to m_W.*

Proof. If $\tilde{h}$ is the another choice, then the map $H\colon [0,1] \times (W_0, W_0^{--}) \to (W_0, W_0^{--})$, defined by

$$H_t(x) = m^{tT} \circ \tilde{m}_{tT},$$

is a homotopy between m_W and $\tilde{m}_W$. □

Example 3.3. The periodic segment W over $[0, 2\pi]$ for the equation

$$\dot{z} = \bar{z}e^{it}, \quad z \in \mathbb{C},$$

is presented in Figure 4. The monodromy map $m_W\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$ is the rotation by π.

4 Periodic Segments and Relative Nielsen Numbers

Let W be a periodic segment over $[0, T]$ for the T-periodic local semi-process Φ on X and let $P = \Phi_{(0,T)}$ be a Poincaré map. Assume that $\sigma\colon W^* \to [0, +\infty)$ is the exit time map. Since $W^* = W$, σ is defined and continuous on the whole segment W.

We define a map $w\colon W_0 \to W_0$ by

$$w(x) = m_{\sigma(0,x)}(\Phi_{(0,\sigma(0,x))}(x)), \quad x \in W_0.$$

Observe that

$$\overline{w} = w|_{W_0^{--}} = m|_{W_0^{--}} = \overline{m}\colon W_0^{--} \to W_0^{--},$$

so we can treat w as a relative map, $w\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$.

Lemma 4.1. *The maps $w, m_W\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$ are homotopic. In particular,*

$$N(m_W, \overline{W_0 \setminus W_0^{--}}) = N(w, \overline{W_0 \setminus W_0^{--}}).$$

Proof. Consider a homotopy $H\colon [0,1] \times W_0 \to W_0$ defined by

$$H_t(x) := \begin{cases} m_{\sigma(0,x)}(\Phi_{(0,\sigma(0,x))}(x)) & \text{if } \sigma(0,x) \le (1-t)T, \\ m_{(1-t)T}(\Phi_{(0,(1-t)T)}(x)) & \text{if } \sigma(0,x) \ge (1-t)T. \end{cases}$$

In particular, $H_1 = m_W$ and $H_0 = w$. Moreover,

$$H_t(x) = m(x), \quad x \in W_0^{--},\ t \in [0,1],$$

hence $H_t(W_0^{--}) = W_0^{--}$ for $t \in [0,1]$, so $H\colon [0,1] \times (W_0, W_0^{--}) \to (W_0, W_0^{--})$ is a homotopy of relative maps. This completes the proof. □

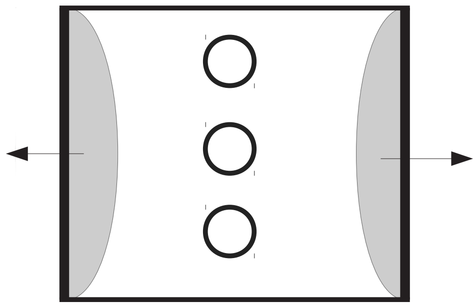

Figure 5. The set W_0 is a square with three holes. The open set $V = \{x \in W_0 : \sigma(0,x) < T\} \subset W_0^{--}$ is shaded in grey. On the complement $W_0 \setminus V$ the map w coincides with the Poincaré map P.

Lemma 4.2. *Let*

$$U = \{x \in W_0 : \Phi_{(0,t)}(x) \in W_t \setminus W_t^{--} \text{ for } t \in [0,T]\}.$$

Then the following hold:
- *U is open in W_0.*
- $w|_U = P|_U$.
- $\mathrm{Fix}(w|_{W_0\setminus W_0^{--}}) = \mathrm{Fix}(P|_U) \subset U$ *is compact.*
- $\mathrm{Fix}(w) = \mathrm{Fix}(P|_U) \cup \mathrm{Fix}(\overline{m})$.
- *There exists V open neighborhood of A such that $w(V) = W_0^{--}$.*

Proof. Let us observe that $w(x) = \Phi_{(0,T)}(x) = P(x)$ if $\sigma(0,x) = T$, hence $w|_U = P|_U$. One can check that

$$U = \{x \in W_0 : \sigma(0,x) = T, P(x) \in W_0 \setminus W_0^{--}\},$$

so $U = w^{-1}(W_0 \setminus W_0^{--})$, and hence U is open in W_0.

If $x \in \mathrm{Fix}(w|_{W_0\setminus W_0^{--}})$, then $\sigma(0,x) = T$, because otherwise $w(x) \in W_0^{--}$. Hence, $x \in U$ and thus

$$\mathrm{Fix}(P|_U) = \mathrm{Fix}(w) \cap \{x \in W_0 : \sigma(0,x) = T\}.$$

In particular, $\mathrm{Fix}(P|_U)$ is compact.

Put $V = \{x \in W_0 : \sigma(0,x) < T\}$. It follows that V is open in W_0, $W_0^{--} \subset V$ and $w(V) = W_0^{--}$ (see Figure 5). One can easily check that

$$\mathrm{Fix}(w) = \mathrm{Fix}(P|_U) \cup \mathrm{Fix}(\overline{m}).$$

□

Lemma 4.3. *Assume that F is a Nielsen class of w. Then F does not assume its index in W_0^{--} if and only if $F \cap U$ is an isolated set of fixed points of $P|_U$ and* $\mathrm{ind}(P|_U, F \cap U) \neq 0$.

Proof. Let F be a Nielsen class of w. By Proposition 2.7, $F \cap (W_0 \setminus W_0^{--})$ is an isolated set of fixed points of w and

$$\mathrm{ind}(w, F \cap (W_0 \setminus W_0^{--})) = \mathrm{ind}(w, F) - \mathrm{ind}(\overline{w}, F \cap W_0^{--}).$$

It follows, by Lemma 4.2, that

$$\mathrm{ind}(w, F \cap (W_0 \setminus W_0^{--})) = \mathrm{ind}(w, F \cap U) = \mathrm{ind}(P|_U, F \cap U),$$

hence

$$\mathrm{ind}(P|_U, F \cap U) = \mathrm{ind}(w, F) - \mathrm{ind}(\overline{w}, F \cap W_0^{--}),$$

so the result follows. □

Theorem 4.4. *Let W be a periodic segment over $[0,T]$. Then $P|_U$ has at least $N(m_W, \overline{W_0 \setminus W_0^{--}})$ fixed points. Moreover, if $L(m) \neq L(\overline{m})$, then $k := N(m_W, \overline{W_0 \setminus W_0^{--}}) \geq 1$. If $F_1, \dots, F_k$ are the Nielsen classes of w that do not assume their indices in W_0^{--}, then*

$$\mathrm{ind}(P|_U, \mathrm{Fix}(P|_U)) = \sum_{i=1}^{k} \mathrm{ind}(P|_U, F_i \cap U) = L(m) - L(\overline{m}) = L(m_W).$$

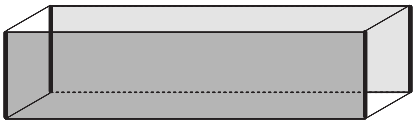

Figure 6. The T-periodic segment for $\dot{z} = \overline{z}$, $z \in \mathbb{C}$. The essential exit set W^{--} is shaded. The monodromy map m_W is the identity, so $N(m_W; W_0, W_0^{--}) = 2$, and the Poincaré map P has exactly one fixed point.

Proof. Since w and m_W are homotopic as relative maps, w has at least $N(m_W, \overline{W_0 \setminus W_0^{--}})$ fixed points. It follows, by Lemma 4.3, that $P|_U$ has at least $N(m_W, \overline{W_0 \setminus W_0^{--}})$ fixed points.

By [18], we have that

$$\operatorname{ind}(P|_U, \operatorname{Fix}(P|_U)) = L(m) - L(\overline{m}).$$

Let $F_1, \ldots, F_n$ be Nielsen classes of w ($n \geq k$) and $F_1, \ldots, F_k$ the Nielsen classes that do not assume their indices in W_0^{--}. By the additivity property of the fixed point index, Lemmas 4.2 and 4.3, we get that

$$\operatorname{ind}(P|_U, \operatorname{Fix}(P|_U)) = \sum_{i=1}^{n} \operatorname{ind}(P|_U, F_i \cap U) = \sum_{i=1}^{k} \operatorname{ind}(P|_U, F_i \cap U),$$

so

$$L(m) - L(\overline{m}) = \sum_{i=1}^{k} \operatorname{ind}(P|_U, F_i \cap U).$$

In particular, if $L(m_W) = L(m) - L(\overline{m}) \neq 0$, then $k \geq 1$. □

Example 4.5. If $N(\overline{m}) = 0$, then $N(m_W, \overline{W_0 \setminus W_0^{--}}) = N(m)$. Indeed, if F is an essential fixed point class of m, then F does not assume its index in W_0^{--}. Otherwise, $\operatorname{ind}(\overline{m}, F \cap W_0^{--}) = \operatorname{ind}(m, F) \neq 0$, and $F \cap W_0^{--}$ has to contain an essential fixed point class of $\overline{m}$, a contradiction. Obviously, if F is an inessential fixed point class of m, then it has to assume its index in W_0^{--}, because $N(\overline{m}) = 0$. Let us mention that the condition $N(\overline{m}) = 0$ holds, for example, if $\overline{m}$ is a fixed point free.

