Research Article

Sami Baraket, Souhail Chebbi, Nejmeddine Chorfi and Vicenţiu D. Rădulescu*

Non-autonomous Eigenvalue Problems with Variable (p_1, p_2)-Growth

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Abstract: We are concerned with the study of a class of non-autonomous eigenvalue problems driven by two non-homogeneous differential operators with variable (p_1, p_2)-growth. The main result of this paper establishes the existence of a continuous spectrum consisting in an unbounded interval and the nonexistence of eigenvalues in a neighbourhood of the origin. The abstract results of this paper are described by two Rayleigh-type quotients and the proofs rely on variational arguments.

Keywords: Non-autonomous Eigenvalue Problem, Continuous Spectrum, Non-homogeneous Differential Operator, Variable Exponent

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1 Introduction

The recent study of various nonlinear models described by partial differential equations with variable exponent is motivated by the rigorous mathematical description of many phenomena in applied sciences. In some cases the standard approach based on the theory of classical L^p and $W^{1,p}$ Lebesgue and Sobolev spaces is not adequate in the framework of material with non-homogeneities. For instance, electro-rheological fluids (sometimes referred to as "smart fluids") or phenomena in image processing are described in a correct manner by mathematical models in which the exponent p is allowed to vary. This leads us to the study of variable exponents Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function. We refer to the work by Diening, Hästo, Harjulehto, and Ruzicka [6] for the abstract framework describing these spaces as well as to the monograph by Rădulescu and Repovš [15], which includes a thorough variational and topological analysis of several classes of problems with variable exponent (see also the survey paper by Rădulescu [14] and the papers by Colasuonno and Pucci [3] and Pucci and Zhang [13]).

We are interested in the study of a class of non-autonomous stationary problems, which are characterized by the fact that the associated energy density changes its ellipticity and growth properties according to the point. Problems of this type have been intensively studied starting with the pioneering contributions of Halsey [7] and Zhikov [17–19] in relationship with the analysis of the behaviour of strongly anisotropic materials in the context of the homogenization and nonlinear elasticity.

Sami Baraket: Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia, e-mail: sami.baraket@fst.rnu.tn
Souhail Chebbi, Nejmeddine Chorfi: Department of Mathematics, College of Sciences, King Saud University, Box 2455, Riyadh 11451, Saudi Arabia, e-mail: schebbi@ksu.edu.sa, nchorfi@ksu.edu.sa
***Corresponding author: Vicenţiu D. Rădulescu:** Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest; and Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania, e-mail: vicentiu.radulescu@math.cnrs.fr

The study of non-homogeneous elliptic problems has been recently extended by Kim and Kim [8] to a new class of differential operators. Their contribution is a step forward in the analysis of nonlinear problems with variable exponent since it enables the understanding of problems with possible lack of uniform convexity. In the present paper, we extend this study to problems involving several non-homogeneous operators (as introduced in [8]) and we describe some spectral properties in relationship with two Rayleigh-type quotients. Section 2 includes some basic properties of function spaces with variable exponents. The main result is described in Section 3 while the proofs and some perspectives are presented in Section 4 of this paper.

2 Basic Properties of Spaces with Variable Exponent

Throughout this paper, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

Assume that $p \in C_+(\overline{\Omega})$ and let

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

We define the Lebesgue space with variable exponent by

$$L^{p(x)}(\Omega) = \Big\{u : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}.$$

This function space is a Banach space if it is endowed with the norm

$$|u|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \le 1\Big\}.$$

This norm is also called the Luxemburg norm. Then $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$, and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

The standard inclusion between Lebesgue spaces generalizes to the framework of spaces with variable exponent, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \le p_2$ in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)}.$$

An important role in analytic arguments on Lebesgue spaces with variable exponent is played by the *modular* of $L^{p(x)}(\Omega)$, which is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following properties hold:

$$\begin{aligned} |u|_{p(x)} > 1 \quad &\Longrightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \\ |u|_{p(x)} < 1 \quad &\Longrightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \\ |u_n - u|_{p(x)} \to 0 \quad &\Longleftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \end{aligned} \tag{2.1}$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\mu}\Big|^{p(x)} + \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\Big)\,dx \le 1\Big\}.$$

