DE GRUYTER Adv. Nonlinear Stud. 2016; 16 (4):843–861

Research Article

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Concentration Phenomena for Fractional Elliptic Equations Involving Exponential Critical Growth

DOI: 10.1515/ans-2016-0097
Received July 9, 2016; revised August 26, 2016; accepted August 28, 2016

Abstract: In this paper, we deal with the singular perturbed fractional elliptic problem $\varepsilon(-\Delta)^{1/2}u + V(z)u = f(u)$ in $\mathbb{R}$, where $(-\Delta)^{1/2}u$ is the square root of the Laplacian and $f(s)$ has exponential critical growth. Under suitable conditions on $f(s)$, we construct a localized bound state solution concentrating at an isolated component of the positive local minimum points of the potential of V as ε goes to 0.

Keywords: Trudinger–Moser Inequality, Nonlinear Schrödinger Equations, Variational Methods, Mountain Pass Theorem, Lack of Compactness, Critical Growth, Trudinger–Moser Inequality, Fractional Laplacian

MSC 2010: 35J60, 35B09, 35B33, 35R11

Communicated by: Patrizia Pucci

1 Introduction

In this paper, we are concerned with existence and concentration of positive solutions for the following singular perturbed fractional elliptic problem:

$$\begin{cases} \varepsilon(-\Delta)^{1/2}u + V(z)u = f(u) & \text{in } \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \quad u > 0 & \text{on } \mathbb{R}, \end{cases} \tag{P_ε}$$

where ε is a small positive parameter, the potential V is bounded away from zero, the non-linearity $f(s)$ has exponential critical growth, and $(-\Delta)^{1/2}u$ is the square root of the Laplacian, which may be defined for smooth functions as

$$\mathcal{F}((-\Delta)^{1/2}u)(\xi) = |\xi|\mathcal{F}(u)(\xi),$$

where $\mathcal{F}$ is the Fourier transform, that is,

$$\mathcal{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\cdot x}\phi(x)\,dx$$

for functions ϕ in the Schwartz class. Also, for sufficiently smooth u, $(-\Delta)^{1/2}u$ can be equivalently represented, see [18, 24], as

$$(-\Delta)^{1/2}u = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2}\,dy,$$

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and, by [18, Propostion 3.6],

$$\|(-\Delta)^{1/4}u\|_{L^2}^2 := \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,dx\,dy \quad \text{for all } u \in H^{1/2}(\mathbb{R}).$$

Here $H^{1/2}(\mathbb{R})$ is the fractional Sobolev space

$$H^{1/2}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \|(-\Delta)^{1/4}u\|_{L^2}^2 < \infty\},$$

endowed with the norm

$$\|u\|_{H^{1/2}} = (\|u\|_{L^2}^2 + \|(-\Delta)^{1/4}u\|_{L^2}^2)^{1/2}.$$

We suppose that the potential $V : \mathbb{R} \to \mathbb{R}$ is bounded and satisfies the following hypotheses:

(V1) V is locally Hölder continuous and there exists $V_0 > 0$ such that

$$V(z) \ge V_0 \quad \text{for all } z \in \mathbb{R}.$$

(V2) There exists a bounded interval $\Lambda \subset \mathbb{R}$ such that

$$V_0 \equiv \inf_{\Lambda} V(z) < \min_{\partial\Lambda} V(z).$$

The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the so-called Ambrosetti–Rabinowitz condition, introduced in [5], namely,

$$\text{there exists } \theta > 2 \text{ with } 0 < \theta F(s) \le sf(s) \text{ for all } s > 0, \tag{AR}$$

where $F(s) = \int_0^s f(t)\,dt$. In addition to the above condition we make the following assumptions on f:

(f1) $f : \mathbb{R} \to \mathbb{R}^+$ is a C^1 function with $f(s) = 0$ if $s < 0$.

(f2) $f(s) = o(s)$ near the origin.

(f3) $f(s)/s$ is an increasing function in $\mathbb{R}^+$.

