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Research Article

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Solutions for the Problems Involving Fractional Laplacian and Indefinite Potentials

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Abstract: In this paper, we consider a class of Schrödinger equations involving fractional Laplacian and indefinite potentials. By modifying the definition of the Nehari–Pankov manifold, we prove the existence and asymptotic behavior of least energy solutions. As the fractional Laplacian is nonlocal, when the bottom of the potentials contains more than one isolated components, the least energy solutions may localize near all the isolated components simultaneously. This phenomenon is different from the Laplacian.

Keywords: Fractional Laplacian, Least Energy Solutions, Indefinite Potentials

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1 Introduction and Main Results

We consider the following Schrödinger equations involving fractional Laplacian and indefinite potentials:

$$(-\Delta)^s u + (\lambda a(x) - \delta)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{$\mathcal{P}_\lambda$}$$

where $0 < s < 1$, $N > 2s$, $2 < p \le 2_s^* := \frac{2N}{N-2s}$, $a(x)$ is a nonnegative continuous function and the zero set $a^{-1}(0) := \{x \in \mathbb{R}^N : a(x) = 0\}$ has a nonempty, smooth and connected interior part $\operatorname{int} a^{-1}(0)$.

The fractional Laplacian has been studied extensively in recent years. In [9], Caffarelli and Silvestre showed that the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^d$ can be expressed as the Dirichlet-to-Neumann operator for a suitable *local* problem on the upper half space $\mathbb{R}_+^{d+1}$. For one-dimensional case, Frank and Lenzmann [16] proved the uniqueness and nondegeneracy of ground states for a class of semilinear fractional Laplacian equations with subcritical growth. The authors also obtained similar results in [17] for the higher-dimensional case. Via a Lyapunov–Schmidt reduction method, Dávila, del Pino and Wei [13] proved the existence and concentration of standing waves for a class of semilinear fractional Laplacian equations. For more details of fractional Laplacian, please see [3, 7, 8, 10, 11, 19, 28, 30] and the references therein.

Many papers deal with Schrödinger equations involving Laplacian and potential wells. For λ large enough, the operator $-\Delta + \lambda a(x) - \delta$ is positive definite if $\delta < \lambda_1$, where λ_1 is the principle eigenvalue for $-\Delta$ on functions defined in $\operatorname{int} a^{-1}(0)$. If $\delta > \lambda_1$, the operator $-\Delta + \lambda a(x) - \delta$ is indefinite. Take $\delta = -1$, the positive definite case, Bartsch and Wang [6] considered ($\mathcal{P}_\lambda$) with a more general nonlinearity $f(x, u)$ satisfies some conditions. The authors proved that there exists $\Lambda_1 > 0$ such that ($\mathcal{P}_\lambda$) has a positive and a negative solution for $\lambda > \Lambda_1$. Furthermore, if $f(x, u)$ is odd in u, the authors also proved that for any $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that ($\mathcal{P}_\lambda$) has at least k pairs $\pm u_1, \pm u_2, \dots, \pm u_k$ of nontrivial solutions for $\lambda \ge \Lambda_k$. For more details about the positive definite case, please see [1, 2, 4, 21] and the references therein.

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For the indefinite case, Ding and Wei [15] considered the following problem:

$$\begin{cases} -\Delta u(x) + \lambda V(x)u(x) = \lambda |u(x)|^{p-2}u(x) + \lambda g(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $V(x)$ can be negative in some domains in $\mathbb{R}^N$, $g(x,u)$ is a perturbation term. Both for subcritical growth and critical growth, using variational methods, the authors proved that there exists $\Lambda > 0$ such that for any $\lambda > \Lambda$, problem (1.1) admits at least one nontrivial solution. In [29], Szulkin and Weth gave a new minimax characterization of the corresponding critical value and hence reduced the indefinite problem to a definite one. The authors also presented a precise description of the Nehari–Pankov manifold which is useful even for other problems. Using this type of manifold, Bartsch and the first author [5] proved the existence of multi-bump solutions for semilinear equations with indefinite potentials. For more details about indefinite case, please see [14, 18] and the references therein.

In present paper, we are interested in the existence and concentration phenomenon of least energy solutions for fractional Laplacian with indefinite potentials. Motivated by Servadei and Valdinoci [24–27], we consider our problem directly in the corresponding fractional Sobolev space without taking s-harmonic extension. We will prove that $(\mathcal{P}_\lambda)$ has a least energy solution which localizes near the potential well and changes sign for λ large. Our assumptions about $a(x)$ and δ are as follows.

(A1) $a(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $a(x) \geq 0$ and $\Omega := \operatorname{int}\{x \in \mathbb{R}^N : a(x) = 0\}$ is nonempty, the boundary of Ω is smooth and $\bar{\Omega} = \{x \in \mathbb{R}^N : a(x) = 0\}$.

(A2) $a_0 = \frac{1}{2}\liminf_{|x|\to\infty} a(x) > 0$.

(A3) The operator $(-\Delta)^s - \delta$ defined in $\operatorname{int} a^{-1}(0)$ is indefinite and nondegenerate. Namely, $\lambda_k < \delta < \lambda_{k+1}$ for some $k \geq 1$. Here $\{\lambda_k\}$ is the class of all eigenvalues for $(-\Delta)^s$ on functions defined in $\operatorname{int} a^{-1}(0)$.

Remark 1.1. Assume (A1) and (A2) hold; we see that Ω is bounded in $\mathbb{R}^N$. In fact, there exists $R > 0$ such that

$$\Omega \subseteq B_R(0), \quad a(x) > a_0 \quad \text{for all } |x| > R, \tag{1.2}$$

where $B_R(0)$ is the ball centered at 0 with radii R. Of course, we can replace (A2) by the following one.

(A2') There exists a constant $M > 0$ such that the Lebesgue measure of the set $\{x \in \mathbb{R}^N : a(x) < M\}$ is finite, i.e. $|\{x \in \mathbb{R}^N : a(x) < M\}| < +\infty$.

Let $H^s(\mathbb{R}^N)$ be the standard fractional Sobolev space defined by

$$H^s(\mathbb{R}^N) := \Big\{u : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < +\infty,\ \int_{\mathbb{R}^N} u^2\,dx < +\infty\Big\}$$

with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} uv\,dx,$$

and the induced norm is

$$\|u\| := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} u^2\,dx\Big)^{\frac{1}{2}}.$$

Let

$$E_0 := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

be a subspace of $H^s(\mathbb{R}^N)$ endowed with a new norm

$$\|u\|_0 = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{1}{2}}, \quad u \in E_0.$$

As described in Section 2.2, $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent and $(E_0, \|\cdot\|_0)$ is a Hilbert space.

Denote

$$V_\lambda = \lambda a(x) - \delta, \quad V_\lambda^+ = \max\{0, V_\lambda\}, \quad V_\lambda^- = \max\{0, -V_\lambda\}.$$

Let

$$E_\lambda := \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2\,dx < \infty\Big\}$$

be the Hilbert space endowed with the norm

$$\|u\|_\lambda := \bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx\bigg)^{\frac{1}{2}} \quad \text{for all } u \in E_\lambda.$$

The energy functional corresponding to $(\mathcal{P}_\lambda)$ is

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx \quad \text{for all } u \in E_\lambda.$$

with Fréchet derivative

$$I_\lambda'(u)v := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda uv\,dx - \int_{\mathbb{R}^N} |u|^{p-2}uv\,dx \quad \text{for all } v \in E_\lambda.$$

We say $u_\lambda \in E_\lambda$ is a solution to $(\mathcal{P}_\lambda)$ if $u_\lambda \in E_\lambda$ is a critical point of I_λ, i.e. $I_\lambda'(u_\lambda) = 0$.

Let us denote $L_\lambda := (-\Delta)^s + \lambda a(x) - \delta$ and $L_0 := (-\Delta)^s - \delta$. Take $\{\mu_k(L_0)\}$ be the class of all eigenvalues of L_0 in E_0(each eigenvalue is repeated according to its multiplicity). As proved in Section 2.3, if $\mu_1(L_0) > 0$, then $(-\Delta)^s + \lambda a(x) - \delta$ is positive definite and the ground state of $(\mathcal{P}_\lambda)$ is positive for λ large enough. If $\mu_1(L_0) < 0$ and $\mu_k(L_0) \neq 0$ for any k, then $(-\Delta)^s + \lambda a(x) - \delta$ is indefinite and the least energy solutions of $(\mathcal{P}_\lambda)$ are sign changing for λ large enough.

Let e_k be an eigenfunction corresponding to $\mu_k(L_0)$ for $k = 1, 2, 3, \dots$. We may assume that $\{e_k\}_{k\geq 1}$ is an orthogonal base of E_0 and $L^2(\Omega)$ (see Lemma 2.3 in Section 2.3). By (A3), E_0 can be split as an orthogonal sum $E_0^- \oplus E_0^+$ according to the positive and negative eigenvalues of L_0, i.e.

$$E_0 = E_0^- \oplus E_0^+,$$

where

$$E_0^- = \operatorname{span}\{e_1, e_2, \dots, e_k\}, \quad E_0^+ = \Big\{u \in E_0 : \int_\Omega uv\,dx = 0 \text{ for all } v \in E_0^-\Big\}.$$

As proved in Section 2.3, we see that $\inf \sigma_e(L_\lambda) \geq \lambda a_0 - \delta$, where $\sigma_{\mathrm{ess}}(L_\lambda)$ is the essential spectrum of L_λ. Moreover, there exist a finite number of k such that $\mu_k(L_\lambda) < 0$, i.e. L_λ thus has finite Morse index. For large λ fixed, L_λ has finite eigenvalues below $\sigma_{\mathrm{ess}}(L_\lambda)$. By the study of asymptotic behavior of eigenvalues for L_λ as $\lambda \to +\infty$, we see that 0 is not an eigenvalue of L_λ for λ large. Therefore, E_λ can be split as an orthogonal sum $E_\lambda = E_\lambda^- \oplus E_\lambda^+$ according to the negative and positive eigenvalues of L_λ.

Note that for λ large enough, the space E_λ^- are close to E_0^- (please see Section 2.3). In order to study the asymptotic behavior of least energy solutions for $(\mathcal{P}_\lambda)$, we need to modify the Nehari–Pankov manifold. To do that, let $P_\lambda^- : E_\lambda \mapsto E_0^-$ and $P_0^- : E_0 \mapsto E_0^-$ be two orthogonal projections. Define the modified Nehari–Pankov manifold of I_λ by

$$\mathcal{N}_\lambda := \{u \in E_\lambda \setminus \{0\} : P_\lambda^- \nabla I_\lambda(u) = 0,\ I_\lambda'(u)u = 0\} \subset E_\lambda \setminus E_0^-,$$

with the corresponding level

$$c_\lambda := \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u).$$

As $\lambda \to +\infty$, the *limit* problem of $(\mathcal{P}_\lambda)$ is

$$\begin{cases} (-\Delta)^s u - \delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.3}$$

with the energy functional

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2}\int_\Omega \delta u^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \quad \text{for all } u \in E_0.$$

The Nehari–Pankov manifold of $I_0(u)$ is defined by

$$\mathcal{N}_0 = \{u \in E_0 \setminus \{0\} : P_0^- \nabla I_0(u) = 0,\ I_0'(u)u = 0\}$$

with the corresponding level

$$c_0 := \inf_{\mathcal{N}_0} I_0(u),$$

where $I_0'(u)$ is the Fréchet derivative of $I_0(u)$.

Remark 1.2. The Nehari–Pankov manifold was firstly introduced by Pankov [22]. This type of manifold coincides with the Nehari manifold if all eigenvalues are positive, i.e. the negative eigenfunction space is $\{0\}$. Moreover, the minimizer for c_0 is a least energy solution of problem (1.3). By the definition of the modified Nehari–Pankov manifold $\mathcal{N}_\lambda$, it is easy to see that $P_\lambda^- u = P_0^- u$ for any $u \in E_0$. As proved in Lemma 2.9, we see that the minimizer for c_λ is indeed a least energy solution of $(\mathcal{P}_\lambda)$, the solution with smallest energy among all nontrivial solutions.

Our main results are as follows.

Theorem 1.3 (Existence). *Assume* (A1), (A2), (A3) *and one of the following conditions hold:*
(i) $N > 2s$, *p is subcritical, i.e.* $2 < p \le 2_s^*$,
(ii) $N \ge 4s$, *p is critical, i.e.* $p = 2_s^*$.
Then there exists $\Lambda > 0$ *such that for any* $\lambda \ge \Lambda$, $(\mathcal{P}_\lambda)$ *admits a least energy solution* $u_\lambda \in E_\lambda$ *which achieves* c_λ.

Theorem 1.4 (Asymptotic Behavior). *Under the assumptions of Theorem 1.3, let* u_λ *be a least energy solution to* $(\mathcal{P}_\lambda)$ *for* $\lambda \ge \Lambda$. *Then up to a subsequence, for any sequence* $\lambda_n \to +\infty$, *the sequence* $\{u_{\lambda_n}\}$ *converges to a least energy solution of the limit problem* (1.3) *in* $H^s(\mathbb{R}^N)$.

The following is an extension of Theorem 1.4.

Theorem 1.5. *Under the assumptions of Theorem 1.3, let* u_λ *be a nontrivial solution to* $(\mathcal{P}_\lambda)$ *for* $\lambda \ge \Lambda$. *Suppose that one of the following conditions holds:*
(i) $\limsup_{\lambda\to+\infty} I_\lambda(u_\lambda) < +\infty$ *for* $2 < p < 2_s^*$,
(ii) $\limsup_{\lambda\to+\infty} I_\lambda(u_\lambda) < \frac{s}{N} S^{\frac{N}{2}}$ *for* $p = 2_s^*$, *where*

$$S := \inf_{u\in H^s(\mathbb{R}^N),\, \|u\|_{L^{2_s^*}(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Then up to a subsequence, for any sequence $\lambda_n \to +\infty$, *the sequence* $\{u_{\lambda_n}\}$ *converges to a solution of the limit problem* (1.3) *in* $H^s(\mathbb{R}^N)$.