Let $W_0^{1,p(x)}(\Omega)$ denote the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|u\|_{p(x)}$. When smooth functions are dense, we can also use the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Using the Poincaré inequality, we can define the space $W_0^{1,p(x)}(\Omega)$, in an equivalent manner, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The vector space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. Moreover, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω, then there exists the continuous embedding

$$W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega).$$

Set

$$\varrho_{p(x)}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx.$$

If $(u_n), u \in W_0^{1,p(x)}(\Omega)$, then the following properties hold:

$$\begin{aligned} \|u\| > 1 &\quad\Longrightarrow\quad \|u\|^{p^-} \le \varrho_{p(x)}(u) \le \|u\|^{p^+}, \\ \|u\| < 1 &\quad\Longrightarrow\quad \|u\|^{p^+} \le \varrho_{p(x)}(u) \le \|u\|^{p^-}, \\ \|u_n - u\| \to 0 &\quad\Longleftrightarrow\quad \varrho_{p(x)}(u_n - u) \to 0. \end{aligned}$$

Set

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

We point out that if $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

For a constant function p the variable exponent Lebesgue and Sobolev spaces coincide with the standard Lebesgue and Sobolev spaces. As pointed out in [15], the function spaces with variable exponent have some striking properties such as the following:

(i) If $1 < p^- \le p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p\,dx = p \int_0^\infty t^{p-1} \big|\{x \in \Omega : |u(x)| > t\}\big|\,dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do *not* have the *mean continuity property*. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \le C|f|_{p(x)}\|g\|_{L^1}$$

holds if and only if p is constant.

3 Spectrum Consisting in an Unbounded Interval

Assume $p_1, p_2 \in C_+(\overline{\Omega})$ and consider the functions $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(H1) The mappings $\phi(\cdot\,,\xi)$ and $\psi(\cdot\,,\xi)$ are measurable on Ω for all $\xi \geq 0$ and $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$.

(H2) There exist functions $a_1 \in L^{p_1'}(\Omega)$ and $a_2 \in L^{p_2'}(\Omega)$ and $b > 0$ such that

$$|\phi(x,|v|)v| \leq a_1(x) + b|v|^{p_1(x)-1}, \quad |\psi(x,|v|)v| \leq a_2(x) + b|v|^{p_2(x)-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(H3) There exists $c > 0$ such that

$$\phi(x,\xi) \geq c\xi^{p_1(x)-2}, \quad \phi(x,\xi) + \xi\frac{\partial\phi}{\partial\xi}(x,\xi) \geq c\xi^{p_1(x)-2}$$

and

$$\psi(x,\xi) \geq c\xi^{p_2(x)-2}, \quad \psi(x,\xi) + \xi\frac{\partial\psi}{\partial\xi}(x,\xi) \geq c\xi^{p_2(x)-2}$$

for almost all $x \in \Omega$ and for all $\xi > 0$.

Assume that $q \in C_+(\overline{\Omega})$ and

(Q) $p_1(x) < q^- \leq q^+ < p_2(x) < p_1^*(x)$ for all $x \in \overline{\Omega}$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that the following assumptions are fulfilled:

(f1) We have $tf(x,t) \geq 0$ for a.a. $(x,t) \in \Omega \times \mathbb{R}$ and there exists $m \in L^\infty(\Omega)_+ \setminus \{0\}$ such that

$$|f(x,t)| \leq m(x)|t|^{q(x)-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

(f2) There exist $M > 0$ and $\theta > p_1^+$ such that

$$0 < \theta F(x,t) \leq tf(x,t) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \setminus \{0\},$$

where $F(x,t) := \int_0^t f(x,s)\,ds$.

