Research Article

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Existence and Asymptotic Profile of Nodal Solutions to Supercritical Problems

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Abstract: We establish the existence of nodal solutions to the supercritical problem

$$-\Delta u = |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

in a symmetric bounded smooth domain Ω of $\mathbb{R}^N$, $N \geq 3$, for $p > 2_N^* := \frac{2N}{N-2}$, up to some range which depends on the symmetries, and we study their asymptotic behavior as $p \to 2_N^*$. We exhibit solutions u_p to this problem in symmetric domains with a shrinking hole, which concentrate at a single point as the hole shrinks and p approaches 2_N^* from above, and whose limit profile is a rescaling of a nonradial sign-changing solution to the limit problem

$$-\Delta u = |u|^{2_N^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N).$$

Keywords: Semilinear Elliptic Boundary Value Problem, Critical and Supercritical Nonlinearity, Nodal Solutions, Asymptotic Analysis

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Communicated by: Paul Rabinowitz
Dedicated to Abbas Bahri, a dear friend and an outstanding mathematician.

1 Introduction

We study the semilinear elliptic problem

$$-\Delta u = |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, and p is either the critical Sobolev exponent $2_N^* := \frac{2N}{N-2}$ or it is supercritical, i.e., $p > 2_N^*$.

It is well known that, as a consequence of the classical Pohozaev identity, problem (1.1) does not admit, neither positive, nor sign-changing solutions, for these values of the exponent p, when Ω is starshaped.

For the critical case, many results concerning the existence and qualitative properties of solutions in non-starshaped domains have been obtained in the last decades. A major breakthrough was achieved thanks to Abbas Bahri's deep theory of critical points at infinity, which lead to the derivation of a fundamental existence result for problem (1.1) in domains with nontrivial topology, and to a delicate blow-up analysis for positive solutions to this problem, see [2, 3].

It is by now well understood that the solutions to the analogous problem in the whole of $\mathbb{R}^N$,

$$-\Delta u = |u|^{2_N^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N), \tag{1.2}$$

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play a crucial role in the study of the critical problem (1.1), $p = 2^*_N$. All positive solutions of (1.2) are known. They are obtained by rescaling a single radial solution, the so-called standard bubble, which is given explicitly. This explains why most existence results for the critical problem in bounded domains rely on constructions performed using the standard bubble as the basic cell, via variational or perturbation methods; see, e.g., [14] and the references therein.

Standard bubbles have been also used to construct positive solutions to the slightly supercritical problem (1.1), with $p = 2^*_N + \varepsilon$, for small enough $\varepsilon > 0$, in domains with a small fixed hole [10, 11, 20] or with a shrinking hole [16]. As $\varepsilon \to 0$, they blow up at two or more points of the domain in the first case, and at a single point inside the shrinking hole in the second case, and their limit profile at each blow-up point is a rescaling of the standard bubble. On the other hand, Ben Ayed and Bouh [4] showed that sign-changing solutions blowing up at two or three points which resemble a sum of positive and negative bubbles do not exist.

The existence of infinitely many sign-changing solutions to problem (1.2) was first established by Ding [13]. These solutions are invariant under the conformal action of groups, all of whose orbits are positive dimensional. Del Pino, Musso, Pacard and Pistoia [12] constructed other sign-changing solutions to (1.2) using positive and negative rescalings of the standard bubble as basic cells. In their recent paper [17], Musso and Wei used these solutions, in turn, as basic cells to construct sign-changing solutions to the slightly supercritical problem (1.1), with $p = 2^*_N + \varepsilon$, in a domain with a fixed small hole, for $\varepsilon > 0$ sufficiently small. These solutions blow up at two different points of the domain as $\varepsilon \to 0$, and their limit profile at each of these points is a rescaling of one of the sign-changing solutions of (1.2) constructed in [12].

Recently, new types of solutions to problem (1.2) were exhibited in [6], and it was shown that they arise as the limit profiles of solutions to subcritical problems (1.1), with particular symmetries, as $p \to 2^*_N$ from below. These solutions are nonradial and change sign, and are quite different from those in [12]. In this paper we will show that they also arise as limit profiles of nodal solutions to the critical problem (1.1), $p = 2^*_N$, in symmetric domains with a shrinking hole, see Theorem 4.2. These solutions are, therefore, different from the sign-changing solutions obtained in [8] for other types of symmetric domains, and from the bubble-towers constructed in [14]. Further, we will show that solutions to problem (1.2) of the type exhibited in [6] also arise as limit profiles of solutions to the supercritical problems (1.1), $p > 2^*_N$, as $p \to 2^*_N$ from above.

To this end, we need first to consider the question of existence of solutions to the supercritical problem (1.1), $p > 2^*_N$. A fruitful approach, which has been applied in recent years to treat supercritical problems, consists in reducing them to some anisotropic critical or subcritical problem, either by considering rotational symmetries, or by means of maps which preserve the Laplace operator, or by a combination of both; see [9] for a detailed overview. These reductions apply only to some very specific types of domains Ω. A common feature of these domains is that they are invariant under the action of some group Γ whose orbits are positive dimensional. It turns out that this last condition suffices to establish existence. We will show that in a Γ-invariant domain Ω, problem (1.1) has infinitely many solutions for all $p \in (2, 2^*_{N-m})$, where m is the smallest dimension of a Γ-orbit in Ω and $2^*_n := \frac{2n}{n-2}$ is the critical Sobolev exponent in dimension n. Note that $2^*_N < 2^*_{N-m}$ if $m > 0$.

To avoid technicalities, here we only state a special case of this result. The general statement is given in Theorem 2.3 below.

