Research Article

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Positive Solutions of Elliptic Kirchhoff Equations

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Abstract: We prove several existence results for some nonlinear elliptic Kirchhoff equations.

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Dedicated to the memory of our colleague and friend Abbas Bahri

1 Introduction

Several recent papers have been devoted to study boundary value problems like

$$\begin{cases} -M\Big(\int\limits_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

under appropriate hypotheses on f and M. See e.g. [1, 9, 10, 12–15] which contain further references.

When $n = 1$, $\Omega = (0, L)$, $M(s) = 1 + s$ and $\lambda f(x, u) = f(x)$, solutions of (1.1) are the stationary states of the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \Big(1 + \int\limits_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = f, \quad u(0,t) = u(L,t) = 0,$$

introduced by Kirchhoff in [16] to describe the small transversal oscillations of an elastic clamped string which is not homogeneous. For this reason (1.1) is often called the elliptic Kirchhoff equation.

We will show that, in spite of the presence of the non-local coefficient $M(\int_\Omega |\nabla u|^2\,dx)$, the Kirchhoff equations do not require special tools, but can be handled by standard arguments of nonlinear functional analysis.

First, in Sections 2 and 3 we deal with asymptotically linear f and use global bifurcation theory to prove the existence of branches of positive solutions. The main results we find are stated in Theorems 2.6, 3.1 and 3.3 and mainly depend on the interplay between the behavior of f and M at zero and infinity. More precisely, in Theorem 2.6, we assume that $M(t) > 0$ for $t \geq 0$ and $\lim_{t\to+\infty} M(t) = M_\infty > 0$ or else $\lim_{t\to+\infty} M(t) = +\infty$. In Theorems 3.1 and 3.3, we allow the coefficient M to vanish at zero and/or at infinity. Let us point out that these degenerate cases are not covered in any of the aforementioned papers on elliptic Kirchhoff equations.

In Section 4, we consider superlinear f. Roughly, we take $M(t) \sim 1 + \gamma t^{q-1}$, $f(u) \sim u^{p-1}$, with $2 < p < 2^*$ and $q \geq 1$ and use critical point theory. In this case the existence results depend upon the parameter γ and the relationship between p and q. In Theorem 4.1, we will show that if $2q < p$, then (1.1) has a positive solution for all $\gamma > 0$, while, if $2q \geq p$, then solutions exist provided γ is small enough. More precisely, if $\gamma \ll 1$ and

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$2q > p$, then (1.1) has at least two positive solutions, see Theorem 4.4. It is worth mentioning that these results are sharp, in the sense that if $2q \geq p$ and $\gamma \gg 1$, then (1.1) can have the trivial solution $u = 0$ only, see Remark 4.5.

In the final Section 5, we shortly discuss a perturbative problem on all of $\mathbb{R}^n$.

Notation. In the sequel we will use the following notation:

- If u, v belong to the Sobolev space $H_0^1(\Omega)$, then
$$(u, v) = \int_\Omega \nabla u \nabla v dx \quad \text{and} \quad \|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}$$
denote the scalar product and norm, respectively. If $r > 0$, we set
$$B_r = \{u \in E : \|u\| < r\} \quad \text{and} \quad \overline{B}_r = \{u \in E : \|u\| \leq r\}.$$
- $F(u)$ denotes the Nemitski operator associated to the function $f(x, u)$.
- If for some $r > N/2$, $q \in L^r(\Omega)$, $q(x) > 0$, we let $\lambda_m[q]$ $(m = 1, 2, \ldots)$ denote the eigenvalues of $-\Delta u = \lambda q(x)u$ in Ω, $u_{|\partial\Omega} = 0$.

2 Global Branch of Positive Solutions

In the sequel, to avoid technicalities, it is always understood that $M : \mathbb{R}^+ \mapsto \mathbb{R}$ and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ are smooth.

Even if we can handle other nonlinearities, see the end of the section, we will focus on the case in which f is asymptotically linear. Precisely, the following specific assumptions are in order.

(f1) $f(x, 0) = 0$ for all $x \in \Omega$, $a(x) := D_u f(x, 0) > 0$.

(f2) $f(x, u) = b(x)u + g(x, u)$, where $b(x) > 0$ and $\lim_{u\to+\infty} g(x, u)/u = 0$.

Moreover, on M we assume:

(M1) There exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$.

Let $E = H_0^1(\Omega)$ and let us consider the Green function $L = (-\Delta)^{-1}$, i.e. the linear operator defined as the unique (weak) solution $Lv \in E$ for every $v \in L^2(\Omega)$. Since $M(\|u\|^2) > 0$, it makes sense to define the operator $T : E \to E$ by setting
$$T(u) = \frac{LF(u)}{M(\|u\|^2)}, \quad u \in E.$$
With this notation, we see that the solutions of
$$u = \lambda T(u), \quad u \in E \tag{2.1}$$
are weak and, by regularity, classical solutions of (1.1).

Remark 2.1. From (f2) it follows that the Nemitski operator F is continuous from $L^2(\Omega)$ into itself (see e.g. [2, Theorem 1.2.1]). Moreover, taking into account that E is compactly embedded into $L^2(\Omega)$, we infer that the restriction of $L \circ F$ to E is compact as a map from E into itself. In addition, if $u_k \in E$ is any weakly convergent sequence, one has that $0 < c_1 \leq M(\|u_k\|^2) \leq c_2$ and hence, up to a subsequence, $M(\|u_k\|^2)$ converges to a positive value. Therefore, T is a continuous, compact operator in E.

Remark 2.2. The linearized equation of (2.1) at $u = 0$ gives rise to
$$-M(0)\Delta v = \lambda a(x)v \quad \text{in } \Omega, \quad v_{|\partial\Omega} = 0. \tag{2.2}$$
The eigenvalues of (2.2) are $M(0)\lambda_m[a]$ and have the same multiplicity of $\lambda_m[a]$ and the same eigenfunctions. In particular, $\Lambda_1 := M(0)\lambda_1[a]$ is simple with eigenfunctions that do not change sign. In the sequel we will denote by ϕ the eigenfunction associated to ($\lambda_1[a]$ and) Λ_1 such that $\phi > 0$, normalized by $\|\phi\| = 1$.

Let Σ denote the closure of all solutions $(\lambda, u) \in \mathbb{R} \times E$ of (1.1) with $u \neq 0$. A subset Γ of Σ is called a *branch of positive solutions* if Γ is a closed connected subset of Σ maximal with respect to the inclusion and such that if $(\lambda, u) \in \Gamma$ and $u \neq 0$ then $u > 0$ in Ω.