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) - \operatorname{div}(\psi(x,|\nabla u|)\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Problem (3.1) is driven by non-homogeneous operators of the type $\operatorname{div}(\phi(x,|\nabla u|)\nabla u)$. If $\phi(x,\xi) = \xi^{p(x)-2}$, then we obtain the standard $p(x)$-Laplace operator, that is, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Our abstract setting includes the case $\phi(x,\xi) = (1+|\xi|^2)^{(p(x)-2)/2}$, which corresponds to the generalized mean curvature operator

$$\operatorname{div}[(1+|\nabla u|^2)^{(p(x)-2)/2}\nabla u].$$

The capillarity equation corresponds to

$$\phi(x,\xi) = \left(1 + \frac{\xi^{p(x)}}{\sqrt{1+\xi^{2p(x)}}}\right)\xi^{p(x)-2}, \quad x \in \Omega,\ \xi > 0,$$

hence the corresponding capillary phenomenon is described by the differential operator

$$\operatorname{div}\left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right].$$

We say that $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ is a solution of problem (3.1) if

$$\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)]\nabla u \cdot \nabla v\,dx = \lambda \int_\Omega f(x,u)v\,dx$$

for all $v \in W_0^{1,p_2(x)}(\Omega)$.

In this case, u is an *eigenfunction* of problem (3.1) and the corresponding $\lambda \in \mathbb{R}$ is an *eigenvalue* of (3.1). The choice of $W_0^{1,p_2(x)}(\Omega)$ as a suitable function space for problem (3.1) is dictated by our hypothesis (Q).

For ϕ and ψ described in hypotheses (H1)–(H3) we set

$$A_0(x,t) := \int_0^t [\phi(x,s) + \psi(x,s)]s\,ds. \tag{3.2}$$

An important role in the proof of our main result is played by the following assumption, which is also used in [8] for the existence of weak solutions in a different framework:

(H4) For all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$ the following estimate holds:

$$0 \le [\phi(x,|\xi|) + \psi(x,|\xi|)]|\xi|^2 \le p_1^+ A_0(x,|\xi|).$$

We notice that our hypothesis (f1) implies that

$$0 \le F(x,t) \le \frac{m(x)}{q(x)}|t|^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.3}$$

We define the following Rayleigh-type quotients:

$$\lambda^* := \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega A_0(x,|\nabla u|)\,dx}{\int_\Omega F(x,u)\,dx}$$

and

$$\lambda_* := \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (\phi(x,|\nabla u|) + \psi(x,|\nabla u|))|\nabla u|^2\,dx}{\int_\Omega uf(x,u)\,dx}. \tag{3.4}$$

Theorem 3.1. *Assume that hypotheses* (H1)–(H4), (f1), (f2), (Q) *are fulfilled. Then the following properties hold:*

(i) *Problem* (3.1) *has solutions for all* $\lambda \ge \lambda^*$.

(ii) *Problem* (3.1) *does not have any solution, provided that* $\lambda < \lambda_*$.

We do not have any information about the contribution of real parameters satisfying $\lambda \in [\lambda_*, \lambda^*)$ even in simple cases, for instance if Ω is a ball or for particular values of ϕ, ψ and f.

Related concentration properties are established in Kim and Kim [8], Mihăilescu and Rădulescu [11, 12], Rădulescu [14] and Rădulescu and Repovš [15, Chapter 3], see also Cencelj, Repovš, Virk [2] and Repovš [16] for recent contributions to anisotropic elliptic problems.

4 Proof of Theorem 3.1 and Perspectives

We first give the proof of our main result. For this purpose we establish several auxiliary results.

Lemma 4.1. *We have* $\lambda^* > \lambda_* > 0$.

Proof. Using hypothesis (H4) we obtain

$$A_0(x,|\xi|) \ge \frac{1}{p_1^+}[\phi(x,|\xi|) + \psi(x,|\xi|)]|\xi|^2$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^N$. Thus

$$\int_\Omega A_0(x,|\nabla u|)\,dx \ge \frac{1}{p_1^+}\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)]|\nabla u|^2\,dx \tag{4.1}$$

for all $u \in W_0^{1,p_2(x)}(\Omega)$. Using now hypothesis (f2), we deduce that for all $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ we have

$$0 < \int_\Omega F(x,u)\,dx \le \frac{1}{\theta}\int_\Omega uf(x,u)\,dx. \tag{4.2}$$

