Research Article

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A Determining Form for a Nonlocal System

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Abstract: This work is concerned with constructing a finite dimensional form (named determining form) by adding a feedback control term through an interpolation operator. The dynamics of the determining form is consistent with those of the original system.

Keywords: Nonlocal Laplacian, Determining Form, Interpolation Operator, Attractor

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1 Introduction

We consider the nonlocal system

$$\frac{du}{dt} = A_\alpha u + f(u) + g(x), \quad x \in (-1, 1), \tag{1.1}$$

with the boundary condition

$$u|_{(-1,1)^c} = 0,$$

and the initial condition

$$u(x, 0) = u_0.$$

Here the operator $A_\alpha = -(-\Delta)^{\alpha/2}$ is the nonlocal Laplacian. The nonlinear function is $f(u) = u - u^3$ and the inhomogeneous forcing $g \in L^2(-1, 1)$.

Our aim is to construct a finite dimensional form named determining form, the dynamics of which is consistent with those of the original infinite dimensional system (1.1).

The nonlocal Laplacian $(-\Delta)^{\alpha/2}$ arises in non-Gaussian stochastic systems. For a stochastic differential equation with symmetric α-stable Lévy motion L_t^α, for $\alpha \in (0, 2)$, its Fokker–Planck equation contains the nonlocal Laplacian $(-\Delta)^{\alpha/2}$ (see [6, 7]). For $u \in C_0^\infty(\mathbb{R})$ and $\alpha \in (0, 2)$, define

$$(-\Delta)^{\alpha/2} u = C_\alpha \mathrm{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+\alpha}} \, dy,$$

where the Cauchy principal value (P.V.) is taken as the limit of the integral over $\mathbb{R} \setminus B_\varepsilon(x)$ as $\varepsilon \to 0$, with $B_\varepsilon(x)$ the ball of radius ε centered at x, and

$$C_\alpha = \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi} |\Gamma(-\frac{\alpha}{2})|}.$$

Here Γ is the Gamma function defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ for every $r > 0$. For more information see [2, 6, 15].

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The nonlocal Laplacian provides us with an interesting tool for mathematical modeling when traditional approaches appear to fail. For complex systems, the nonlocal Laplacian arises in modeling heat transfer processes in fractal and disordered media, and acoustic propagation in porous media [14]. In mechanics, the nonlocal Laplacian describes the motion of a chain [15]. A nonlocal diffusion equation also arises in pricing derivative securities in financial markets [1, 3].

A system restricted to the inertial manifold is a finite dimensional system, even if the original system was infinite dimensional [17]. In [18], we know that the inertial manifold of the nonlocal system exists for $1 < \alpha < 2$, as well as for $\alpha = 1$, when the Lipschitz constant of nonlinearity is less than $\frac{1}{4}$. But for $\alpha < 1$, the nonlocal system does not satisfy the spectral gap condition, so the existence of the inertial manifold is unknown. In particular, the Lipschitz constant of $f(u) = u - u^3$ is bigger than $\frac{1}{4}$. Then, in the case of $\alpha = 1$, the existence of the inertial manifold of system (1.1) is also unknown. Fortunately, system (1.1) has a global attractor for $\alpha \in [\frac{1}{2}, 2)$. Hence, we show that for $\alpha \in [\frac{1}{2}, 1)$, we can construct a finite dimensional system (i.e. determining form) to capture the dynamics of system (1.1).

The determining form starts with the property of the determining modes [10, 12] which can be described as a finite subset of Fourier modes. For the case of Fourier modes, a projector P_m is called determining if whenever two solutions $u_1(\cdot), u_2(\cdot) \in \mathcal{A}$ have the same projection $P_m u_1(\cdot) = P_m u_2(\cdot)$ for all $t \in \mathbb{R}$, then they are in fact the same solution. And m is said to be the number of the determining modes [12].

There are two different constructions of determining forms. In [8], determining modes were used to find a determining form for the 2D Navier–Stokes equations. The trajectories in the global attractor of the 2D Navier–Stokes equations are identified with traveling wave solutions of the determining form in [8]. Another type of determining form was found in [9] for the 2D Navier–Stokes equations. It is done by adding a feedback control term through a general interpolation operator. The trajectories in the global attractor of the 2D Navier–Stokes equations are precisely the steady states of this type of determining form. It is more general in that it can be induced by a variety of determining parameters such as nodal values and finite volumes, as well as Fourier modes. In this paper we adopt the second approach to construct a determining form for the nonlocal system (1.1).