Example 4.6. If W_0 is simply connected or m is homotopic to $\operatorname{id}_{(W_0, W_0^{--})}$, then

$$N(m_W, \overline{W_0 \setminus W_0^{--}}) = \begin{cases} 1 & \text{if } L(m) \neq L(\overline{m}), \\ 0 & \text{if } L(m) = L(\overline{m}). \end{cases}$$

Indeed, m has exactly one fixed point class $F = \operatorname{Fix}(m)$ and $\operatorname{ind}(m, F) = L(m)$. Then

$$L(\overline{m}) = \operatorname{ind}(\overline{m}, \operatorname{Fix}(\overline{m})) = \operatorname{ind}(\overline{m}, F \cap W_0^{--}).$$

In particular, in that case the relative Nielsen number $N(m_W, \overline{W_0 \setminus W_0^{--}})$ does not give more information about the fixed points of the Poincaré map than the relative Lefschetz number $L(m_W) = L(m) - L(\overline{m})$.

Example 4.7. The Nielsen number $N(m_W, \overline{W_0 \setminus W_0^{--}})$ in Theorem 4.4 cannot be replaced by the Nielsen number $N(m; W_0, W_0^{--})$. Consider a local flow defined by the planar ordinary differential equation

$$\dot{z} = \overline{z}^n, \quad z \in \mathbb{C},\ n \geq 1.$$

One can check (see [18]) that for fixed $T > 0$, in the extended phase space, there exists a T-periodic segment over $[0, T]$ such that $W_0 = W_t$ ($t \in [0, T]$) is a regular $2(n+1)$-gon and W_0^{--} consists of $n + 1$ disjoint contractible parts (see Figure 6). Obviously, $m = \operatorname{id}_{(W_0, W_0^{--})}$ and $\Phi_{(0,T)}$ has exactly one fixed point. It follows, by the previous example, that $N(m, \overline{W_0 \setminus W_0^{--}}) = 1$, because $L(m) = 1$ and $L(\overline{m}) = n + 1$. On the other hand, $N(m) = 1$, $N(\overline{m}) = n + 1$, $N(m, \overline{m}) = 1$, hence

$$N(m_W; W_0, W_0^{--}) = n + 1,$$

and so $N(m_W; W_0, W_0^{--})$ is not a lower bound for the number of fixed points of the Poincaré map P.

Example 4.8. Since $N'(m_W, \overline{W_0 \setminus W_0^{--}}) \geq N(m_W, \overline{W_0 \setminus W_0^{--}})$, one can try to replace $N(m_W, \overline{W_0 \setminus W_0^{--}})$ in Theorem 4.4 by $N'(m_W, \overline{W_0 \setminus W_0^{--}})$. Unfortunately, $N'(m_W, \overline{W_0 \setminus W_0^{--}})$ is not a lower bound of the fixed points of $P|_U$ inside the segment. For the segment W, in the previous example we have $N'(m_W, \overline{W_0 \setminus W_0^{--}}) = n + 1$.

5 Periodic Points by Periodic Segments

Let Φ be a T-periodic ($T > 0$) semi-process on X. We define an operation of gluing periodic segments. If W and Z are periodic segments over $[0, T]$ having the same cross-section at 0, i.e.,

$$(W_0, W_0^{--}) = (Z_0, Z_0^{--}),$$

then we put

$$WZ := \{(t, x) \in [0, 2\pi] \times X : x \in W_t \text{ if } t \in [0, T],\ x \in Z_{t-T} \text{ if } t \in [T, 2T]\}.$$

This is a periodic segment over $[0, 2T]$.

If $Z_1, \dots, Z_r$ are periodic segments over $[0, T]$ having the same cross-sections at 0, then we define recurrently another periodic segment over $[0, rT]$ by

$$Z_1 \cdots Z_r := (Z_1 \cdots Z_{r-1})Z_r.$$

If $Z_i = W$ for each $i = 1, \dots, r$, then we put $W^n = Z_1 \cdots Z_r$. It follows that if m_W is a monodromy map for W, then m_W^n is a monodromy map for W^n. Obviously, $W_0^n = W_0$ and $(W^n)_0^{--} = W_0^{--}$. Moreover, if $P = \Phi_{(0,T)}$ is the Poincaré map, then $P^n = \Phi_{(0,nT)}$.

Corollary 5.1. *Let W be a periodic segment over $[0, T]$ for the T-periodic semi-process Φ. The set*

$$U_{W^n} = \{x \in W_0 : \Phi_{(0,t)}(x) \in (W^n)_t \setminus (W^n)_t^{--} \text{ for all } t \in [0, nT]\}$$

is open in W_0, and P^n has at least $N(m_W^n, \overline{W_0 \setminus W_0^{--}})$ fixed points in U_{W^n}.

The *growth rate* of a sequence of complex numbers (a_n) is defined by

$$\text{Growth } a_n = \max\Big\{1, \limsup_{n\to\infty} |a_n|^{1/n}\Big\}.$$

We say that the sequence *grows exponentially* if Growth $a_n > 1$.

Let $f : X \to X$ be a continuous map of compact ENR. We define the *asymptotic Nielsen number* of f to be the growth rate of the Nielsen numbers

$$N^\infty(f) := \text{Growth } N(f^n).$$

For a relative map $f : (X, A) \to (X, A)$, we put

$$N^\infty(f, \overline{X \setminus A}) := \text{Growth } N(f^n, \overline{X \setminus A}).$$

Theorem 5.2. *If $N(\overline{m}^n)$ is bounded and $N^\infty(m) > 1$, then the Poincaré map P has infinitely many periodic points whose trajectories are contained in the segment W.*

Proof. By Theorem 4.4, it is sufficient to show that $N^\infty(m_W, \overline{W_0 \setminus W_0^{--}}) > 1$. It follows, by the properties of the relative Nielsen numbers, that

$$N(m_W^n, \overline{W_0 \setminus W_0^{--}}) \ge N(m^n) - N(m^n, \overline{m}^n), \quad N(m^n, \overline{m}^n) \le N(\overline{m}^n).$$

Since $N(\overline{m}^n)$ is bounded, for some $M > 0$, we have

$$N(m_W^n, \overline{W_0 \setminus W_0^{--}}) \ge N(m^n) - M,$$

so the result follows because $N^\infty(m) > 1$. □

By $h(f)$ we denote the topological entropy of the self map $f : X \to X$ of the compact metric space. We refer the reader to [5] for the definition and basic properties of the topological entropy.

Theorem 5.3. *Let W_0 be a compact polyhedron. Then there exists a compact set $I \subset W_0 \setminus W_0^{--}$, invariant for the Poincaré map, such that*

$$\max\{h(P|_I), h(\overline{m})\} \geq \log N^{\infty}(m).$$

Proof. Let $w\colon W_0 \to W_0$ be defined by

$$w(x) = m_{\sigma(0,x)}(\Phi_{(0,\sigma(0,x))}), \quad x \in W_0,$$

where $\sigma\colon W \to [0, +\infty)$ is the exit time function associated to the segment W. It follows, by [6, Theorem 2.7] and Lemma 4.1, that

$$h(w) \geq \log N^{\infty}(w) = \log N^{\infty}(m).$$

We define

$$I = \{x \in W_0 : \Phi_{(0,t+kT)}(x) \in W_t \setminus W_t^{--} \text{ for } t \in [0, T),\ k \geq 0\}.$$

Geometrically, I is the set of points in W_0 whose trajectories are contained in the translated copies of the segment W for $t \geq 0$. One can check that I is compact and invariant for the Poincaré map P (compare [17, 18]).

Let $x \in W_0 \setminus I$. Then there exists $t \in [0, T)$ and $k \geq 0$ such that $\Phi_{(0,t+kT)}(x) \in W_t^{--}$. It follows that

$$\sigma(0, P^i(x)) = T, \quad i = 1, \ldots, k-1, \quad \sigma(0, P^k(x)) = t,$$

so

$$w^i(x) = P^i(x), \quad i = 1, \ldots, k, \quad w^{k+1}(x) = m_t(\Phi_{(0,t)}(P^k(x))) \in W_0^{--}.$$

The map $w\colon W_0 \to W_0$ has the following properties:

- I and W_0^{--} are disjoint, compact and invariant for w.
- $w|_{W_0^{--}} = \overline{m}$.
- For each $x \in W_0 \setminus I$, there exists $n \geq 0$ such that $w^n(x) \in W_0^{--}$.
- $w|_I = P|_I$.