Let $\mathbb{S}^1 := \{e^{i\vartheta} : \vartheta \in [0, 2\pi)\}$ be the group of unit complex numbers, let $O(m)$ be the group of linear isometries of $\mathbb{R}^m$ and let $A_{\delta,R} := \{z \in \mathbb{R}^N : 0 < \delta < |z| < R\}$ be an annulus. For $N \geq 4$, we write the points in $\mathbb{R}^N \equiv \mathbb{C}^2 \times \mathbb{R}^{N-4}$ as (z, y) with $z = (z_1, z_2) \in \mathbb{C}^2$, $y \in \mathbb{R}^{N-4}$.

Theorem 1.1. *Let $N = 4$ or $N \geq 6$. Then, for every $p \in (2, 2^*_{N-1})$, the problem*

$$-\Delta u = |u|^{p-2}u \quad \text{in } A_{\delta,R}, \qquad u = 0 \quad \text{on } \partial A_{\delta,R}, \tag{1.3}$$

has infinitely many solutions which satisfy

$$u(z, y) = u(e^{i\vartheta}z, \varrho y) \quad \text{and} \quad u(z_1, z_2, y) = -u(-\bar{z}_2, \bar{z}_1, y) \tag{1.4}$$

for all $e^{i\vartheta} \in \mathbb{S}^1$, $\varrho \in O(N-4)$, $z = (z_1, z_2) \in \mathbb{C}^2$ and $y \in \mathbb{R}^{N-4}$.

It is well known that problem (1.3) has one positive and infinitely many sign-changing radial solutions for every $p \in (2, \infty)$. Note, however, that the solutions provided by Theorem 1.1 are nonradial and change sign. For subcritical $p \in (2, 2_N^*)$, the existence of infinitely many solutions satisfying (1.4) can be obtained by standard variational methods. For critical $p = 2_N^*$, this was shown in [6]. As in the critical case, for supercritical $p \in (2_N^*, 2_{N-1}^*)$, the symmetries play a crucial role.

The next theorem describes the asymptotic profile of the solutions provided by Theorem 1.1 as p approaches 2_N^* from above and the hole shrinks. It is a special case of a more general result stated below; see Theorem 4.3.

Theorem 1.2. *Fix $R > 0$. For $p \in (2_N^*, 2_{N-1}^*)$, let $u_{\delta,p}$ be a solution to* (1.3) *which satisfies* (1.4) *and has minimal energy among all solutions to* (1.3) *with these symmetry properties. Then there exist sequences (p_k) in $(2_N^*, 2_{N-1}^*)$, and (δ_k) and (λ_k) in $(0, \infty)$, and a nontrivial solution ω to problem* (1.2) *such that the following hold:*

(i) $p_k \to 2_N^*$, $\delta_k \to 0$ *and* $\lambda_k \to 0$.

(ii) $\omega(z, y) = \omega(e^{i\vartheta} z, \varrho y)$ *and* $\omega(z_1, z_2, y) = -\omega(-\bar{z}_2, \bar{z}_1, y)$ *for all* $e^{i\vartheta} \in \mathbb{S}^1$, $\varrho \in O(N-4)$, $z = (z_1, z_2) \in \mathbb{C}^2$ *and* $y \in \mathbb{R}^{N-4}$. *Also, ω has minimal energy among all solutions to* (1.2) *which have these symmetry properties.*

(iii) u_{δ_k,p_k} *has the following asymptotic profile:*

$$u_{\delta_k,p_k} = \lambda_k^{\frac{2-N}{2}} \omega\Big(\frac{\cdot}{\lambda_k}\Big) + o(1) \quad \text{in } D^{1,2}(\mathbb{R}^N).$$

We stress that the limit profile of the solutions $u_{\delta,p}$ given by Theorem 1.2 is a rescaling of a nonradial sign-changing solution to problem (1.2), like those exhibited in [6]. So the solutions $u_{\delta,p}$ are different from those constructed in [16], which resemble a rescaling of the standard bubble, and they are also different from the sign-changing solutions constructed in [17].

This paper is organized as follows. In Section 2, we describe the symmetries involved and we establish existence of infinitely many sign-changing solutions to problem (1.1) for supercritical exponents in some range which depends on the symmetries. In Section 3, we analyze the behavior of symmetric minimizing sequences for the critical problem. In Section 4, we describe the asymptotic profile of least energy symmetric solutions to the critical and the slightly supercritical problem in domains with a shrinking hole.

2 Symmetries and Compactness

Let Γ be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$, let Ω be a Γ-invariant bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and let $\phi \colon \Gamma \to \mathbb{Z}/2 := \{1, -1\}$ be a continuous homomorphism of groups such that $\{\gamma \in \Gamma : \gamma x_0 = x_0\} \subset \ker \phi$ for some $x_0 \in \Omega$. This last condition guarantees that the space

$$D_0^{1,2}(\Omega)^\phi := \{u \in D_0^{1,2}(\Omega) : u(\gamma x) = \phi(\gamma) u(x) \text{ for all } \gamma \in \Gamma,\ x \in \Omega\}$$

is infinite dimensional, cf. [5]. Here, as usual, $D_0^{1,2}(\Omega)$ denotes the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm

$$\|u\| := \Big(\int_\Omega |\nabla u|^2\Big)^{\frac{1}{2}}.$$

Let $G := \ker \phi$ and set

$$D_0^{1,2}(\Omega)^G := \{u \in D_0^{1,2}(\Omega) : u(gx) = u(x) \text{ for all } g \in G,\ x \in \Omega\}.$$

Note that $D_0^{1,2}(\Omega)^\phi \subset D_0^{1,2}(\Omega)^G$. Note also that if ϕ is surjective and $u \in D_0^{1,2}(\Omega)^\phi$ is nontrivial, then u changes sign.