This paper is organized as follows. In Section 2, we recall some basic concepts about the nonlocal Laplacian, the space $H^{\alpha/2}(-1, 1)$ and the attractor. In Section 3, we present our main results on the determining form.

2 Preliminaries

In this section we briefly review the eigenvalues of the nonlocal Laplacian and the basic concepts about the space $H^{\alpha/2}(-1, 1)$. For more details see [4, 5, 13]. A conclusion about the existence of the global attractor is also given. Let $\|\cdot\|$ be the norm of $L^2(-1, 1)$.

Let $\alpha \in (0, 2)$. The eigenvalues of the nonlocal Laplacian $(-\Delta)^{\alpha/2}$ are given in the lemma below.

Lemma 2.1 ([13]). *The eigenvalues of the spectral problem*

$$\begin{cases} (-\Delta)^{\alpha/2} e(x) = \lambda e(x), & x \in (-1, 1), \\ e(x) = 0, & x \in (-1, 1)^c, \end{cases}$$

where $e(\cdot) \in L^2(-1, 1)$, *are*

$$\lambda_n = \Big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big)^{\alpha} + O\Big(\frac{1}{n}\Big).$$

Moreover,

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots \quad \textit{for } n = 1, 2, \ldots.$$

Furthermore, we note that $(-A_\alpha)^{-1}$ is a bounded linear operator on $L^2(-1, 1)$, and also a compact, self-adjoint operator. Owing to the Hilbert–Schmidt theorem [13, 16], the eigenfunctions $e_j(x)$ of A_α form an orthonormal basis in $L^2(-1, 1)$.

In order to better review the properties of nonlocal Laplacian A_α, we need the Sobolev spaces $H^{\alpha/2}(-1,1)$ and $H_0^{\alpha/2}(-1,1)$. We have

$$H^{\alpha/2}(-1,1)=\left\{u\in L^2(-1,1):\int_{-1}^{1}\int_{-1}^{1}\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}dxdy<\infty\right\}$$

with the norm

$$\|u\|_{H^{\alpha/2}}=\left(\int_{-1}^{1}|u|^2dx+\int_{-1}^{1}\int_{-1}^{1}\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}dxdy\right)^{1/2}$$

and the inner product

$$(u,v)_{H^{\alpha/2}}=(u,v)_{L^2}+\int_{-1}^{1}\int_{-1}^{1}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}dxdy.$$

The space $H_0^{\alpha/2}(-1,1)$ is the completion of the space $C_c^\infty(-1,1)$ in $H^{\alpha/2}(-1,1)$ with the norm

$$\|u\|_{H_0^{\alpha/2}}=\left(\int_{-1}^{1}\int_{-1}^{1}\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}dxdy\right)^{1/2}$$

and the inner product

$$(u,v)_{H_0^{\alpha/2}}=\int_{-1}^{1}\int_{-1}^{1}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}dxdy.$$

For convenience, we let $\|\cdot\|_\alpha=\|\cdot\|_{H_0^{\alpha/2}}$ and $(\cdot,\cdot)_\alpha=(\cdot,\cdot)_{H_0^{\alpha/2}}$. By the nonlocal Green's first identity (see [4, 5]), we have

$$(-A_\alpha u,v)=\int_{-1}^{1}\int_{-1}^{1}(\mathcal{D}^*v)\cdot(\Theta\cdot\mathcal{D}^*u)dydx,$$

where $\mathcal{D}^*$ denotes the adjoint of the nonlocal divergence operator $\mathcal{D}$ and Θ denotes a second-order tensor satisfying $\Theta=\Theta^T$. Especially when $\Theta=\frac{1}{2}$, we have

$$(A_\alpha u,u)=-\int_{-1}^{1}\int_{-1}^{1}(\mathcal{D}^*u)^2dydx=-\frac{1}{2}\|u\|_\alpha^2.$$