Starting with the bifurcation from the trivial solution, we show:

Lemma 2.3. *If* (f1), (f2) *and* (M1) *hold, then* $\Lambda_1 = M(0)\lambda_1[a]$ *is a bifurcation point of positive solutions for* (1.1)*, and the only one. Furthermore, an unbounded branch* $\Gamma \subset (0, +\infty) \times E$ *of positive solutions emanates from* $(\Lambda_1, 0)$.

Proof. The arguments are rather standard and are outlined below for completeness and for the reader's convenience.

In the previous Remarks 2.1 and 2.2 we have pointed out that T is compact and $\lambda = \Lambda_1$ is a simple eigenvalue. Then we can apply the Rabinowitz global bifurcation result [17, Theorem 1.3] to $u - \lambda T(u)$ yielding a branch $\mathcal{C} \subset \Sigma$ emanating from $(\Lambda_1, 0)$. More precisely, according to [11, Theorem 2], there exist

$$\mathcal{C}^+ \subseteq \Big\{u \in \mathcal{C} : \int_\Omega u\phi \geq 0\Big\}, \quad \mathcal{C}^- \subseteq \Big\{u \in \mathcal{C} : \int_\Omega u\phi \leq 0\Big\},$$

where $\phi > 0$ is the eigenfunction introduced in Remark 2.2 before, such that

(i) $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ and $(\Lambda_1, 0) \in \mathcal{C}^+ \cap \mathcal{C}^-$,

(ii) $\mathcal{C}^+$ and $\mathcal{C}^-$ are closed and connected,

(iii) either $\mathcal{C}^+$ and $\mathcal{C}^-$ are both unbounded or $\mathcal{C}^+ \cap \mathcal{C}^-$ meets another bifurcation point from the trivial solution of (2.1).

We follow now the arguments of [7, 8]. Observe that if $(\lambda_n, u_n) \in \mathcal{C}^+$, $u_n \neq 0$, is a sequence converging to $(\Lambda_1, 0)$, then the normalized sequence $z_n := u_n/\|u_n\|$ satisfies $z_n = \lambda_n T(z_n)$, namely

$$-M(\|u_n\|^2)\Delta z_n = \lambda_n \frac{f(x, u_n)}{\|u_n\|}. \tag{2.3}$$

It follows that z_n converges to some z in E and, by regularity, in $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$ such that $\|z\| = 1$. Passing to the limit in (2.3), taking into account that $u_n \to 0$ and using (f1), we deduce that z satisfies

$$-M(0)\Delta z = \Lambda_1 a(x) z = M(0)\lambda_1[a]a(x)z.$$

Recalling that $\int_\Omega z_n\phi \geq 0$, we infer that $z = \phi$. In particular, z belongs to $\mathring{P}$, the interior of the cone P in $C_0^1(\overline{\Omega})$ of the nonnegative functions. Thus we find that (z_n and hence) u_n also belong to $\mathring{P}$ for $n \gg 1$. We claim that $u \in \mathring{P}$ for every $(\lambda, u) \in \mathcal{C}^+ \setminus \{(\Lambda_1, 0)\}$. Otherwise, by the connectedness of $\mathcal{C}^+$, there exists $(\Lambda, u_0) \in \mathcal{C}^+ \setminus \{(\Lambda_1, 0)\}$ with $u_0 \in \partial P$. The strong maximum principle implies that $u_0 = 0$ and thus $\mathcal{C}^+$ contains a bifurcation point from zero $(\Lambda, 0)$ with $\Lambda \neq \Lambda_1$. According to Remark 2.2 we have that $\Lambda = M(0)\lambda_k[a]$, for some $k \geq 2$. As before, if we consider now a sequence $(\lambda_n, u_n) \in \mathcal{C}^+$, $u_n \neq 0$, converging to $(\Lambda, 0) = (M(0)\lambda_k[a], 0)$, then we can prove that the normalized sequence $u_n/\|u_n\|$ converges to an eigenfunction associated to $\lambda_k[a]$. Since $k \geq 2$, this eigenfunction has to change the sign. This is in contradiction with the fact that $(\Lambda, 0) \in \mathcal{C}^+ \cap \partial P$, proving the claim. Notice that the preceding argument also proves that $(\Lambda_1, 0)$ is the only bifurcation point from zero of positive solutions and that, according to (iii), $\mathcal{C}^+$ is unbounded. Moreover, since $M(\|u\|^2) > 0$, then $M(\|u\|^2)\Delta u = 0$ has only the trivial solution. Therefore $\mathcal{C}^+$ cannot cross the axis $\lambda = 0$ and hence $\mathcal{C}^+ \subset (0, +\infty) \times E$.

To complete the proof it suffices to take $\Gamma = \mathcal{C}^+$. □

Remark 2.4. To show that Λ_1 is merely a bifurcation point for (1.1), it suffices to assume (f1) and $M(0) > 0$. Actually, we can apply the theory of bifurcation from a simple eigenvalue (see e.g. [2, Chapter 6]) to the equation $M(\|u\|^2)\Delta u + \lambda F(u) = 0$, $u \in C_0^2(\Omega)$, which is differentiable with respect to u at $u = 0$ with derivative $v \mapsto M(0)\Delta v + \lambda a(x)v$.

Next we deal with the bifurcation from infinity. By definition, we say that $\lambda^* \in \mathbb{R}$ is a bifurcation point from infinity for (1.1) if there exists a sequence $(\lambda_k, u_k) \in \mathbb{R} \times C_0^2(\Omega)$ with

$$\lambda_k \to \lambda^*, \quad \|u_k\| \to +\infty,$$

such that u_k is a solution of (1.1) with $\lambda = \lambda_k$.

Here we suppose the following:

(M∞) There exists $M_\infty > 0$ such that $\lim_{t\to+\infty} M(t) = M_\infty$.

Lemma 2.5. *Suppose that* (f2) *and* (M∞) *hold. Then* $\Lambda_\infty = M_\infty \lambda_1[b]$ *is a bifurcation point from infinity of positive solutions to* (1.1)*, and the only one. Furthermore, a branch of positive solutions emanates from* (Λ_∞, ∞).