Let Ω be a set of non-wandering points of w, i.e., the set of points $x \in W_0$ such that for any open neighborhood of x there exists $N > 0$ such that $w^N(U) \cap U \neq \emptyset$. It follows that $\Omega \subset I \cup W_0^{--}$. By the properties of the topological entropy, we have

$$h(w) = h(w|_\Omega) \leq h(w|_{I\cup W_0^{--}}) = \max\{h(w|_I), h(w|_{W_0^{--}})\} = \max\{h(P|_I), h(\overline{m})\},$$

hence the result follows. □

Corollary 5.4. *If $h(\overline{m}) = 0$, then $h(P|_I) \geq \log N^{\infty}(m)$. In particular, the conclusion holds if $\overline{m} : W_0^{--} \to W_0^{--}$ is a periodic homeomorphism, i.e., $\overline{m}^n = \mathrm{id}_{W_0^{--}}$ for some $n \geq 1$.*

6 Braid-Like Periodic Segment

Definition 6.1 (The Algebraic Braid Group). For a given integer $n > 0$, the Artin *braid group B_n* is the group defined by the generators $\sigma_1, \ldots, \sigma_{n-1}$ and the relations

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$,
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \leq i, j \leq n-1$ such that $|i-j| \geq 2$.

An element in this group is called a *braid*.

Example 6.2. The 1-braid group $B_1 = \{1\}$ is a trivial group. The 2-braid group is a free group with one generator isomorphic to $\mathbb{Z}$. The 3-braid group B_3 has the representation (see Figures 7 and 8)

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

and B_4 has the representation

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,\ \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle.$$

Figure 7. The generators σ_1 and σ_2 of B_3.

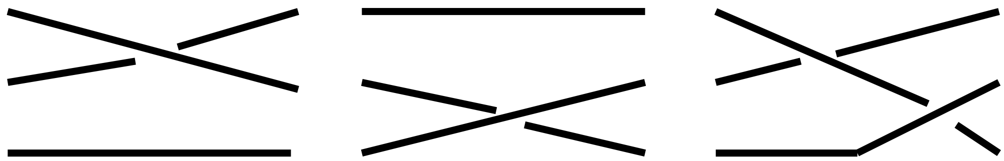

Figure 8. The braids σ_1, σ_2^{-1} and $\sigma_1\sigma_2^{-1}$ in B_3.

A geometrical representation of the Artin braid group is given by the following construction (see Figure 9). Let S be the set of n distinct points in the interior of a 2-dimensional disk $\mathbb{D}$. We call a subset G of $[0,1]\times\mathbb{D}$ a *geometric n-braid* (compare [3, 10]) if the following conditions hold:

- G is the union of mutually disjoint n arcs.
- Each arc joins a point $(0,x)\in\{0\}\times S$ to $(1,\tau(x))\in\{1\}\times S$, where τ is a permutation defined on S.
- Each arc intersects every $\{t\}\times\mathbb{D}$ ($t\in[0,1]$) exactly once.

Two geometric braids are *equivalent* if there exists a continuous deformation from one to the other through geometric braids. The set of equivalence classes of geometric braids is the Artin braid group B_n, where $n = \operatorname{card} S$. The composition law is given by concatenation of geometric braids.

The following theorem, due to Birman, shows the relation between braids and dynamics.

Theorem 6.3 ([3]). *Let M be the group of authomorphisms of the fundamental group $\pi_1(\mathbb{D}\setminus S)$, which are induced by homeomorphisms of $\mathbb{D}\setminus S$, which in turn keep the boundary of $\mathbb{D}$ fixed pointwise. Then M is precisely the group B_n.*

Remark 6.4 ([3]). Let $f\colon \mathbb{D}\setminus S\to\mathbb{D}\setminus S$ be an a homeomorphism which keeps the boundary of $\mathbb{D}$ fixed pointwise. Thus, f represents an element of M. Then f has a unique extension $m_{\mathbb{D}}$ to $\mathbb{D}$ which permutes the points of S. The map $m_{\mathbb{D}}$ is isotopic to the identity in $\mathbb{D}$. This isotopy may be used to define a homeomorphism $\tilde{h}\colon [0,T]\times\mathbb{D}\to[0,T]\times\mathbb{D}$ such that

$$\tilde{h}|\{0\}\times\mathbb{D} = \mathrm{id}_{\{0\}\times\mathbb{D}}, \quad \pi_1 = \pi_1\circ\tilde{h}.$$

The associated monodromy mapping is equal to the homeomorphism

$$m_{\mathbb{D}}(x) = \pi_2\tilde{h}(T, \pi_2\tilde{h}^{-1}(0,x)), \quad x\in\mathbb{D}.$$

The image $\tilde{h}(I\times S)$ is a geometric braid.

Let D_k denote a k-punctured disk, i.e., D_k is the 2-dimensional disk $\mathbb{D}$ with k disjoint open disks removed. More precisely, $\mathbb{D}\setminus D_k$ is the union of k-disjoint open disks $D(1),\dots,D(k)$, and $\overline{D(1)},\dots,\overline{D(k)}$ are disjoint such that $\overline{D(i)}\subset\mathbb{D}\setminus\partial\mathbb{D}$ for $i=1,\dots,k$. For $k=0$, we put $D_0=\mathbb{D}$.

Let $W\subset[0,T]\times\mathbb{D}$ be a periodic segment over $[0,T]$ for a T-periodic local semi-process on $\mathbb{R}^2$. We say that W is a *k-braid-like periodic segment* if the following conditions hold (see Figure 10):

- $W_0 = D_k$.
- The homeomorphism $h\colon [0,T]\times(W_0, W_0^{--})\to(W, W^{--})$ in the definition of a periodic segment W has an extension $\tilde{h}\colon[0,T]\times\mathbb{D}\to[0,T]\times\mathbb{D}$ such that

$$\tilde{h}|\{0\}\times\mathbb{D} = \mathrm{id}_{\{0\}\times\mathbb{D}}, \quad \pi_1 = \pi_1\circ\tilde{h}.$$

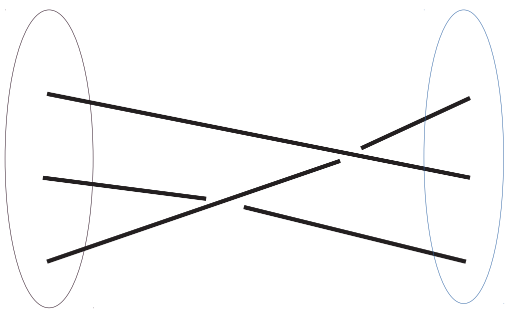

Figure 9. The geometric 3-braid.

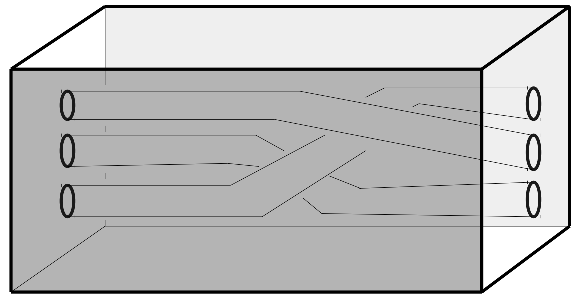

Figure 10. The braid-like periodic segment W. The essential exit set W^{--} has two components shaded in grey.

Since W_0^{--} is a compact ENR contained in the boundary of D_k, it is a finite union of the circles and the sets homeomorphic to closed intervals in $\mathbb{R}$. The associated monodromy map of $\tilde{h}$

$$m_{\mathbb{D}}(x) = \pi_2\tilde{h}(T, \pi_2\tilde{h}^{-1}(0, x)), \quad x \in \mathbb{D},$$

is a relative homeomorphism $m_{\mathbb{D}}\colon (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}, \partial\mathbb{D})$, where $\partial\mathbb{D}$ is the boundary $\mathbb{D}$. Moreover, $m_{\mathbb{D}}$ is isotopic to the identity by the homotopy $H\colon \mathbb{D} \times [0, 1] \to \mathbb{D}$, given by

$$H_t(x) = \pi_2\tilde{h}(tT, \pi_2\tilde{h}^{-1}(0, x)), \quad x \in \mathbb{D},\ t \in [0, 1].$$

The monodromy map $m_W\colon (D_k, W_0^{--}) \to (D_k, W_0^{--})$ associated to the k-braid-like periodic segment W is the orientation-preserving homeomorphism and $m_W(\partial\mathbb{D}) = \partial\mathbb{D}$. The isotopy $H\colon \mathbb{D} \times [0, 1] \to \mathbb{D}$, defined above, has the following properties:

$$H_0|_{D_k} = \mathrm{id}_{D_k}, \quad H_1|_{D_k} = m\colon D_k \ni x \to m_W(x) \in D_k,$$

and

$$H_t(\partial\mathbb{D}) = \partial\mathbb{D}, \quad H_t(W_0^{--}) = W_0^{--}.$$