Recall that $(-\Delta)^s$ is a nonlocal operator, please see the definition of $(-\Delta)^s$ in Section 2.1. Compared with local operators, such as the Laplacian $-\Delta$, nonlocal operators seem much more difficult to deal with. There are some different phenomena between the local and nonlocal operators. For instance:

Remark 1.6. Assume $\Omega := \operatorname{int}\{x \in \mathbb{R}^N : a(x) = 0\} = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$. Here Ω_1 and Ω_2 are two smooth domains in $\mathbb{R}^N$. We see that the least energy solutions to $(\mathcal{P}_\lambda)$ may localize near Ω_1 and Ω_2 simultaneously for λ large enough (see Lemma 3.6). This phenomena cannot happen for local operators such as the Laplacian.

In the following sections, without especially stated, we always assume $\lambda_k < \delta < \lambda_{k+1}$ for some $k \ge 1$, i.e. $\mu_k(L_0) < 0$ and $\mu_{k+1}(L_0) > 0$. Our paper is organized as follows. In Section 2, we give some preliminary results. More precisely, in Section 2.1, we introduce some definitions of fractional Laplacian. In Section 2.2, we introduce some fractional Sobolev spaces. The spectrum of L_λ and L_0 is studied in Section 2.3 and some properties of the Modified Nehari–Pankov manifold are proved in Section 2.4. In Section 3, we prove the existence of least energy solutions for the limit problem (1.3). Section 4 is dedicated to the proof of our main results in the subcritical case and Section 5 focus on the proof of the main results in the critical case.

2 Preliminary Results

In this section, for the convenience of the readers, we present some definitions and notations. Firstly, we introduce some definitions of the fractional Laplacian $(-\Delta)^s$. Secondly, we consider some fractional Sobolev spaces, such as E_λ and E_0. Then we discuss the spectrum of L_λ and L_0. At last, we give some properties of the modified Nehari–Pankov manifold.

2.1 Introduction to Fractional Laplacian Operators

Let $\mathfrak{I}(\mathbb{R}^N) := \{\phi \in C_0^\infty(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N}(1+|x|^2)^{\frac{k}{2}}|D^\alpha\phi(x)| < +\infty,\, k, |\alpha| = 0, 1, 2, \ldots\}$ be the so-called Schwartz space with the seminorms

$$p_n(\phi) = \sup_{x\in\mathbb{R}^N} \sum_{|\alpha|\le n} |D^\alpha\phi(x)|, \quad n = 1, 2, \ldots .$$

We denote the topological dual of $\mathfrak{I}(\mathbb{R}^N)$ by $\mathfrak{I}'(\mathbb{R}^N)$. As usual, for any $\phi \in \mathfrak{I}$, we denote by

$$\mathfrak{F}\phi(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\phi(x)\,dx$$

the Fourier transform of ϕ. For any $\phi \in \mathfrak{I}$, $0 < s < 1$, we define the fractional Laplacian operator $(-\Delta)^s$ as

$$(-\Delta)^s\phi(x) = C_{N,s}\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy = C_{N,s}\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy. \tag{2.1}$$

Here P.V. is a commonly used abbreviation for “in the principle value sense” and $C_{N,s}$ is a dimensional constant that depends on N and s, precisely given by

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi\right)^{-1}.$$

According to (2.1), it is easy to see that $(-\Delta)^s$ is a nonlocal operator. Due to the singularity of the kernel, the right-hand side of (2.1) is not well defined in general. In the case $0 < s < \frac{1}{2}$ the integral (2.1) is not really singular near x, see [20]. In order to get rid of P.V. in (2.1), one can redefine the fractional Laplacian operator $(-\Delta)^s$ as

$$(-\Delta)^s\phi(x) = -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy \quad \text{for all } \phi \in \mathfrak{I},\ x \in \mathbb{R}^N. \tag{2.2}$$

Now we take into account an alternative definition of the space $H^s(\mathbb{R}^N)$ via the Fourier transform. Precisely, we define

$$\widehat{H}^s := \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^{2s})|\mathfrak{F}\phi(\xi)|^2\,d\xi < +\infty\right\}.$$

The fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$, namely for $0 < s < 1$,

$$(-\Delta)^s\phi(x) = \mathfrak{F}^{-1}(|\xi|^{2s}(\mathfrak{F}u)) \quad \text{for all } u \in \mathfrak{I},\ \xi \in \mathbb{R}^N. \tag{2.3}$$

As proved in [20], $H^s(\mathbb{R}^N)$ and $\widehat{H}^s$ are coincide and the above definitions of $(-\Delta)^s$ are also coincide.

Recently Caffarelli and Silvestre [7] considered the following boundary value problem in the half space $\mathbb{R}^{N+1}_+$:

$$\begin{cases} \nabla\cdot(t^{1-2s}\nabla\widetilde{\phi}) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \widetilde{\phi}(x,0) = \phi(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Here $\widetilde{\phi}(x,t)$ is the so-called s-harmonic extension of ϕ, explicitly given as a convolution integral with the s-Poisson kernel $p_s(x,y)$,

$$\widetilde{\phi}(x,t) = \int_{\mathbb{R}^N} p_s(x-z,t)\phi(z)\,dz,$$

where

$$p_s(x,t) = d_{N,s}\frac{t^{4s-1}}{(|x|^2+|t|^2)^{\frac{N-1+4s}{2}}}$$

and $d_{N,s}$ achieves $\int_{\mathbb{R}^N} p_s(x,t)\,dx = 1$. Then under suitable regularity, $(-\Delta)^s$ is the Dirichlet-to-Neumann map for this problem, namely

$$(-\Delta)^s\phi(x) = -\lim_{t\to 0^+} t^{1-2s}\partial_t\widetilde{\phi}(x,t). \tag{2.4}$$

Thus by this new transformation, the nonlocal operator $(-\Delta)^s$ may be reduced to a local, possibly singular or degenerate operator on functions sitting in the higher-dimensional half space $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,+\infty)$.

The definitions (2.1), (2.2), (2.3) and (2.4) of the fractional Laplacian $(-\Delta)^s$ are all equivalent for instance in Schwartz's space of rapidly decreasing smooth functions. For more details about the fractional Laplacian, one refers to [9, 20] and the references therein.

In the present paper, we use (2.1) as the definition of $(-\Delta)^s$ and for the simplicity and without loss of generality, we replace the constant $C_{N,s}$ by 1 throughout this paper.

2.2 Some Fractional Sobolev Spaces

As we know, $H^s(\mathbb{R}^N)$ can be continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \le p \le 2^*_s$, and compactly embedded into $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le p < 2^*_s$. In particular, the following fractional Sobolev inequality holds:

$$S = \inf\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy : u \in H^s(\mathbb{R}^N)\setminus\{0\},\ \|u\|_{L^{2^*_s}(\mathbb{R}^N)=1}\right\} > 0. \tag{2.5}$$

In [12], Cotsiolis and Tavoularis showed that S is achieved by

$$v = \frac{V(x)}{\|V\|_{L^{2^*_s}(\mathbb{R}^N)}},$$

where

$$V(x) = \kappa(\mu^2+|x-x_0|^2)^{-\frac{N-2s}{2}}$$

for $\kappa \in \mathbb{R}\setminus\{0\}$, $\mu \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ fixed.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain in $\mathbb{R}^N$, and let $H^s(\Omega)$ be the standard fractional Sobolev space. The embedding $H^s(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le 2^*_s$, and compact for $1 \le p < 2^*_s$. For more details about the fractional Sobolev spaces, one refers to [20] and the references therein.

Recall that $E_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$. In [12], Cotsiolis and Tavoularis gave a sort of Poincaré–Sobolev inequality as follows:

$$\int_\Omega |u|^2\,dx \le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \quad \text{for all } u \in E_0,$$

where $C > 0$ depends on Ω and N. According to the definitions of the norms $\|\cdot\|_0$ and $\|\cdot\|$, we see that the two norms are equivalent in E_0. Moreover, $(E_0, \|\cdot\|_0)$ is also a Hilbert space (see [12, Lemma 7]). The embedding $E_0 \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le 2^*_s$, and compact for $1 \le p < 2^*_s$ (see [12, Lemma 8]).

We finish this subsection by giving the following uniformly imbedding lemma.

Lemma 2.1. *There exists a constant $\lambda_0 > 0$ such that the embedding $E_\lambda \hookrightarrow H^s(\mathbb{R}^N)$ is continuous, uniformly in λ for $\lambda > \lambda_0$.*

Proof. By (1.2) in Remark 1.1, $\Omega \subseteq B_R(0)$ for large $R > 0$ and $a(x) > a_0$ for each $x \in \mathbb{R}^N \setminus B_R(0)$. Let $\xi : \mathbb{R}^N \to \mathbb{R}$ be a smooth cutoff function satisfying

$$0 \le \xi \le 1, \qquad \xi = 1 \quad \text{in } B_R(0), \qquad \xi = 0 \quad \text{in } \mathbb{R}^N \setminus B_{R+1}(0). \tag{2.6}$$

Take $\lambda_0 = \frac{a_0+\delta}{a_0}$; then by (2.6), for $\lambda \ge \lambda_0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (1-\xi^2)u^2\,dx &= \int_{\mathbb{R}^N\setminus B_R(0)} (1-\xi^2)u^2\,dx \\ &\le \frac{1}{a_0}\int_{\mathbb{R}^N\setminus B_R(0)} (\lambda a(x)-\delta)u^2\,dx \\ &= \frac{1}{a_0}\int_{\mathbb{R}^N\setminus B_R(0)} V_\lambda^+ u^2\,dx \\ &\le \frac{1}{a_0}\int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx. \end{aligned}\tag{2.7}$$

By (2.6), Hölder's inequality and the definition of S, we find that

$$\begin{aligned}\int_{\mathbb{R}^N} \xi^2 u^2\,dx &= \int_{B_{R+1}(0)} \xi^2 u^2\,dx \\ &\le |B_{R+1}(0)|^{1-\frac{2}{2_s^*}}\Big(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \\ &\le \frac{|B_{R+1}(0)|^{1-\frac{2}{2_s^*}}}{S}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.\end{aligned}\tag{2.8}$$

According to (2.7), (2.8) and the definition of the norm $\|\cdot\|$, we obtain that

$$\begin{aligned}\|u\|^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} u^2\,dx \\ &\le \big(|B_{R+1}(0)|^{1-\frac{2}{2_s^*}}+1\big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{a_0}\int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx \\ &\le C\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx\Big) \le C\|u\|_\lambda^2.\end{aligned}$$

Thus $\|u\| \le C\|u\|_\lambda$, where $C > 0$ is independent of λ. □

Remark 2.2. For $\lambda > \lambda_0$, since the embedding constant for $E_\lambda \hookrightarrow H^s(\mathbb{R}^N)$ does not dependent on λ, E_λ keeps all properties of $H^s(\mathbb{R}^N)$. Precisely, for $\lambda > \lambda_0$, E_λ can be continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \le p \le 2_s^*$ and compactly into $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le p < 2_s^*$. Moreover, all embedding constants do not depend on λ.

2.3 Eigenvalue Problems with Fractional Laplacian Operators

Firstly, we consider the spectrum of the operator L_0 in E_0. Recall that $\Omega = \operatorname{int} a^{-1}(0)$ is a smooth bounded domain in $\mathbb{R}^N$. For $0 < s < 1$, we say λ is an eigenvalue of $(-\Delta)^s$ in E_0 if there exists $u \in E_0$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \lambda\int_\Omega u(x)v(x)\,dx \tag{2.9}$$

for any $v \in E_0$, i.e. (u, λ) is a weak solution to the equation

$$\begin{cases} (-\Delta)^s u(x) = \lambda u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.10}$$

A function $u \in E_0$ satisfying (2.9) is called an eigenfunction corresponding to λ. For convenience, we also say λ is an eigenvalue of (2.10).

The following lemma described the eigenvalues and eigenfunctions to (2.10).

Lemma 2.3 ([26, Proposition 9]). *Let $0 < s < 1$, $N > 2s$, and let Ω be an open bounded set of $\mathbb{R}^N$.*

(i) *Problem (2.10) admits an eigenvalue λ_1 which is positive and that can be characterized as*

$$\lambda_1 = \min\bigg\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy : u \in E_0,\ \|u\|_{L^2(\Omega)=1} \bigg\}. \tag{2.11}$$

(ii) *There exists a nonnegative function $e_1 \in E_0$, which is an eigenfunction corresponding to λ_1, attaining the minimum in (2.11), that is $\|e_1\|_{L^2(\Omega)} = 1$ and*

$$\lambda_1 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_1(x) - e_1(y)|^2}{|x - y|^{N+2s}}\,dx\,dy.$$

(iii) *λ_1 is simple, that is if $u \in E_0$ is a solution to the equation*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}}\,dx\,dy = \lambda_1 \int_{\Omega} u(x)\phi(x)\,dx, \quad u \in E_0, \quad \text{for all } \phi(x) \in E_0,$$

then $u = ke_1$, with $k \in \mathbb{R}$.