(f4) There exist constants $p > 2$ and $C_p > 0$ such that

$$f(s) \ge C_p s^{p-1} \quad \text{for all } s > 0,$$

where

$$C_p > \Big[\beta_p\Big(\frac{2\theta}{\theta-2}\Big)\frac{1}{\min\{1, V_0\}}\Big]^{(p-2)/2},$$

with

$$\beta_p = \inf_{\mathcal{N}_0} \tilde{J}_0, \quad \mathcal{N}_0 = \{v \in X^1(\mathbb{R}_+^2) \setminus \{0\} : \tilde{J}_0'(v)v = 0\}$$

and

$$\tilde{J}_0(v) = \frac{1}{2}\int_{\mathbb{R}_+^2} |\nabla v|^2\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}} V_0|v(x,0)|^2\,dx - \frac{1}{p}\int_{\mathbb{R}} |v(x,0)|^p\,dx,$$

where $X^1(\mathbb{R}_+^2)$ is defined in (2.1).

We are interested in a bound state solution of (P_ε) (solution with finite energy), when f has the maximal growth, which allows us to treat problem (P_ε) variationally in the fractional Sobolev space $H^{1/2}(\mathbb{R})$ motivated by the following Trudinger–Moser type inequality due to T. Ozawa [26].

Theorem 1.1. *There exists $0 < \omega \le \pi$ such that, for all $\alpha \in (0, \omega)$, there exists $H_\alpha > 0$ with*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,dx \le H_\alpha \|u\|_{L^2}^2 \tag{1.1}$$

for all $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_{L^2}^2 \le 1$.

In view of (1.1), we say that f has exponential critical growth at $+\infty$, if there exist $\omega \in (0, \pi)$ and $\alpha_0 \in (0, \omega)$, such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = 0 \quad \text{for all } \alpha > \alpha_0, \quad \text{and} \quad \lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = +\infty \quad \text{for all } \alpha < \alpha_0.$$

1.1 Statement of the Main Result

The following theorem contains our main result.

Theorem 1.2. *Assume* (V1), (V2), (AR), *and* (f1)–(f4) *hold. Then there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$, *problem* (P_ε) *possesses a positive bound state solution* $u_\varepsilon(z)$ *verifying the following conditions:*
(i) u_ε *has at most one local (hence global) maximum* z_ε *in* $\mathbb{R}$ *and* $z_\varepsilon \in I$.
(ii) $\lim_{\varepsilon\to 0^+} V(z_\varepsilon) = V_1 = \inf_I V$.

Theorem 1.2 may be considered as the extension of the main result for the Laplacian in [20] to the case of the square root of the Laplacian. The proof is made combining the Ozawa inequality [26] with the Del Pino and Felmer truncation argument [17] and a recent approach developed by Alves and Miyagaki [4]. In [12, 16, 27, 29], existence results in non-local situation were established, while in [13, 14, 21, 28], concentration phenomena were proved imposing a global condition in V. Recently, Felmer and Torres [23] studied a new class of the problem involving a non-linear Schrödinger equation with non-local regional diffusion, which generalizes, for instance, the fractional Laplacian operator. In [31], Zhang, do Ó and Squassina treated some problems involving fractional Schrödinger–Poisson systems.

Remark 1.3. (i) We recall that condition (AR) imposes some superquadratic growth condition on the non-linearity F.

(ii) Condition (f4) appeared first in [11], then for instance in [2, 20]. For the non-local situation it was used, e.g., in [19].

(iii) Critical growth of Trudinger–Moser type was used in [15] and also in [1, 2, 20]. The Ozawa inequality to discuss the non-local problem in bounded and unbounded domain was used in [25] and [19], respectively.

(iv) Notice that, if $f(s)$ has exponential critical growth, instead of assumption (f4), it is enough to assume that there exist $p > 2$ and $\mu > 0$ such that

$$\liminf_{s\to 0^+} \frac{f(s)}{s^{p-1}} \geq \mu.$$

Throughout the paper, unless explicitly stated, the symbol C will denote a generic positive constant, which may vary from line to line.