The Sobolev embedding theorem and the Rellich–Kondrachov theorem imply that $D_0^{1,2}(\Omega)$ is embedded in $L^p(\Omega)$ for every $p \in [1, 2_N^*]$ and that this embedding is compact for $p \in [1, 2_N^*)$, where $2_n^* := \frac{2n}{n-2}$ is the critical Sobolev exponent in dimension n. If every Γ-orbit in Ω has positive dimension and if we restrict to the space of G-invariant functions, the range of p's for which this occurs increases. This was proved by Hebey and Vaugon in [15]. Their result is the core of the following proposition. Set

$$m := \min\{\dim(\Gamma x) : x \in \Omega\},$$

where $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ is the Γ-orbit of the point x.

Proposition 2.1. *If $N - m > 2$, then $D_0^{1,2}(\Omega)^G$ and $D_0^{1,2}(\Omega)^\phi$ are embedded in $L^p(\Omega)$ for every $p \in [1, 2^*_{N-m}]$ and the embedding is compact for $p \in [1, 2^*_{N-m})$.*

Proof. Note that $\dim(Gx) = \dim(\Gamma x)$ for every $x \in \Omega$. Hebey and Vaugon showed that the space $D_0^{1,2}(\Omega)^G$ is embedded in $L^p(\Omega)$ for every $p \in [1, 2^*_{N-m}]$ and that the embedding is compact for $p \in [1, 2^*_{N-m})$, see [15, Corollary 2]. Since $D_0^{1,2}(\Omega)^\phi$ is a subspace of $D_0^{1,2}(\Omega)^G$, this is also true for $D_0^{1,2}(\Omega)^\phi$. □

Proposition 2.1 guarantees that the functional $J_p : D_0^{1,2}(\Omega)^G \to \mathbb{R}$, given by

$$J_p(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p,$$

and the Nehari manifold

$$\mathcal{N}_p^\phi := \{u \in D_0^{1,2}(\Omega)^\phi : u \neq 0,\ \|u\|^2 = |u|_p^p\},$$

are well defined for $p \in (2, 2^*_{N-m}]$, where $|\cdot|_p$ denotes the L^p-norm. It also guarantees that

$$c_p^\phi := \inf_{u \in \mathcal{N}^\phi} J_p(u) > 0.$$

Now we consider problem (1.1) for $p \in (2, 2^*_{N-m}]$. A (weak) solution to this problem is a function $u \in D_0^{1,2}(\Omega) \cap L^p(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega |u|^{p-2} u \varphi = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Lemma 2.2. *If $p \in (2, 2^*_{N-m}]$ and u is a critical point of $J_p : D_0^{1,2}(\Omega)^\phi \to \mathbb{R}$, then u is a solution to problem* (1.1) *which satisfies*

$$u(\gamma x) = \phi(\gamma) u(x) \quad \textit{for all } \gamma \in \Gamma,\ x \in \Omega. \tag{2.1}$$

Proof. For a function $v : \Omega \to \mathbb{R}$ and $\gamma \in \Gamma$, we set $v_\gamma(x) := \phi(\gamma) v(\gamma^{-1} x)$. Since $\phi(\gamma^{-1} g \gamma) = 1$ for every $g \in G$, we have that $\gamma^{-1} g \gamma \in G$. So, if $v \in D_0^{1,2}(\Omega)^G$, then, as

$$v_\gamma(gx) = \phi(\gamma) v(\gamma^{-1} g x) = \phi(\gamma) v(\gamma^{-1} g \gamma \gamma^{-1} x) = \phi(\gamma) v(\gamma^{-1} x) = v_\gamma(x)$$

for every $g \in G$, we have that $v_\gamma \in D_0^{1,2}(\Omega)^G$. This shows that $D_0^{1,2}(\Omega)^G$ is invariant under the action of Γ given by $(\gamma, v) \mapsto v_\gamma$. The set of fixed points of this action is the space $D_0^{1,2}(\Omega)^\phi$. Hence, by the principle of symmetric criticality, if u is a critical point of $J_p : D_0^{1,2}(\Omega)^\phi \to \mathbb{R}$, then u is also a critical point of $J_p : D_0^{1,2}(\Omega)^G \to \mathbb{R}$, see [18].

For $\varphi \in \mathcal{C}_c^\infty(\Omega)$, set

$$\widetilde{\varphi}(x) := \frac{1}{\mu(G)} \int_G \varphi(gx)\, d\mu,$$

where μ is the Haar measure. Then, $\widetilde{\varphi} \in \mathcal{C}_c^\infty(\Omega)^G := \mathcal{C}_c^\infty(\Omega) \cap D_0^{1,2}(\Omega)^G$. Since u is G-invariant, an easy computation shows that

$$\int_\Omega \nabla u \cdot \nabla \widetilde{\varphi} = \int_\Omega \nabla u \cdot \nabla \varphi \quad \text{and} \quad \int_\Omega |u|^{p-2} u \widetilde{\varphi} = \int_\Omega |u|^{p-2} u \varphi.$$

As u is a critical point of $J_p : D_0^{1,2}(\Omega)^G \to \mathbb{R}$, this implies that

$$0 = J_p'(u)\widetilde{\varphi} = \int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega |u|^{p-2} u \varphi \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega),$$

as claimed. □

Theorem 2.3. *If $p \in (2, 2^*_{N-m})$, then problem* (1.1) *has a solution u_p, which satisfies* (2.1) *such that $J_p(u_p) = c_p^\phi$. Moreover,* (1.1) *has an unbounded sequence of solutions which satisfy* (2.1).