(iv) *The set of the eigenvalues of problem (2.10) consists of a sequence $\{\lambda_k\}_{k\geq 1}$ with*

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots$$

and

$$\lambda_k \to +\infty \quad \text{as } k \to +\infty.$$

Moreover, for any $k \geq 1$, the eigenvalues can be characterized as

$$\lambda_{k+1} = \min\bigg\{ \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy : u \in \mathcal{P}_{k+1},\ \|u\|_{L^2(\Omega)} = 1 \bigg\}, \tag{2.12}$$

where

$$\mathcal{P}_{k+1} = \bigg\{ u \in E_0 : \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(e_j(x) - e_j(y))}{|x - y|^{N+2s}} = 0,\ j = 1, \dots, k \bigg\}.$$

(v) *For any $k \geq 1$, there exists a function e_{k+1}, which is an eigenfunction corresponding to λ_{k+1}, attaining the minimum in (2.12), that is $\|e_{k+1}\|_{L^2(\Omega)} = 1$ and*

$$\lambda_{k+1} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_{k+1}(x) - e_{k+1}(y)|^2}{|x - y|^{N+2s}}\,dx\,dy.$$

(vi) *The sequence $\{e_k\}_{k\geq 1}$ of eigenfunctions corresponding to λ_k is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of E_0.*

(vii) *Each eigenvalue λ_k has finite multiplicity; more precisely, if λ_k is such that*

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+h} < \lambda_{k+h+1}$$

for some $h \geq 0$, then the set of all eigenfunctions corresponding to λ_k agrees with

$$\operatorname{span}\{e_k, e_{k+1}, \dots, e_{k+h}\}.$$

Next, we study the spectrum of L_λ in E_λ. It is easy to see that L_λ is self-adjoint in $L^2(\mathbb{R}^N)$ and bounded from below by $-\delta$. Denote

$$U_\lambda(\psi_1, \psi_2, \dots, \psi_m) = \inf\{L_\lambda(\psi)\psi : \psi \in E_\lambda,\ \|\psi\|_{L^2(\mathbb{R}^N)} = 1,\ (\psi, \psi_j) = 0,\ j = 1, 2, \dots, m\},$$

where

$$L_\lambda(\psi)\psi := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda \psi^2\,dx,$$

$$\|\psi\|^2_{L^2(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \psi^2\,dx,$$

$$(\psi,\psi_j) := \int_{\mathbb{R}^N} \psi\psi_j\,dx = 0.$$

For $k \in \mathbb{N}$ fixed, we define spectral values of L_λ by the k-th Rayleigh quotient

$$\mu_k(L_\lambda) := \sup\{U_\lambda(\psi_1,\psi_2,\dots,\psi_{k-1}) : \psi_1,\psi_2,\dots,\psi_{k-1} \in E_\lambda\}.$$

Obviously, $\mu_k(L_\lambda)$ is nondecreasing respect to k and λ. By [23, Theorem XIII.1 and Theorem XIII.2], we know that either $\mu_k(L_\lambda)$ is an eigenvalue of L_λ or $\mu_k(L_\lambda) = \mu_{k+1}(L_\lambda) = \cdots = \inf \sigma_{\mathrm{ess}}(L_\lambda)$. Denote $\mu_k(L_0) := \lambda_k - \delta$. By Lemma 2.3, it is easy to see that $\{\mu_k(L_0)\}_{k\ge1}$ is the class of all eigenvalues of $L_0 := (-\Delta)^s - \delta$ in E_0. Moreover, $\mu_k(L_0)$ can be characterized as

$$\mu_k(L_0) = \max_{S\in\sum_{k-1}} \min_{\psi\in S^\perp,\, \|\psi\|_{L^2(\Omega)=1}} \{L_0(\psi)\psi\},$$

where

$$L_0(\psi)\psi := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega \psi^2\,dx,$$

and $\sum_{k-1}$ denotes the collection of $(k-1)$-dimensional subspaces of E_0. Since $L_\lambda(\psi)\psi = L_0(\psi)\psi$ for each $\psi \in E_0$, we have $\mu_k(L_\lambda) \le \mu_k(L_0)$. Thus in order to prove the existence of eigenvalues of L_λ in E_λ, we just need to prove that $\inf\sigma_{\mathrm{ess}}(L_\lambda) \to +\infty$ as $\lambda \to +\infty$.

Lemma 2.4. *Under the assumptions* (A1)–(A3)*, the essential spectrum $\sigma_{\mathrm{ess}}(L_\lambda)$ is contained in $[\lambda a_0 - \delta, +\infty)$ for $\lambda > \lambda_0$. Furthermore,* $\inf\sigma_{\mathrm{ess}}(L_\lambda) \to +\infty$ *as $\lambda \to +\infty$.*

Proof. The proof of this lemma is similar to [4, Proposition 2.3]. For the convenience of the reader, we give the details.

We set $W_\lambda = V_\lambda - \lambda a_0 + \delta = \lambda(a(x) - a_0)$ and write $W^1_\lambda = \max\{W_\lambda, 0\}$ and $W^2_\lambda = \min\{W_\lambda, 0\}$. Obviously, for $\lambda > \lambda_0$,

$$\sigma((-\Delta)^s + W^1_\lambda + \lambda a_0 - \delta) \subset [\lambda a_0 - \delta, +\infty) \tag{2.13}$$

for $W^1_\lambda \ge 0$. Let $H_\lambda := (-\Delta)^s + W^1_\lambda + \lambda a_0 - \delta$; then $L_\lambda = H_\lambda + W^2_\lambda$.

We claim that W^2_λ is a relative form compact perturbation of L_λ. Since W^2_λ is bounded, the form domain of H_λ is the same as the form domain E_λ of L_λ. Thus we have to show that $E_\lambda \mapsto E^*_\lambda$, $u \mapsto W^2_\lambda \cdot u$ is compact. Select a bounded sequence $\{u_n\}_{n\ge1}$ in $\langle E_\lambda, \|\cdot\|_\lambda\rangle$; then according to Lemma 2.1, $\{u_n\}_{n\ge1}$ is also a bounded sequence in $H^s(\mathbb{R}^N)$. Thus there exists a function $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^s(\mathbb{R}^N),\\ u_n \to u & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N),\\ u_n \to u & \text{a.e. in } \mathbb{R}^N \end{cases} \tag{2.14}$$

as $n \to +\infty$. Notice that $a(x) > a_0$ for each $x \in \mathbb{R}^N \setminus B_R$ (see (1.2) in Remark 1.1), then the support set of W^2_λ is contained in B_R. Thus by Hölder's inequality, and Lemma 2.1, for $\lambda > \lambda_0$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} W^2_\lambda(u_n-u)v\,dx\right| &= \left|\int_{B_R} W^2_\lambda(u_n-u)v\,dx\right| \le \delta\int_{B_R} |(u_n-u)v|\,dx\\ &\le \delta\Big(\int_{B_R}(u_n-u)^2\,dx\Big)^{\frac12}\|v\| \le C\Big(\int_{B_R}(u_n-u)^2\,dx\Big)^{\frac12}\|v\|_\lambda.\end{aligned}$$

Hence by (2.14), we have

$$\|W_\lambda^2 u_n - W_\lambda^2 u\|_{E_\lambda^*} \le C\Big(\int_{B_R} (u_n - u)^2\,dx\Big)^{\frac{1}{2}} \to 0$$

as $n \to +\infty$. Thus W_λ^2 is a relative form compact perturbation of L_λ.

According to the Classical Weyl Theorem (see [23, Example 3, p. 117]), $\sigma_{\mathrm{ess}}(L_\lambda) = \sigma_{\mathrm{ess}}(H_\lambda)$. Thus by (2.13), for $\lambda > \lambda_0$, $\sigma_{\mathrm{ess}}(L_\lambda) \subset [\lambda a_0 - \delta, +\infty)$. Moreover, $\inf \sigma_{\mathrm{ess}}(L_\lambda) \to +\infty$ as $\lambda \to +\infty$. □

At last, we deal with the asymptotic behavior of the eigenvalues of L_λ in E_λ. Let $\{\mu_k^\lambda\}_{k\ge1}$ be the class of all distinct eigenvalues of $L_\lambda := (-\Delta)^s + \lambda a(x) - \delta$ in E_λ, and let $\{\mu_k\}_{k\ge1}$ be the class of all distinct eigenvalues of $L_0 := (-\Delta)^s - \delta$ in E_0. Without loss of generality, we can order these eigenvalues as follows:

$$\mu_1^\lambda < \mu_2^\lambda < \mu_3^\lambda < \cdots < \mu_{k_\lambda}^\lambda < \inf \sigma_{\mathrm{ess}}(L_\lambda),$$

and

$$\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_k < 0 < \mu_{k+1} < \cdots .$$

Moreover, $\mu_{k_\lambda}^\lambda \to +\infty$ as $\lambda \to +\infty$, and $\mu_k \to +\infty$ as $k \to +\infty$. Let V_j^λ be the eigenfunction space corresponding to μ_j^λ and let V_j be the eigenfunction space corresponding to μ_j. We say V_k^λ converges to V_k, i.e. $V_k^\lambda \to V_k$ as $k \to +\infty$, if for any sequence $\lambda_i \to \infty$ and normalized eigenfunctions $\psi_i \in V_k^{\lambda_i}$, there exists a normalized eigenfunction $\psi \in V_k$ such that $\psi_i \to \psi$ strongly in $H^s(\mathbb{R}^N)$ along a subsequence. We have the following lemma.

Lemma 2.5. *We have $\mu_k^\lambda \to \mu_k$ and $V_k^\lambda \to V_k$ as $\lambda \to \infty$.*

Proof. We prove this lemma by induction.

Step 1. We firstly prove the case for $k = 1$, i.e. up to a subsequence, we will prove that

$$\lambda_n \to +\infty, \quad \mu_1^{\lambda_n} \to \mu_1, \quad V_1^{\lambda_n} \to V_1 \quad \text{as } n \to +\infty.$$

According to the definition of $\mu_1^{\lambda_n}$, it is easy to see that $\mu_1^{\lambda_n} \le \mu_1$. Now we assume that $\psi_n \in E_{\lambda_n}$ is such that

$$\int_{\mathbb{R}^N} \psi_n^2\,dx = 1, \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} (\lambda_n a(x) - \delta)\psi_n^2\,dx = \mu_1^{\lambda_n}. \tag{2.15}$$

By (2.15), we have

$$\|\psi_n\|_{\lambda_n}^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_{\lambda_n}\psi_N^2\,dx + \int_{\mathbb{R}^N} V_{\lambda_n}^-\psi_n^2\,dx = \mu_1^{\lambda_n} + \int_{\mathbb{R}^N} V_{\lambda_n}^-\psi_n^2\,dx \le \mu_1 + \delta.$$

Then by Lemma 2.1, $\{\psi_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Up to a subsequence, there is $\psi \in H^s(\mathbb{R}^N)$ such that

$$\begin{cases} \psi_n \rightharpoonup \psi & \text{weakly in } H^s(\mathbb{R}^N), \\ \psi_n \to \psi & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \\ \psi_n \to \psi & \text{a.e. in } \mathbb{R}^N \end{cases} \tag{2.16}$$

as $\lambda_n \to +\infty$.

Claim 1. *We have $\psi(x) = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.*

In fact, let $C_m := \{x \in \mathbb{R}^N : a(x) > \frac{1}{m}\}$. For fixed positive integer m, by (2.15), we have

$$\begin{aligned} \int_{C_m} \psi^2\,dx &\le \frac{m}{\lambda_n}\int_{\mathbb{R}^N} \lambda_n a(x)\psi_n^2\,dx \\ &\le \frac{m}{\lambda_n}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} \lambda_n a(x)\psi_n^2\,dx\Big) \\ &\le \frac{m}{\lambda_n}(\mu_1 + \delta) \to 0, \end{aligned}$$

as $\lambda_n \to +\infty$. Thus $u(x) = 0$ a.e. in C_m for $m = 1, 2, 3, \ldots$. Since $\bigcup_{m=1}^\infty C_m = \mathbb{R}^N \setminus \Omega$, it follows that $u(x) = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

Claim 2. *We have* $\int_\Omega \psi^2\,dx = 1$.

As in Section 1 (see (1.2) in Remark 1.1), $\Omega \subset B_R(0)$ and $a(x) > a_0$ for all $|x| > R$. Then by (2.15), we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R(0)} \psi_n^2\,dx &\le \frac{1}{a_0\lambda_n}\int_{\mathbb{R}^N\setminus B_R(0)} \lambda_n a(x)\psi_n^2\,dx\\ &\le \frac{1}{a_0\lambda_n}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}\lambda_n a(x)\psi_n^2\,dx\bigg)\\ &= \frac{1}{a_0\lambda_n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{a_0\lambda_n}\int_{\mathbb{R}^N}\lambda_n V_{\lambda_n}\psi_n^2\,dx + \frac{\delta}{a_0\lambda_n}\int_{\mathbb{R}^N}|\psi_n|^2\,dx\\ &\le \frac{1}{a_0\lambda_n}(\mu_1+\delta) \to 0\end{aligned}$$

as $\lambda_n \to +\infty$. Thus

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_R(0)} \psi_n^2\,dx = 0. \tag{2.17}$$

Combine (2.15), (2.16), (2.17) and Claim 1 to have

$$\int_\Omega \psi^2\,dx = \int_{B_R(0)} \psi^2\,dx = \lim_{n\to+\infty}\int_{B_R(0)} \psi_n^2\,dx = \lim_{n\to+\infty}\int_{\mathbb{R}^N}\psi_n^2\,dx - \lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\psi_n^2\,dx = 1.$$

According to Claim 1 and Claim 2, we find that

$$\int_{\mathbb{R}^N}\psi^2\,dx = \int_\Omega \psi^2\,dx = 1. \tag{2.18}$$

Let $\psi_n^1 = \psi_n - \psi$. By (2.16) and Brézis–Lieb's lemma,

$$\int_{\mathbb{R}^N}\psi_n^2\,dx = \int_{\mathbb{R}^N}\psi^2\,dx + \int_{\mathbb{R}^N}(\psi_n^1)^2\,dx + o(1) \tag{2.19}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n^1(x)-\psi_n^1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + o(1) \tag{2.20}$$

as $n \to +\infty$. Thus by (2.15), (2.18) and (2.19), we have $\psi_n \to \psi$ strongly in $L^2(\mathbb{R}^N)$ as $n \to +\infty$. By Claim 1, we obtain that $\psi \in E_0$. Thus by (2.11), (2.15), (2.18) and (2.20), we have

$$\begin{aligned}\mu_1 &:= \inf_{u\in E_0,\,\|u\|_{L^2(\Omega)}=1}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega \delta u^2\,dx\bigg)\\ &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega \delta\psi^2\,dx\\ &\le \lim_{n\to\infty}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_{\mathbb{R}^N}\delta\psi_n^2\,dx\bigg)\\ &\le \lim_{n\to\infty}\mu_1^{\lambda_n} \le \mu_1,\end{aligned}$$

which implies that $\mu_1^{\lambda_n} \to \mu_1$ and $\psi_n \to \psi$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$.