1.2 Outline

The sequel of the paper is organized as follows. Section 2 contains some technical results, which are crucial tools to prove our main theorem. In Section 3, we adapt a method due to L. Caffarelli and L. Silvestre to obtain a local realization of the fractional Laplacian via a Dirichlet-to-Neumann operator. As a consequence of this argument we transform our non-local problem (P_ε) into one local problem (LP_ε) defined on the upper half plane. Using variational techniques combined with Del Pino and Felmer's truncation argument, we prove Theorem 1.2 in Section 4.

2 Preliminary Results

In this section we collect preliminary facts for future reference. First of all, let us set the standard notations to be used in the paper. We denote the upper half-space in $\mathbb{R}^2$ by $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. In the sequel, $X^1(\mathbb{R}^2_+)$ denotes the completion of $C_0^\infty(\overline{\mathbb{R}^2_+})$ with relation to the norm $\|v\|_\varepsilon$:

$$X^1(\mathbb{R}^2_+) := \overline{C_0^\infty(\overline{\mathbb{R}^2_+})}^{\|\cdot\|_\varepsilon}, \tag{2.1}$$

where

$$\|v\|_\varepsilon := \Big(\int_{\mathbb{R}^2_+} |\nabla v(x, y)|^2 \, dx \, dy + \int_{\mathbb{R}} V(\varepsilon x)|v(x, 0)|^2 \, dx \Big)^{1/2}.$$

Moreover, we denote by $\|\cdot\|$ the usual norm in $X^1(\mathbb{R}^2_+)$, that is,

$$\|v\| = \Big(\int_{\mathbb{R}^2_+} |\nabla v(x,y)|^2 \, dx \, dy + \int_{\mathbb{R}} |v(x,0)|^2 \, dx\Big)^{1/2}.$$

Since the potential V is bounded from above and below, it is easy to see that $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent norms in $X^1(\mathbb{R}^2_+)$ with

$$\min\{1, V_0\}\|v\| \le \|v\|_\varepsilon \le \min\{1, |V|_\infty\}\|v\| \quad \text{for all } v \in X^1(\mathbb{R}^2_+).$$

Using the above definition, we see that if $v \in X^1(\mathbb{R}^2_+)$, then $u(x) = v(x,0)$ belongs to $H^{1/2}(\mathbb{R})$ and

$$\|v\| = \|u\|_{H^{1/2}}.$$

Since $H^{1/2}(\mathbb{R})$ is continuously embedded into $L^q(\mathbb{R})$ for all $q \ge 2$, cf. [18, Theorem 6.9], it follows that $X^1(\mathbb{R}^2_+)$ is also continuously embedded into $L^q(\mathbb{R})$ for all $q \ge 2$. Moreover, the embedded

$$X^1(\mathbb{R}^2_+) \hookrightarrow L^q(A)$$

are compact for any bounded measurable set $A \subset \mathbb{R}$, see [24, Proposition 3.6] and [19, Remark 2.1].

Our first lemma is an important Trudinger–Moser inequality on $X^1(\mathbb{R}^2_+)$, which was proved in [19, Lemma 2.4].

Lemma 2.1. *Let $(v_n) \subset X^1(\mathbb{R}^2_+)$ be a bounded sequence and assume $\sup_{n\in\mathbb{N}}\|v_n\|^2 = M$. Then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha|v_n(x,0)|^2} - 1) \, dx < \infty \quad \textit{for every } 0 < \alpha < \frac{\omega}{M^2}.$$

In particular, if $M \in (0,1)$, then there exists $\alpha_M < \omega$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} (e^{\alpha_M|v_n(x,0)|^2} - 1) \, dx < \infty.$$

Using the above lemma, we are able to prove some technical lemmas. The first of them is crucial in the study of the behavior of Palais–Smale sequences.