Lemma 6.5. *Let $(L(m^n))_{n\geq 0}$ be the sequence of Lefschetz numbers of the iterations of the monodromy map $m\colon D_k \ni x \to m_W(x) \in D_k$ associated to the k-braid-like periodic segment W. Then $(L(m^n))_{n\geq 0}$ is periodic with period less than or equal to k.*

Proof. The result is trivial if $k = 0$, because then $L(m^n) = 1$ for $n \geq 0$. Assume that $k \geq 1$. Since $m|_{\partial D(i)} = m|_{\partial D(i)}$ restricted to each $\partial D(i)$ ($i = 1, \dots, k$) is an orientation preserving homeomorphism of the circle, it is isotopic to the identity map on $\partial D(i)$, so without lost of generality we can assume that $m_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ is a periodic point free on $\partial D(i)$. By the Lefschetz fixed point theorem (applied to $m_{\mathbb{D}}^n\colon \mathbb{D} \to \mathbb{D}$), we get that

$$1 = L(m_{\mathbb{D}}^n) = \mathrm{ind}(m_{\mathbb{D}}^n, \mathbb{D}).$$

Let $C = \overline{D(1)} \cup \cdots \cup \overline{D(k)}$. Since $m_{\mathbb{D}}$ is a periodic point free on $\partial D(i)$, by the additivity of the fixed point index, we get that

$$1 = \mathrm{ind}(m_{\mathbb{D}}^n, \mathbb{D}) = \mathrm{ind}(m_{\mathbb{D}}^n, D_k) + \mathrm{ind}(m_{\mathbb{D}}^n, C).$$

Since $m_W^n(D_k) = D_k$ and $m_W^n(C) = (C)$, we have

$$\mathrm{ind}(m_D^n, D_k) = L(m_W^n), \quad \mathrm{ind}(m_{\mathbb{D}}^n, C) = L(m_{\mathbb{D}}^n|_C),$$

hence

$$1 = L(m_W^n) + L(m_{\mathbb{D}}^n|_C).$$

It is sufficient to show that the sequence $L(m_{\mathbb{D}}^n|_C)$ is periodic. Since each component of C is contractible, the Lefschetz number $L(m_{\mathbb{D}}^n|_C)$ is the Lefschetz number of the iteration of the associated permutation $\tau\colon \{1, \dots, k\} \to \{1, \dots, k\}$. Obviously, $L(m_{\mathbb{D}}^n|_C) = L(\tau^n)$ is periodic with period less than or equal to k. □

Lemma 6.6. *The sequence of Lefschetz numbers $L(m_W^n|_{W_0^{--}})$ is periodic.*

Proof. Indeed, if C is a component of W_0^{--} homeomorphic to a circle, then $L(m_W^n|_C) = 0$ since $m_W|_C$ is orientation preserving, hence it is homotopic to the identity on C. It follows that $L(m_W^n|_{W_0^{--}})$ is the sequence of Lefschetz numbers of m_W^n restricted to the components of W_0^{--} homeomorphic to closed intervals, so it is a sequence of Lefschetz numbers of the permutation of a finite set, hence it is a periodic sequence. □

Corollary 6.7. *Assume that W is a braid-like periodic segment over $[0, T]$ for some T-periodic (local) semi-process Φ on $\mathbb{R}^2$. If $m_W\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$ is a monodromy map associated to W, then the sequence of relative Lefschetz numbers $(L(m_W^n))_{n\geq 0}$ is periodic.*

Proposition 6.8. *Assume that W is a periodic segment over $[0, T]$ for some T-periodic local semi-process Φ on X.*

- *If $\chi(W_0, W_0^{--}) = \chi(W_0) - \chi(W_0^{--}) \neq 0$ then there exists $n > 0$ such that $L(m_W^n) \neq 0$, so the Poincaré map P has an n-periodic point whose trajectory is contained in W^n.*
- *If $X = \mathbb{R}^2$, W is a braid-like periodic segment and $H(W_0, W_0^{--}) \neq 0$, then the Poincaré map P has a periodic point (not necessarily contained in W).*

Proof. For the periodic segment W over $[0, T]$, one can define the *Lefschetz zeta function of W* by

$$\zeta_W(t) = \exp\Big(\frac{L(m_W^n)t^n}{n}\Big).$$

It follows that

$$\zeta_W(t) = \prod_{k=0}^{\infty} \det(I - H_k(m_W)t)^{(-1)^{k+1}},$$

where $H_k(m_W)\colon H_k(W_0, W_0^{--}) \to H_k(W_0, W_0^{--})$ is the isomorphism induced in the singular homology with $\mathbb{Q}$-coefficients. It is easy to check that $\chi(W_0, W_0^{--}) \neq 0$ implies $\zeta_W(t) \neq 1$, hence there exists $n > 0$ such that $L(m_W^n) \neq 0$.

Assume that W is a k-braid-like periodic segment for the local semi-process on $\mathbb{R}^2$. Let $m \geq 0$ be the number of components of W_0^{--} homeomorphic to a closed interval in $\mathbb{R}$. It follows that

$$\chi(W_0) = 1 - k, \quad \chi(W_0^{--}) = m.$$

If $\chi(W_0) < 0$, then $\chi(W_0, W_0^{--}) < 0$, so the result follows. If $\chi(W_0) = 0$ and $m \geq 1$, then again $\chi(W_0, W_0^{--}) \neq 0$. Assume that $\chi(W_0) = 0$ and $m = 0$. Then, obviously, $\chi(W_0, W_0^{--}) = 0$. Since $m = 0$, we have that $W_0^{--} \cap \partial\mathbb{D}$ is either the circle $\partial\mathbb{D}$ or the empty set. It follows, by the definition of braid-like segments, that $U = [0, T] \times \mathbb{D}$ is a periodic segment over $[0, T]$ with the essential exit set U^{--} being $[0, T] \times \partial\mathbb{D}$ or the empty set. In both cases, $\chi(U_0, U_0^{--}) = 1$, so the result follows. If $\chi(W_0) = 1$, then $m \neq 1$, because otherwise $H(W_0, W_0^{--}) = 0$. In particular, $\chi(W_0, W_0^{--}) \neq 0$ and the proof is finished. □

Proposition 6.9. *Assume that W is a braid-like periodic segment over $[0, T]$ for some T-periodic semi process Φ on $\mathbb{R}^2$. If $N^\infty(m) > 1$, then the Poincaré map P has infinitely many periodic points. Moreover, there exists a compact set I, invariant for the Poincaré map, such that*

$$h(P|_I) \geq \log N^\infty(m) > 0.$$

Proof. Since every component of W_0^{--} is homeomorphic to either a circle or a closed interval, it is easy to check that $N(\overline{m}^n)$ is periodic, hence it is bounded and the result follows by Theorem 5.2 and Corollary 5.4. □

7 Example of Braid-Like Periodic Segment with $N^\infty(m) > 1$

Assume that W is a k-braid-like periodic segment such that $k > 1$ and $\chi(D_k) < 0$. Let $m\colon D_k \ni x \to m_W(x) \in D_k$ be the associated monodromy map. Following [8, 9], we recall a recipe on how to estimate $N^\infty(m)$.

Since $m\colon D_k \to D_k$ is a homeomorphism and $\chi(D_k) < 0$, by [8, 9], it follows that

$$N^\infty(m) = L^\infty(m) = \lambda,$$

where λ is the largest stretching factor of the pseudo-Anosov pieces in the Thurston canonical form of m ($\lambda := 1$ if there is no pseudo-Anosov piece).

Let $a_1, \dots, a_k$ be generators of $G = \pi_1(D_k)$ and $G = \langle a_1, \dots, a_k\rangle$ a free group. Let $\Gamma = \pi_1(T_m)$. Then Γ is described by the generators

$$\Gamma = \langle a_1, \dots, a_r, z \mid a_i z = z a_i',\ i = 1, \dots, r\rangle,$$

where $a_i' = m_*(a_i)$ and $m_*\colon G \to G$ is induced by m.