Combining relations (4.1) and (4.2), we obtain

$$\frac{\int_\Omega A_0(x, |\nabla u|)\,dx}{\int_\Omega F(x,u)\,dx} \geq \frac{\theta}{p_1^+}$$

for all $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$. Taking the infimum in this inequality for $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ and using (f2), we deduce that

$$\lambda^* \geq \frac{\theta}{p_1^+}\lambda_* > \lambda_*.$$

It remains to show that $\lambda_* > 0$. Using (f1), we deduce that for all $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ we have

$$\begin{aligned} 0 < \int_\Omega uf(x,u)\,dx \leq \int_\Omega m(x)|u|^{q(x)}\,dx &\leq \|m\|_{L^\infty} \int_\Omega |u|^{q(x)}\,dx \\ &\leq \|m\|_{L^\infty} \int_\Omega (|u|^{q^+} + |u|^{q^-})\,dx. \end{aligned} \tag{4.3}$$

Next, using hypothesis (H3), we deduce that for all $u \in W_0^{1,p_2(x)}(\Omega)$ we have

$$\begin{aligned} \int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)]|\nabla u|^2 &\geq c \int_\Omega [|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}]\,dx \\ &\geq \frac{c}{2} \int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\,dx \\ &\geq C_1 \int_\Omega (|u|^{q^+} + |u|^{q^-})\,dx, \end{aligned} \tag{4.4}$$

where C_1 is a positive constant depending only on Ω, q^+, q^-, and c (given by (H3)).

Relations (4.3) and (4.4) imply that

$$\frac{\int_\Omega (\phi(x,|\nabla u|) + \psi(x,|\nabla u|))|\nabla u|^2\,dx}{\int_\Omega uf(x,u)\,dx} \geq \frac{C_1}{\|m\|_{L^\infty}}$$

for all $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$. Hence $\lambda_* > 0$. □

Define the functional $A : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ by

$$A(u) := \int_\Omega A_0(x, |\nabla u|)\,dx,$$

where A_0 is defined in (3.2).

Then $A \in C^1(W_0^{1,p_2(x)}(\Omega), \mathbb{R})$ and for all $u, v \in W_0^{1,p_2(x)}(\Omega)$ we have

$$A'(u)(v) = \int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla v|)]\nabla u \cdot \nabla v\,dx.$$

Moreover, the operator $A : W_0^{1,p_2(x)}(\Omega) \to (W_0^{1,p_2(x)}(\Omega))^*$ is strictly monotone and is a mapping of type (S_+), that is, if

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p_2(x)}(\Omega) \quad \text{as } n \to \infty$$

and

$$\limsup_{n\to\infty} \langle A'(u_n) - A'(u), u_n - u \rangle \leq 0,$$

then

$$u_n \to u \quad \text{in } W_0^{1,p_2(x)}(\Omega) \quad \text{as } n \to \infty.$$

Standard arguments also show that A is weakly lower semicontinuous. We refer to [8, Lemmas 3.2 and 3.4] for details and proofs.

Set

$$B(u) := \int_\Omega F(x,u)\,dx, \quad u \in W_0^{1,p_2(x)}(\Omega).$$

Then $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ is a solution of problem (3.1) if and only if $A'(u) = \lambda B'(u)$.

Lemma 4.2. *We have*

$$\lim_{u\to 0} \frac{A(u)}{B(u)} = \lim_{\|u\|\to\infty} \frac{A(u)}{B(u)} = +\infty.$$

Proof. Using hypothesis (3.3), we deduce that

$$F(x,u) \le \int_\Omega \frac{m(x)}{q(x)}|u|^{q(x)}\,dx \le \frac{\|m\|_{L^\infty}}{q^-}\int_\Omega |u|^{q(x)}\,dx$$

for all $u \in W_0^{1,p_2(x)}(\Omega)$. But there holds

$$|u|^{q(x)} \le |u(x)|^{q^+} + |u(x)|^{q^-}.$$

It follows that for all $u \in W_0^{1,p_2(x)}(\Omega)$ we have

$$\begin{aligned} B(u) &\le \frac{\|m\|_{L^\infty}}{q^-}\int_\Omega (|u|^{q^+} + |u|^{q^-})\,dx \\ &\le C_2(\|u\|^{q^+} + \|u\|^{q^-}), \end{aligned} \tag{4.5}$$

where C_2 is a positive constant depending only on m, q and the continuous embeddings of $W_0^{1,p_2(x)}(\Omega)$ into $L^{q^+}(\Omega)$ and $L^{q^-}(\Omega)$.