Then we get

$$\frac{1}{2}\frac{\|u\|_\alpha^2}{\|u\|^2}=\frac{(-A_\alpha u,u)}{(u,u)}=\frac{(\sum_{n=1}^\infty\lambda_n(u,e_n)e_n,\sum_{n=1}^\infty(u,e_n)e_n)}{(\sum_{n=1}^\infty(u,e_n)e_n,\sum_{n=1}^\infty(u,e_n)e_n)}\geq\lambda_1,$$

that is,

$$\|u\|_\alpha\geq\sqrt{2\lambda_1}\|u\|. \tag{2.1}$$

We also need the following lemma [11].

Lemma 2.2. *For $\alpha\in[\frac{1}{2},2)$, system (1.1) has a weak solution: for any $T>0$ given $u_0\in L^2(-1,1)$, there exists a solution u with*

$$u\in L^2(0,T;H_0^{\alpha/2}(-1,1))\cap L^4(0,T;L^4(-1,1)),\quad u\in C^0([0,T];L^2(-1,1)).$$

Equation (1.1) holds as an equality in $L^{4/3}(0,T;H^{-\alpha/2}(-1,1))$, this means that for any $v\in L^4(0,T;H_0^{\alpha/2}(-1,1))$, we have

$$\left\langle\frac{du}{dt},v\right\rangle=(A_\alpha u,v)+\langle f(u),v\rangle+(g(x),v)$$

for almost every $t\in[0,T]$. Moreover, system (1.1) has a global attractor $\mathcal{A}$ in $L^2(-1,1)$, obviously, there is a constant $\rho_1>0$, such that

$$\mathcal{A}\subset\{u\in L^2:\|u\|\leq\rho_1\}. \tag{2.2}$$

3 Determining Form

In this section, our main aim is to construct the determining form induced by an interpolation operator $J_h : L^2(-1, 1) \to C^\infty(-1, 1)$. The operator J_h approximates the identity. In addition, we assume that

$$\|J_h\phi - \phi\| \le h\|\phi\|, \tag{3.1}$$

where h is a small enough parameter that determines the order of approximating. The most straightforward example of such interpolation operator is the projection operator $J_h = P_N$ onto $\operatorname{span}\{e_1, e_2, \dots, e_N\}$, where $h = \lambda_N^{-1}$. According to Lemma 2.1 we know that $\lambda_n \to +\infty$ as $n \to +\infty$. Then there is always an $N \in \mathbb{Z}$ such that $h = \lambda_N^{-1}$ small enough satisfies (3.1).

First we introduce the Banach space

$$X = C_b(\mathbb{R}; J_h L^2(-1, 1)) = \{v : \mathbb{R} \to J_h L^2(-1, 1) : v \text{ is continuous and bounded}\},$$

with the norm

$$\|v\|_X = \sup_{t\in\mathbb{R}} \|v(t)\|.$$

We state our main result below.

Theorem 3.1. *Let $\rho = 4R$, where $R := (h + 1)\rho_1$. Let $v \in \mathcal{B}_X^\rho(0) = \{v \in X : \|v\|_X < \rho\}$. There exists a Lipschitz mapping $W : \mathcal{B}_X^\rho(0) \to C(\mathbb{R}, L^2)$ for every $v \in \mathcal{B}_X^\rho(0)$. If u^* is a steady state of the nonlocal system (1.1), then the determining form is*

$$\frac{dv}{dt} = -\|v - JW(v)\|_X^2 (v - J_h u^*), \tag{3.2}$$

with the following properties:

(i) *The vector field in the determining form (3.2) is a Lipschitz map from the ball $\mathcal{B}_X^\rho(0) = \{v \in X : \|v\|_X < \rho\}$ into X. Hence, equation (3.2) has a short time existence and uniqueness for initial data in $\mathcal{B}_X^\rho(0)$. Moreover, equation (3.2) has global existence and uniqueness for all initial data in the forward invariant set $\mathcal{B}_X^{3R}(J_h u^*) = \{v \in X : \|v - J_h(u^*)\|_X < 3R\}$.*

(ii) *Every solution of (3.2), with initial data in $\mathcal{B}_X^{3R}(J_h u^*)$, converges to the set of steady states of (3.2).*