Proof. Since the embedding $D_0^{1,2}(\Omega)^\phi \hookrightarrow L^p(\Omega)$ is compact for $p \in (2, 2^*_{N-m})$, a standard argument shows that J_p satisfies the Palais–Smale condition on $D_0^{1,2}(\Omega)^\phi$. Therefore, c_p^ϕ is attained on $\mathcal{N}_p^\phi$ and, as $J_p : D_0^{1,2}(\Omega)^\phi \to \mathbb{R}$ is even and has the mountain pass geometry, the symmetric mountain pass theorem (see [1]) guarantees the existence of an unbounded sequence of critical values of J_p on $\mathcal{N}_p^\phi$. By Lemma 2.2, the corresponding critical points are solutions to (1.1). □

The following example shows that this result is optimal. Let $\Omega := \{(y, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{N-m-1} : (|y|, z) \in B\}$, where B is an open ball centered in $(0, \infty) \times \{0\}$, whose closure is contained in the half-space $(0, \infty) \times \mathbb{R}^{N-m-1}$. Note that Ω is invariant under the action of $O(m + 1)$ on the y-coordinate and m is the smallest dimension of an $O(m + 1)$-orbit in Ω. In [19], Passaseo showed that (1.1) does not have a nontrivial solution for any $p \in [2^*_{N-m}, \infty)$.

Proof of Theorem 1.1. For $N \geq 4$, let Γ be the subgroup of $O(N)$ generated by $\mathbb{S}^1 \cup O(N-4) \cup \{\tau\}$, where $e^{\mathrm{i}\vartheta} \in \mathbb{S}^1$, $\varrho \in O(N-4)$ and τ act on a point $(z_1, z_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \equiv \mathbb{R}^N$ by

$$\begin{aligned} e^{\mathrm{i}\vartheta}(z_1, z_2, y) &:= (e^{\mathrm{i}\vartheta} z_1, e^{\mathrm{i}\vartheta} z_2, \varrho y), \\ \varrho(z_1, z_2, y) &:= (e^{\mathrm{i}\vartheta} z_1, e^{\mathrm{i}\vartheta} z_2, \varrho y), \\ \tau(z_1, z_2, y) &:= (-\bar{z}_2, \bar{z}_1, y). \end{aligned}$$

Let $\phi : \Gamma \to \mathbb{Z}/2$ be the homomorphism given by $\phi(e^{\mathrm{i}\vartheta}) := 1 =: \phi(\varrho)$, $\phi(\tau) := -1$.

If $N = 4$, then $\dim(\Gamma x) = 1$ for every $x = (z_1, z_2, y) \in A_{r,R}$, whereas for $N \geq 5$, we have that

$$\dim(\Gamma x) = \begin{cases} N-4 & \text{if } z \neq 0 \text{ and } y \neq 0, \\ 1 & \text{if } y = 0, \\ N-5 & \text{if } z = 0. \end{cases}$$

Hence, $m := \min\{\dim(\Gamma x) : x \in A_{r,R}\} = 1$ if $N = 4$ and $N \geq 6$, and Theorem 1.1 follows from Theorem 2.3. □

To conclude this section, we study the continuity of c_p^ϕ with respect to $p \in (2, 2^*_{N-m})$. We start with the following lemma.

Lemma 2.4. *If $p_k, q \in (2, 2^*_{N-m})$, $p_k \to q$, and (u_k) is a bounded sequence in $D_0^{1,2}(\Omega)^\phi$, then*

$$\lim_{k\to\infty} \int_\Omega (|u_k|^{p_k} - |u_k|^q) = 0.$$

Proof. By the mean value theorem, for each $x \in \Omega$, there exists $q_k(x)$ between p_k and q such that

$$||u_k(x)|^{p_k} - |u_k(x)|^q| = |\ln|u_k(x)||\,|u_k(x)|^{q_k(x)}|p_k - q|.$$

Fix $\eta > 0$ so that $[q - \eta, q + \eta] \subset (2, 2^*_{N-m})$. Then, for some positive constant C and k large enough,

$$|\ln|u_k||\,|u_k|^{q_k} \leq \begin{cases} \ln|u_k|\,|u_k|^{q+\eta} \leq C|u_k|^{2^*_{N-m}} & \text{if } |u_k| \geq 1, \\ (\ln \frac{1}{|u_k|})\,|u_k|^{q-\eta} \leq C|u_k|^2 & \text{if } |u_k| \leq 1. \end{cases}$$

Therefore, using Proposition 2.1, we obtain

$$\begin{aligned} \int_\Omega (|u_k|^{p_k} - |u_k|^q) &= \int_{|u_k|\leq 1} (|u_k|^{p_k} - |u_k|^q) + \int_{|u_k|\geq 1} (|u_k|^{p_k} - |u_k|^q) \\ &\leq C|p_k - q| \int_\Omega (|u_k|^2 + |u_k|^{2^*_{N-m}}) \\ &\leq \bar{C}|p_k - q| \|u_k\|^{2^*_{N-m}} \end{aligned}$$

for some positive constant $\bar{C}$. As (u_k) is bounded in $D_0^{1,2}(\Omega)$, we conclude that $\lim_{k\to\infty} \int_\Omega (|u_k|^{p_k} - |u_k|^q) = 0$, as claimed. □

For $p \in (2, 2^*_{N-m})$, let $u_p \in \mathcal{N}^\phi_p$ be a solution to (1.1) such that $J_p(u_p) = c^\phi_p$. Fix $q \in (2, 2^*_{N-m})$ and let $t_{q,p} \in (0, \infty)$ be such that $\tilde{u}_p := t_{q,p}u_p \in \mathcal{N}^\phi_q$, i.e.,

$$t_{q,p} = \left(\frac{\|u_p\|^2}{|u_p|_q^q}\right)^{\frac{1}{q-2}} = \left(\frac{|u_p|_p^p}{|u_p|_q^q}\right)^{\frac{1}{q-2}}. \tag{2.2}$$