Fadell and Husseini [4] devised a method of computing $L_\Gamma(m^n)$. Let

$$D = \left(\frac{\partial a_i'}{\partial a_j}\right)$$

be the Jacobian in Fox calculus, an $n \times n$ matrix in $\mathbb{Z}\Gamma$. We recall that if G is a free group with an identity element e and generators g_i, then the Fox derivative with respect to g_i is a function from G into the integral group ring $\mathbb{Z}G$, which is denoted $\frac{\partial}{\partial g_i}$, and obeys the following axioms:

$$\frac{\partial g_j}{\partial g_i} = \delta_{ij}, \qquad \frac{\partial e}{\partial g_i} = 0, \qquad \frac{\partial (uv)}{\partial g_i} = \frac{\partial u}{\partial g_i} + u\frac{\partial v}{\partial g_i}, \qquad u, v \in G,$$

where δ_{ij} is the Kronecker delta. It follows that

$$\frac{\partial u^{-1}}{\partial g_i} = -u^{-1}\frac{\partial u}{\partial g_i}.$$

It was proved in [4] that the matrices of the lifted chain map $\tilde{m}$ (compare Appendix A) are given by

$$\tilde{F}_0 = (1), \quad \tilde{F}_1 = D,$$

and consequently,

$$L_\Gamma(m) = [z] - \sum_{i=1}^{n}\left[z\frac{\partial(a_i')}{\partial a_i}\right] \in \mathbb{Z}\Gamma, \quad L_\Gamma(m^n) = [z^n] - \sum_{i=1}^{n}[\operatorname{tr}(zD)^n] \in \mathbb{Z}\Gamma.$$

If $\rho\colon \Gamma \to \mathrm{GL}_l(\mathbb{C})$ is a group representation, then the ρ-twisted zeta function of m is given by

$$\zeta_\rho(m) = \frac{\det(I - t(zD)^\rho)}{\det(I - tz^\rho)} \in \mathbb{C}(t),$$

where $(zD)^\rho$ is the block matrix obtained from the matrix zD by replacing each entry (in $\mathbb{Z}\Gamma$) with its ρ-image (an $l \times l$-matrix), and I is a suitable identity matrix. Note that ρ extends to a ring representation $\rho\colon \mathbb{Z}\Gamma \to M_l(\mathbb{C})$, so the formula for $\zeta_\rho(m)$ is well defined. Moreover, if $\rho\colon \Gamma \to U_l(\mathbb{C})$ is a unitary representation and r is the minimum modulus of the zeros and poles of the rational function $\zeta_\rho(m)$, then (compare [8, 9])

$$L^\infty(m) = \operatorname{Growth}\|L_\Gamma(m^n)\| \geq \frac{1}{r}.$$

Following [8, 9], we show an example of a braid-like periodic segment W such that $N^\infty(m) > 1$. In particular, in that case P has infinitely many periodic points and $h(P|_I) \geq \log N^\infty(m) > 0$ for some invariant set I (compare Corollary 6.9).

Example 7.1 ([8]). Assume that m is a homeomorphism corresponding to a braid $\sigma = \sigma_1\sigma_2^{-1} \in B_3$. Then $G = \langle a_1, a_2, a_3\rangle$ is a free group of rank 3. Let $m_*\colon G \to G$ be induced by m and $a_i' = m_*(a_i)$. Then

$$a_1' = a_1 a_3 a_1^{-1}, \quad a_2' = a_3, \quad a_3' = a_3^{-1} a_2 a_3,$$

hence

$$\Gamma = \langle a_1, a_2, a_3, z \mid a_i z = z a_i', \ i = 1, 2, 3\rangle.$$

The matrix D in Fox calculus is given by

$$D = \begin{bmatrix} 1 - a_1 a_3 a_1^{-1} & 0 & a_1 \\ 1 & 0 & 0 \\ 0 & a_3^{-1} & -a_3^{-1} + a_3^{-1} a_2 \end{bmatrix}.$$

A representation $\rho\colon \Gamma \to U(1)$ is given in the following way:

- The first step is to abelianize Γ and let $z \to 1$.
- Then we get a projection $\Gamma \to H = \langle a\rangle$ for $a_i \to a$.
- We take $a \in \mathbb{S}^1$.

Thus,

$$(zD)^\rho = \begin{bmatrix} 1 - a & 0 & a \\ 1 & 0 & 0 \\ 0 & a^{-1} & 1 - a^{-1} \end{bmatrix},$$

so

$$\zeta_\rho(m) = \frac{\det(I - t(zD)^\rho)}{\det(I - tz^\rho)} = 1 - (1 - a - a^{-1})t + t^2.$$

Take $a = -1$. Then $\zeta_\rho(m) = 1 - 3t + t^2$, and its smallest root is $r = \frac{3-\sqrt{5}}{2} < \frac{2}{5}$.

It follows that $N^\infty(m) > \frac{5}{2}$.

8 The Number of Periodic Points of the Poincaré Map by Lefschetz Numbers

Let W be a periodic segment over $[0, T]$. If W_0 is simply connected, then

$$N(m_W, \overline{W_0 \setminus W_0^{--}}) = \begin{cases} 1 & \text{if } L(m_W) \neq 0, \\ 0 & \text{if } L(m_W) = 0, \end{cases}$$

so the relative Nielsen number $N(m_W, \overline{W_0 \setminus W_0^{--}})$ does not give more information about the fixed points of the Poincaré map than the relative Lefschetz number $L(m_W) = L(m) - L(\overline{m})$. One need some additional information concerning the behavior of the system inside the segment to prove the existence of more periodic solutions.

In this section we assume that there exists another periodic segment $Z \subset W$ over $[0, T]$. Let us observe that if $L(m_W) \neq L(m_Z)$, then by Theorem 4.4 there exists a fixed point $x \in W_0$ for the Poincaré map P with trajectory contained in W but not contained in Z. If additionally $L(m_Z) \neq 0$, then P has at least two fixed points in W_0. Their trajectories are contained in W, but one of them is contained in Z and the other one leaves Z in time less than T. Usually, one cannot get more than two fixed points in W_0.

We will say that $(\Phi; Z, W)$ is a *chaotic triple* if Φ is T-periodic (local) semi-process on X, and Z and W are periodic segments over $[0, T]$ such that

$$Z \subset W, \quad Z_0 = W_0, \quad Z_0^{--} = W_0^{--}$$

and

$$m_Z = \mathrm{id}_{(W_0, W_0^{--})}, \quad L(\mu_W) \neq \chi(W_0, W_0^{--}) = L(\mu_Z),$$

where $\mu_W\colon H(W_0, W_0^{--}) \to H(W_0, W_0^{--})$ is the authomorphism induced by $m_W\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$, a homeomorphism in singular homologies (with rational coefficients).

Assume that $(\Phi; Z, W)$ is a chaotic triple with the Poincaré map $P = P_\Phi = \Phi_{(0,T)}$. Let ϕ be a semiflow associated to Φ. We define

$$I := I(\Phi) = \{x \in W_0 : \Phi_{(0,t+kT)}(x) \in W_t \setminus W_t^{--} \text{ for } t \in [0, T),\ k \geq 0\},$$

so I is the set of points x in W_0 such that positive trajectories (with respect to ϕ) of $(0, x)$ are contained in the bigger segment W.

Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$. We define $g := g_\Phi : I \to \Sigma_2$ by the following rule:

- If on the time interval $[iT, (i+1)T]$ the trajectory of $(0, x)$ is contained in Z, then $g(x)_i = 0$.
- If $(0, P^i(x))$ leaves Z in time less than T, then $g(x)_i = 1$.

It follows (see [18]) that $g\colon I \to \Sigma_2$ is continuous and $\sigma \circ g = g \circ P$, where $\sigma\colon \Sigma_2 \to \Sigma_2$ is the shift map.

Let $c = (c_0, \dots, c_{n-1}) \in \Sigma_2$ be n-periodic sequence. We define the set $(W_0 \setminus W_0^{--})_c(\Phi)$ as a set of points x satisfying the following conditions:

- $P^l(x) \in W_0 \setminus W_0^{--}$ for $l \in \{0, \dots, n\}$.
- $\Phi_{(0,t+lT)}(x) \in W_t \setminus W_t^{--}$ for $t \in [0, T]$ and $l \in \{0, \dots, n-1\}$.
- For each $i \in \{0, \dots, n-1\}$, if $c_i = 0$, then $\Phi_{(0,iT+t)}(x) \in Z_t \setminus Z_t^{--}$ for $t \in (0, T)$.
- For each $i \in \{0, \dots, n-1\}$, if $c_i = 1$, then $(0, P^i(x))$ leaves Z in time less than T.

It follows that $(W_0 \setminus W_0^{--})_c(\Phi)$ is open in $W_0 \setminus W_0^{--}$ and

$$K_c(\Phi) = \mathrm{Fix}(P^n) \cap (W_0 \setminus W_0^{--})_c(\Phi)$$

is compact. In particular, the fixed point index $\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})_c}, K_c)$ is well defined and one can prove (see [18, 20]) that

$$\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})_c(\Phi)}, K_c(\Phi)) = L((\mu_W - I)^k),$$

where the symbol 1 appears in c exactly k-times.