(iii) *All the steady states of the determining form (3.2) that are contained in the ball $\mathcal{B}_X^\rho(0)$ are given by the form $v(s) = J_h u(s)$ for all $s \in \mathbb{R}$, where $u(s)$ is a trajectory that lies on the global attractor $\mathcal{A}$ of the nonlocal system (1.1).*

Proof. For the nonlinear term $f(u) = u - u^3$, there are positive constants k, β such that

$$f(u)u \le k - \beta|u|^4. \tag{3.3}$$

To prove (i), we need the following claim:

Claim 1. *Assume μ satisfies*

$$\mu\Big(h - \frac{1}{2}\Big) < 1 - \lambda_1 \quad \text{and} \quad \frac{1}{\mu} + h < \frac{1}{2}. \tag{3.4}$$

Then for every $v \in \mathcal{B}_X^\rho(0)$, the equation

$$\frac{dw}{dt} = A_\alpha w + f(w) + g(x) - \mu(J_h w - v) \tag{3.5}$$

has a unique solution $w(t)$ for all $t \in \mathbb{R}$, and satisfies the estimate

$$\sup_{t\in\mathbb{R}} \|w(t)\| \le \sqrt{\frac{\mu}{\lambda_1 - 2\mu K}}\, G.$$

Moreover, suppose $v_1, v_2 \in X$. Let w_1, w_2 be the corresponding solutions of (3.5), and $\delta = w_1 - w_2$, $\gamma = v_1 - v_2$. Then

$$\sup_{t\in\mathbb{R}} \|\delta(t)\| \le \sqrt{\frac{\mu}{2(\mu K - 1)}}\, \|\gamma\|,$$

where

$$K = \frac{\lambda_1}{\mu} - \Big(h - \frac{1}{2}\Big) \quad \textit{and} \quad G = \sqrt{\frac{\|g\|^2 + \lambda_1\rho^2 + 4k\lambda_1}{\mu\lambda_1}}.$$

By Claim 1, we have a Lipschitz mapping

$$W : \mathcal{B}_X^{\rho}(0) \to C(\mathbb{R}, L^2) \quad \text{for every } v \in \mathcal{B}_X^{\rho}(0).$$

There exists a w such that $W(v)(t) = w(t)$ for all $t \in \mathbb{R}$, where $w(t)$ is the unique solution of (3.5).

Next, we prove that equation (3.2) has a local solution through verifying its nonlinear term is Lipschitz. Let

$$F(v) = -\|v - J_h W(v)\|_X^2 (v - J_h u^*).$$

For $v_1, v_2 \in \mathcal{B}_X^{\rho}(0)$ we have

$$\begin{aligned} \|F(v_1) - F(v_2)\|_X &= \big\| \|v_2 - J_h W(v_2)\|_X^2 (v_2 - J_h u^*) - \|v_1 - J_h W(v_1)\|_X^2 (v_1 - J_h u^*) \big\|_X \\ &\le \big| \|v_1 - J_h W(v_1)\|_X^2 - \|v_2 - J_h W(v_2)\|_X^2 \big| \, \|v_1 - J_h u^*\|_X \\ &\quad + \|v_2 - J_h W(v_2)\|_X^2 \|v_1 - v_2\|_X. \end{aligned} \tag{3.6}$$

Thanks to (3.1) and (3.12) we have

$$\begin{aligned} \|J_h W(v_1) - J_h W(v_2)\| &= \|J_h\delta\| \le \|J_h\delta - \delta\| + \|\delta\| \le (h+1)\|\delta\| \\ &\le \frac{\sqrt{\mu}(h+1)}{\sqrt{2(\mu K - 1)}}\|\gamma\| = \frac{\sqrt{\mu}(h+1)}{\sqrt{2(\mu K - 1)}}\|v_1 - v_2\| \end{aligned}$$

and

$$\|J_h W(v)\| \le \|J_h W(v) - W(v)\| + \|W(v)\| \le \frac{(h+1)G}{\sqrt{\lambda_1 - 2\mu K}}.$$