Proposition 2.5. *For $q \in (2, 2^*_{N-m})$, we have that*

$$\lim_{p\to q} c^\phi_p = c^\phi_q, \quad \lim_{p\to q} t_{q,p} = 1, \quad \lim_{p\to q} J_q(\tilde{u}_p) = c^\phi_q.$$

Proof. Set

$$S^\phi_p := \inf_{u\in D^{1,2}_0(\Omega)^\phi\setminus\{0\}} \frac{\|u\|^2}{|u|_p^2}.$$

From Hölder's inequality, we obtain that

$$S^\phi_r \geq |\Omega|^{\frac{2(r-p)}{rp}} S^\phi_p \quad \text{if } p > r.$$

So, as p approaches q from the right, we get

$$\limsup_{p\to q^+} S^\phi_p \leq S^\phi_q.$$

Assume that $\liminf_{p\to q^+} S^\phi_p < S^\phi_q$. Then there exist $\varepsilon > 0$ and sequences (p_k) in $(q, 2^*_{N-m})$ and (u_k) in $D^{1,2}_0(\Omega)^\phi$, with $|u_k|_{p_k} = 1$, such that $\|u_k\|^2 < S^\phi_q - \varepsilon$. Lemma 2.4 implies that $|u_k|_q \to 1$. Hence, $\|u_k\|^2/|u_k|_q^2 < S^\phi_q$ for k large enough, contradicting the definition of S^ϕ_q. This proves that

$$\lim_{p\to q^+} S^\phi_p = S^\phi_q.$$

The corresponding statement when p approaches q from the left is proved in a similar way. Therefore, $\lim_{p\to q} S^\phi_p = S^\phi_q$. An easy computation shows that $c^\phi_p = \frac{p-2}{2p}(S^\phi_p)^{\frac{p}{p-2}}$. It follows that

$$c^\phi_q = \lim_{p\to q} c^\phi_p.$$

Since $J_p(u_p) = \frac{p-2}{2p}\|u_p\|^2 = c^\phi_p$, we have that (u_p) is bounded in $D^{1,2}_0(\Omega)^\phi$ for p close to q. The expression (2.2), together with Lemma 2.4, yields $\lim_{p\to q} t_{q,p} = 1$ which, in turn yields

$$\lim_{p\to q} J_q(\tilde{u}_p) = \lim_{p\to q} \frac{q-2}{2q}\|t_{q,p}u_p\|^2 = \lim_{p\to q} t^2_{q,p}c^\phi_p = c^\phi_q,$$

as claimed. □

3 Minimizing Sequences for the Critical Problem

Let $\Omega^\Gamma := \{x \in \Omega : \Gamma x = \{x\}\}$ be the set of Γ-fixed points in Ω. Throughout this section, we will assume that $\Omega \setminus \Omega^\Gamma$ and Ω^Γ are nonempty, and that every Γ-orbit in $\Omega \setminus \Omega^\Gamma$ has positive dimension. We consider the critical problem

$$\begin{cases} -\Delta u = |u|^{2^*_N-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\gamma x) = \phi(\gamma)u(x) & \text{for all } \gamma \in \Gamma,\ x \in \Omega. \end{cases}$$

The solutions to this problem are the critical points of the energy functional $J_*: D^{1,2}_0(\Omega)^\phi \to \mathbb{R}$, given by

$$J_*(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*_N}|u|^{2^*_N}_{2^*_N}.$$

The nontrivial solutions lie on the Nehari manifold

$$\mathcal{N}_*^\phi := \left\{u \in D_0^{1,2}(\Omega)^\phi : u \neq 0,\ \|u\|^2 = |u|_{2_N^*}^{2_N^*}\right\}.$$

We also consider the problem

$$\begin{cases} -\Delta u = |u|^{2_N^*-2}u, \\ u \in D^{1,2}(\mathbb{R}^N), \\ u(\gamma x) = \phi(\gamma)u(x) \quad \text{for all } \gamma \in \Gamma,\ x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

and we denote by $J_\infty\colon D^{1,2}(\mathbb{R}^N)^\phi \to \mathbb{R}$ its associated energy functional, and by $\mathcal{N}_\infty^\phi$ the corresponding Nehari manifold. We set

$$c_*^\phi := \inf_{u\in\mathcal{N}_*^\phi} J_*(u) \quad \text{and} \quad c_\infty^\phi := \inf_{u\in\mathcal{N}_\infty^\phi} J_\infty(u).$$

The following nonexistence result was shown in [6].

Theorem 3.1. *If $\Omega^\Gamma \neq \emptyset$, then $c_*^\phi = c_\infty^\phi$ and c_*^ϕ is not attained by J_* on $\mathcal{N}_*^\phi$.*

Proof. See [6, Theorem 2.3]. □

Theorem 3.3 below describes the shape of minimizing sequences for J_* on $\mathcal{N}_*^\phi$. Its proof is similar to that of [6, Theorem 2.5]. We give a sketch of it here for the reader's convenience.

Recall that the Γ-orbit Γx of a point $x \in \mathbb{R}^N$ is Γ-homeomorphic to the homogeneous space Γ/Γ_x, where

$$\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$$

is the isotropy group of x. In particular, $\#\Gamma x = |\Gamma/\Gamma_x|$, the index of the subgroup Γ_x in Γ.