Proof. Using a standard procedure, we set $z = u/\|u\|^2$ and transform (1.1) into

$$z = \lambda \|z\|^2 T\Big(\frac{z}{\|z\|^2}\Big), \quad u \in E$$

equivalent to the equation

$$-M\Big(\frac{1}{\|z\|^2}\Big)\Delta z = \lambda \|z\|^2 f\Big(x, \frac{z}{\|z\|^2}\Big) = \lambda\Big[b(x)z + \|z\|^2 g\Big(x, \frac{z}{\|z\|^2}\Big)\Big]. \tag{2.4}$$

Arguing as in Remark 2.2, we see that the linearization at $z = 0$ is

$$-M_\infty \Delta v = \lambda b(x) v \quad \text{in } \Omega, \quad v_{|\partial\Omega} = 0,$$

whose first eigenvalue is $M_\infty \lambda_1[b]$. Since the bifurcation points from the trivial solution of (2.4) correspond to the bifurcation from infinity of (1.1), the lemma follows. □

The next result provides different global bifurcations depending on the behavior of the function M. We say that the branch Γ of positive solutions meets $(\mu, +\infty)$ if there is a sequence $(\lambda_k, u_k) \in \Gamma$ such that $\lambda_k \to \mu$ and $\|u_k\| \to +\infty$.

Theorem 2.6. *Let* (f1), (f2) *and* (M1) *hold and consider the branch* Γ *of positive solutions of* (1.1) *bifurcating from* $(\Lambda_1, 0)$.

(i) *If condition* (M∞) *is also satisfied and*

(f3) *there exists* $\kappa > 0$ *such that* $f(x,u) \ge \kappa u$ *for all* $(x,u) \in \Omega \times \mathbb{R}^+$,

then Γ *meets* $(\Lambda_\infty, +\infty)$.

(ii) *If* $\lim_{t\to\infty} M(t) = \infty$*, then the projection of* Γ *on the* λ *axis is an unbounded interval* $[\ell, +\infty)$ *with* $\ell \le \Lambda_1$.

Proof. (i) By Lemmas 2.3 and 2.5, we know that there is an unbounded branch Γ bifurcating from zero at $(\Lambda_1, 0)$ as well as a branch bifurcating from infinity at (Λ_∞, ∞). We recall that $(\lambda, u) \in \Gamma$ implies $\lambda > 0$ (see Lemma 2.3). Thus, to conclude the proof of (i), it suffices to prove the nonexistence of positive solutions of (1.1) for $\lambda \gg 1$. Actually, if (λ, u) is a positive solution of (1.1), one has

$$\lambda \int_\Omega \phi f(x,u) = -M(\|u\|^2) \int_\Omega \nabla\phi \nabla u = \lambda_1 M(\|u\|^2) \int_\Omega \phi u,$$

where $\lambda_1 = \lambda_1[1]$ and ϕ stands for the corresponding positive eigenfunction. By hypotheses (M∞) and (M1) the function M is bounded from above. Letting $\overline{M} = \sup_{t\ge 0} M$, we get

$$\lambda \int_\Omega \phi f(x,u) \le \lambda_1 \overline{M} \int_\Omega \phi u.$$

Then, using (f3), it follows

$$\lambda\kappa \int_\Omega \phi u \le \lambda \int_\Omega \phi f(x,u) \le \lambda_1 \overline{M} \int_\Omega \phi u,$$

with the first inequality strict unless $f(x, u) \equiv \kappa u$. This implies

$$\lambda \leq \frac{\lambda_1 \overline{M}}{\kappa}, \tag{2.5}$$

with strict inequality unless $f(x, u) \equiv \kappa u$.

(ii) We still have an unbounded branch Γ of positive solutions of (1.1) bifurcating from $(\Lambda_1, 0)$. We claim that if $\lim_{t\to+\infty} M(t) = +\infty$ then there is no bifurcation from infinity. For this, we will show an a-priori estimate for every positive solution u of (1.1).

Let $\lambda \geq 0$ and assume that $u \in H_0^1(\Omega)$ is a positive solution of (1.1) for this λ. Taking u as a test function in (1.1), we obtain

$$M(\|u\|^2)\|u\|^2 = \lambda \int_\Omega f(x, u)u.$$

Using (f2), there exists $K > 0$ such that $f(x, u) \leq Ku$ for every $u \geq 0$, and thus

$$M(\|u\|^2)\|u\|^2 \leq \lambda K \int_\Omega u^2 \leq \frac{\lambda K}{\lambda_1[a]}\|u\|^2,$$

i.e.

$$M(\|u\|^2) \leq \frac{\lambda K}{\lambda_1[a]}.$$

Since $\lim_{t\to\infty} M(t) = \infty$, it follows that there is $C_\lambda > 0$ such that $\|u\|^2 \leq C_\lambda$. This completes the proof of the claim and thus of (ii). □

In the following corollary we state explicitly some straight consequences of Theorem 2.6.

Corollary 2.7. *Assume* (f1), (f2) *and* (M1).

(i) *If conditions* (M∞) *and* (f3) *hold and* $\Lambda_1 < \Lambda_\infty$, *resp.* $\Lambda_1 > \Lambda_\infty$, *then* (1.1) *has a positive solution for all* $\lambda \in (\Lambda_1, \Lambda_\infty)$, *resp.* $\lambda \in (\Lambda_\infty, \Lambda_1)$.

(ii) *If* $\lim_{t\to\infty} M(t) = \infty$, *then there exists a positive solution of* (1.1) *for every* $\lambda > \Lambda_1$.

Two examples are shown in Figure 1 (in this and in the next figures the bifurcation diagrams are suggestive only).

It is clear that, depending on the behavior of Γ near $(\Lambda_1, 0)$ or $(\Lambda_\infty, +\infty)$, one finds multiple positive solutions. A particular case is reported in the next example in the case that $\Lambda_1 = \Lambda_\infty$.

Example 2.8. Let (f1), (f2), (f3), (M1) and (M∞) hold and suppose also that $M(0)\lambda_1[a] = M_\infty\lambda_1[b]$ so that $\Lambda_1 = \Lambda_\infty$. Furthermore, if in (f3) we can take

$$\kappa = \frac{\lambda_1[1]\,\overline{M}}{\lambda_1[b]\,M_\infty},$$

then (2.5) becomes

$$\lambda \leq \frac{\lambda_1[1]\,\overline{M}}{\kappa} = \lambda_1[b]\,M_\infty = \Lambda_\infty\,(= \Lambda_1).$$

In particular, the bifurcations from zero and infinity are both on the left of the bifurcation point, except for the linear case $f(x, u) \equiv \kappa u$. As a consequence, there exists $\lambda^* < \Lambda_1$ such that (1.1) has at least two positive solutions for all $\lambda \in (\lambda^*, \Lambda_1)$. See Figure 2.

As anticipated before, we can also consider other nonlinearities. To be short, we will limit ourselves to sketch two specific cases. Further examples can be discussed in a similar way.