Then

$$\|J_h W(v_1) - J_h W(v_2)\|_X \le \frac{\sqrt{\mu}(h+1)}{\sqrt{2(\mu K - 1)}}\|v_1 - v_2\|_X \tag{3.7}$$

and

$$\|J_h W(v)\|_X \le \frac{(h+1)G}{\sqrt{\lambda_1 - 2\mu K}}. \tag{3.8}$$

Similarly, we have

$$\|J_h u^*\|_X \le (h+1)\rho_1. \tag{3.9}$$

For the first term on the right-hand side of inequality (3.6), with inequalities (3.7) and (3.8), we have

$$\begin{aligned} \|v_1 - J_h W(v_1)\|_X + \|v_2 - J_h W(v_2)\|_X &\le \|v_1\|_X + \|v_2\|_X + \|J_h W(v_1)\|_X + \|J_h W(v_2)\|_X \\ &\le 2\rho + \frac{2(h+1)G}{\sqrt{\lambda_1 - 2\mu K}} \end{aligned}$$

and

$$\begin{aligned} \big| \|v_1 - J_h W(v_1)\|_X - \|v_2 - J_h W(v_2)\|_X \big| &\le \|v_1 - v_2\|_X + \|J_h W(v_1) - J_h W(v_2)\|_X \\ &\le \Big(\frac{\sqrt{\mu}(h+1)}{\sqrt{2(\mu K - 1)}} + 1 \Big)\|v_1 - v_2\|_X. \end{aligned}$$

With (3.9), we have

$$\|v_1 - J_h u^*\|_X \le \|v_1\|_X + \|J_h u^*\|_X \le \rho + (h+1)\rho_1.$$

Then the estimation of first term on the right-hand side of inequality (3.6) is

$$\begin{aligned} &\big| \|v_1 - J_h W(v_1)\|_X^2 - \|v_2 - J_h W(v_2)\|_X^2 \big| \, \|v_1 - J_h u^*\|_X \\ &\quad \le \Big(2\rho + \frac{2(h+1)G}{\sqrt{\lambda_1 - 2\mu K}} \Big)(\rho + (h+1)\rho_1)\Big(\frac{\sqrt{\mu}(h+1)}{\sqrt{2(\mu K - 1)}} + 1 \Big)\|v_1 - v_2\|_X \\ &\quad \le C_1\|v_1 - v_2\|_X. \end{aligned}$$

For the estimation of the second term on the right-hand side of inequality (3.6), we have

$$\|v_2 - J_h W(v_2)\|_X^2 \le \|v_2\|_X^2 + \|J_h W(v_2)\|_X^2 \le \rho^2 + \frac{(h+1)^2 G^2}{\lambda_1 - 2\mu K} = C_2.$$

Then we obtain

$$\|F(v_1) - F(v_2)\|_X \le C_1\|v_1 - v_2\|_X + C_2\|v_1 - v_2\|_X = C\|v_1 - v_2\|_X.$$

So $F(v) = -\|v - J_h W(v)\|_X^2 (v - J_h u^*)$ is Lipschitz, and equation (3.2) has a local solution. Moreover, by (3.9) we can get

$$\mathcal{B}_X^{3R}(J_h u^*) = \{v \in X : \|v - J_h(u^*)\|_X < 3R\} \subset \mathcal{B}_X^{\rho}(0) = \{v \in X : \|v\|_X < \rho\}.$$

The dissipative property of equation (3.2) implies that

$$\mathcal{B}_X^{3R}(J_h u^*) = \{v \in X : \|v - J_h(u^*)\|_X < 3R\} \tag{3.10}$$

is forward invariant for all $t \ge 0$, which proves that equation (3.2) has a global solution, that is, (i) holds.

Assertion (ii) follows directly from the forward invariance of (3.10) and the dissipative property of (3.2).

To prove (iii), we note that when either $v = J_h u^*$, or $v \in \mathcal{B}_X^{\rho}(0)$ such that $\|v - J_h W(v)\|_X = 0$, the right-hand side of equation (3.2) is zero. In the first case, since u^* is a steady state of the nonlocal system (1.1), we have $u^* \in \mathcal{A}$.

In the second case, to understand better, we need the following claim which states a connection between equation (1.1) and the determining form.