Step 2. Suppose $k \ge 2$ and the results hold up to $k-1$. We need to prove the same results hold for the k-th eigenvalues.

Since $L_\lambda\psi = L_0\psi$ for all $\psi \in E_0$, by the k-th Rayleigh quotient descriptions of μ_k^λ and μ_k, we have

$$\limsup_{\lambda\to+\infty}\mu_k^\lambda \le \mu_k.$$

Just like in the case $k = 1$, we can select $\lambda_n \to +\infty$ and the normalized eigenfunctions $\psi_n \in V_k^{\lambda_n}$ which is the eigenfunction space corresponding to $\mu_k^{\lambda_n}$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} \lambda_n(a(x)-\delta)\psi_n^2\,dx = \mu_k^{\lambda_n}$$

and

$$\int_{\mathbb{R}^N} \psi_n^2\,dx = 1, \qquad \psi_n \perp V_j^{\lambda_n}, \quad j = 1,2,3,\dots,k-1.$$

Similar to the proof in Step 1, we have

$$\begin{cases} \psi_n \rightharpoonup \psi & \text{weakly in } H^s(\mathbb{R}^N),\\ \psi_n \to \psi & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N),\\ \psi_n \to \psi & \text{a.e. in } \mathbb{R}^N \end{cases}$$

for some $\psi \in E_0$ satisfying $\int_\Omega |\psi|^2\,dx = 1$. Since $\psi_n \perp V_j^{\lambda_n}$, $j = 1,2,\dots,k-1$, and $V_j^{\lambda_n} \to V_j$ as $n \to +\infty$, it follows that $\psi \perp V_j$, $j = 1,2,\dots,k-1$, and

$$\begin{aligned} \mu_k &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega \delta\psi^2\,dx\\ &\le \varliminf_{n\to\infty}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega \delta\psi_n^2\,dx\Big)\\ &\le \varlimsup_{n\to\infty} \mu_k^{\lambda_n} \le \mu_k. \end{aligned}$$

This induces that $\mu_k^{\lambda_n} \to \mu_k$, $V_k^{\lambda_n} \to V_k$ as $n \to +\infty$. □

Corollary 2.6. *For λ large, the operator $(-\Delta)^s + \lambda a + a_0$ on E_λ is nondegenerate and has finite Morse index $d_j := \dim E_\lambda^-$ uniformly in λ.*

2.4 Modified Nehari–Pankov Manifold

Recall the modified Nehari–Pankov manifold

$$\mathcal{N}_\lambda := \{u \in E_\lambda \setminus \{0\} : P_\lambda^- \nabla I_\lambda(u) = 0,\ I_\lambda'(u)u = 0\} \subset E_\lambda \setminus E_0^-$$

and the Nehari–Pankov manifold

$$\mathcal{N}_0 := \{u \in E_0 \setminus \{0\} : P_0^- \nabla I_0(u) = 0,\ I_0'(u)u = 0\} \subset E_0 \setminus E_0^-.$$

The corresponding levels are

$$c_\lambda := \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u) \quad \text{and} \quad c_0 := \inf_{u\in\mathcal{N}_0} I_0(u).$$

For the Nehari–Pankov manifold $\mathcal{N}_0$, we have the following lemma due to Szulkin and Weth [29].

Lemma 2.7. *For any $\omega \in E_0 \setminus E_0^-$, set*

$$H_\omega := \{v + t\omega : v \in E_0^-,\ t > 0\}.$$

The following properties hold:
(i) $\mathcal{N}_0 = \{w \in E_0 \setminus E_0^- : \nabla(I_0|H_w) = 0\}$.
(ii) *For every $w \in E_0^+ \setminus \{0\}$ there exist $t_w > 0$ and $\varphi(w) \in E_0^-$ such that*

$$H_w \cap \mathcal{N}_0 = \{\varphi(w) + t_w \cdot w\}.$$

(iii) *For every $w \in \mathcal{N}_0$ and every $u \in H_w \setminus \{w\}$ there holds $I_0(u) < I_0(w)$.*
(iv) $c_0 = \inf_{u\in\mathcal{N}_0} I_0(u) > 0$.

Proof. The proof of this lemma is similar to the proofs of [29, Lemma 2.2–2.6]; we omit the details. □

For the modified Nehari–Pankov manifold, we have the following lemma.

Lemma 2.8. *For any $\omega \in E_\lambda \setminus E_0^-$, set*

$$\hat{H}_\omega := \{v + t\omega : v \in E_0^-,\ t > 0\}.$$

The following properties hold:

(i) $\mathcal{N}_\lambda = \{w \in E_\lambda \setminus E_0^- : \nabla(I_\lambda|\hat{H}_w) = 0\}$.

(ii) *Let*

$$\hat{E}_\lambda^+ := \Big\{\omega \in E_\lambda^+ : \int_{\mathbb{R}^N} \omega e_i \, dx = 0,\ i = 1, 2, \ldots, k\Big\}.$$

Then for any $w \in \hat{E}_\lambda^+ \setminus \{0\}$ there exist $t_w > 0$ and $\varphi(w) \in E_0^-$ such that

$$\hat{H}_w \cap \mathcal{N}_\lambda = \{\varphi(w) + t_w \cdot w\}.$$

(iii) *For any $w \in \mathcal{N}_\lambda$ and $u \in \hat{H}_w \setminus \{w\}$ there holds $I_\lambda(u) < I_\lambda(w)$.*

(iv) $c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) \geq \tau > 0$ *for some small $\tau > 0$ independent of large λ.*

Proof. The proofs of (i)–(iii) are obvious. We just need to prove (iv).

Firstly, we claim that there exists a $\mu_0 > 0$ such that for any $\lambda > \mu_0$ and $u \in E_\lambda^+$, we have

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda u^2 \, dx \geq C\Big(\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2 \, dx\Big)$$

for some $C > 0$ which is independent of λ.

In fact, for any $u \in E_\lambda^+$, we have $u = u_\lambda^+ + u_\lambda^-$, $u_\lambda^+ \in E_\lambda^+$, $u_\lambda^- \in E_\lambda^-$. Note that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{(u_\lambda^+(x) - u_\lambda^+(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda^+ (u_\lambda^+)^2 \, dx &= \int_{\mathbb{R}^{2N}} \frac{(u_\lambda^+(x) - u_\lambda^+(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda (u_\lambda^+)^2 \, dx + \int_{\mathbb{R}^N} V_\lambda^- (u_\lambda^+)^2 \, dx \\ &\leq \frac{\mu_{k+1}(L_\lambda) + \delta}{\mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^{2N}} \frac{(u_\lambda^+(x) - u_\lambda^+(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda (u_\lambda^+)^2 \, dx.\end{aligned}$$

Then a simple computation shows

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda u^2 \, dx &= \int_{\mathbb{R}^{2N}} \frac{(u_\lambda^+(x) - u_\lambda^+(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda (u_\lambda^+)^2 \, dx \\ &\quad + \int_{\mathbb{R}^{2N}} \frac{(u_\lambda^-(x) - u_\lambda^-(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda (u_\lambda^-)^2 \, dx \\ &\geq \frac{\mu_{k+1}(L_\lambda)}{\mu_{k+1}(L_\lambda) + \delta}\Big(\int_{\mathbb{R}^{2N}} \frac{(u_\lambda^+(x) - u_\lambda^+(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda^+ (u_\lambda^+)^2 \, dx\Big) \\ &\quad + \int_{\mathbb{R}^{2N}} \frac{(u_\lambda^-(x) - u_\lambda^-(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda (u_\lambda^-)^2 \, dx \\ &\geq \frac{\mu_{k+1}(L_\lambda)}{\mu_{k+1}(L_\lambda) + \delta}\Big(\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2 \, dx\Big) \\ &\quad + \Big\{\frac{\delta}{\mu_{k+1}(L_\lambda) + \delta}\Big(\int_{\mathbb{R}^{2N}} \frac{(u_\lambda^-(x) - u_\lambda^-(y))^2}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} V_\lambda^+ (u_\lambda^-)^2 \, dx\Big) \\ &\quad - \frac{\mu_{k+1}(L_\lambda)}{\mu_{k+1}(L_\lambda) + \delta} \int_{\mathbb{R}^N} V_\lambda^- [(u_\lambda^-)^2 + 2u_\lambda^+ u_\lambda^-] \, dx\Big\},\end{aligned}$$

since

$$u_\lambda^- = \sum_{i=1}^{k}\bigg(\int_{\mathbb{R}^N} ue_{\lambda,i}\,dx\bigg)e_{\lambda,i} = \sum_{i=1}^{k}\bigg[\int_{\mathbb{R}^N} u(e_{\lambda,i}-e_i)\,dx\bigg]e_{\lambda,i},$$

where e_i and $e_{\lambda,i}$ are the eigenfunction of L_0 and L_λ, respectively. Note that $e_{\lambda,i} \to e_i$ in $H^s(\mathbb{R}^N)$ for each i as $\lambda \to +\infty$. Then we can easily get

$$\int_{\mathbb{R}^{2N}} \frac{(u_\lambda^-(x)-u_\lambda^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda(u_\lambda^-)^2\,dx + \int_{\mathbb{R}^N} V_\lambda^-[(u_\lambda^-)^2 + 2u_\lambda^+u_\lambda^-]\,dx = o(1)\|u\|_\lambda^2.$$

Since $\mu_{k+1}(L_\lambda) \to \mu_{k+1}(L_0)$ as $\lambda \to +\infty$, there exists $\mu_0 > \lambda_0$ such that for any $\lambda > \mu_0$, we have

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u^2\,dx \ge \frac{\mu_{k+1}(L_0)}{\mu_{k+1}(L_0)+\delta}\bigg(\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx\bigg).$$

Secondly, let

$$S_\alpha := \bigg\{u \in E_\lambda^+ : \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u^2\,dx = \alpha^2\bigg\}.$$

For any $u \in S_\alpha$, by the Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^p &\le C\bigg[\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u^2\,dx\bigg]^{\frac{p}{2}} \\ &\le C\bigg[\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u^2\,dx\bigg]^{\frac{p}{2}} \le C\alpha^p.\end{aligned}$$

Therefore, for $\alpha > 0$ small enough, we have $I_\lambda(u) \ge \frac{1}{2}\alpha^2 - C\alpha^p \ge \frac{1}{4}\alpha^2 > 0$. This implies that $\inf_{S_\alpha} I_\lambda(u) > 0$.

Finally, for any $\omega \in \mathcal{N}_\lambda$, $\omega^+ = \omega - \omega^- \in \hat{H}_\omega$, take $t > 0$ small enough such that $t\omega^+ \in H_\omega \cap S_\alpha$, thus take $\tau - \frac{1}{4}\alpha^2$. By (iii), we have

$$I_\lambda(\omega) > I_\lambda(t\omega^+) \ge \inf_{S_\alpha} I_\lambda(u) \ge \tau > 0,$$

which implies that $c_\lambda \ge \tau > 0$. □

At the end of this subsection, in the following lemma we will prove that the minimizer for c_λ is indeed a least energy solution for $(\mathcal{P}_\lambda)$.

Lemma 2.9. *For λ large enough, $u \in \mathcal{N}_\lambda$ is an achieved function for c_λ, i.e. $c_\lambda = I_\lambda(u)$. Then u is a least energy solution of* $(\mathcal{P}_\lambda)$.

Proof. Note that each solution for $(\mathcal{P}_\lambda)$ belongs to N_λ; thus we just need to show that the achieved function u is indeed a solution.