Example 2.9. First of all we consider the case in which the function M satisfies (M1) and (M∞), and the nonlinearity f satisfies (f1), (f2) and the following assumption:

(f3′) There exist $0 < s' < s''$ such that $f(x, u) \leq 0$ for every $u \in [s', s'']$, and $f(x, u) > 0$ provided that $u \in (0, s') \cup (s'', +\infty)$.

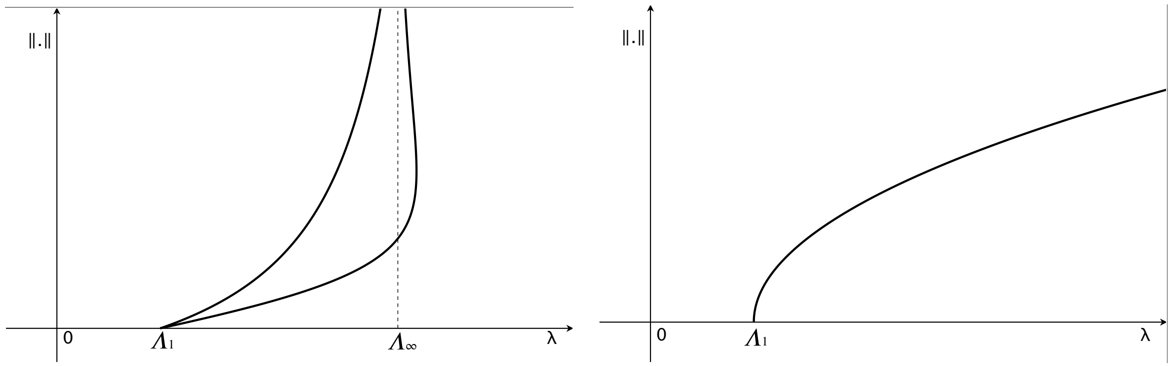

Figure 1. Bifurcation diagrams in the case of Corollary 2.7 (i) with $\Lambda_1 < \Lambda_\infty$ (left) and (ii) (right).

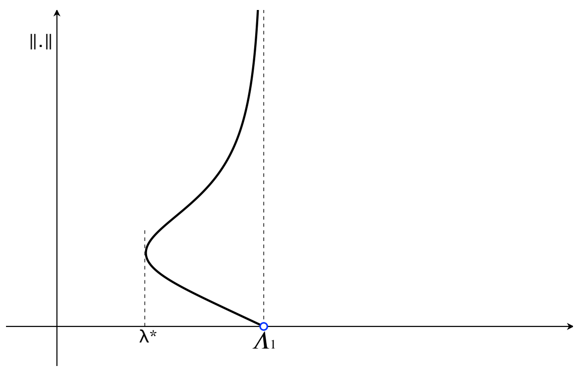

Figure 2. Bifurcation diagram in the case of Example 2.8.

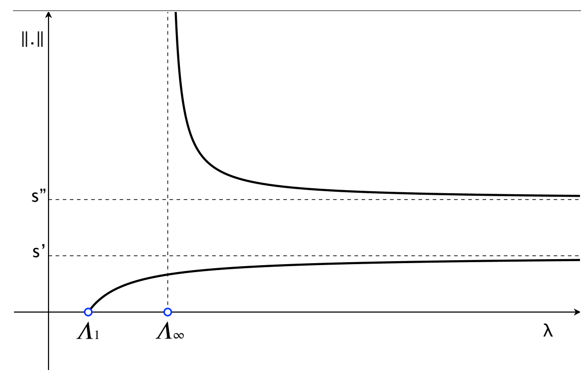

Figure 3. Bifurcation diagram in the case of Example 2.9, with $\Lambda_1 < \Lambda_\infty$.

The maximum principle implies that $\max_\Omega u \notin [s', s'']$ for every positive solution of (1.1). This means that any pair (λ, u) in the branch emanating from $(\Lambda_1, 0)$ satisfies $|u|_\infty = \max u(x) < s'$. Similarly, $|u|_\infty > s''$ for any pair (λ, u) in the branch emanating from $(\Lambda_\infty, 0)$. In particular, the two branches are distinct.

A possible bifurcation diagram in this case is reported in Figure 3, where we have taken $\Lambda_1 < \Lambda_\infty$ and we have considered a supercritical bifurcation, i.e. on the right of the bifurcation point. However, in any case we see that (1.1) has at least two positive solutions for all $\lambda > \max\{\Lambda_1, \Lambda_\infty\}$.

Example 2.10. As a further example, for a function satisfying condition (M1), we consider the case of a logistic nonlinearity $f(x, u) = f(u) \equiv au - u^{p+1}$, i.e., the problem

$$-M(\|u\|^2)\Delta u = \lambda(au - u^{p+1}), \quad u_{|\partial\Omega} = 0, \tag{2.6}$$

where $a > 0$ and $p > 0$.

In this case, $f(u) \leq 0$ for every $u \geq a^{1/p}$ and, as in the previous example, applying the maximum principle, $\max_\Omega u(x) \leq a^{1/p}$ for every positive solution (2.6). Therefore, the projection on the λ axis of the branch of positive solutions to (2.6) emanating from $\Lambda_1 = M(0)\lambda_1[a]$ contains the interval $[\Lambda_1, +\infty)$, and consequently problem (2.6) has at least one positive solution for all $\lambda > \Lambda_1$ (see Figure 4).

Remark 2.11. Since in this case $\max_\Omega u(x) \leq a^{1/p}$, we can weaken (M1) by merely assuming that $M(t) > 0$ for all $0 \leq t \leq a^{1/p}$.

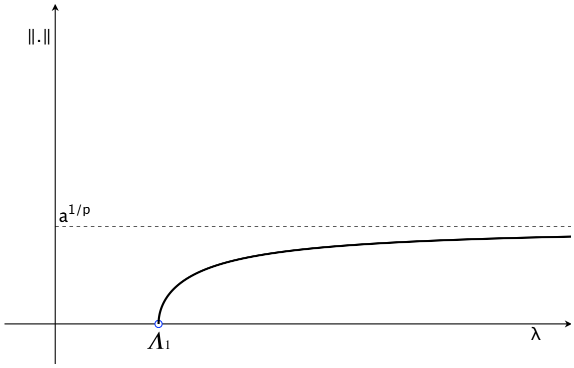

Figure 4. Bifurcation diagram in the case of Example 2.10.