Claim 2. *Let $u(t)$ be a solution of nonlocal equation* (1.1)*, which lies in the global attractor $\mathcal{A}$. Suppose $w \in C(\mathbb{R}, H^{\alpha/2}) \cap L^2(0, T; D(A^\alpha))$ with $\frac{dw}{dt} \in L^2(0, T; L^2)$ satisfies the equation*

$$\frac{dw}{dt} = A_\alpha w + f(w) + g(x) - \mu J_h(w - u), \tag{3.11}$$

providing h satisfies (3.4)*. Then $w = u$ in $C(\mathbb{R}, L^2)$.*

Now we have $v(t) = J_h W(v)(t)$, i.e., $v(t) = J_h(w(t))$, for all $t \in \mathbb{R}$, where $w(t)$ is a solution of (3.5). In this case, the solution of equation (3.5), $w(t)$, is a bounded solution of system (1.1). Therefore, from (2.2) we know that $w(\cdot)$ is a trajectory on the global attractor $\mathcal{A}$ of system (1.1). Conversely, since $\rho = 4R$, it follows from (3.9) that $J(\mathcal{A}) \subset \mathcal{B}_X^{3R}(J_h u^*) \subset \mathcal{B}_X^{\rho}(0)$. Thus, for every trajectory $u(\cdot) \subset \mathcal{A}$ it follows from Claim 2 and (3.5) that $u(t) = W(J_h u)(t)$ for all $t \in \mathbb{R}$. In particular, $J_h u = J_h W(J_h u)$. Consequently, $J_h u$ is a steady state of (3.2) in $\mathcal{B}_X^{\rho}(0)$. Thus (iii) holds. □

Proof of Claim 1. We divide the proof into two steps.

Step 1: A priori estimate. We take the inner product of equation (3.5) with w to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 &\le -\frac{1}{2}\|w\|_\alpha^2 + \int_{-1}^{1} f(w)w\,dx + \|g\|\|w\| - \mu(J_h w, w) + \mu\int_{-1}^{1} wv\,dx \\
&\le -\lambda_1\|w\|^2 + \int_{-1}^{1}(k - \beta|w|^4)\,dx + \|g\|\|w\| + \mu h\|w\|^2 - \mu\|w\|^2 + \mu\|w\|\|v\| \\
&\le \Big[\mu\Big(h - \frac{1}{2}\Big) - \frac{\lambda_1}{2}\Big]\|w\|^2 + \frac{1}{2\lambda_1}\|g\|^2 + \frac{\mu}{2}\|v\|^2 + \int_{-1}^{1}(k - \beta|w|^4)\,dx,
\end{aligned}$$

where we used inequalities (2.1), (3.1), (3.3), the Cauchy–Schwarz inequality and Young's inequality. Dropping the term in $|w|^4$, we get

$$\frac{d}{dt}\|w\|^2 \le -2\mu K\|w\|^2 + \frac{1}{\lambda_1}\|g\|^2 + \mu\|v\|^2 + 4k.$$

By Gronwall's inequality [17] and assuming that $\|w\|$ is bounded, we obtain

$$\begin{aligned} \|w(t)\|^2 &\leq \frac{1}{\lambda_1 - 2\mu K}\left(\frac{1}{\lambda_1}\|g\|^2 + \mu\|v\|^2 + 4k\right) \\ &\leq \frac{1}{\lambda_1 - 2\mu K}\left(\frac{\|g\|^2 + \mu\lambda_1\rho^2 + 4k\lambda_1}{\lambda_1}\right) \\ &\leq \frac{G^2}{\lambda_1 - 2\mu K}. \end{aligned} \tag{3.12}$$

It follows that

$$\sup_{t\in\mathbb{R}}\|w(t)\| \leq \frac{G}{\sqrt{\lambda_1 - 2\mu K}}.$$

For $\|\delta\|$, let $\gamma = v_1 - v_2$. Then we have

$$\frac{d\delta}{dt} = A_\alpha\delta + f(w_1) - f(w_2) - \mu(J_h\delta - \gamma).$$

Taking the inner product with δ, we obtain

$$\left(\frac{d\delta}{dt}, \delta\right) = (A_\alpha\delta, \delta) + (f(w_1) - f(w_2), \delta) - \mu(J_h\delta - \delta, \delta) - \mu(\delta, \delta) + \mu(\gamma, \delta).$$