Lemma 2.2. *Let (v_n) be a sequence in $X^1(\mathbb{R}^2_+)$ with*

$$\limsup_{n\to+\infty}\|v_n\|^2 < 1. \tag{2.2}$$

Then, there exist $t > 1$ sufficiently close to 1 and $C > 0$ satisfying

$$\int_{\mathbb{R}} (e^{\omega|v_n(x,0)|^2} - 1)^t \, dx \le C \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Using (2.2), there are $m > 0$ and $n_0 \in \mathbb{N}$ verifying

$$\|v_n\|^2 < m < 1 \quad \text{for all } n \ge n_0.$$

Fix $t > 1$ sufficiently close to 1 and $\beta > t$ satisfying $\beta m < 1$. Then, there exists $C = C(\beta) > 0$ such that

$$\int_{\mathbb{R}} (e^{\omega|v_n(x,0)|^2} - 1)^t \, dx \le C \int_{\mathbb{R}} (e^{\beta m\omega(|v_n(x,0)|/\|v_n\|)^2} - 1) \, dx$$

for every $n \ge n_0$. Hence, by Lemma 2.1,

$$\int_{\mathbb{R}} (e^{\omega|v_n(x,0)|^2} - 1)^t \, dx \le C_1 \quad \text{for all } n \ge n_0$$

for some positive constant C_1. Now, the lemma follows fixing

$$C = \max\Big\{C_1, \int_{\mathbb{R}} (e^{\omega|v_1|^2} - 1)^t \, dx, \dots, \int_{\mathbb{R}} (e^{\omega|v_{n_0}|^2} - 1)^t \, dx\Big\}. \qquad \square$$

Corollary 2.3. *Let (v_n) be a sequence in $X^1(\mathbb{R}^2_+)$ satisfying (2.2). If $v_n \rightharpoonup v$ weakly in $X^1(\mathbb{R}^2_+)$ and $v_n(x,0) \to v(x,0)$ a.e in $\mathbb{R}$, as $n \to \infty$, then,*

$$F(v_n(x,0)) \to F(v(x,0)) \qquad \text{in } L^1(-R,R), \tag{2.3}$$

$$f(v_n(x,0))v_n(x,0) \to f(v(x,0))v(x,0) \qquad \text{in } L^1(-R,R), \tag{2.4}$$

$$\int_{-R}^{R} f(v_n(x,0))\phi(x,0) \to \int_{-R}^{R} f(v(x,0))\phi(x,0), \tag{2.5}$$

as $n \to \infty$ for all $\phi \in X^1(\mathbb{R}^2_+)$ and $R > 0$.

Proof. By (f1), for each $\beta > 1$ and $\alpha > \alpha_0$, there is $C > 0$ such that

$$|F(s)| \le C\big(|s|^2 + (e^{\alpha\beta|s|^2} - 1)\big) \quad \text{for all } s \in \mathbb{R},$$

from where it follows that

$$|F(v_n(x,0))| \le C\big(|v_n(x,0)|^2 + (e^{\alpha\beta|v_n(x,0)|^2} - 1)\big) \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

Setting

$$h_n(x) = C(e^{\alpha_0\beta|v_n(x,0)|^2} - 1),$$

we can fix $\beta, q > 1$ sufficiently close to 1 and α sufficiently close to α_0 such that

$$h_n \in L^q(\mathbb{R}) \quad \text{and} \quad \sup_{n\in\mathbb{N}} |h_n|_q < +\infty,$$

which is an immediate consequence of Lemma 2.2. Therefore, up to subsequence, we derive that

$$h_n \rightharpoonup h = C\big(e^{\alpha_0\beta|v(x,0)|^2} - 1\big) \quad \text{weakly in } L^q(\mathbb{R}) \quad \text{as } n \to \infty.$$

Since $h_n, h \ge 0$, the last limit yields

$$h_n \to h \quad \text{in } L^1(-R,R) \quad \text{for all } R > 0 \quad \text{as } n \to \infty.$$