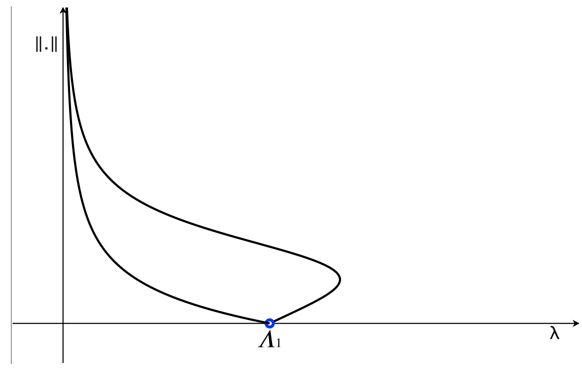

Figure 5. Bifurcation diagram when (M2) holds.

3 Degenerate Coefficients *M*

In this section we extend the previous results to some case in which M vanishes at zero and/or at infinity.

First we study the case of a function M which vanishes at infinity. More precisely, we assume:

(M2) $M(t) > 0$ for every $t \geq 0$ and $\lim_{t\to+\infty} M(t) = 0$.

Theorem 3.1. *Suppose* (f1), (f2), (f3) *and* (M2) *hold. Then there is an unbounded branch* $\overline{\Gamma}$ *of positive solutions of* (1.1) *bifurcating from* $(\Lambda_1, 0)$ *that meets* $(0, +\infty)$ *(see Figure 5). In particular,* (1.1) *has at least one positive solution for all* $0 < \lambda < \Lambda_1$.

Proof. For each integer $k > 0$, let us consider the perturbed equation

$$-\Big(\frac{1}{k} + M(\|u\|^2)\Big)\Delta u = \lambda f(x, u) \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0. \tag{3.1}$$

Since $\frac{1}{k} + M(\|u\|^2) \geq \frac{1}{k} > 0$, we can apply Theorem 2.6 (i) to deduce the existence of a branch Γ_k of positive solutions of (3.1) bifurcating from $(\Lambda_{1,k}, 0)$, where $\Lambda_{1,k} = (\frac{1}{k} + M(0))\lambda_1[a]$, that meets $(\frac{1}{k}, \infty)$. In addition, the projection of Γ_k on the λ axis is a closed interval contained into $(0, \lambda_1(\frac{1}{k} + \overline{M})/\kappa]$ (see (2.5)), and in particular there exists $\varepsilon_k > 0$ such that

$$0 < \varepsilon_k \leq \lambda \leq \Lambda := \frac{\lambda_1(1 + \overline{M})}{\kappa} \quad \text{for all } (\lambda, u) \in \Gamma_k,\ k > 0.$$

We now use the following topological lemma, see e.g. [5, Lemma A.3.1].

Lemma 3.2 (Whyburn). *Let X be a complete metric space and let $Y_k \subset X$ be connected and such that* (i) $\bigcup_{k>0} Y_k$ *is precompact and* (ii) $\liminf_{k\to\infty} Y_k \neq \emptyset$. *Then* $\limsup_{k\to\infty} Y_k$ *is not empty, compact and connected.*

For any $R > 0$ we take $X = \mathbb{R} \times E$, and let $Y_k = Y_k^R$ be the connected component of $\Gamma_k \cap ([0, \Lambda] \times \overline{B}(0, R))$ which contains $(\Lambda_{1,k}, 0)$.

Since the sequence $(\Lambda_{1,k}, 0) \in Y_k^R$ converges to $(\Lambda_1, 0) = (M(0)\lambda_1[a], 0)$ we deduce that

$$(\Lambda_1, 0) \in \liminf_{k\to\infty} Y_k^R \quad \text{and} \quad \liminf_{k\to\infty} Y_k^R \neq \emptyset.$$

On the other hand, if $(\lambda_k, u_k) \in \bigcup_{k>0} Y_k^R$, then there exists an integer $k_* > 0$ such that

$$(\lambda_k, u_k) \in Y_{k_*}^R \subset [\varepsilon_{k_*}, \Lambda] \times \overline{B}(0, R) \quad \text{for all } k \geq k_*.$$

In particular, u_k is bounded and $M(\|u_k\|^2)$ is bounded between positive constants. This, together with the equation

$$-\Big(\frac{1}{k} + M(\|u_k\|^2)\Big)\Delta u_k = \lambda_k f(x, u_k) \quad \text{in } \Omega, \quad (u_k)_{|\partial\Omega} = 0$$

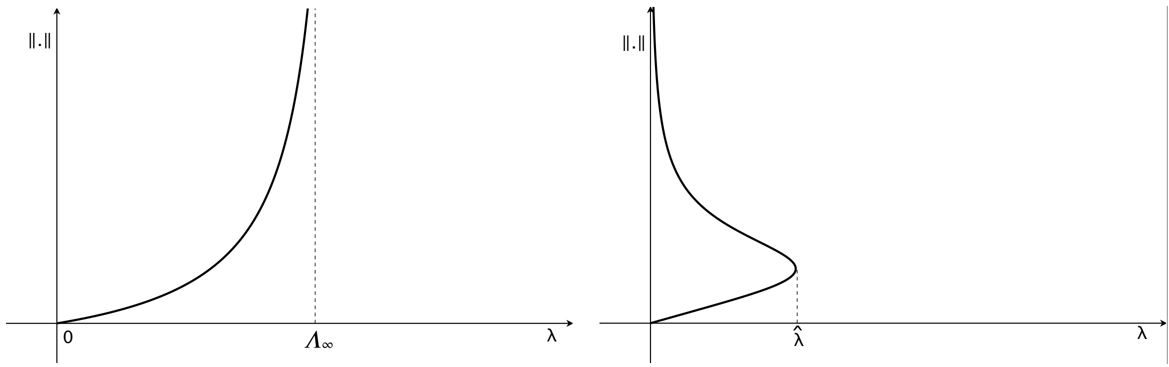

Figure 6. Bifurcation diagrams when $M(0) = 0$ and $M_\infty > 0$ (left) or $M_\infty = 0$ (right).

satisfied by u_k, implies that, up to a sequence, u_k is strongly convergent, proving that $\bigcup_{k>0} Y_k^R$ is precompact. Applying Lemma 3.2, we see that $\widehat{\Gamma}^R = \limsup_{k\to\infty} Y_k^R$ is compact, connected and contains $(\Lambda_1, 0)$. Notice that each branch Γ_k connects $(\Lambda_{1,k}, 0)$ and $(\frac{1}{k}, \infty)$, which are the unique bifurcation points from zero and infinity of (3.1). Thus there are $(\lambda_R, u_R) \in \widehat{\Gamma}^R$ such that

$$\|u_R\| = R \quad \text{and} \quad \lim_{R\to\infty} \lambda_R = 0. \tag{3.2}$$

In particular, $\widehat{\Gamma}^R \setminus \{(\Lambda_1, 0)\} \neq \emptyset$ for all $R > 0$.