Lemma 3.2. *Given sequences (λ_k) in $(0,\infty)$ and (ξ_k) in $\mathbb{R}^N$, there exist a sequence (ζ_k) in $\mathbb{R}^N$ and a closed subgroup K of Γ such that, after passing to a subsequence, the following statements hold true:*
(a) *The sequence $(\lambda_k^{-1}\operatorname{dist}(\Gamma\xi_k,\zeta_k))$ is bounded.*
(b) *$\Gamma_{\zeta_k} = K$ for all $k \in \mathbb{N}$.*
(c) *If $|\Gamma/K| < \infty$, then $\lambda_k^{-1}|\alpha\zeta_k - \beta\zeta_k| \to \infty$ for any $\alpha,\beta \in \Gamma$ with $\alpha^{-1}\beta \notin K$.*
(d) *If $|\Gamma/K| = \infty$, then there is a closed subgroup K' of Γ such that $K \subset K'$, $|\Gamma/K'| = \infty$ and $\lambda_k^{-1}|\alpha\zeta_k - \beta\zeta_k| \to \infty$ for any $\alpha,\beta \in \Gamma$ with $\alpha^{-1}\beta \notin K'$.*

Proof. See [7, Lemma 3.3]. □

Theorem 3.3. *Assume that $\Omega \setminus \Omega^\Gamma$ and Ω^Γ are nonempty, and that every Γ-orbit in $\Omega \setminus \Omega^\Gamma$ has positive dimension. Let(u_k) be sequence in $\mathcal{N}_*^\phi$ such that $J_*(u_k) \to c_*^\phi$. Then, after passing to a subsequence, there exist a nontrivial solution ω to problem* (3.1)*, a sequence (ζ_k) in Ω^Γ and a sequence (λ_k) in $(0,\infty)$ with the following properties:*
(i) $\lambda_k^{-1}\operatorname{dist}(\zeta_k,\partial\Omega) \to \infty$,
(ii) $J_\infty(\omega) = c_\infty^\phi$,
(iii) $\lim_{k\to\infty}\|u_k - \lambda_k^{\frac{2-N}{2}}\omega(\frac{\cdot-\zeta_k}{\lambda_k})\| = 0$.

Proof. By Ekeland's variational principle, we may assume that (u_k) is a Palais–Smale sequence. Then, (u_k) is bounded in $D_0^{1,2}(\Omega)$ and, after passing to a subsequence, $u_k \rightharpoonup u$ weakly in $D_0^{1,2}(\Omega)$. If $u \neq 0$, an easy argument shows that $u \in \mathcal{N}_*^\phi$ and $J_*(u) = c_*^\phi$, contradicting Theorem 3.1. Therefore, $u = 0$.

Fix $\delta \in (0, \frac{N}{2}c_*^\phi)$. Then there exist bounded sequences (λ_k) in $(0,\infty)$ and (ξ_k) in $\mathbb{R}^N$ such that, after passing to a subsequence,

$$\sup_{x\in\mathbb{R}^N}\int_{B_{\lambda_k}(x)}|v_k|^{2^*} = \int_{B_{\lambda_k}(\xi_k)}|v_k|^{2^*} = \delta.$$

For (λ_k) and (ξ_k), we choose K and (ζ_k) as in Lemma 3.2. Then, $\Gamma_{\zeta_k} = K$ and $\operatorname{dist}(\Gamma\xi_k,\zeta_k) < C\lambda_k$ for some positive constant C and all $k \in \mathbb{N}$. Therefore, (ζ_k) is bounded and, as $|v_k|$ is Γ-invariant, we have that

$$\delta = \int_{B_{\lambda_k}(\xi_k)}|v_k|^{2^*} \leq \int_{B_{(C+1)\lambda_k}(\zeta_k)}|v_k|^{2^*}. \tag{3.2}$$

Set $\Omega_k := \{z \in \mathbb{R}^N : \lambda_k z + \zeta_k \in \Omega\}$ and, for $z \in \Omega_k$, define

$$w_k(z) := \lambda_k^{\frac{N-2}{2}} v_k(\lambda_k z + \zeta_k).$$

The sequence (w_k) is bounded in $D^{1,2}(\mathbb{R}^N)$ so, after passing to a subsequence, $w_k \rightharpoonup \omega$ weakly in $D^{1,2}(\mathbb{R}^N)$, $w_k \to \omega$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $w_k \to \omega$ a.e. in $\mathbb{R}^N$. A standard argument, using inequality (3.2), shows that $\omega \neq 0$. Moreover, since $\Gamma_{\zeta_k} = K$, we have that $w_k(\gamma z) = \phi(\gamma) w_k(z)$ for all $\gamma \in K$. Hence,

$$\omega(\gamma z) = \phi(\gamma)\omega(z) \quad \text{for all } \gamma \in K \text{ and all } z \in \mathbb{R}^N. \tag{3.3}$$

Using the fact that the equation $-\Delta u = |u|^{2_N^*-2}u$ does not have a nontrivial solution in a half-space, it is also standard to show, after passing to a subsequence, that $\lambda_k^{-1}\operatorname{dist}(\zeta_k, \partial\Omega) \to \infty$, $\zeta_k \in \Omega$ and ω is a solution to $-\Delta u = |u|^{2_N^*-2}u$ in $\mathbb{R}^N$.

Now, if $\zeta_k \notin \Omega^\Gamma$, then $\dim(\Gamma\zeta_k) > 0$, and hence $|\Gamma/K| = \infty$. But then, for any given $m \in \mathbb{N}$, statement (d) of Lemma 3.2 allows us to choose m elements $\alpha_1, \alpha_2, \dots, \alpha_m \in \Gamma$ such that $\lambda_k^{-1}|\alpha_j\zeta_k - \alpha_i\zeta_k| \to \infty$ for $i \neq j$. Arguing as in the proof of [6, inequality (2.7)], we obtain that

$$c_*^\phi \geq m\frac{1}{N}\|\omega\|^2.$$

This is a contradiction. Therefore, $\zeta_k \in \Omega^\Gamma$ and $K = \Gamma$. From (3.3), we conclude that ω is a nontrivial solution to (3.1).