On the other hand, we know that

$$v_n(\cdot,0) \to v(\cdot,0) \quad \text{in } L^2(-R,R) \quad \text{as } n \to \infty.$$

Gathering the above limits with (2.6), we get (2.3):

$$F(v_n(x,0)) \to F(v(x,0)) \quad \text{in } L^1(-R,R) \quad \text{for all } R > 0 \quad \text{as } n \to \infty,$$

Limits (2.4) and (2.5) follow with the same type of arguments. □

The next lemma is a Lions type result, which is crucial in our approach. Since it follows with the same arguments found in [2, Proposition 2.3], we will omit its proof.

Lemma 2.4. *Let $(v_n) \subset X^1(\mathbb{R}^2_+)$ be a sequence with $\limsup_{n\to+\infty}\|v_n\|^2 < 1$. If there is $R > 0$ such that*

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}} \int_{z-R}^{z+R} |v_n(x,0)|^2\,dx = 0,$$

then

$$\lim_{n\to+\infty} \int_{\mathbb{R}} F(v_n(x,0))\,dx = \lim_{n\to+\infty} \int_{\mathbb{R}} f(v_n(x,0))v_n(x,0)\,dx = 0.$$

3 Caffarelli and Silvestre's Method

First of all, using the change of variable $u(x) = v(\varepsilon x)$, it is possible to prove that problem (P_ε) is equivalent to the problem

$$\begin{cases} (-\Delta)^{1/2}u + V(\varepsilon z)u = f(u) & \text{in } \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \quad u > 0 & \text{on } \mathbb{R}. \end{cases} \tag{P'_ε}$$

Hereafter, to get a solution for (P'_ε), we will use a version of the Caffarelli–Silvestre extension [10] due to Frank and Lenzmann [24] defined on the whole real line. In both papers, a local interpretation of the fractional Laplacian given in $\mathbb{R}$ was developed considering a Dirichlet-to-Neumann type operator in the domain $\mathbb{R}^2_+ = \{(x, t) \in \mathbb{R}^2 : t > 0\}$. For a similar extension in a bounded domain see, e.g., [6, 8, 9]. For $u \in H^{1/2}(\mathbb{R})$, the solution $w \in X^1(\mathbb{R}^2_+)$ of

$$\begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}^2_+, \\ w = u & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

is called 1/2-harmonic extension $w = E_{1/2}(u)$ of u, and it is proved in [10] that

$$\lim_{y \to 0^+} \frac{\partial w}{\partial y}(x, y) = -(-\Delta)^{1/2}u(x).$$

To get a solution for the non-local problem (P'_ε), we will study the existence of solutions for the local problem defined on the upper half plane:

$$\begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}^2_+, \\ -\dfrac{\partial w}{\partial \nu} = -V(\varepsilon x)w + f(w) & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{LP_ε}$$

where

$$\frac{\partial w}{\partial \nu} = \lim_{y \to 0^+} \frac{\partial w}{\partial y}(x, y),$$

since if w is a solution for (LP_ε), the function $u(x) = w(x, 0)$ is a solution for (P'_ε).

Associated with (LP_ε), we have $J_\varepsilon : X^1(\mathbb{R}^2_+) \to \mathbb{R}$ defined by

$$J_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla v|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x)|v(x, 0)|^2 \, dx - \int_{\mathbb{R}} F(v(x, 0)) \, dx,$$

which is $C^1(X^1(\mathbb{R}^2_+), \mathbb{R})$ with derivative given by

$$J'_\varepsilon(v)\phi = \frac{1}{2} \int_{\mathbb{R}^2_+} \nabla v . \nabla \phi \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x)v(x, 0)\phi(x, 0) \, dx - \int_{\mathbb{R}} f(v(x, 0))\phi(x, 0) \, dx \quad \text{for all } \phi \in X^1(\mathbb{R}^2_+).$$

We would like to point out that u is a solution of (P'_ε) if, and only if, $u = v(x, 0)$ for all $x \in \mathbb{R}$, for some critical point v of J_ε.