In addition, we claim that $\widehat{\Gamma}^R \subset \Sigma$. Indeed, if $(\lambda, u) \in \widehat{\Gamma}^R \setminus \{(\Lambda_1, 0)\}$, then there exists a subsequence $(\lambda_{k_l}, u_{k_l}) \in \Gamma_{k_l}^R$ such that $\lambda_{k_l} \to \lambda$ and $u_{k_l} \to u$. In particular, $\|u_{k_l}\| \le R$ (and thus $\|u\| \le R$), $u_{k_l} > 0$ and

$$-\Delta u_{k_l} = \frac{\lambda_{k_l}}{\tau_{k_l}} f(x, u_{k_l}), \quad (u_{k_l})_{|\partial\Omega} = 0,$$

where

$$\tau_{k_l} = \left(\frac{1}{k_l} + M(\|u_{k_l}\|^2)\right)$$

is converging to $M(\|u\|^2)$. Consequently, as limit of positive solutions, (λ, u) is a solution of (1.1). Furthermore, $u > 0$ because, arguing by contradiction, if $u = 0$, then λ is a bifurcation point from zero of positive solutions, which, recalling that Λ_1 is the unique bifurcation point from zero of positive solutions of (1.1), implies that $\lambda = \Lambda_1$; i.e., $(\lambda, u) = (\Lambda_1, 0)$. This is a contradiction which proves that $u > 0$ and thus the claim.

It follows that the set $\bigcup_{R>0} \widehat{\Gamma}^R \subset \Sigma$ is a connected set that contains $(\Lambda_1, 0)$. Moreover, using that $\lambda_R \to 0$ and $\|u_R\| \to +\infty$ (see (3.2)), we find that $\bigcup_{R>0} \widehat{\Gamma}^R$ is unbounded and meets $(0, \infty)$.

Therefore, the branch we are searching is nothing but the component $\widehat{\Gamma}$ of Σ that contains $(\Lambda_1, 0)$.

Finally, as before we can show that every $(\lambda, u) \in \widehat{\Gamma} \setminus \{(\Lambda_1, 0)\}$ is a positive solution of (1.1). This completes the proof. □

The next result deals with the case in which $M(0) = 0$. See Figure 6 for an example of bifurcation diagrams.

Theorem 3.3. *Suppose* (f1), (f2), (f3) *hold. Assume also that* $M(0) = 0$, $M(t) > 0$ *for all* $t > 0$, *and that there exists* $M_\infty \ge 0$ *such that* $\lim_{t\to+\infty} M(t) = M_\infty$. *Then there is a branch* $\widetilde{\Gamma}$ *of positive solutions of* (1.1) *emanating from* $(0, 0)$ *which meets* $(M_\infty \lambda_1[b], \infty)$. *In particular,*

(i) *if* $M_\infty > 0$, *then* (1.1) *has at least one positive solution for all* $0 < \lambda < M_\infty \lambda_1[b]$;

(ii) *if* $M_\infty = 0$, *then there exists* $\hat{\lambda} > 0$ *such that* (1.1) *has at least two positive solutions for all* $0 < \lambda < \hat{\lambda}$.

Proof. The proof is based on the same approximation argument used in the previous theorem. The only difference is that, when we prove the compactness of the sequences $(\lambda_k, u_k) \in \bigcup_{k>0} Y_k^R$, two cases can occur: either, as before, $\|u_k\| \ge c > 0$, or $\|u_k\| \to 0$ (up to a subsequence). In both cases, u_k converges and hence $\bigcup_{k>0} Y_k^R$ is precompact. □

4 Problems with Superlinear f: A Variational Approach

Setting $M(t) = 1 + \gamma G'(t)$, we deal here with the problem

$$\begin{cases} -(1+\gamma G'(\|u\|^2))\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\gamma > 0$ is a parameter,

$$2 < p < 2^* = \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3, \\ +\infty & \text{if } n = 1, 2, \end{cases} \tag{4.2}$$

and G (is smooth and) satisfies:

(G0) $G(0) = G'(0) = 0$ and $G'(t) \geq 0$ for all $t > 0$.

(G1) There exist $q \geq 1$ and $C > 0$ such that $(0 \leq)\ G(t) \leq C\,t^q$ for all $t \geq 0$.

(G2) $G(t) \geq \frac{2}{p} G'(t)t$ for all $t \geq 0$.

Notice that if $G(t) = t^q/q$, then (G2) implies $2q \leq p$.

Solutions of (4.1) are the stationary points on $E = H_0^1(\Omega)$ of the functional

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{\gamma}{2} G(\|u\|^2) - \frac{1}{p}\int_\Omega |u|^p,$$

which is well defined and smooth because $2 < p < 2^*$.

It is convenient to consider separately the cases $2q \leq p$ and $2q > p$. Let

$$S(p, \Omega) = \max_{u \in E\setminus\{0\}} \frac{\int_\Omega |u|^p}{\|u\|^p}$$

denote the best Sobolev constant of the embedding of $H^1(\Omega)$ into $L^p(\Omega)$.

Using the mountain pass theorem, we prove that the following result holds true.

Theorem 4.1. *Suppose that* (4.2) *and* (G0), (G1), (G2) *hold. Then* (4.1) *has a positive solution provided that either the condition $2q < p$, or the conditions $2q = p$, $C = \frac{1}{q}$ in* (G1) *and $\gamma < S(3q, \Omega)$ hold.*

To prove Theorem 4.1, two lemmas are in order. Let us underline that assumption (G2) is not used in the first one.

Lemma 4.2. *J satisfies the mountain pass geometry, namely $J(0) = 0$, J has a local minimum at $u = 0$ and there exists $e \in E$ such that $J(e) < 0$.*

Proof. Since $G'(t) \geq 0$, we have $G(t) \geq G(0) = 0$. This implies that

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

It is well known that the functional on the right-hand side has a local minimum at zero and therefore the same holds for J.

Furthermore, let $\bar{z} \in E$ be such that $\bar{z} > 0$ and $\|\bar{z}\| = 1$. We prove that we can choose $e = t\bar{z}$ for some $t > 0$. For $t \geq 0$ we find

$$J(t\bar{z}) = \frac{1}{2}t^2 + \frac{\gamma}{2}G(t^2) - \frac{1}{p}t^p\int_\Omega \bar{z}^p.$$

Since $G(t) \leq Ct^q$, we get

$$J(t\bar{z}) \leq \frac{1}{2}t^2 + \frac{\gamma}{2}Ct^{2q} - \frac{1}{p}t^p\int_\Omega \bar{z}^p.$$

If $2q < p$, it immediately follows that $J(t\bar{z}) \to -\infty$ as $t \to +\infty$.