Remark 8.1. It follows that if $(\Phi; Z, W)$ is a chaotic triple, then $\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})_c(\Phi)}, K_c(\Phi))$ is independent of Φ, i.e., if $(\Psi; Z, W)$ is another chaotic triple, then

$$\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})_c(\Phi)}, K_c(\Phi)) = \mathrm{ind}(P_\Psi^n|_{(W_0 \setminus W_0^{--})_c(\Psi)}, K_c(\Psi)).$$

Corollary 8.2. *If $(\Phi; Z, W)$ is a chaotic triple, then the Poincaré map P has infinitely many periodic points.*

Proof. It is sufficient to consider the n-periodic sequences with $k = 1$. ☐

Let K be a positive integer and let $E(1), \dots, E(K)$ be disjoint closed subsets of the essential exit set Z^{--} which are T-periodic, i.e., $E(l)_0 = E(l)_T$, and such that $Z^{--} = \bigcup_{l=1}^K E(l)$.

For $p \in \Sigma_{K+1} = \{0, 1, \dots, K\}^{\mathbb{N}}$, we define the set $(W_0 \setminus W_0^{--})(p, \mathrm{rel}\,\Phi)$ by the following rules:

- $P^l(x) \in W_0 \setminus W_0^{--}$ for $l \in \{0, \dots, n\}$.
- $\Phi_{(0,t+lT)}(x) \in W_t \setminus W_t^{--}$ for $t \in [0, T]$ and $l \in \{0, \dots, n-1\}$.
- For each $i \in \{0, \dots, n-1\}$, if $p_i = 0$, then $\Phi_{(0,iT+t)}(x) \in Z_t \setminus Z_t^{--}$ for $t \in (0, T)$.
- For each $i \in \{0, \dots, n-1\}$, if $p_i > 0$, then $(0, P^i(x))$ leaves Z in time less than T through $E(p_i)$.

The set $(W_0 \setminus W_0^{--})(p, \mathrm{rel}\,\Phi)$ is open in W_0 and the set

$$K(p, \mathrm{rel}\,\Phi) = (W_0 \setminus W_0^{--})(p, \mathrm{rel}\,\Phi) \cap \mathrm{Fix}(P^n)$$

is compact for every n-periodic sequence $p \in \Sigma_{K+1}$. In particular, the fixed point index

$$\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})(p, \mathrm{rel}\,\Phi)}, K(p, \mathrm{rel}\,\Phi))$$

is well defined for every chaotic triple $(\Phi; Z, W)$ and each n-periodic sequence $c \in \Sigma_{K+1}$.

Problem 8.3. *Is the fixed point index* $\mathrm{ind}(P^n|_{(W_0 \setminus W_0^{--})(p, \mathrm{rel}\,\Phi)}, K(p, \mathrm{rel}\,\Phi))$ *independent of the semi-process Φ in the chaotic triple $(\Phi; Z, W)$?*

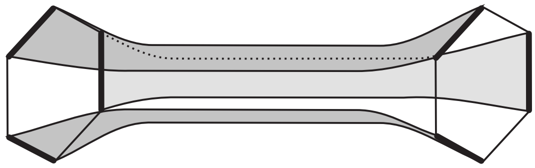

Figure 11. The periodic segment $Z(2)$ over $[0, T]$. The monodromy map is the identity. The essential exit set $Z(2)^{--}$ has three components. One can choose the identity as the monodromy map $m_{Z(2)}$, so $L(m_{Z(2)}) = -2$.

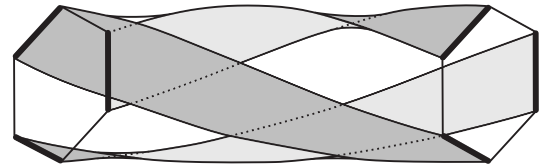

Figure 12. The periodic segment $W(2)$ over $[0, T]$. One can choose the rotation by $\frac{2\pi}{3}$ as the monodromy map $m_{W(2)}$, so $L(m_{W(2)}) = L(m) = 1$, because $\overline{m}$ has no fixed points in W_0^{--} and W_0 is contractible.

Remark 8.4. Let us mention that it was proved in [18, 21] that if (Φ^λ, Z, W) $(\lambda \in [0, 1])$ is an *admissible* continuous family of chaotic triples, then

$$\operatorname{ind}((P^0)^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi^0)}, K(p, \mathrm{rel}\,\Phi^0)) = \operatorname{ind}((P^1)^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi^1)}, K(p, \mathrm{rel}\,\Phi^1)).$$

This result works in metric spaces X, and the admissibility of the family $(\Phi^\lambda; Z, W)$ means that there exists $\eta > 0$ such that for every $w \in W^{--}$ and $z \in Z^{--}$, there exists $t > 0$ such that for $0 < \tau < t$ and $\lambda \in [0, 1]$, we have

$$\phi_\tau^\lambda(w) \notin W, \quad d(\phi_t^\lambda(w), W) > \eta, \quad \phi_\tau^\lambda(z) \notin Z, \quad d(\phi_t^\lambda(z), Z) > \eta,$$

where d is a corresponding metric on $\mathbb{R} \times X$.

For the n-periodic sequence $c = (c_0, \dots, c_{n-1}) \in \Sigma_2$, by Π_c we denote the set of all n-periodic sequences $p = (p_0, \dots, p_{n-1}) \in \Sigma_{K+1}$ such that $c_i = 0$ implies $p_i = 0$.

Corollary 8.5. *Let $(\Phi; Z, W)$ be a chaotic triple such that* $\operatorname{ind}(P^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi)}, K(p, \mathrm{rel}\,\Phi)) \in \{0, \pm 1\}$ *for every $p \in \Pi_c$. Then*

$$\operatorname{card} K_c(\Phi) \geq |L((\mu_W - I)^k)|.$$

Proof. The set $K_c(\Phi)$ splits into a finite, disjoint union of sets $K(p, \mathrm{rel}\,\Phi)$ $(p \in \Pi_c)$, so by the additivity property of the fixed point index we get that

$$L((\mu_W - I)^k) = \operatorname{ind}(P^n|_{(W_0\setminus W_0^{--})_c(\Phi)}, K_c(\Phi)) = \sum_{p\in\Pi_c} \operatorname{ind}(P^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi)}, K(p, \mathrm{rel}\,\Phi)),$$

hence the result follows. □

We will say that a chaotic triple $(\Phi^M; Z, W)$ is a *regular model* (resp. *weak regular model*) for a chaotic triple $(\Phi; Z, W)$ if for every n-periodic sequence $p \in \Sigma_{K+1}$,

$$\operatorname{ind}(P^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi)}, K(p, \mathrm{rel}\,\Phi)) = \operatorname{ind}((P^M)^n|_{(W_0\setminus W_0^{--})(p,\mathrm{rel}\,\Phi^M)}, K(p, \mathrm{rel}\,\Phi^M)) \in \{\pm 1\} \quad (\text{resp. } \{0, \pm 1\}).$$

In particular, if a chaotic triple $(\Phi; Z, W)$ has a weak regular model, then

$$\operatorname{card} K_c(\Phi) \geq |L((\mu_W - I)^k)|$$

for every n-periodic sequence $c \in \Sigma_2$.

As an application we consider the local process Φ_m generated by the $T = \frac{2\pi}{\kappa}$-periodic planar equation

$$\dot{z} = \overline{z}^m(1 + e^{i\kappa t}|z|^2), \quad z \in \mathbb{C},\ m \geq 1,$$

where $\kappa > 0$ is a real parameter.

It was proved in [20] that for sufficiently small $\kappa > 0$ (depending on m), there exists a chaotic triple $(\Phi_m; Z(m), W(m))$ associated to Φ_m. The time t-section of the segment $Z(m)$ is a regular $2(m+1)$-gon based prism centered at the origin. The essential exit set $Z(m)^{--}$ consists of $n+1$ disjoint parts. The segment $W(m)$ is a twisted prism with a $2(m+1)$-gon base also centered at the origin. Its time sections, are obtained by rotating the base with angular velocity $\frac{\kappa}{m+1}$ over the time interval $[0, \frac{2\pi}{\kappa}]$. The essential exit set $W(m)^{--}$ consists of $m+1$ disjoint ribbons winding around the prism (see Figures 11 and 12).