In what follows, we will not work directly with functional J_ε, because we have some difficulties to prove that it verifies the Palais–Smale compactness condition. We recall that a C^1-functional $\Psi : X \to \mathbb{R}$ defined on a Banach space X satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ condition for short), if each sequence $(u_n) \subset X$ such that (i) $\Psi(u_n) \to c$ and (ii) $\Psi'(u_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence (u_n) satisfying (i) and (ii) is called a Palais–Smale sequence at level c (a $(\mathrm{PS})_c$ for short), see [30].

Hereafter, we will use the same approach explored in [17], modifying the non-linearity of a suitable way. The idea is the following:

First of all, without loss of generality, we will assume that

$$0 \in \Lambda \quad \text{and} \quad V(0) = V_0 = \inf_{x \in \mathbb{R}} V(x).$$

We recall that in assumption (f1) we imposed that $f(t) = 0$ for all $t \leq 0$, because we are looking for positive solutions. Moreover, let us choose $k > 2\theta/(\theta - 2)$ and $a > 0$ verifying

$$\frac{f(a)}{a} = \frac{V_0}{k},$$

where $V_0 > 0$ was given in (V1). Using these numbers, we set the functions

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \leq a, \\ \frac{V_0}{k} t & \text{if } t \geq a \end{cases}$$

and

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda)\tilde{f}(t) \quad \text{for all } (x,t) \in \mathbb{R}^2,$$

where Λ was given in (V2) and χ_Λ denotes the characteristic function associated with Λ, that is,

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \Lambda, \\ 0 & \text{if } x \in \Lambda^c. \end{cases}$$

Using the above functions, we will study the existence of positive solutions for the following problem:

$$\begin{cases} (-\Delta)^{1/2} u + V(\varepsilon x) u = g_\varepsilon(x,u), & x \in \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \end{cases} \tag{AP}$$

where

$$g_\varepsilon(x,t) = g(\varepsilon x, t) \quad \text{for all } (x,t) \in \mathbb{R}^2.$$

We recall from [10] that, to get a solution for problem (AP), it is enough to study the existence of solutions for the following problem:

$$\begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial w}{\partial \nu} = V(\varepsilon x) w - g_\varepsilon(x,w) & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{AP$'$}$$

because if w is a solution of (AP′), the function $u(x) = w(x,0)$ is a solution for (AP).

Here, we would like to point out that if $v_\varepsilon \in X^1(\mathbb{R}^2_+)$ is a solution of (AP′) with

$$v_\varepsilon(x,0) < a \quad \text{for all } x \in \Lambda^c_\varepsilon,$$

where $\Lambda_\varepsilon = \Lambda/\varepsilon$, then $u_\varepsilon(x) = v_\varepsilon(x,0)$ is a solution of (P'_ε).

Associated with (AP′), we have the energy functional $E_\varepsilon : X^1(\mathbb{R}^2_+) \to \mathbb{R}$ given by

$$E_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla v|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x) |v(x,0)|^2 \, dx - \int_{\mathbb{R}} G_\varepsilon(x, v(x,0)) \, dx,$$

where

$$G_\varepsilon(x,t) = \int_0^t g_\varepsilon(x,\tau) \, d\tau \quad \text{for all } (x,t) \in \mathbb{R}^2.$$

Using the definition of g, it follows that

$$\theta G_\varepsilon(x,t) \leq g_\varepsilon(x,t) t \quad \text{for all } (x,t) \in \Lambda_\varepsilon \times \mathbb{R}, \tag{3.1}$$

and

$$2 G_\varepsilon(x,t) \leq g_\varepsilon(x,t) t \leq \frac{V_0}{k} |t|^2 \quad \text{for all } (x,t) \in (\Lambda_\varepsilon)^c \times \mathbb{R}. \tag{3.2}$$