We denote

$$G_0(u) = I_\lambda'(u)u, \qquad G_i(u) = I_\lambda'(u)e_i, \quad i = 1, 2, \dots, k.$$

Then the modified Nehari–Pankov manifold

$$\mathcal{N}_\lambda = \{u \in E_\lambda \setminus \{0\} : G_i(u) = 0,\ i = 1, 2, \dots, k\}.$$

According to the Lagrange Multiplier Theorem, there exists $(\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ such that

$$I_\lambda'(u) + \lambda_0 G_0'(u) + \lambda_1 G_1'(u) + \cdots + \lambda_k G_k'(u) = 0.$$

Multiplying u and e_i on both sides of the above equation, respectively, for $i = 1, 2, \dots, k$, we have the following system:

$$\begin{cases} a_{00}\lambda_0 + a_{0,1}\lambda_1 + \cdots + a_{0k}\lambda_k = 0, \\ a_{10}\lambda_0 + a_{1,1}\lambda_1 + \cdots + a_{1k}\lambda_k = 0, \\ \qquad\vdots \\ a_{k0}\lambda_0 + a_{k,1}\lambda_1 + \cdots + a_{kk}\lambda_k = 0, \end{cases}$$

where

$$a_{00} = (p-2)\int_{\mathbb{R}^N} |u|^p\,dx, \qquad a_{ii} = (p-1)\int_{\mathbb{R}^N} |u|^{p-2}ue_i^2\,dx - \mu_i(L_0)\int_{\mathbb{R}^N} e_i^2\,dx,$$

$$a_{0,i} = a_{i,0} = (p-2)\int_{\mathbb{R}^N} |u|^{p-2}ue_i\,dx, \quad a_{ij} = a_{ji} = (p-1)\int_{\mathbb{R}^N} |u|^{p-2}e_ie_j\,dx$$

for $i, j = 1, 2, \dots, k$. It is easy to show that the coefficient matrix of the above system is positive definite. Thus the solution of the above system is $(\lambda_0, \lambda_1, \dots, \lambda_k) = (0, 0, \dots, 0)$ which implies that u is indeed a solution to $(\mathcal{P}_\lambda)$. □

3 Existence of Least Energy Solutions to (1.3)

In this section, we present an existence result for the least energy solutions to (1.3). Recall the energy functional of (1.3):

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2}\int_\Omega \delta u^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx \quad \text{for all } u \in E_0.$$

And the Nehari–Pankov manifold related to I_0 is

$$\mathcal{N}_0 = \{u \in E_0 \setminus \{0\} : P_0^-\nabla I_0(u) = 0,\ I_0'(u)u = 0\},$$

with the corresponding least level

$$c_0 := \inf_{\mathcal{N}_0} I_0(u).$$

We state our main result in this section as follows.

Proposition 3.1. *Assume that* $\Omega := \operatorname{int}\{x \in \mathbb{R}^N : a(x) = 0\}$ *is a smooth bounded domain in* $\mathbb{R}^N$, $c_0 := \inf_{\mathcal{N}_0} I_0(u)$ *and one of the following conditions hold:*

(i) $N > 2s$, *p is subcritical, i.e.* $2 < p < 2_s^*$,

(ii) $N \geq 4s$, *p is critical, i.e.* $p = 2_s^*$.

Then c_0 *is achieved by a sign-changing function which is a least energy solution to* (1.3).

To show that, we get a $(\mathrm{PS})_{c_0}$ sequence by Ekeland's variational principle and study the boundedness and compactness of the $(\mathrm{PS})_{c_0}$ sequence. We have the following lemmas.

Lemma 3.2. *Let* $2 < p \leq 2_s^*$, $N > 2s$. *Then there exists a* $(\mathrm{PS})_{c_0}$ *sequence* $\{u_n\}_{n\geq 1}$, *i.e.*

$$I_0(u_n) \to c_0, \quad I_0'(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Proof. Indeed, similar arguments can be fund in the proof of [22, Theorem 5.7]. This is a direct result of Ekeland's variational principle. For the convenience of the reader, here we give a sketch of the argument.

By the definition of c_0, we can easily get a minimizing sequence $\{v_n\} \subset \mathcal{N}_0$ satisfying

$$I_0(v_n) \to c_0 \quad \text{as } n \to \infty.$$

Take

$$\epsilon_n = \begin{cases} I_0(v_n) - c_0, & \text{if } I_0(v_n) > c_0, \\ \frac{1}{n}, & \text{if } I_0(v_n) = c_0. \end{cases}$$

Then by Ekeland's variational principle, there exists a sequence $\{u_n\} \subset E_0$ such that

$$I_0(u_n) \leq I_0(v_n), \quad \|u_n - v_n\|_0 \leq \epsilon_n^{\frac{1}{2}}, \quad \|(I_0'(u_n))^1\|_{E_0'} \leq \epsilon_n^{\frac{1}{2}},$$

where E_0' is the dual space of E_0 and $(I_0'(v_n))^1$ denotes the orthogonal projection of $I_0'(u_n)$ onto the tangent space of $\mathcal{N}_0$ at u_n. One can show that there is a constant $C > 0$ which is independent of n such that

$$\|I_0'(u_n)\|_{E_0'} \leq C\|(I_0'(u_n))^1\|_{E_0'}.$$

Let $n \to +\infty$; we have $I_0(u_n) \to c_0$, $I_0'(u_n) \to 0$ as $n \to +\infty$. □

Lemma 3.3. *Let $2 < p \le 2_s^*$, $N > 2s$, $\{u_n\}_{n\ge1}$ be a $(\mathrm{PS})_{c_0}$ sequence. Then $\{u_n\}_{n\ge1}$ is bounded in E_0.*

Proof. According to the definition of $(\mathrm{PS})_{c_0}$ sequences, there exists a positive integer number M such that for each $n > M$, we have

$$c_0 + 1 + \|u_n\|_0 \ge I_0(u_n) - \frac{1}{2} I_0'(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_\Omega |u_n|^p \, dx \tag{3.1}$$

and

$$c_0 + 1 + \|u_n\|_0 \ge I_0(u_n) - \frac{1}{p} I_0'(u_n)u_n = \frac{p-2}{2p}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \, dx\, dy - \delta \int_\Omega u^2 \, dx \bigg). \tag{3.2}$$

By Hölder's inequality, we have

$$\int_\Omega u^2 \, dx \le |\Omega|^{1-\frac{2}{p}} \bigg(\int_\Omega |u|^p \, dx \bigg)^{\frac{2}{p}}. \tag{3.3}$$

Combining (3.1)–(3.3), we can easily get that for each $n > M$,

$$\|u_n\|_0^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \, dx\, dy \le C(c_0 + 1 + \|u_n\|_0),$$

where the constant C does not depend on n. Thus $\{u_n\}_{n\ge1}$ is bounded in E_0. □

Lemma 3.4. *Assume that one of the following conditions holds:*
(i) *$N > 2s$, p is subcritical, i.e. $2 < p < 2_s^*$,*
(ii) *$N \ge 4s$, p is critical, i.e. $p = 2_s^*$.*
Then the $(\mathrm{PS})_{c_0}$ condition holds. Namely, let $\{u_n\}_{n\ge1}$ be a $(\mathrm{PS})_{c_0}$ sequence. Then there must exist $u \in E_0$ such that, up to a subsequence, $u_n \to u$ strongly in E_0 as $n \to +\infty$.

Proof. We consider two cases.

Case 1: p is subcritical, i.e. $2 < p < 2_s^*$. In fact, by Lemma 3.3, there exists a function $u \in E_0$, up to a subsequence, still denoted by $\{u_n\}_{n\ge1}$, such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } E_0, \\ u_n \to u & \text{strongly in } L^2(\Omega) \text{ and } L^p(\Omega), \\ u_n \to u & \text{a.e. in } \Omega \end{cases} \tag{3.4}$$

as $n \to +\infty$. According to the definition of $(\mathrm{PS})_{c_0}$ sequence and (3.4), we have

$$\begin{aligned} o(1) &= (I_0'(u_n) - I_0'(u))(u_n - u) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{((u_n-u)(x) - (u_n-u)(y))^2}{|x-y|^{N+2s}} \, dx\, dy - \delta \int_\Omega (u_n-u)^2 \, dx - \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u) \, dx. \end{aligned} \tag{3.5}$$

By (3.4) and the Lebesgue Dominated Theorem, we can easily get that

$$|u_n|^{p-2}u_n \to |u|^{p-2}u \quad \text{strongly in } L^{\frac{p}{p-1}}(\Omega).$$

By Hölder's inequality and (3.4), we have

$$\bigg| \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\, dx \bigg| \le \bigg(\int_\Omega \big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{\frac{p}{p-1}} \, dx \bigg)^{\frac{p-1}{p}} \bigg(\int_\Omega |u_n - u|^p \, dx \bigg)^{\frac{1}{p}} \to 0 \tag{3.6}$$

as $n \to +\infty$. Combining (3.4), (3.5) and (3.6), we have as $n \to +\infty$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{((u_n-u)(x) - (u_n-u)(y))^2}{|x-y|^{N+2s}} \, dx\, dy \to 0.$$

Thus $\lim_{n\to+\infty} \|u_n - u\|_0 = 0$.

Case 2: p is critical, i.e. $p = 2_s^*$. To begin with, by the next lemma, we see

$$0 < c_0 < \frac{s}{N} S^{\frac{N}{2s}}. \tag{3.7}$$

As proved in Lemma 3.3, $\{u_n\}$ in bounded in E_0. Taking if necessary a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_0 \text{ and } L^{2_s^*}(\Omega), \\ u_n \to u & \text{strongly in } L^2(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{3.8}$$

By (3.8) and the definition of $(\mathrm{PS})_{c_0}$ sequences, for each $v \in E_0$, we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty} I_0'(u_n)v \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \delta \lim_{n\to+\infty}\int_\Omega u_n v\,dx - \lim_{n\to+\infty}\int_\Omega |u_n|^{2_s^*-2}u_n v\,dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega uv\,dx - \int_\Omega |u|^{2_s^*-2}uv\,dx, \end{aligned} \tag{3.9}$$

i.e. $I_0'(u) = 0$. By (3.9), we have

$$I_0(u) = I_0(u) - \frac{1}{2}I_0'(u)u = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_\Omega |u|^{2_s^*}\,dx \geq 0. \tag{3.10}$$

Let $v_n = u_n - u$, due to Brézis–Lieb's lemma and (3.8), we have as $n \to +\infty$,

$$\|u_n\|_0^2 = \|v_n\|_0^2 + \|u\|_0^2 + o(1), \quad \|u_n\|_{L^{2_s^*}(\Omega)}^{2_s^*} = \|v_n\|_{L^{2_s^*}(\Omega)}^{2_s^*} + \|u\|_{L^{2_s^*}(\Omega)}^{2_s^*}. \tag{3.11}$$

By (3.8) and (3.11), we have

$$\begin{aligned} o(1) &= \langle I_0'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega u_n^2\,dx - \int_\Omega |u_n|^{2_s^*}\,dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega u^2\,dx - \int_\Omega |u|^{2_s^*}\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |v_n|^{2_s^*}\,dx + o(1) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |v_n|^{2_s^*}\,dx + o(1) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_0(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\delta}{2}\int_\Omega u_n^2\,dx - \frac{1}{2_s^*}\int_\Omega |u_n|^{2_s^*}\,dx \\ &= \left(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\delta}{2}\int_\Omega u^2\,dx - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx\right) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2_s^*}\int_\Omega |v_n|^{2_s^*}\,dx + o(1) \\ &= I_0(u) + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2_s^*}\int_\Omega |v_n|^{2_s^*}\,dx + o(1). \end{aligned} \tag{3.13}$$

We assume that $b = \lim_{n\to+\infty}\int_\Omega |v_n|^{2_s^*}\,dx$.

(i) If $b = 0$, we complete the proof by (3.12).

(ii) If $b > 0$, by (3.12) and Sobolev inequality, we have

$$b = \lim_{n\to+\infty}\int_\Omega |v_n|^{2_s^*}\,dx = \lim_{n\to+\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \geq S\left(\lim_{n\to+\infty}\int_\Omega |v_n|^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}} = Sb^{\frac{2}{2_s^*}}.$$

Thus $b \geq S^{\frac{N}{2s}}$. On the other hand, by (3.10), (3.12) and (3.13), we have

$$c_0 = \lim_{n\to+\infty} I_0(u_n) \geq \frac{1}{2}\lim_{n\to+\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2_s^*}\lim_{n\to+\infty}\int_{\Omega}|v_n|^{2_s^*}\,dx = \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)b.$$

Thus $b < S^{\frac{N}{2s}}$ due to the estimate of c_0, see (3.7). This leads to a contradiction. □

We prove (3.7) by the following lemma.

Lemma 3.5. *Assume that $N \geq 4s$, $p = 2_s^*$ is critical and S is defined as* (2.5) *in Section 2.2. Then*

$$0 < c_0 < \frac{s}{N}S^{\frac{N}{2s}}.$$

Proof. According to (iv) in Lemma 2.7, we know that $c_0 > 0$. We just need to verify that $c_0 < \frac{s}{N}S^{\frac{N}{2s}}$. Without loss of generality, we may assume that $0 \in \Omega$ and $B_{4\alpha} \subseteq \Omega$ for some $\alpha > 0$. Let

$$\hat{u} = \frac{U(x)}{\|U(x)\|_{L^{2_s^*}(\mathbb{R}^N)}},$$

where

$$U(x) := \kappa(\mu^2 + |x|^2)^{-\frac{N-2s}{2}}.$$

As we have described in Section 2.2, $\hat{u}$ is a minimizer for S. Let

$$U_\epsilon(x) = \epsilon^{\frac{2s-N}{2}}u^*\Big(\frac{x}{\epsilon}\Big),$$

where $u^*(x) = \hat{u}(\frac{x}{S^{\frac{1}{2s}}})$ and $\epsilon > 0$ is small. Put $u_\epsilon = \eta(x)U_\epsilon(x)$, where $\eta(x)$ is a smooth cutoff function satisfying

$$\eta = 1 \quad \text{in } B_\alpha, \qquad \eta = 0 \quad \text{in } \mathbb{R}^N \setminus B_{2\alpha}, \qquad 0 \leq \eta \leq 1.$$

By a standard argument which is similar to [25, Proposition 7.2], we have for $N \geq 4s$ and $\epsilon > 0$ small enough,

$$\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\epsilon(x)-u_\epsilon(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega |u_\epsilon(x)|^2\,dx}{(\int_\Omega |u_\epsilon|^{2_s^*}\,dx)^{\frac{2}{2_s^*}}} < S. \tag{3.14}$$

Let $H_{u_\epsilon} := \operatorname{span}\{E_0^-, u_\epsilon\}$. According to [25, Proposition 7.3] and some calculation, we have

$$M_\epsilon := \max\Big\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega |u(x)|^2\,dx : u \in H_{u_\epsilon}\setminus\{0\},\ \int_\Omega |u_\epsilon|^{2_s^*}\,dx = 1\Big\} < S$$

for $\epsilon > 0$ small and $N \geq 4s$. For each $u \in H_{u_\epsilon} \setminus \{0\}$, we have

$$I_0(u) \leq \max_{t\geq 0} I_0(tu) = \frac{s}{N}\Bigg(\frac{\|u\|_0^2 - \delta\|u\|_{L^2(\Omega)}^2}{\|u\|_{L^{2_s^*}(\Omega)}^2}\Bigg).$$

Thus

$$\max_{u\in H_{u_\epsilon}} I_0(u) \leq \frac{s}{N}M_\epsilon^{\frac{N}{2}} < \frac{s}{N}S^{\frac{N}{2s}}$$

for $\epsilon > 0$ small and $N \geq 4s$. Lemma 2.7 immediately implies that $c_0 < \frac{s}{N}S^{\frac{N}{2s}}$ for $N \geq 4s$. □

Now we come to give the proof of Proposition 3.1.