Corollary 8.6. *Let $c \in \Sigma_2$ be an n-periodic sequence such that the symbol 1 appears k-times ($0 \le k \le n$) in c. Then*

$$\mathrm{card}(K_c(\Phi_m)) \ge |\mathrm{tr}(I - A_m)^k|,$$

where

$$A_m = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in M_{m+1}(\mathbb{Z}).$$

Proof. We first explain that

$$|L((\mu_{W(m)} - I)^k)| = |\mathrm{tr}(I - A_m)^k|.$$

Since the monodromy map $m_{W(m)} \colon (W(m)_0, W(m)_0^{--}) \to (W(m)_0, W(m)_0^{--})$ is a rotation by $\frac{2\pi}{m+1}$, we have

$$L(\mu_{W(m)}) = \cdots = L(\mu^m_{W(m)}) = 1, \quad L(\mu^{m+1}_{W(m)}) = \chi(W_0) - \chi(W_0^{--}) = 1 - (m+1) = -m.$$

It follows that

$$\begin{aligned} L((I - \mu_{W(m)})^k)) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} L(\mu^i_{W(m)}) \\ &= \Big(\sum_{m+1|s} (-1)^s \binom{k}{s} \Big) (L(\mu^{m+1}_{W(m)}) - L(\mu_{W(m)})) \\ &= \Big(\sum_{m+1|s} (-1)^s \binom{k}{s} \Big) (-m-1) \\ &= -\sum_{t=0}^{m} (1 - \omega^t)^k \\ &= -\mathrm{tr}(I - A_m)^k, \end{aligned}$$

where $\omega = e^{\frac{2\pi i}{m+1}}$ is the $m+1$-primitive root of unity. The last equation holds because $1, \omega, \dots, \omega^m$ are the eigenvalues of A_m.

By Corollary 8.5, it is sufficient to show that a chaotic triple $(\Phi_m; Z(m), W(m))$ has a weak regular model $(\Phi^M_m; Z(m), W(m))$, and this is the case by results in [22], where the appropriate model semi-processes were constructed for every $m \ge 1$. For the convenience of the reader we very briefly describe below this construction for $m = 1$ and $m = 2$. □

Assume that $m = 1$. We write $Z = Z(1)$ and $W = W(1)$, $\Phi = \Phi_1$. Then

$$L(\mu_W) = 1, \quad L(\mu_W^2) = -1, \quad |L((\mu_W - I)^k)| = 2^k.$$

Let $W_0 = [-R, R] \times [-R, R]$. For $0 < c < a < b < R$, we put $J_{-1} = [-b, -a]$, $J_0 = [-c, c]$, $J_1 = [a, b]$. Consider the function $f \colon J_{-1} \cup J_0 \cup J_1 \to [-R, R]$ having the graph in Figure 13.

Let Z^{+1}, Z^{-1} be two connected components of Z^{--} (right and left, respectively). The chaotic triple $(\Phi; Z, W)$ has a regular model $(\Phi^M; Z, W)$ (see [22]) such that the following hold:

- $\{J_{-1} \cup J_0 \cup J_1\} \times [-R, R] = \{z \in W_0 : \Phi^M_{(0,t)}(z) \in W_t$ for all $t \in [0, T]\}$.
- $J_0 \times [-R, R] = \{z \in W_0 : \Phi^M_{(0,t)}(z) \in Z_t$ for all $t \in [0, T]\}$.
- For $l = +1, -1$,
$$J_0 \times [-R, R] = \{z \in W_0 : z \text{ leaves } Z \text{ through } Z^l \text{ in time } \le T\}.$$
- For $z = (x, y) \in \{J_{-1} \cup J_0 \cup J_1\} \times [-R, R]$, the Poincaré map P^M is given by
$$P^M(x, y) = (f(x), 0).$$

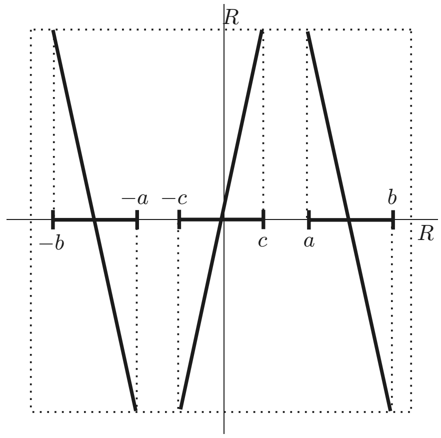

Figure 13. The graph of function $f\colon J_{-1} \cup J_0 \cup J_1 \to [-R, R]$ being a one-dimensional topological horseshoe.

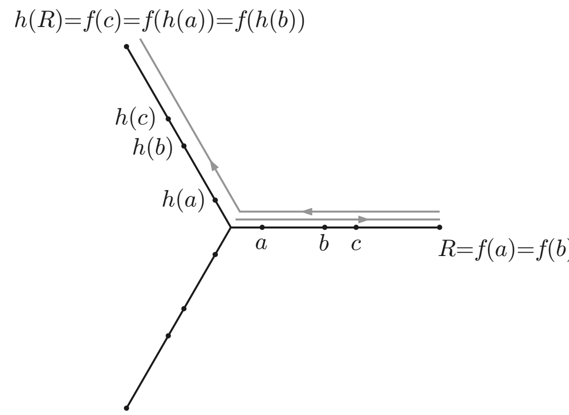

Figure 14. The graph of function $f\colon J_0 \cup J_1 \cup J_2 \cup J_3 \to S$.

Let $m = 2$ and $Z = Z(2)$, $W = W(2)$, $\Phi = \Phi_2$. It follows that $Z_0 = W_0$ is a hexagon centered at the origin, and the exit set Z^{--} has three components E_1, E_2 and E_3. Let R be equal to the radius of the inscribed circle Z_0 and let $h\colon \mathbb{C} \ni z \to ze^{\frac{2\pi i}{3}} \in \mathbb{C}$ be the rotation. We define

$$S = [0, R] \cup h([0, R]) \cup h^2([0, R]), \quad J_k = h^{k-1}([b, c]), \quad k \in \{1, 2, 3\}.$$

Let $f\colon J \cup J_1 \cup J_2 \cup J_3 \to S$ be a continuous map, symmetric with respect to h, with its graph shown in Figure 14.

Let $r\colon W_0 \to S$ be the retraction shown in Figure 15. We put

$$K = r^{-1}(J \cup J_1 \cup J_2 \cup J_3).$$

The chaotic triple $(\Phi; Z, W)$ has a weak regular model $(\Phi^M; Z, W)$ (see [22]) such that the following hold:

- $K = \{z \in W_0 : \Phi^M_{(0,t)}(z) \in W_t \text{ for all } t \in [0, T]\}$.
- $r^{-1}(J) = \{z \in W_0 : \Phi^M_{(0,t)}(z) \in Z_t \text{ for all } t \in [0, T]\}$.
- For $k \in \{1, 2, 3\}$,
$$r^{-1}(J_k) = \{z \in K : z \text{ leaves } Z \text{ through } Z^l \text{ in time } \le T\}.$$
- The Poincaré map P^M for Φ^M satisfies
$$P^M(z) = f(r(z)), \quad z \in K.$$

Remark 8.7. According to Corollary 8.5, it is interesting to study the behavior of the sequence $L((I - \mu_W)^k)$. Let us first focus on the sequence $a_k = \operatorname{tr}(I - A_m)^k$, where A_m is defined in Corollary 8.6. Let $\rho > 1$ be the spectral radius of $I - A_m$. Then there exists a sequence $n_k \to \infty$ and $q > 0$ such that

$$\lim_{k\to\infty} \frac{a_{n_k}}{\rho^{n_k}} = q.$$

In particular, a_k has exponential growth so it is unbounded. In general, one can prove that the same conclusion holds if the sequence $(L(\mu_W^k))_{k\ge 0}$ is l-periodic with period $l \ne 6$.

Problem 8.8. *Does every chaotic triple $(\Phi; Z, W)$ have a weak regular model? If the answer is not, characterize the chaotic triples that have weak regular models. What one can say about chaotic triples having regular models?*

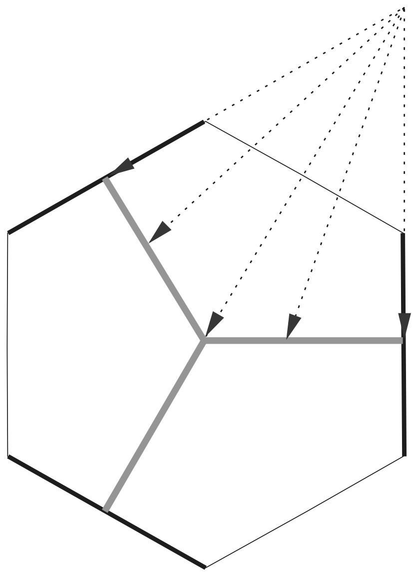

Figure 15. The deformation retraction $r\colon W_0 \to S$.

A Nielsen Numbers Theory: The Mapping Torus Approach

A.1 Generalized Lefschetz Number

The *mapping torus* T_f of $f\colon X \to X$ is the space obtained from $X \times \mathbb{R}_+$ by identifying $(x, s+1)$ with $(f(x), s)$ for all $x \in X$, $s \in \mathbb{R}_+$. On T_f there exists a natural suspension semiflow

$$\phi\colon T_f \times \mathbb{R}_+ \to T_f, \quad \phi_t([x,s]) = [x, s+t], \quad t \geq 0.$$

We may identify X with $X \times \{0\} \subset T_f$. Then the map $f\colon X \to X$ is the return map of the semiflow ϕ. A point $x \in X$ and a positive number $\tau > 0$ determine the time-τ orbit curve

$$\mathrm{orb}_x^\tau := \{\phi_t(x) = \phi_t([x,0]) : t \in [0,\tau]\} \subset T_f.$$

It follows that $x \in \mathrm{Fix}(f)$ if and only if orb_x^1 is a closed curve.