Next, using (H4), we have

$$A(u) \ge \frac{1}{p_1^+}\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)]|\nabla u|^2\,dx$$

for all $u \in W_0^{1,p_2(x)}(\Omega)$. By (H3) we deduce that

$$\begin{aligned} A(u) &\ge \frac{c}{p_1^+}\int_\Omega [|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}]\,dx \\ &\ge \frac{c}{p_1^+}\int_\Omega |\nabla u|^{p_1(x)}\,dx. \end{aligned} \tag{4.6}$$

Let us first assume that $(u_n) \subset W_0^{1,p_2(x)}(\Omega)$ and $u_n \to 0$ as $n \to \infty$. Without loss of generality we can assume that $\|u_n\| < 1$. Relation (4.6) implies that

$$A(u_n) \ge \frac{c}{p_1^+}\int_\Omega |\nabla u_n|^{p_1^+}\,dx = \frac{c}{p_1^+}\|u_n\|^{p_1^+}.$$

Combining this information with (4.5), we obtain for all n that

$$\frac{A(u_n)}{B(u_n)} \ge \frac{c}{C_2 p_1^+}\,\frac{\|u_n\|^{p_1^+}}{\|u_n\|^{q^+} + \|u_n\|^{q^-}}.$$

Using hypothesis (Q), we deduce that $A(u_n)/B(u_n) \to +\infty$ as $n \to \infty$.

For the second limit in the statement of the Lemma we observe that relation (4.6) also yields

$$A(u) \ge \frac{c}{p_1^+}\int_\Omega |\nabla u|^{p_2(x)}\,dx.$$

Since $\|u\| \to \infty$, we can assume that $\|u\| > 1$. It follows that

$$A(u) \geq \frac{c}{p_1^+} \int_\Omega |\nabla u|^{p_2^-}\,dx = \frac{c}{p_1^+} \|u\|^{p_2^-},$$

hence

$$\frac{A(u)}{B(u)} \geq \frac{c}{C_2 p_1^+} \frac{\|u\|^{p_2^-}}{\|u\|^{q^+} + \|u\|^{q^-}}.$$

Using again assumption (Q), we deduce that $A(u)/B(u) \to +\infty$ as $\|u\| \to \infty$. □

The next step is to show that the infimum in $W_0^{1,p_2(x)}(\Omega)$ of the Rayleigh quotient $A(u)/B(u)$ is attained.

Recall that

$$\lambda^* := \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{A(u)}{B(u)}.$$

Lemma 4.3. *There exists $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ such that*

$$\lambda^* = \frac{A(u)}{B(u)}.$$

Moreover, u is a solution of problem (3.1) *for $\lambda = \lambda^*$.*

Proof. Let $(u_n) \subset W_0^{1,p_2(x)}(\Omega)$ be such that

$$\lambda^* = \lim_{n\to\infty} \frac{A(u_n)}{B(u_n)}.$$

By Lemma 4.2, the sequence (u_n) is bounded. Thus, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p_2(x)}(\Omega), \tag{4.7}$$

$$u_n \to u \quad \text{strongly in } L^{q(x)}(\Omega). \tag{4.8}$$

Using the weak lower semicontinuity of A, we obtain

$$A(u) \leq \liminf_{n\to\infty} A(u_n). \tag{4.9}$$

Using now (2.1) in combination with (4.8) and the mean value theorem, we deduce that

$$B(u) = \lim_{n\to\infty} B(u_n). \tag{4.10}$$

Relations (4.9) and (4.10) yield

$$\lambda^* = \frac{A(u)}{B(u)}.$$

We now prove that $u \neq 0$. Arguing by contradiction, we obtain that relations (4.7) and (4.8) imply that (u_n) converges weakly to 0 in $W_0^{1,p_2(x)}(\Omega)$ and strongly in $L^{q(x)}(\Omega)$. In particular, we have

$$\lim_{n\to\infty} B(u_n) = 0. \tag{4.11}$$

By Lemma 4.1 we have $\lambda^* > 0$. Fix $0 < \varepsilon < \lambda^*$. Thus, for all $n \in \mathbb{N}$ large enough, we have

$$\left| \frac{A(u_n)}{B(u_n)} - \lambda^* \right| < \varepsilon,$$

hence

$$(\lambda^* - \varepsilon) B(u_n) < A(u_n) < (\lambda^* + \varepsilon) B(u_n).$$