Proof of Proposition 3.1. According to the above lemmas, there is a sequence $\{u_n\} \subset E_0$ such that $I_0(u_n) \to c_0$, $I_0'(u_n) \to 0$, and $u_n \to u$ strongly in E_0. This implies that $I_0(u) = c_0$, $I_0'(u) = 0$. Thus u is a least energy solution to (1.3).

For $\delta \geq \lambda_1$, u is sign-changing. In fact, let e_1 be the principle eigenfunction corresponding to the principle eigenvalue λ_1 of L_0 defined in E_0. We may assume that e_1 is positive. Multiplying e_1 on both sides of equation (1.3) and integrate over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(e_1(x)-e_1(y))}{|x-y|^{N+2s}}\,dx\,dy - \delta\int_\Omega ue_1\,dx = \int_\Omega |u|^{p-2}ue_1\,dx.$$

This implies that

$$(\lambda_1 - \delta) \int_{\Omega} ue_1 \, dx = \int_{\Omega} |u|^{p-2} ue_1 \, dx. \tag{3.15}$$

If u keeps sign, we may assume that $u \geq 0$. Thus the left side of (3.15) is nonpositive, but the right side is positive. This leads to a contradiction. Thus u changes sign. This completes the proof of Proposition 3.1. □

Remark 3.6. Assume $\Omega = \Omega_1 \cup \Omega_2$ is the interior part of the zero set $a^{-1}(0)$, where Ω_1 and Ω_2 are two smooth, connected and bounded domains in $\mathbb{R}^N$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose u is a least energy solution of (1.3) with $u(x) = 0$ in Ω_1 and $u(x) \neq 0$ in Ω_2. Then for any $x \in \Omega_1$,

$$(-\Delta)^s u(x) = \int_{\Omega_2} \frac{-u(y)}{|x-y|^{N+2s}} \, dy$$

might be nonzero. For example, for $0 < \delta < \lambda_1$, u is nontrivial and keeps sign in Ω_2. On one hand,

$$(-\Delta)^s u(x) = \int_{\Omega_2} \frac{-u(y)}{|x-y|^{N+2s}} \, dy$$

is positive or negative. But on the other hand, for $x \in \Omega_1$,

$$(-\Delta)^s u(x) = \delta u(x) + u(x)^{p-1} = 0.$$

Thus $u(x) \neq 0$ in both Ω_1 and Ω_2. However, for the Laplacian, the local case, $u = 0$ in Ω if and only if $\Delta u = 0$. This is the difference between the local and nonlocal problems.

Remark 3.7. Recently, Raffaella Servadei and Enrico Valdinoci considered the following problem:

$$\begin{cases} (-\Delta)^s u(x) - \delta u(x) = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.16}$$

where $0 < s < 1$, $N > 2s$, and $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and the nonlinearity $f(x, u)$ satisfies some growth conditions. When $f(x, u)$ is subcritical, if $\delta = 0$, they proved in [24] the existence of ground states to (3.16) by the Mountain Pass Theorem. If $\delta \neq \lambda_1$, they obtained in [26] a positive solution to (3.16) by the Mountain Pass Theorem for $\delta < \lambda_1$ and a sign-changing solution by the Linking Theorem for $\delta \geq \lambda_1$. When $f(x, u)$ is critical, the authors also obtained the similar results to (3.16). For more details, please see [27] and [25].

4 The Proof of Main Results in the Subcritical Case

In this section, we focus on the proof of the main results in the subcritical case and we divide it into several subsections. To be precise, Theorem 1.3 is proved in Section 4.1 and Theorems 1.4–1.5 are proved in Section 4.2. Throughout this section, we always assume $2 < p < 2_s^*$ and $N > 2s$.

4.1 The Proof of Theorem 1.3 in the Subcritical Case

In this subsection, we prove the existence and asymptotic behavior of least energy solutions to $(\mathcal{P}_\lambda)$ in the subcritical case. Similar to the proof of the previous section, we consider the existence, boundedness and the compactness of a $(\mathrm{PS})_{c_\lambda}$ sequence. We have the following lemmas.

Lemma 4.1 (Existence of $(\mathrm{PS})_{c_\lambda}$ Sequences). *Let $\lambda \geq \Lambda_0$ be fixed. Then there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}_{n\geq 1}$.*

Proof. The proof of this lemma is similar to Lemma 3.2; we omit the details. □

Lemma 4.2 (Boundedness of $(\mathrm{PS})_{c_\lambda}$ Sequences). *Let $\lambda \geq \Lambda_0$ be fixed and let $\{u_n\}_{n\geq 1}$ be a $(\mathrm{PS})_{c_\lambda}$ sequence. Then $\{u_n\}_{n\geq 1}$ is bounded in E_λ.*

Proof. According to the definition of $(\mathrm{PS})_{c_\lambda}$ sequence, for n large,

$$\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^N}|u_n|^p\,dx = I_\lambda(u_n) - \frac{1}{2}I'_\lambda(u_n)u_n \leq c_\lambda + 1 + \frac{1}{2}\|\nabla I'_\lambda(u_n)\|_\lambda\|u_n\|_\lambda \leq c_\lambda + 1 + \|u_n\|_\lambda. \tag{4.1}$$

On the other hand, as in Section 1 (see (1.2) in Remark 1.1),

$$\Omega \subseteq B_R(0),\ \ a(x) > a_0 \quad \text{for all } |x| > R.$$

Then for any $\lambda \geq \Lambda_0 > \lambda_0$, we have

$$V_\lambda(x) = \lambda a(x) - \delta > a_0 > 0.$$

Thus the support of V_λ^- is contained in $B_R(0)$, i.e.

$$\operatorname{supp} V_\lambda^-(x) \subseteq B_R(0). \tag{4.2}$$

Since $V_\lambda^- < \delta$, it follows that for $\lambda \geq \Lambda_0$, by (4.2) and Hölder's inequality,

$$\int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx = \int_{B_R(0)} V_\lambda^- u_n^2\,dx \leq \delta \int_{B_R(0)} u_n^2\,dx \leq C\Big(\int_{B_R(0)}|u_n|^p\,dx\Big)^{\frac{2}{p}} \leq C\Big(\int_{\mathbb{R}^N}|u_n|^p\,dx\Big)^{\frac{2}{p}}, \tag{4.3}$$

where C is a constant which does not depend on n. By the definition of $(\mathrm{PS})_{c_\lambda}$ sequences, for n large we have

$$\begin{aligned}
&\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u_n^2\,dx\Big)\\
&\quad = \frac{p-2}{2p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u_n^2\,dx\Big) - \frac{p-2}{2p}\int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx\\
&\quad = I_\lambda(u_n) - \frac{1}{p}I'_\lambda(u_n)u_n \leq c_\lambda + 1 + \|u_n\|_\lambda.
\end{aligned} \tag{4.4}$$

Combining (4.1), (4.3) and (4.4), we have for n large,

$$\|u_n\|_\lambda^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda^+ u_n^2\,dx \leq C(c_\lambda + 1 + \|u_n\|_\lambda),$$

where $C > 0$ does not depend on n. Thus $\{u_n\}_{n\geq 1}$ is bounded in E_λ. □

Lemma 4.3 (Converges of $(\mathrm{PS})_{c_\lambda}$ Sequences). *There exists a positive constant $\Lambda_1 > \Lambda_0$ such that, for $\lambda \geq \Lambda_1$, any $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}$ of I_λ with $c_\lambda \leq c_0$ converges strongly in E_λ along a subsequence to a least energy solution u_λ of $(\mathcal{P}_\lambda)$ with $I_\lambda(u_\lambda) = c_\lambda$, where c_0 is the least energy to problem* (1.3) *with subcritical exponent, i.e.* $2 < p < 2_s^*$.

Proof. Let $\{u_n\}_{n\geq 1} \subseteq E_\lambda$ be a $(\mathrm{PS})_{c_\lambda}$ sequence of I_λ, where $\lambda > \Lambda_0$. Then due to Lemma 4.2, the sequence $\{u_n\}_{n\geq 1}$ is bounded in E_λ. Thus up to a subsequence, there exists a function $u_\lambda \in E_\lambda$ such that

$$\begin{cases}
u_n \rightharpoonup u_\lambda & \text{weakly in } E_\lambda,\\
u_n \rightharpoonup u_\lambda & \text{weakly in } L^p(\mathbb{R}^N),\\
u_n \to u_\lambda & \text{strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^N),\ 2 \leq q < 2_s^*,\\
u_n \to u_\lambda & \text{a.e. in } \mathbb{R}^N.
\end{cases} \tag{4.5}$$

We first claim that u_λ is a solution to $(\mathcal{P}_\lambda)$. Indeed, for any $v \in E_\lambda$, we have

$$\begin{aligned}
I'_\lambda(u_n)v &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u_n v\,dx - \int_{\mathbb{R}^N}|u_n|^{p-2}u_n v\,dx\\
&= \langle u_n, v\rangle_\lambda - \int_{\mathbb{R}^N} V_\lambda^- u_n v\,dx - \int_{\mathbb{R}^N}|u_n|^{p-2}u_n v\,dx.
\end{aligned} \tag{4.6}$$

Notice that supp $V_\lambda^- \subseteq B_R(0)$ for $\lambda > \Lambda_0$ and $V_\lambda^- \leq \delta$ (see (1.2) in Remark 1.1). Then by Hölder's inequality and the third equality in (4.5), we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} V_\lambda^-(u_n - u_\lambda)v\,dx\,dy\Big| &= \Big|\int_{B_R(0)} V_\lambda^-(u_n - u_\lambda)v\,dx\Big| \\ &\leq \delta \int_{B_R(0)} |(u_n - u_\lambda)v|\,dx \leq \delta\Big(\int_{B_R(0)} |u_n - u_\lambda|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} v^2\,dx\Big)^{\frac{1}{2}} \to 0 \qquad (4.7)\end{aligned}$$

as $n \to +\infty$. Hence by (4.5), (4.6) and (4.7), we have

$$I_\lambda'(u_n)v \to I_\lambda'(u_\lambda)v \quad \text{as } n \to +\infty.$$

Since $I_\lambda'(u_n) \to 0$ strongly in E_λ' as $n \to +\infty$, thus $I_\lambda'(u_\lambda) = 0$ in E_λ. This implies that u_λ is a solution to $(\mathcal{P}_\lambda)$. Moreover,

$$I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{2}I_\lambda'(u_\lambda)u_\lambda = \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^N} |u_\lambda|^p\,dx \geq 0. \qquad (4.8)$$

Let $v_n = u_n - u_\lambda$, by Brézis–Lieb's lemma, we have

$$\|u_n\|_\lambda^2 = \|u_\lambda\|_\lambda^2 + \|v_n\|_\lambda^2 + o(1), \quad \|u_n\|_{L^p(\mathbb{R}^N)}^p = \|u_\lambda\|_{L^p(\mathbb{R}^N)}^p + \|v_n\|_{L^p(\mathbb{R}^N)}^p + o(1). \qquad (4.9)$$

For $\lambda > \Lambda_0$, since $\{u_n\}_{n\geq 1}$ is bounded in E_λ, it follows from Lemma 2.1 that $\{u_n\}_{n\geq 1}$ is also bounded in $H^s(\mathbb{R}^N)$. Replace v by $u_n + u_\lambda$ in (4.7), it is easy to get that as $n \to +\infty$,

$$\int_{\mathbb{R}^N} V_\lambda^- v_n^2\,dx \to 0, \quad \int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx \to \int_{\mathbb{R}^N} V_\lambda^- u_\lambda^2\,dx. \qquad (4.10)$$

Thus by (4.9) and (4.10), as $n \to +\infty$ we have

$$\begin{aligned} I_\lambda(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda u_n^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p\,dx \\ &= \frac{1}{2}\|u_n\|_\lambda^2 - \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p\,dx \\ &= \frac{1}{2}\|u_n\|_\lambda^2 - \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda^- u_\lambda^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p\,dx + o(1) \\ &= I_\lambda(u_\lambda) + I_\lambda(v_n) + o(1) \end{aligned} \qquad (4.11)$$

and

$$\begin{aligned} I_\lambda'(u_n)u_n &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u_n^2\,dx - \int_{\mathbb{R}^N} |u_n|^p\,dx \\ &= \|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx - \int_{\mathbb{R}^N} |u_n|^p\,dx \\ &= \|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} V_\lambda^- u_\lambda^2\,dx - \int_{\mathbb{R}^N} |u_n|^p\,dx + o(1) \\ &= I_\lambda'(v_n)v_n + o(1). \end{aligned} \qquad (4.12)$$

We may assume that

$$b = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^p\,dx \geq 0.$$

If $b = 0$, then $u_n \to u_\lambda$ in $L^p(\mathbb{R}^N)$ as $n \to +\infty$, hence

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^p \, dx = \int_{\mathbb{R}^N} |u_\lambda|^p \, dx.$$

Since $I'_\lambda(u_\lambda) = 0$, it follows from (4.9) and (4.10) that

$$\begin{aligned} c_\lambda = \lim_{n\to+\infty} I_\lambda(u_n) &= \lim_{n\to+\infty}\Big(I_\lambda(u_n) - \frac{1}{2}I'_\lambda(u_n)u_n\Big) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\lim_{n\to+\infty}\int_{\mathbb{R}^N} |u_n|^p \, dx \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^N} |u_\lambda|^p \, dx = I_\lambda(u_\lambda) - \frac{1}{2}I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

Thus u_λ is a least energy solution to $(\mathcal{P}_\lambda)$. Furthermore, by (4.9) and (4.10), we have

$$\begin{aligned} c_\lambda = \lim_{n\to+\infty} I_\lambda(u_n) &= \lim_{n\to+\infty}\Big(I_\lambda(u_n) - \frac{1}{p}I'_\lambda(u_n)\Big) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\lim_{n\to+\infty}\Big(\|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} V_\lambda^- |u_n|^2 \, dx\Big) \\ &\geq \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\|u_\lambda\|_\lambda^2 - \int_{\mathbb{R}^N} V_\lambda^- |u_\lambda|^2 \, dx\Big) = I_\lambda(u_\lambda) = c_\lambda, \end{aligned}$$

which implies that $\lim_{n\to+\infty} \|u_n\|_\lambda^2 = \|u_\lambda\|_\lambda^2$. Hence $u_n \to u_\lambda$ strongly in E_λ by (4.9).