Lemma A.1. *Two fixed points $x_0, x_1 \in \mathrm{Fix}(f)$ belong to the same fixed point class (Nielsen class) if and only if the closed curves $\mathrm{orb}_{x_0}^1$ and $\mathrm{orb}_{x_1}^1$ are freely homotopic in T_f. (The term freely homotopic means homotopic as maps from $\mathbb{S}^1$ into T_f.)*

In the mapping torus T_f take the base point $v \in X \subset T_f$. Let $w\colon [0,1] \to X$ be a fixed path from v to $f(v)$. Let $\Gamma = \pi_1(T_f, v)$ be a fundamental group and let $\pi = \pi_1(X, v)$. By the Seifert–van Kampen theorem,

$$\Gamma = \langle \pi, z \mid \alpha z = z f_\pi(\alpha) \text{ for all } \alpha \in \pi \rangle,$$

where z is represented by the loop $(\mathrm{orb}_v^1)w^{-1}$.

Two elements in $g, g' \in \Gamma$ are *conjugated* if there exists $a \in \Gamma$ such that $g' = aga^{-1}$. Let Γ_c denote the set of conjugacy classes in Γ and $\Gamma \ni \gamma \to [\gamma] \in \Gamma_c$.

Suppose $x \in \mathrm{Fix}(f)$. Pick any path c from v to x, and let $\alpha = [w(f \circ c)c^{-1}] \in \pi$. Then $c_x := [c(\mathrm{orb}_x^1)c^{-1}] \in \Gamma$. One can check that $c_x = z\alpha \in \Gamma$. The Γ-coordinate $\mathrm{cd}_\Gamma(x, f)$ of x is defined to be the conjugacy class $[c_x] = [z\alpha] \in \Gamma_c$. It is equal to the free homotopy class of the closed curve orb_x^1. It follows that the two fixed points are in the same fixed point (Nielsen) class if and only if they have the same Γ-coordinate.

The Γ-*coordinate* $\mathrm{cd}_\Gamma(F, f)$ of a non-empty Nielsen class F is defined to be the common Γ-coordinate of its members

$$\mathrm{cd}_\Gamma(F, f) = \mathrm{cd}_\Gamma(x, f), \quad x \in F.$$

Let $\mathbb{Z}\Gamma$ be the integral ring of group Γ, and let $\mathbb{Z}\Gamma_c$ be a free abelian group with basis Γ_c. The norm in $\mathbb{Z}\Gamma_c$ is defined by

$$\left\| \sum_i k_i \gamma_i \right\| = \sum_i |k_i|,$$

when $\gamma_i \in \Gamma_c$ are all different.

We define the *generalized* Γ*-Lefschetz number* by

$$L_\Gamma(f) := \sum_F \operatorname{ind}(f, F) \cdot \operatorname{cd}_\Gamma(F, f) \in \mathbb{Z}\Gamma_c,$$

the summation being over all Nielsen classes F of f.

The Nielsen number of f is nothing but the number of non-zero terms in $L_\Gamma(f)$. $L_\Gamma(f)$ is a homotopy invariant of f.

A.2 Reidemeister Trace Formula

Assume that X is a finite cell complex and $f\colon X \to X$ is a cellular map. Pick a cellular decomposition $\{e_j^d\}$ of X, the base point v being a 0-cell. This lifts to a π-invariant cellular structure on the universal covering $\tilde{X}$. Choose an arbitrary lift $\tilde{e}_j^d$ for each e_j^d. These lifts form a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\tilde{X}$. The lift $\tilde{f}$ of f is also a cellular map. In every dimension d, the cellular chain map $\tilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\tilde{F}_d$ with respect to the above basis, i.e., $\tilde{F}_d = (a_{ij})$ if

$$\tilde{f}(\tilde{e}_j^d) = \sum_i a_{ij} \tilde{e}_j^d, \quad a_{ij} \in \mathbb{Z}\pi.$$

Theorem A.2 (Reidemeister Trace Formula). *The generalized Lefschetz number is given by*

$$L_\Gamma(f) = \sum_d (-1)^d [\operatorname{tr}(z\tilde{F}_d)] \in \mathbb{Z}\Gamma_c,$$

where $z\tilde{F}_d$ is regarded in $\mathbb{Z}\Gamma$, and the brackets denote the linear map $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma_c$.

A.3 Periodic Orbit Classes

Let $f\colon X \to X$. Observe that $x \in \operatorname{Fix}(f^n)$ if and only if on the mapping torus T_f the time n-orbit curve orb_x^n is a closed curve. We define $x, y \in \operatorname{Fix}(f^n)$ to be in the same *n-orbit class* if and only if orb_x^n and orb_y^n are in the same free homotopy class of closed curves in T_f. The set $\operatorname{Fix}(f^n)$ splits into a disjoint union of n-orbit classes.

Let O^n be an n-orbit class. If $x \in O^n$, then its n-orbit is $\{x, f(x), \dots, f^{n-1}(x)\} \subset O^n$, because the closed orbit curves of these points are the same curve with different base points.

Remark A.3. If $x \in O^n$ and F^n is a Nielsen class containing x, then

$$O^n = F^n \cup f(F^n) \cup \cdots \cup f^{n-1}(F^n).$$

Since for all $x \in O^n$, the closed curves orb_x^n are freely homotopic in T_f, they represent a well-defined conjugacy class $[\operatorname{orb}_x^n]$ in Γ. It will be called the *coordinate* of O^n in Γ, and it is written as

$$\operatorname{cd}_\Gamma(O^n) = [\operatorname{orb}_x^n] \in \Gamma_c.$$

Every n-orbit class O^n is an isolated subset of $\operatorname{Fix}(f^n)$. Its *index* is defined by $\operatorname{ind}(f^n, O^n)$. An n-orbit class O^n is called *essential* if its index is non-zero.

We define the (generalized) *Lefschetz number* (with respect to Γ) by

$$L_\Gamma(f^n) := \sum_{O^n} \operatorname{ind}(f^n, O^n) \cdot \operatorname{cd}_\Gamma(O^n) \in \mathbb{Z}\Gamma_c,$$

the summation being over all n-orbit classes O^n of f^n. The *asymptotic absolute Lefschetz number* is defined by

$$L^\infty(f) := \mathrm{Growth}\|L_\Gamma(f^n)\|.$$

The number $N_\Gamma(f^n)$ of non-zero terms in $L_\Gamma(f^n)$ is called the *n-orbit Nielsen number* of f. It follows that

$$N_\Gamma(f^n) \le N(f^n) \le \|L_\Gamma(f^n)\|.$$

Moreover,

$$L_\Gamma(f^n) = \sum_d (-1)^d [\mathrm{tr}(z\tilde{F})^n] \in \mathbb{Z}\Gamma_c.$$

A.4 Twisted Zeta Function

Suppose a group representation $\rho\colon \Gamma \to \mathrm{GL}_l(R)$ (i.e., $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma$) is given, where R is a commutative ring with unity. Then ρ extends to a ring representation $\rho\colon \mathbb{Z}\Gamma \to M_l(R)$. Define the *ρ-twisted Lefschetz number*

$$L_\rho(f^n) := \mathrm{tr}(L_\Gamma(f^n))^\rho = \sum_{O^n} \mathrm{ind}(f^n, O^n)\ \mathrm{tr}(\mathrm{cd}_\Gamma(O^n))^\rho \in R.$$

The *ρ-twisted Lefschetz zeta function* of f is the formal power series

$$\zeta_\rho(f) := \exp\Big(\sum_{n=1}^{\infty} L_\rho(f^n)\frac{t^n}{n}\Big).$$

It follows that it is a rational function

$$\zeta_\rho(f) = \prod_d \det(I - t(z\tilde{F}_d)^\rho)^{(-1)^{d+1}} \in R(t),$$

where $(z\tilde{F}_d)^\rho$ is the block matrix obtained from the matrix $z\tilde{F}_d$ by replacing each entry (in $\mathbb{Z}\Gamma$) with its ρ-image (an $l \times l$-matrix), and I is a suitable identity matrix.

Example A.4. If $R = \mathbb{Z}$ and $\rho\colon \Gamma \to GL_1(\mathbb{Z}) = \mathbb{Z}$ is trivial (sending everything to 1), then $L_\rho(f) \in \mathbb{Z}$ is the ordinary Lefschetz number, and $\zeta_\rho(f)$ is the classical Lefschetz zeta function.

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