Note that

$$w_k\Big(y - \frac{\zeta_k}{\lambda_k}\Big) = \lambda_k^{\frac{N-2}{2}} u_k(\lambda_k y).$$

Since $w_k \rightharpoonup \omega$ weakly in $D^{1,2}(\mathbb{R}^N)$, we have that

$$\|w_k\|^2 = \|w_k - \omega\|^2 + \|\omega\|^2 + o(1)$$

and, performing the change of variable $z = y - \frac{\zeta_k}{\lambda_k}$, we obtain

$$Nc_*^\phi = \lim_{k\to\infty}\|u_k\|^2 = \lim_{k\to\infty}\Big\|u_k - \lambda_k^{\frac{2-N}{2}}\omega\Big(\frac{\cdot - \zeta_k}{\lambda_k}\Big)\Big\|^2 + \|\omega\|^2 \geq \|\omega\|^2 \geq Nc_\infty^\phi.$$

From Theorem 3.1, it follows that

$$\lim_{k\to\infty}\Big\|u_k - \lambda_k^{\frac{2-N}{2}}\omega\Big(\frac{\cdot - \zeta_k}{\lambda_k}\Big)\Big\|^2 = 0$$

and $J_\infty(\omega) = \frac{1}{N}\|\omega\|^2 = c_\infty^\phi$, as claimed. □

Theorem 3.3 asserts, in particular, the existence of a least energy nontrivial solution ω to problem (3.1). If $\phi \equiv 1$, ω is simply the standard bubble. On the other hand, if $\phi\colon \Gamma \to \mathbb{Z}/2$ is surjective, then ω is nonradial and sign-changing. Solutions of this type were recently exhibited in [6].

4 Minimizers of Critical and Slightly Supercritical Problems

Throughout this section, we continue to assume that $\Omega \setminus \Omega^\Gamma$ and Ω^Γ are nonempty, and that every Γ-orbit in $\Omega \setminus \Omega^\Gamma$ has positive dimension. Set

$$d := \min\{\dim(\Gamma x) : x \in \Omega \setminus \Omega^\Gamma\}.$$

Note that $d \geq 1$. Hence, $2_{N-d}^* > 2_N^*$. Set

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \Omega^\Gamma) > \delta\},$$

and fix $\delta_0 > 0$ so that $\Omega_{\delta_0} \neq \emptyset$.

For $\delta \in (0, \delta_0)$ and $\varepsilon \in [0, 2^*_{N-d} - 2^*_N)$ we consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*_N - 2 + \varepsilon} u & \text{in } \Omega_\delta, \\ u = 0 & \text{on } \partial\Omega_\delta, \\ u(\gamma x) = \phi(\gamma) u(x) & \text{for all } \gamma \in \Gamma,\ x \in \Omega_\delta. \end{cases} \tag{4.1}$$

This problem is critical for $\varepsilon = 0$ and supercritical for $\varepsilon > 0$. We write $J_{\delta,\varepsilon} : D_0^{1,2}(\Omega_\delta)^\phi \to \mathbb{R}$ for its associated energy functional,

$$\mathcal{N}^\phi_{\delta,\varepsilon} := \left\{ u \in D_0^{1,2}(\Omega_\delta)^\phi : u \neq 0,\ \|u\|^2 = |u|^{2^*_N+\varepsilon}_{2^*_N+\varepsilon} \right\}$$

for its Nehari manifold, and set

$$c^\phi_{\delta,\varepsilon} := \inf_{u \in \mathcal{N}^\phi_{\delta,\varepsilon}} J_{\delta,\varepsilon}(u).$$

Extending each function in $\mathcal{N}^\phi_{\delta,0}$ by 0 outside Ω_δ, we have that $\mathcal{N}^\phi_{\delta,0} \subset \mathcal{N}^\phi_*$ and $J_{\delta,0}(u) = J_*(u)$ for every $u \in \mathcal{N}^\phi_{\delta,0}$, where $\mathcal{N}^\phi_*$ and J_* are the Nehari manifold and the energy functional associated to the critical problem in the whole domain Ω, as in the previous section. Hence, $c^\phi_* \leq c^\phi_{\delta,0}$.

Lemma 4.1. *We have that $c^\phi_{\delta,0} \to c^\phi_*$ as $\delta \to 0$.*

Proof. For each $\eta > 0$ there exists $\psi \in \mathcal{N}^\phi_* \cap \mathcal{C}^\infty_c(\Omega)$ such that $J_*(\psi) < c^\phi_* + \frac{\eta}{2}$. Let $V := (\mathbb{R}^N)^\Gamma$ be the space of Γ-fixed points in $\mathbb{R}^N$ and $W := V^\perp$ be its orthogonal complement. Our symmetry assumptions on Ω imply that $\dim(W) \geq 2$. Hence, there are radial functions $\chi_k \in \mathcal{C}^\infty_c(W)$ such that $\chi_k(y) = 1$ if $|y| \leq \frac{1}{k}$, $\chi_k(y) = 0$ if $|y| \geq \frac{2}{k}$ and

$$\lim_{k\to\infty} \int_W |\nabla\chi_k(y)|^2 \, dy = 0.$$

For $(x, y) \in V \times W$, set $\psi_k(x, y) := (1 - \chi_k(y))\psi(x, y)$. Then, $\operatorname{supp}(\psi_k) \subset \Omega_\delta$ if $\delta < \frac{1}{k}$. Note that, since χ_k is radial, we have

$$\begin{aligned} \psi_k(\gamma(x, y)) &= \psi_k(x, \gamma y) = (1 - \chi_k(\gamma y))\psi(x, \gamma y) \\ &= (1 - \chi_k(y))\psi(\gamma(x, y)) = \phi(\gamma)(1 - \chi_k(y))\psi(x, y) \\ &= \phi(\gamma)\psi_k(x, y) \quad \text{for all } \gamma \in \Gamma. \end{aligned}$$