Note that by the bound on the derivative of f we have

$$\left|(f(w_1) - f(w_2), \delta)\right| \leq \int_{-1}^{1} |f(w_1) - f(w_2)||\delta| dx \leq \int_{-1}^{1} |\delta|^2 dx = \|\delta\|^2. \tag{3.13}$$

Using inequalities (2.1), (3.1), the Cauchy–Schwarz inequality and Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 \leq (\mu K - 1)\|\delta\|^2 + \frac{\mu}{2}\|\gamma\|^2.$$

Then we have

$$\frac{d}{dt}\|\delta\|^2 \leq -2(1 - \mu K)\|\delta\|^2 + \mu\|\gamma\|^2.$$

So by Gronwall's inequality, we get

$$\|\delta(t)\|^2 \leq \left(\|\delta(0)\|^2 - \frac{\|\gamma\|^2}{1 - 2K}\right)e^{-2(1-\mu K)t} + \frac{\mu\|\gamma\|^2}{2(\mu K - 1)}.$$

Thus it follows that

$$\sup_{t\in\mathbb{R}}\|\delta(t)\| \leq \sqrt{\frac{\mu}{2(\mu K - 1)}}\|\gamma\|.$$

Step 2: Existence of solution. Applying the Galerkin method (see e.g. [17]), we get the existence and uniqueness of the solution for equation (3.5). Then by the above a priori estimate, we know that equation (3.5) has a global solution. Thus Claim 1 holds. □

Proof of Claim 2. Let $\delta = w - u$. The difference of (3.11) and (1.1) is

$$\frac{d\delta}{dt} = A_\alpha\delta + f(w) - f(u) - \mu J_h\delta. \tag{3.14}$$

Suppose there is a time $t^* \in \mathbb{R}$ such that $\delta(t^*) \neq 0$. Since $\delta(t)$ is continuous, we can find the maximum interval (t_0, t_1) containing t^*, such that $\delta(t) \neq 0$ for all $t \in (t_0, t_1)$ and $\delta(t_0) = 0$. Taking the inner product of (3.14) with δ and using inequality (2.1), we have, for all $t \in (t_0, t_1)$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta\|^2 &= -\frac{1}{2}\|\delta\|_\alpha^2 + (f(w) - f(u), \delta) - \mu(J_h\delta, \delta) \\ &\leq -\lambda_1\|\delta\|^2 + (f(w) - f(u), \delta) - \mu(J_h\delta - \delta, \delta) - \mu\|\delta\|^2. \end{aligned}$$

Using inequalities (3.1) and (3.13), we get

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 \le -\lambda_1\|\delta\|^2 + \|\delta\|^2 + \mu h\|\delta\| - \mu\|\delta\|^2,$$

i.e.

$$\frac{d}{dt}\|\delta\|^2 \le 2\Big[\mu\Big(\frac{1}{\mu} + h - 1\Big) - \lambda_1\Big]\|\delta\|^2.$$

Using Gronwall's inequality, we obtain

$$\|\delta(t)\|^2 \le \|\delta(\sigma_0)\|^2 \exp\Big\{\Big[2\mu\Big(\frac{1}{\mu} + h - 1\Big) - \lambda_1\Big](t - \sigma_0)\Big\},$$

where $t_0 < \sigma_0 < t < t_1$. Taking $\sigma_0 \to t_0^+$, by (3.4) we have $\delta(t) = 0$ for any $t \in (t_0, t_1)$. In particular, $\delta(t^*) = 0$, a contradiction. □

Remark 3.2. Suppose system (1.1) has a more general nonlinearity $f(u)$, which is a C^1 function satisfying

$$-k - \beta_2|u|^p \le f(u)u \le k - \beta_1|u|^p, \quad p > 2,$$

and

$$f'(u) \le l.$$

Then, for the case that $\alpha \in [1 - \frac{2}{p}, 2)$, we could prove that Theorem 3.1 still holds. Note that, in this case, system (1.1) has a weak solution and a global attractor $\mathcal{A}$ in $L^2(-1, 1)$.

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