If $2q = p$ and $C = \frac{1}{q}$, we take $\overline{z} \in E$ such that $\|\overline{z}\| = 1$ and $\int_\Omega |\overline{z}|^{2q} = S(2q, \Omega)$. Then by (G1) (with $C = \frac{1}{q}$) one has

$$J(t\overline{z}) = \frac{1}{2}t^2 + \frac{\gamma}{2}G(t^2) - \frac{S(2q,\Omega)}{2q}t^{2q} \le \frac{1}{2}t^2 + \frac{\gamma}{2q}t^{2q} - \frac{S(2q,\Omega)}{2q}t^{2q}.$$

If $\gamma < S(2q, \Omega)$ and since $2q = p > 2$, it follows again that $J(t\overline{z}) \to -\infty$ as $t \to +\infty$. This completes the proof. □

Lemma 4.3. *J satisfies the* (PS) *condition at level $c > 0$, namely:*
$(\mathrm{PS})_c$ *Any sequence $u_j \in E$ such that $J(u_j) \to c > 0$ and $J'(u_j) \to 0$ has a convergent subsequence.*

Proof. If $J(u_j) \to c > 0$, then

$$\|u_j\|^2 + \gamma G(\|u_j\|^2) \le C_0 + \frac{2}{p}\int_\Omega |u_j|^p$$

for some $C_0 > 0$. Since $J'(u_j) \to 0$, we have $|(J'(u_j), u_j)| \le o(1)\|u_j\|$. From

$$(J'(u_j), u_j) = \|u_j\|^2 + \gamma G'(\|u_j\|^2)\|u_j\|^2 - \int_\Omega |u_j|^p$$

we infer

$$\big|(J'(u_j), u_j)\big| = \Big|\|u_j\|^2 + \gamma G'(\|u_j\|^2)\|u_j\|^2 - \int_\Omega |u_j|^p\Big| \le o(1)\|u_j\|,$$

whence

$$\int_\Omega |u_j|^p \le \|u_j\|^2 + \gamma G'(\|u_j\|^2)\|u_j\|^2 + o(1)\|u_j\|.$$

Thus

$$\|u_j\|^2 + \gamma G(\|u_j\|^2) \le C_0 + \frac{2}{p}[\|u_j\|^2 + \gamma G'(\|u_j\|^2)\|u_j\|^2 + o(1)\|u_j\|],$$

namely

$$\frac{p-2}{p}\|u_j\|^2 + \gamma\Big[G(\|u_j\|^2) - \frac{2}{p}G'(\|u_j\|^2)\|u_j\|^2\Big] \le C_0 + o(1)\|u_j\|.$$

Using (G2), we deduce that $\|u_j\|$ is bounded and then, up to a subsequence, u_j converges weakly in E and strongly in $L^p(\Omega)$. Notice that

$$\nabla J(u_j) = (1 + \gamma G'(\|u_j\|^2))u_j - R(u_j),$$

where R is the weakly continuous operator defined by $(R(u), v) = \int_\Omega |u|^{p-2}uv$. Since $M(t) = 1 + \gamma G'(t) \ge 1$ ($M(t) \ge m > 0$ would suffice), we can write

$$u_j = \frac{J'(u_j) + R(u_j)}{1 + \gamma G'(\|u_j\|^2)}.$$

Passing possibly to a subsequence, $1 + \gamma G'(\|u_j\|^2)$ converges to a positive number and $R(u_j)$ is strongly convergent. Since $J'(u_j) \to 0$, it follows that u_j is also strongly convergent and $(\mathrm{PS})_c$ holds. □

Proof of Theorem 4.1. The preceding two lemmas allow us to apply the mountain pass theorem yielding a stationary point $u \ne 0$ of J and hence a non-trivial solution of (4.1). The fact that $u > 0$ follows in a standard way, substituting $|u|^{p-2}u$ by u_+^{p-1}, where $u_+ = u \vee 0$. □

We now consider the case in which $2q > p$. We will see that in this case multiple solutions occur provided $\gamma \ll 1$.

We do not assume (G2) but strengthen (G1) by requiring the following:
(G1′) There exist $q \ge 1$ and $C, C' > 0$ such that $C't^q \le G(t) \le Ct^q$ for all $t \ge 0$.

Theorem 4.4. *Suppose that* (4.2) *and* (G0), (G1′) *hold and let $2q > p$. Then there exists $\gamma_0 > 0$ such that* (4.1) *has at least two positive solutions for any $\gamma \in (0, \gamma_0)$.*

Proof. As in Lemma 4.2 (where (G2) is not used) J has a local minimum at $u = 0$. From $G(t) \leq Ct^q$ it follows once more that

$$J(t\overline{z}) \leq \frac{1}{2}t^2 + \frac{\gamma}{2}Ct^{2q} - \frac{1}{p}t^p \int_\Omega \overline{z}^p.$$

If $p > 2$, we can choose $t_0 > 0$ such that

$$\frac{1}{2}t_0^2 - \frac{1}{p}t_0^p \int_\Omega \overline{z}^p < 0.$$

Thus there exists $\gamma_0 > 0$ such that

$$\min_{t\geq 0} J(t\overline{z}) \leq J(t_0\overline{z}) < 0 \quad \text{for all } \gamma \in (0, \gamma_0).$$

This shows that J has the mountain pass geometry.

Moreover, using $G(t) \geq C' t^q$ (see (G1′)), we infer

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{\gamma}{2}G(\|u\|^2) - \frac{1}{p}\int_\Omega |u|^p \\ &\geq \frac{1}{2}\|u\|^2 + \frac{\gamma C'}{2}\|u\|^{2q} - \frac{S(p,\Omega)}{p}\|u\|^p. \end{aligned}$$

Since $2q > p$, we infer that J is bounded from below and coercive and this implies that any sequence $u_j \in E$ with $J(u_j) \to c \in \mathbb{R}$ is bounded. Repeating the arguments carried out in the second part of Lemma 4.3 (notice that, in such a lemma, (G2) has been used only to show that the $(PS)_c$ sequence u_j is bounded), we deduce that J satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Therefore, if $\gamma \in (0, \gamma_0)$, then J has a mountain pass critical point u_1 with $J(u_1) > 0$, as well as a global minimum u_2 with $J(u_2) < 0$, giving rise to two different positive solutions of (4.1). □

In the next remark we discuss some improvements of Theorem 4.4 in the model case, namely when $G(t) = \frac{1}{q}t^q$, with $2q > p > 2$.