If $b > 0$, we claim that there exists a positive constant Λ which does not depend on λ such that $b \geq \Lambda$. In fact, since $I_\lambda(u_n) \to 0$ in E'_λ as $n \to +\infty$ and $\{u_n\}_{n\geq 1}$ is bounded in E_λ, we have

$$I'_\lambda(u_n)u_n \to 0 \quad \text{as } n \to +\infty. \tag{4.13}$$

Thus by (4.12), (4.13), (4.10) and Lemma 2.1, we have

$$b = \lim_{n\to+\infty}\int_{\mathbb{R}^N} |v_n|^p \, dx = \lim_{n\to+\infty} \|v_n\|_\lambda^2 \geq C \liminf_{n\to+\infty} \|v_n\|^2 \geq C\Big(\liminf_{n\to+\infty}\int_{\mathbb{R}^N} |v_n|^p \, dx\Big)^{\frac{2}{p}} = Cb^{\frac{2}{p}},$$

where the constant C does not depend on λ. Select $\Lambda := C^{\frac{p}{p-2}}$, thus $b \geq \Lambda$.

At last, we claim that there exists a constant $\Lambda_1 > \Lambda_0$ such that $b < \frac{1}{2}\Lambda$ for any $\lambda > \Lambda_1$. In fact, according to (4.8), (4.11), (4.12) and (4.13), we have

$$\begin{aligned} c_0 \geq c_\lambda \geq \lim_{n\to+\infty} I_\lambda(v_n) &= \lim_{n\to+\infty}\Big(I_\lambda(v_n) - \frac{1}{2}I'_\lambda(v_n)v_n\Big) \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\lim_{n\to+\infty}\int_{\mathbb{R}^N} |v_n|^p \, dx = \Big(\frac{1}{2} - \frac{1}{p}\Big)b. \end{aligned} \tag{4.14}$$

Due to the third equality in (4.5), combining (4.12) and (4.13) we have

$$\begin{aligned} b = \lim_{n\to+\infty}\int_{\mathbb{R}^N} |v_n|^p \, dx &= \lim_{n\to+\infty} \|v_n\|_\lambda^2 \geq \lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_R(0)} V_\lambda |v_n|^2 \, dx \\ &\geq (\lambda a_0 - \delta)\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^2 \, dx. \end{aligned} \tag{4.15}$$

Thus by (4.14) and (4.15), we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^2 \, dx \leq \frac{b}{\lambda a_0 - \delta} \leq \frac{C}{\lambda a_0 - \delta}, \tag{4.16}$$

where $C = \frac{2p}{p-2}c_0$. According to (4.8), (4.11), (4.12), (4.13) and (4.16), we have

$$\begin{aligned} c_0 \geq c_\lambda \geq \lim_{n\to+\infty} I_\lambda(v_n) &= \lim_{n\to+\infty}\Big(I_\lambda(v_n) - \frac{1}{p}I'_\lambda(v_n)v_n\Big) \\ &= \frac{p-2}{2p}\lim_{n\to+\infty}\Big(\|v_n\|_\lambda^2 - \int_{\mathbb{R}^N} V_\lambda^- v_n^2 \, dx\Big) = \frac{p-2}{2p}\lim_{n\to+\infty}\|v_n\|_\lambda^2. \end{aligned} \tag{4.17}$$

Thus by the third equality in (4.5), Hölder's inequality, the definition of S and (4.17), we have

$$\begin{aligned} b &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^p\,dx \le \lim_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |v_n|^2\,dx \Big)^{\frac{p\theta}{2}} \Big(\int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx \Big)^{\frac{p(1-\theta)}{2*}} \\ &\le \lim_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |v_n|^2\,dx \Big)^{\frac{p\theta}{2}} \Big(S^{-1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \Big)^{\frac{p(1-\theta)}{2}} \\ &\le C \lim_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |v_n|^2\,dx \Big)^{\frac{p\theta}{2}} \lim_{n\to+\infty} \|v_n\|_\lambda^{p(1-\theta)} \\ &\le C \lim_{n\to+\infty} \Big(\int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^2\,dx \Big)^{\frac{p\theta}{2}} \le C\Big(\frac{1}{\lambda a_0 - \delta} \Big)^{\frac{p\theta}{2}}, \end{aligned}$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$ and C is independent of λ. Thus there exists $\Lambda_1 > \Lambda_0$ such that $b < \frac{1}{2}\Lambda$ for any $\lambda \ge \Lambda_1$ which leads to a contradiction. □

The following gives the boundedness of c_λ from both sides.

Lemma 4.4 (Boundedness for c_λ). *For $N > 2s$ and $\lambda \ge \Lambda_1$, we have*

$$0 < \tau < c_\lambda \le c_0.$$

Proof. Since $\mathcal{N}_0 \subseteq \mathcal{N}_\lambda$, it follows from the definition of c_λ and c_0 that $c_\lambda \le c_0 < +\infty$, where c_0 is defined as in Section 1. By (iv) in Lemma 2.7, $c_\lambda > \tau > 0$. Therefore $\tau < c_\lambda \le c_0$. □

Proof of Theorem 1.3 in the subcritical case. This is a direct result of Lemma 4.1 to Lemma 4.4. □

4.2 The Proofs of Theorem 1.4 and Theorem 1.5 in the Subcritical Case

In this subsection, we mainly focus on the study of the asymptotic behavior of the least energy solution to $(\mathcal{P}_\lambda)$ in the subcritical case, namely the proof of Theorem 1.4. Since the proof of Theorem 1.5 can be done similarly with the proof of Theorem 1.4, we will not give the details here.

Proof of Theorem 1.4 in the subcritical case. Since $\mathcal{N}_0 \subseteq \mathcal{N}_\lambda$, it follows from the definition of c_λ and c_0 that $c_\lambda \le c_0$. Taking any sequence $\lambda_n (> \Lambda_1) \to +\infty$ as $n \to +\infty$, up to a subsequence we have that $c_{\lambda_n} \to k \le c_0$. Let u_n be a least energy solution to $(\mathcal{P}_{\lambda_n})$; then $I_{\lambda_n}(u_n) = c_n$, $I'_{\lambda_n}(u_n) = 0$. We firstly claim that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Indeed, since

$$c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{2} I'_{\lambda_n}(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\mathbb{R}^N} |u_n|^p\,dx \tag{4.18}$$

and

$$c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{p} I'_{\lambda_n}(u_n)u_n = \frac{p-2}{2p} \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u_n^2\,dx \Big) \tag{4.19}$$

and by Hölder's inequality and (1.2), we have

$$\int_{\mathbb{R}^N} V_{\lambda_n}^- u_n^2\,dx = \int_{B_R(0)} V_{\lambda_n}^- u_n^2\,dx \le \delta \int_{B_R(0)} u_n^2\,dx \le \delta |B_R(0)|^{1-\frac{2}{p}} \Big(\int_{\mathbb{R}^N} |u_n|^p\,dx \Big)^{\frac{2}{p}}. \tag{4.20}$$

Then by (4.18), (4.19) and (4.20), we have

$$\|u_n\|_{\lambda_n}^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda u_n^2\,dx + \int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx \le C\Big(c_{\lambda_n} + c_{\lambda_n}^{\frac{2}{p}}\Big) \le C\Big(c_0 + c_0^{\frac{2}{p}}\Big),$$

where the constant $C > 0$ does not depend on n. Thus by Lemma 2.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Thus up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^s(\mathbb{R}^N), \\ u_n \to u & \text{strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 2 \le q < 2^*_s, \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{4.21}$$

Now we claim that $u|_{\Omega^c} = 0$, where $\Omega^c := \{x : x \in \mathbb{R}^N \setminus \Omega\}$.

Indeed, let $C_m := \{x \in \mathbb{R}^N : a(x) > \frac{1}{m}\}$, $m = 1, 2, \dots$. Since $\{u_n\}_{n\ge1}$ is bounded in $H^s(\mathbb{R}^N)$, we have for any fixed m,

$$\begin{aligned} c_0 \ge c_{\lambda_n} = I_{\lambda_n}(u_n) &= I_{\lambda_n}(u_n) - \frac{1}{p} I'_{\lambda_n}(u_n)u_n \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_{\lambda_n} u_n^2\,dx\Big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\lambda_n \int_{\mathbb{R}^N} a(x)u_n^2\,dx - \delta \int_{\mathbb{R}^N} u_n^2\,dx\Big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\frac{\lambda_n}{m} \int_{C_m} u_n^2\,dx - C\Big). \end{aligned}$$

Let $n \to +\infty$. We have $\int_{C_m} u_n^2\,dx \to 0$. By (4.21) and Fatou's lemma,

$$0 \le \int_{C_m} u^2\,dx \le \liminf_{n\to+\infty} \int_{C_m} u_n^2\,dx = 0.$$

Then $u = 0$ a.e. in C_m. Since $\mathbb{R}^N \setminus \Omega = \bigcup_{m=1}^{+\infty} C_m$, it follows that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Now we come to show that $u \in E_0$ is a solution to (1.3). Indeed, according to (4.21) , we have for each $v \in E_0$,

$$\begin{aligned} I'_{\lambda_n}(u_n)v &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_{\lambda_n} u_n v\,dx - \int_{\mathbb{R}^N} |u_n|^{p-2}u_n v\,dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+2s}}\,dx\,dy - \delta \int_{\Omega} u_n v\,dx - \int_{\Omega} |u_n|^{p-2}u_n v\,dx \\ &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}}\,dx\,dy - \delta \int_{\Omega} uv\,dx - \int_{\Omega} |u|^{p-2}uv\,dx = I'_0(u)v \end{aligned}$$

as $n \to +\infty$. Since $I'_{\lambda_n}(u_n) = 0$, it follows that $I'_0(u) = 0$. Hence, u is a solution to (1.2). Moreover,

$$I_0(u) = I_0(u) - \frac{1}{2} I'_0(u)u = \Big(\frac{1}{2} - \frac{1}{2^*_s}\Big) \int_{\mathbb{R}^N} |u|^{2^*_s}\,dx \ge 0. \tag{4.22}$$

At this moment, we want to show that, up to subsequence, $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ as $n \to +\infty$. We do it by a contradiction argument. Let $v_n = u_n - u$ and we assume that $\lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^p\,dx = b > 0$. By (4.21) and Brézis–Lieb's lemma,

$$\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o(1), \qquad \int_{\mathbb{R}^N} |u_n|^p\,dx = \int_{\mathbb{R}^N} |u|^p\,dx + \int_{\mathbb{R}^N} |v_n|^p\,dx + o(1) \quad \text{as } n \to +\infty.$$

Then we can easily get

$$0 = I'_{\lambda_n}(u_n)u_n = I'_0(u)u + I'_{\lambda_n}(v_n)v_n + o(1) = I'_{\lambda_n}(v_n)v_n + o(1) \tag{4.23}$$

and

$$I_{\lambda_n}(u_n) = I_0(u) + I_{\lambda_n}(v_n) + o(1). \tag{4.24}$$

According to (4.22), (4.23)and (4.24), we have

$$\begin{aligned} c_0 \geq c_{\lambda_n} = I_{\lambda_n}(u_n) \geq I_{\lambda_n}(v_n) + o(1) &= I_{\lambda_n}(v_n) - \frac{1}{2} I'_{\lambda_n}(v_n)v_n + o(1) \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\mathbb{R}^N} |v_n|^p \, dx + o(1) = \Big(\frac{1}{2} - \frac{1}{p}\Big) b + o(1). \end{aligned} \tag{4.25}$$

By (4.21), (4.23) and (1.2), we have

$$b = \int_{\mathbb{R}^N} |v_n|^p \, dx + o(1) = \|v_n\|^2_{\lambda_n} + o(1) \geq \int_{\mathbb{R}^N \setminus B_R(0)} V_{\lambda_n} |v_n|^2 \, dx + o(1) \geq (\lambda_n a_0 - \delta) \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^2 \, dx + o(1).$$

Let $n \to +\infty$. By (4.25), we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^2 \, dx = 0.$$

Then by (4.21), we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^2 \, dx = 0. \tag{4.26}$$

Combining Hölder's inequality, Sobolev imbedding inequality (4.21) and (4.26), we have

$$\int_{\mathbb{R}^N} |v_n|^p \, dx \leq \Big(\int_{\mathbb{R}^N} |v_n|^2 \, dx \Big)^{\frac{p\theta}{2}} \Big(\int_{\mathbb{R}^N} |v_n|^{2^*_s} \, dx \Big)^{\frac{p(1-\theta)}{2*}} \leq C \Big(\int_{\mathbb{R}^N} |v_n|^2 \, dx \Big)^{\frac{p\theta}{2}} \|v_n\|^{\frac{p(1-\theta)}{2}} \to 0$$

as $n \to +\infty$. Thus $b = 0$ which contradict to our assumption $b > 0$.