Therefore, $\psi_k \in D_0^{1,2}(\Omega_\delta)^\phi$ if $\delta < \frac{1}{k}$. Moreover, as

$$\|\psi - \psi_k\|^2 = \int_\Omega |\nabla(\chi_k\psi)|^2 \leq C_1\Bigg[\int_{\Omega\setminus\Omega_{2/k}} \chi_k^2|\nabla\psi|^2 + \int_\Omega \psi^2|\nabla\chi_k|^2\Bigg] \leq C_2\Bigg[|\Omega \setminus \Omega_{2/k}| + \int_W |\nabla\chi_k|^2\Bigg],$$

we have that $\psi_k \to \psi$ in $D_0^{1,2}(\Omega)$. Let $t_k \in (0, \infty)$ be such that $\widetilde{\psi}_k := t_k\psi_k \in \mathcal{N}^\phi_*$. Clearly, $\widetilde{\psi}_k \to \psi$ in $D_0^{1,2}(\Omega)$. Hence, $J_*(\widetilde{\psi}_k) < c^\phi_* + \eta$ for k large enough. Choosing k with this property and $\delta < \frac{1}{k}$, we conclude that $\widetilde{\psi}_k \in \mathcal{N}^\phi_{\delta,0}$ and $c^\phi_{\delta,0} \leq J_*(\widetilde{\psi}_k) < c^\phi_* + \eta$. This finishes the proof. □

Theorem 2.3 asserts that, for $\delta \in (0, \delta_0)$ and $\varepsilon \in [0, 2^*_{N-d} - 2^*_N)$, there exists a solution of (4.1) such that $J_{\delta,\varepsilon}(u_{\delta,\varepsilon}) = c^\phi_{\delta,\varepsilon}$. The following results describe the asymptotic profile of these solutions, in the critical and the supercritical case, as $\delta \to 0$ and $\varepsilon \to 0$.

Theorem 4.2. *Let u_δ be a solution to the critical problem* (4.1), $\varepsilon = 0$, *such that $J_{\delta,0}(u_\delta) = c^\phi_{\delta,0}$. Then there exist sequences (δ_k) and (λ_k) in $(0, \infty)$, a sequence (ζ_k) in Ω^Γ and a nontrivial solution ω to*

$$\Delta u = |u|^{2^*_N-2}u, \quad u \in D^{1,2}(\mathbb{R}^N)^\phi, \tag{4.2}$$

with the following properties:
(i) $\delta_k \to 0$,
(ii) $\lambda_k^{-1} \operatorname{dist}(\zeta_k, \partial\Omega) \to \infty$,
(iii) $J_\infty(\omega) = c^\phi_\infty$,
(iv) $\lim_{k\to\infty} \|u_{\delta_k} - \lambda_k^{\frac{2-N}{2}} \omega(\frac{\cdot-\zeta_k}{\lambda_k})\| = 0$.

Proof. Since $u_\delta \in \mathcal{N}^\phi_{\delta,0} \subset \mathcal{N}^\phi_*$ and, by Lemma 4.1, we have $J_*(u_\delta) \to c^\phi_*$ as $\delta \to 0$, the result follows from Theorem 3.3. □

Theorem 4.3. *For $\varepsilon \in (0, 2^*_{N-d} - 2^*_N)$, assume that $u_{\delta,\varepsilon}$ is a solution to the supercritical problem* (4.1) *such that $J_{\delta,\varepsilon}(u_{\delta,\varepsilon}) = c^\phi_{\delta,\varepsilon}$. Then there exist sequences (δ_k), (ε_k) and (λ_k) in $(0,\infty)$, a sequence (ζ_k) in Ω^Γ and a nontrivial solution ω to problem* (4.2) *with the following properties:*
(i) $\delta_k \to 0$, $\varepsilon_k \to 0$,
(ii) $\lambda_k^{-1}\,\mathrm{dist}(\zeta_k, \partial\Omega) \to \infty$,
(iii) $J_\infty(\omega) = c^\phi_\infty$,
(iv) $\lim_{k\to\infty} \|u_{\delta_k,\varepsilon_k} - \lambda_k^{\frac{2-N}{2}}\,\omega(\frac{\cdot - \zeta_k}{\lambda_k})\| = 0$.

Proof. Let $t_{\delta,\varepsilon} \in (0,\infty)$ be such that $\tilde{u}_{\delta,\varepsilon} := t_{\delta,\varepsilon}u_{\delta,\varepsilon} \in \mathcal{N}^\phi_{\delta,0} \subset \mathcal{N}^\phi_*$. Proposition 2.5 allows us to choose $\varepsilon(\delta)$ in $(0, 2^*_{N-d} - 2^*_N)$ with $\varepsilon(\delta) \to 0$ as $\delta \to 0$, such that $\tilde{u}_\delta := \tilde{u}_{\delta,\varepsilon(\delta)}$ satisfies

$$c^\phi_* \le J_*(\tilde{u}_\delta) = J_{\delta,0}(\tilde{u}_\delta) \le c^\phi_{\delta,0} + \delta.$$

By Lemma 4.1, we have that $J_*(\tilde{u}_\delta) \to c^\phi_*$ as $\delta \to 0$. The result now follows from Theorem 3.3. □

Proof of Theorem 1.2. This result is a special case of Theorem 4.3 applied to the ball $\Omega := \{x \in \mathbb{R}^N : |x| \le R\}$, with the action of the group Γ introduced in the proof of Theorem 1.1. Then $\Omega^\Gamma := \{0\}$ and every Γ-orbit in $\Omega \setminus \Omega^\Gamma$ has positive dimension if $N = 4$ and $N \ge 6$. □

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