Remark 4.5. (i) For each $\gamma \in (0, \gamma_0)$, let u_γ denote a solution of global minimum of J found before. We claim that $\|u_\gamma\| \to +\infty$ as $\gamma \downarrow 0$. Actually, since $J(u_\gamma) < 0$, we find

$$\frac{1}{2}\|u_\gamma\|^2 + \frac{\gamma}{2q}\|u_\gamma\|^{2q} < \frac{1}{p}\int_\Omega |u_\gamma|^p.$$

Moreover, $(J'(u_\gamma), u_\gamma) = 0$ yields

$$\int_\Omega |u_\gamma|^p = \|u_\gamma\|^2 + \gamma\|u_\gamma\|^{2q} \tag{4.3}$$

and thus

$$\frac{1}{2}\|u_\gamma\|^2 + \frac{\gamma}{2q}\|u_\gamma\|^{2q} < \frac{1}{p}(\|u_\gamma\|^2 + \gamma\|u_\gamma\|^{2q}),$$

whence

$$\frac{p-2}{2p} < \frac{\gamma(2q-p)}{2pq} \cdot \|u_\gamma\|^{2q-2},$$

which implies that $\|u_\gamma\| \to +\infty$ as $\gamma \downarrow 0$.

(ii) If γ is large, (4.1) can only have the trivial solution. To see this, let $u \neq 0$ be any non-trivial solution of (4.1), with $G(t) = \frac{1}{q}t^q$. Then, (4.3) implies

$$\|u\|^2 + \gamma\,\|u\|^{2q} = \int_\Omega |u|^p \leq S(p,\Omega)\|u\|^p. \tag{4.4}$$

If $2q > p$ and $\gamma \gg 1$, then $s^2 + \gamma s^{2q} > S(p,\Omega)s^p$ for every $s > 0$ and (4.4) gives rise to a contradiction.

(iii) If $2q = p$, (4.4) yields

$$\gamma\,\|u\|^{2q} < \int_\Omega |u|^{2q} \le S(2q,\Omega)\|u\|^{2q}.$$

Therefore, if $G(t) = \frac{1}{q}t^q$ and $2q = p$, the assumption $\gamma < S(2q,\Omega)$ is a necessary condition for the existence of non-trivial solutions.

We conclude this section by stating some extensions of the above results. The proofs require quite standard arguments and are left to the reader.

Remark 4.6. (i) Since $|u|^{p-2}u$ is odd, one can use the symmetric mountain pass theorem [6] (see also [5, Theorem 10.18]) to find that (4.1) possesses infinitely many solutions with positive energy. Furthermore, if $2q > p$ and $\gamma \ll 1$, the Lusternik–Schnirelman theory (see e.g. [5, Chapter 9]) yields infinitely many solutions with negative energy.

(ii) One can substitute $|u|^{p-2}u$ by any $f(u)$ such that (the dependence on x is understood)

$$f(0) = f'(0) = 0, \quad uf(u) \ge p\int_0^u f(s)ds, \quad \Big|\int_0^u f(s)ds\Big| \le c_1 + c_2|u|^p.$$

In addition, if f is odd, we find infinitely many solutions. We can also take $f(u) = \lambda a(x)u + \tilde f(u)$, where $\tilde f$ is as before. In this case, if $\lambda > (1 + G'(0))\lambda_1[a]$, one uses the linking theorem instead of the mountain pass theorem.

(iii) One can handle sublinear problems like (2.6), namely

$$-(1 + G'(\|u\|^2))\Delta u = \lambda u - h(x,u), \quad u_{|\partial\Omega} = 0. \tag{4.5}$$

If $G(t) \ge c|t|^r$ for some $c > 0$, $r \ge 1$ and $h(x,u) \sim u^{s+1}$ near $u = 0$ and near $u = \infty$, then the corresponding functional is bounded below and coercive and attains its minimum. If $\lambda > (1 + G'(0))\lambda_1[a]$, this minimum value is strictly negative and hence it gives rise to a positive solution of (4.5). Moreover, if the nonlinearity $h(x,.)$ is odd and $\lambda > (1 + G'(0))\lambda_k[a]$, then (2.6) has at least k (pairs of) non-trivial solutions.

5 A Perturbation Problem on $\mathbb{R}^n$

In this final section we look for solutions of the following problem on $\mathbb{R}^n$:

$$-\Big(1 + \varepsilon\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u + u = (1 + \varepsilon\alpha(x))|u|^{p-2}u, \quad u \in W^{1,2}(\mathbb{R}^n), \tag{5.1}$$

where, as before, $2 < p < 2^*$. Also in this case the purpose is to show that well-known results apply, yielding positive solutions of (5.1) for $|\varepsilon|$ small enough.

The solutions of (5.1) are the stationary points of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} [|\nabla u|^2 + u^2]dx - \frac{1}{p}\int_{\mathbb{R}^n} |u|^p dx + \varepsilon\,\Psi(u),$$

where

$$\Psi(u) = \frac{1}{4}\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^n} \alpha(x)|u|^p dx.$$

The unperturbed problem $-\Delta u + u = u^{p-1}$ has a family of positive ground state $U(x-\xi)$, $\xi \in \mathbb{R}^n$. According to well-known perturbation results (see [5, Appendix 5]), solutions of (5.1) can be found by looking for the minima or maxima of the finite dimensional functional $\psi(\xi) = \Psi(U(x-\xi))$. One finds

$$\begin{aligned}\psi(\xi) &= \frac{1}{4}\Big(\int_{\mathbb{R}^n} |\nabla U(x-\xi)|^2 dx\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^n} \alpha(x)U^p(x-\xi)dx \\ &= \frac{1}{4}\Big(\int_{\mathbb{R}^n} |\nabla U(y)|^2 dy\Big)^2 - \frac{1}{p}\int_{\mathbb{R}^n} \alpha(y-\xi)U^p(y)dx \\ &= -\frac{1}{p}\int_{\mathbb{R}^n} \alpha(y-\xi)U^p(y)dx + \text{Const.}\end{aligned}$$

Consider the following assumption:
(α) $\alpha(x)$ is continuous and $\lim_{|x|\to+\infty}\alpha(x) = 0$.
If (α) holds, one can prove that $\psi(\xi) \to 0$ as $|\xi| \to +\infty$, see [4]. Then ψ has either a minimum or a maximum on $\mathbb{R}^n$. Therefore we get the following result.

Theorem 5.1. *If $2 < p < 2^*$ and assumption (α) holds, then (5.1) has a solution provided $|\varepsilon|$ is small enough.*

Added in proof. Further results on elliptic Kirchhoff equations are contained in the paper [3].

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