Thus, by (4.11) we have

$$\lim_{n\to\infty} A(u_n) = 0. \tag{4.12}$$

We use this information in order to prove that (u_n) converges strongly to 0 in $W_0^{1,p_2(x)}(\Omega)$. Indeed, by (H4), we have

$$A(u_n) \geq \frac{1}{p_1^+} \int_\Omega [\phi(x, |\nabla u_n|) + \psi(x, |\nabla u_n|)]|\nabla u_n|^2 \, dx.$$

Using now hypothesis (H3), we deduce that

$$A(u_n) \geq \frac{c}{p_1^+} \int_\Omega (|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)}) \, dx.$$

This inequality and (4.12) imply that $u_n \to 0$ in $W_0^{1,p_2(x)}(\Omega)$. By Lemma 4.2 we now deduce that

$$\lim_{n\to\infty} \frac{A(u_n)}{B(u_n)} = +\infty,$$

which is a contradiction. This contradiction shows that $u \neq 0$.

It remains to show that u is a weak solution of problem (3.1). The basic idea in the proof is that

$$\lambda^* = \frac{A(u)}{B(u)} = \inf_{v \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{A(v)}{B(v)}.$$

Fix arbitrarily $v \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ and consider the map

$$t \mapsto h(t) := \frac{A(u + tv)}{B(u + tv)},$$

which is defined in a neighbourhood of the origin. It follows that $h'(0) = 0$, hence

$$[A'(u + tv)B(u + tv) - A(u + tv)B'(u + tv)]_{|t=0} = 0.$$

Therefore,

$$B(u) \int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla u|)]\nabla u \cdot \nabla v \, dx - A(u) \int_\Omega f(x, u)v \, dx = 0.$$

Since $A(u) = \lambda^* B(u)$, we conclude that u solves (3.1), hence λ^* is an eigenvalue of this problem. □

Lemma 4.4. *Problem* (3.1) *admits a solution for all* $\lambda > \lambda^*$.

Proof. Fix $\lambda > \lambda^*$ and consider the nonlinear map

$$C(u) = A(u) - \lambda B(u).$$

Then C is differentiable and λ is an eigenvalue of problem (3.1) if and only if C admits a nontrivial critical point.

Using hypotheses (H3), (f1), (f2), we obtain

$$\begin{aligned} C(u) &\geq \frac{c}{p_1^+} \int_\Omega [|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}] \, dx - \lambda \int_\Omega \frac{m(x)}{q(x)} |u|^{q(x)} \, dx \\ &\geq \frac{c}{p_1^+} \int_\Omega [|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}] \, dx - \frac{\lambda \|m\|_{L^\infty}}{q^-} \int_\Omega |u|^{q(x)} \, dx. \end{aligned}$$

Using now hypothesis (Q), we deduce that C is coercive, that is, $\lim_{\|u\|\to\infty} C(u) = +\infty$. By the weak lower semicontinuity of C there exists $w \in W_0^{1,p_2(x)}(\Omega)$ such that

$$C(w) = \inf_{u \in W_0^{1,p_2(x)}(\Omega)} C(u).$$

We argue in what follows that $w \neq 0$. Indeed, using the definition of λ^* and the fact that $\lambda^* < \lambda$, we find $v \in W_0^{1,p_2(x)}(\Omega)$ such that $A(v) - \lambda B(v) < 0$, hence $C(v) < 0$. Since w is a global minimum point of C, it follows that $C(w) < 0$, which implies $w \neq 0$. We conclude that λ is an eigenvalue of problem (3.1) with corresponding eigenfunction w. □

We now return to the second Rayleigh-type quotient, which defines λ_*; see relation (3.4).