At last we show that u is indeed a least energy solution to (1.3) and $I_0(u) = c_0$. Indeed,

$$\lim_{n \to +\infty} I_{\lambda_n}(v_n) = \lim_{n \to +\infty} \Big(I_{\lambda_n} v_n - \frac{1}{2} I'_{\lambda_n}(v_n) v_n \Big) = \Big(\frac{1}{2} - \frac{1}{p} \Big) \lim_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^p \, dx = 0.$$

Thus by (4.24), we have $I_0(u) = k \geq \tau > 0$. Therefore, $c_0 \geq k = I_0(u) \geq c_0$, i.e. $I_0(u) = c_0$ and u is a least energy solution to (1.3). This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5 in the subcritical case. Let $\hat{c}_0 := \limsup_{\lambda \to +\infty} I_\lambda(u_\lambda)$, $\hat{c}_\lambda := I_\lambda(u_\lambda)$, where u_λ is a nontrivial solution to $(\mathcal{P}_\lambda)$. Similar to the proof of Theorem 1.4, we complete the proof. □

5 The Proof of Main Results in the Critical Case

In this section, we deal with the proof of the main theorems in the critical case. In Section 5.1 we prove Theorem 1.3, and Theorems 1.4–1.5 are proved in Section 5.2. Throughout this section, we assume $p = 2^*_s$ and $N \geq 4s$ without especially stated.

5.1 The Proof of Theorem 1.3 in the Critical Case

In this subsection, by showing series lemmas, we complete the proof of Theorem 1.3 in the critical case. These lemmas deal with the existence, boundedness, compactness of $(\mathrm{PS})_{c_\lambda}$ sequences in the critical case and we list them as follows.

Lemma 5.1 (Existence of $(\mathrm{PS})_{c_\lambda}$ Sequences). *Let $\lambda \geq \Lambda_0$ fixed. Then there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}_{n \geq 1}$, i.e.*

$$I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Lemma 5.2 (Boundedness of $(\mathrm{PS})_{c_\lambda}$ Sequences). *Let $\lambda \geq \Lambda_0$ be fixed and let $\{u_n\}_{n \geq 1}$ be a $(\mathrm{PS})_{c_\lambda}$ sequence. Then $\{u_n\}_{n \geq 1}$ is bounded in E_λ.*

Remark 5.3. The proofs of Lemma 5.1 and Lemma 5.2 are similar to the proofs of Lemma 4.1 and Lemma 4.2, respectively, just replacing p by 2_s^*; we omit them.

The following lemma is about the compactness of $(\mathrm{PS})_{c_\lambda}$ sequences in the critical case.

Lemma 5.4 (Converges of $(\mathrm{PS})_{c_\lambda}$ Sequences). *For $\lambda \geq \Lambda_0$, $\{u_n\}_{n\geq 1}$ is a $(\mathrm{PS})_{c_\lambda}$ sequence of I_λ with $c_\lambda < \frac{s}{N} S^{\frac{N}{2s}}$, where S is the best embedding constant for $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$. Then up to a subsequence, there exists $u_\lambda \in E_\lambda$ such that $u_n \to u_\lambda$ in E_λ. Furthermore, u_λ is a least energy solution of $(\mathcal{P}_\lambda)$ which satisfies $I_\lambda(u_\lambda) = c_\lambda$.*

Proof. By Lemma 5.2, we know that $\{u_n\}_{n\geq 1}$ is bounded in E_λ. Thus there is a function $u_\lambda \in E_\lambda$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } E_\lambda, \\ u_n \rightharpoonup u_\lambda & \text{weakly in } L^{2^*}(\mathbb{R}^N), \\ u_n \to u_\lambda & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \\ u_n \to u_\lambda & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

Similar to the proof of Lemma 4.3, by (5.1), one can easily check that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) \geq 0$. Then u_λ is a solution to $(\mathcal{P}_\lambda)$. Let $v_n = u_n - u_\lambda$. Then by the first and second equality in (5.1) and Brézis–Lieb's lemma, we have as $n \to +\infty$,

$$\|u_n\|_\lambda^2 = \|u_\lambda\|_\lambda^2 + \|v_n\|_\lambda^2 + o(1), \tag{5.2}$$

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*}\,dx = \int_{\mathbb{R}^N} |u_\lambda|^{2_s^*}\,dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx + o(1). \tag{5.3}$$

Up to a subsequence, we may assume that $b = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx$. By (1.2), the third and fourth equality in (5.1), we can easily get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V_\lambda^- u_n^2\,dx = \int_{\mathbb{R}^N} V_\lambda^- u_\lambda^2\,dx, \qquad \lim_{n\to+\infty} \int_{\mathbb{R}^N} V_\lambda^- (u_n - u_\lambda)^2\,dx = 0.$$

By (5.2) and (5.3), one has

$$o(1) = I'_\lambda(u_n)u_n = I'_\lambda(u_\lambda)u_\lambda + I'_\lambda(v_n)v_n + o(1) = I'_\lambda(v_n)v_n + o(1), \tag{5.4}$$

$$I_\lambda(u_n) = I_\lambda(u_\lambda) + I_\lambda(v_n) + o(1) \tag{5.5}$$

as $n \to +\infty$. Thus on one hand, by (5.4) and (5.5), we have

$$c_\lambda \geq \lim_{n\to+\infty} I_\lambda(v_n) = \lim_{n\to+\infty}\Big(I_\lambda(v_n) - \frac{1}{2}I'_\lambda(v_n)v_n\Big) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\lim_{n\to+\infty}\int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx = \frac{s}{N}b.$$

Taking into account of $c_\lambda < \frac{s}{N} S^{\frac{N}{2s}}$, we have $b < S^{\frac{N}{2s}}$. On the other hand, by (5.1) and (5.4), we have

$$\begin{aligned} b &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx \\ &= \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V_\lambda v_n^2\,dx\Big) \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} V_\lambda v_n^2\,dx + \lim_{n\to+\infty} \int_{B_R(0)} V_\lambda v_n^2\,dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} V_\lambda v_n^2\,dx + 0 \\ &\geq \lim_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \geq S \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}. \end{aligned} \tag{5.6}$$

If $b > 0$, then $b \ge S^{\frac{N}{2s}}$, which contradicts $b < S^{\frac{N}{2s}}$. Hence $b = 0$. Replace b by 0 in (5.6), we can easily get $\lim_{n\to+\infty} \|v_n\|_\lambda = 0$. Thus $u_n \to u_\lambda$ strongly in E_λ. Furthermore, by (5.3), we have

$$I_\lambda(u_\lambda) = I_\lambda(u) - \frac{1}{2}I'_\lambda(u_\lambda)u_\lambda = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) \int_{\mathbb{R}^N} |u_\lambda|^{2_s^*}\,dx = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) \lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*}\,dx = \lim_{n\to+\infty} I_\lambda(u_n) = c_\lambda.$$

Thus u_λ is a least energy solution to (1.3). □

The boundedness of c_λ from two sides is given by the following lemma.

Lemma 5.5 (Boundedness for c_λ). *For $N \ge 4s$ and $\lambda \ge \Lambda_0$, we have*

$$0 < \tau < c_\lambda < \frac{s}{N} S^{\frac{N}{2s}}.$$

Proof. By the definition of c_λ we know that $c_\lambda \le c_0$, where c_0 is defined as in Section 1. Thus by Lemma 5.5, we know that $c_0 < \frac{s}{N} S^{\frac{N}{2s}}$. By (iv) in Lemma 2.8, we have $\tau < c_\lambda < \frac{s}{N} S^{\frac{N}{2s}}$. □

Proof of Theorem 1.3 in the critical case. Select $\Lambda_2 := \Lambda_0$. This is a direct result of Lemma 5.1, Lemma 5.2, Lemma 5.4 and Lemma 5.5. □

5.2 The Proofs of Theorem 1.4 and Theorem 1.5 in the Critical Case

In this subsection, we first prove Theorem 1.4 in the critical case. Since the proof of Theorem 1.5 in the critical case is also similar to Theorem 1.4 in the critical case, we just give a sketch.

Proof of Theorem 1.4 in the critical case. Since $\mathcal{N}_0 \subseteq \mathcal{N}_\lambda$, it follows from the definition of c_λ and c_0 that we can take any sequence $\lambda_n (> \Lambda_2) \to +\infty$ as $n \to +\infty$ so that we have up to a subsequence $c_{\lambda_n} \to k \le c_0$. Let u_n be a least energy solution to $(\mathcal{P}_{\lambda_n})$. Then $I_{\lambda_n}(u_n) = c_n$, $I'_{\lambda_n}(u_n) = 0$. Similar to the subcritical case, we can prove that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Thus, there exists a function u such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^s(\mathbb{R}^N), \\ u_n \rightharpoonup u & \text{weakly in } L^{2_s^*}(\mathbb{R}^N), \\ u_n \to u & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{5.7}$$

Indeed, using similar arguments as in the subcritical case, we have $u|_{\Omega^c} = 0$, where $\Omega^c =: \{x : x \in \mathbb{R}^N \setminus \Omega\}$. On the other hand, $u \in E_0$ is a solution to equation (1.3) with $I_0(u) \ge 0$. Now we come to show that, up to a subsequence, $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^N)$ as $n \to +\infty$.

We prove this by a contradiction argument. Assume $\lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx = b > 0$, where $v_n = u_n - u$. By Brézis–Lieb's lemma,

$$\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o(1), \tag{5.8}$$

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*}\,dx = \int_{\mathbb{R}^N} |u|^{2_s^*}\,dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx + o(1) \tag{5.9}$$

as $n \to +\infty$. Then by the third equality in (5.7), (5.8) and (5.9), we can easily get

$$0 = I'_{\lambda_n}(u_n)u_n = I'_0(u)u + I'_{\lambda_n}(v_n)v_n + o(1) = I'_{\lambda_n}(v_n)v_n + o(1) \tag{5.10}$$

and

$$I_{\lambda_n}(u_n) = I_0(u) + I_{\lambda_n}(v_n) + o(1). \tag{5.11}$$

On the one hand, by (5.10) and (5.11), we have

$$\begin{aligned} k &= \lim_{n\to+\infty} c_{\lambda_n} = \lim_{n\to+\infty} I_{\lambda_n}(u_n) \ge \lim_{n\to+\infty} I_{\lambda_n}(v_n) \\ &= \lim_{n\to+\infty} \Big(I_{\lambda_n}(v_n) - \frac{1}{2}I'_{\lambda_n}(v_n)v_n\Big) = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) \int_{\mathbb{R}^N} |v_n|^{2_s^*}\,dx = \frac{s}{N} b. \end{aligned}$$

Thus by the third equality in (5.7), (5.10) and the definition of S, we have

$$\begin{aligned} b &= \lim_{n\to+\infty}\int_{\mathbb{R}^N}|v_n|^{2_s^*}\,dx = \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}V_{\lambda_n}v_n^2\,dx\Big)\\ &= \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}V_{\lambda_n}^+v_n^2\,dx - \int_{\mathbb{R}^N}V_{\lambda_n}^-v_n^2\,dx\Big)\\ &= \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N}V_{\lambda_n}^+v_n^2\,dx\Big) - 0\\ &\geq \lim_{n\to+\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\ &\geq S\Big(\lim_{n\to+\infty}\int_{\mathbb{R}^N}|v_n|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} = Sb^{\frac{2}{2_s^*}}. \end{aligned} \tag{5.12}$$

Then $b \geq S^{\frac{N}{2s}}$. Therefore, $c_0 \geq k \geq \frac{s}{N}b \geq \frac{s}{N}S^{\frac{N}{2s}} > c_0$ which leads to a contradiction and thus we have that $u_n \to u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$. Furthermore, $u \in E_0$ is a solution of (1.3) and

$$I_0(u) = I_0(u) - \frac{1}{2}I_0'(u)u = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx = \lim_{n\to+\infty}I_{\lambda_n}(u_n) = \lim_{n\to+\infty}c_{\lambda_n} = k.$$

Since $k > 0$, it follows that $I_0(u) \geq c_0$. Thus $c_0 \geq k = I_0(u) \geq c_0$, i.e. $I_0(u) = c_0$. This implies that u is a least energy solution to (1.3). □

Proof of Theorem 1.5 in the critical case. Let $\hat{c}_0 := \limsup_{\lambda\to+\infty} I_\lambda(u_\lambda)$ and $\hat{c}_\lambda := I_\lambda(u_\lambda)$, where u_λ is a nontrivial solution to $(\mathcal{P}_\lambda)$. The left arguments are the same as the proof of Theorem 1.4 in the critical case and we omit it. □

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