Lemma 4.5. *For all $\lambda < \lambda_*$ problem* (3.1) *does not have a solution.*

Proof. Recall that

$$\lambda_* = \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{S(u)}{T(u)},$$

where

$$S(u) := \int_\Omega (\phi(x, |\nabla u|) + \psi(x, |\nabla u|))|\nabla u|^2 \, dx, \quad T(u) := \int_\Omega uf(x, u) \, dx.$$

Fix $\lambda < \lambda_*$. We argue by contradiction and assume that λ is an eigenvalue of problem (3.1). Thus, there exists $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ such that for all $v \in W_0^{1,p_2(x)}(\Omega)$ there holds

$$\int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla v|)]\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega f(x, u)v \, dx,$$

that is, $S'(u) = \lambda T'(u)$.

Taking $v = u$, we obtain

$$S(u) = \lambda T(u).$$

Therefore,

$$\lambda = \frac{S(u)}{T(u)} \geq \inf_{v \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{S(v)}{T(v)} = \lambda_*,$$

which contradicts the choice of λ. □

Combining Lemmas 4.1–4.5, we obtain the conclusion of Theorem 3.1.

4.1 Motivation and Perspectives

The variable exponents $p_1(x)$ and $p_2(x)$ dictate the geometry of a composite that changes its hardening exponent according to the point. We point out that the abstract setting developed in this paper extends the nonstandard growth conditions of (p, q) type. We refer to Marcellini [9, 10] who is interested in integral functionals of the type

$$u \mapsto \int_\Omega F(x, \nabla u) \, dx,$$

where the integrand $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies unbalanced polynomial growth conditions of the type

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q \quad \text{with } 1 < p < q$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

We consider that a very interesting research direction is to extend the approach developed in this paper to the abstract setting recently studied by Baroni, Colombo and Mingione [1] and Colombo and Mingione [4, 5], namely non-autonomous problems with associated energies of the type

$$u \mapsto \int_\Omega [|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}] \, dx \tag{4.13}$$

and

$$u \mapsto \int_\Omega [|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e + |x|)] \, dx, \tag{4.14}$$

where $p_1(x) \leq p_2(x)$, $p_1 \neq p_2$ and $a(x) \geq 0$. If we consider two different materials with power hardening exponents $p_1(x)$ and $p_2(x)$, respectively, the coefficient $a(x)$ dictates the geometry of a composite of the two materials. When $a(x) > 0$, then $p_2(x)$-material is present, otherwise the $p_1(x)$-material is the only one making the composite. On the other hand, since the integral functional defined in (4.14) is degenerate on the zero set of the gradient, it is natural to ask oneself what happens if we modify the integrand in such a way that, when additionally $|\nabla u|$ is small, there is an unbalance between the two terms of the integrand. For instance, we can consider the functional

$$u \mapsto \int_\Omega [|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(1 + |x|)]\, dx.$$

For the isotropic case we refer for further comments to Baroni, Colombo and Mingione [1, pp. 376–377], including remarks on degeneracy phenomena at the phase transition.

According to the terminology used in this paper, the study of the integral functionals defined in (4.13) and (4.14) corresponds to the analysis of the differential operators

$$-\operatorname{div}(\phi(x, |\nabla u|)\nabla u) - \operatorname{div}(a(x)\psi(x, |\nabla u|)\nabla u)$$

and

$$-\operatorname{div}(\phi(x, |\nabla u|)\nabla u) - \operatorname{div}(a(x)\psi(x, |\nabla u|) \log(e + |x|)\nabla u).$$

This approach can be developed not only in Sobolev spaces with variable exponents (like in the present work) but also in the more general framework of Musielak–Orlicz spaces (see Rădulescu and Repovš [15, Chaper 4] for a collection of stationary problems studied in these function spaces).

The problem analyzed in this paper corresponds to a subcritical setting, as described in hypothesis (Q). We appreciate that valuable research directions correspond either to the critical or to the supercritical framework (in the sense of Sobolev variable exponents). No results are known even for the “almost critical” case with lack of compactness, namely assuming that hypothesis (Q) is replaced with

(Q’) $p_1(x) < q^- \le q^+ < p_2(x) \preceq p_1^*(x)$ for all $x \in \overline{\Omega}$,

where $p_2(x) \preceq p_1^*(x)$ means that there exists $z \in \Omega$ such that $p_2(z) = p_1^*(z)$ and $p_2(x) < p_1^*(x)$ for all $x \in \overline{\Omega} \setminus \{z\}$.

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