From assumption (3.2), we deduce

$$L(x,t) = V(x) - G_\varepsilon(x,t) \geq \Big(1 - \frac{1}{2k}\Big) V(x) |t|^2 \geq 0 \quad \text{for all } (x,t) \in (\Lambda_\varepsilon)^c \times \mathbb{R}, \tag{3.3}$$

and

$$M(x,t) = V(x) - g_\varepsilon(x,t) t \geq \Big(1 - \frac{1}{k}\Big) V(x) |t|^2 \geq 0 \quad \text{for all } (x,t) \in (\Lambda_\varepsilon)^c \times \mathbb{R}. \tag{3.4}$$

Lemma 3.1. *The functional E_ε verifies the mountain pass geometry, that is,*
(i) *there are $r, \rho > 0$ such that $E_\varepsilon(v) \geq \rho$ for $\|v\| = r$;*
(ii) *there is $e \in X^1(\mathbb{R}^2_+)$ with $\|e\| > r$ and $E_\varepsilon(e) < 0$.*

Proof. By (3.1)–(3.4), there exist $c_1, c_2 > 0$ verifying

$$E_\varepsilon(v) \geq c_1\|v\|^2 - c_2\|v\|^q \quad \text{for all } v \in X^1(\mathbb{R}^2_+).$$

From the above inequality, we deduce that there are $r, \rho > 0$ such that

$$E_\varepsilon(v) \geq \rho \quad \text{for } \|v\|_{1,s} = r,$$

showing (i).

To prove (ii), fix $\varphi \in X^1(\mathbb{R}^2_+)$ with $\operatorname{supp}\varphi \subset \Lambda_\varepsilon \times \mathbb{R}$. Then, for $t > 0$,

$$E_\varepsilon(t\varphi) = \frac{t^2}{2}\|\varphi\|^2 - \int_{\mathbb{R}} F(t\varphi(x,0))\,dx.$$

From (f3), we know that there are $c_3, c_4 \geq 0$ verifying

$$F(t) \geq c_1|t|^\theta - c_2 \quad \text{for all } t \geq 0.$$

Using the above inequality, we derive

$$\lim_{t\to+\infty} E_\varepsilon(t\varphi) = -\infty.$$

Thereby, (ii) follows with $e = t\varphi$ and t large enough. □

In what follows, we denote by c_ε the mountain pass level associated with E_ε. Related to the case $\varepsilon = 0$, it is possible to prove that there is $w_0 \in X^1(\mathbb{R}^2_+)$ such that

$$J_0(w_0) = c_0 \quad \text{and} \quad J_0'(w_0) = 0.$$

The existence of w_0 can be obtained repeating the same approach explored in [2].

Lemma 3.2. *The minimax level c_0 verifies*

$$0 < c_0 < \min\{1, V_0\}\Big(\frac{1}{2} - \frac{1}{\theta}\Big).$$

Proof. Consider $w_* \in X^1(\mathbb{R}^2_+)$ verifying

$$\tilde{J}_0(w_*) = \beta_p \quad \text{and} \quad \tilde{J}_0'(w_*) = 0.$$

By characterization of c_0,

$$c_0 \leq \max_{t\geq 0} J_0(tw_*).$$

Consequently, by (f4),

$$c_0 \leq \max_{t\geq 0}\Big\{\frac{t^2}{2}\int_{\mathbb{R}^2_+} |\nabla w_*|^2\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}} V_0|w_*(x,0)|^2\,dx - \frac{C_p t^p}{p}\int_{\mathbb{R}} |w_*(x,0)|^p\,dx\Big\},$$

which implies that

$$c_0 \leq C_p^{2/(2-p)}\beta_p.$$

Hence, by (f4),

$$0 < c_0 < \min\{1, V_0\}\Big(\frac{1}{2} - \frac{1}{\theta}\Big).$$ □

Hereafter, we will assume that k is large enough such that

$$0 < c_0 < \min\{1, V_0\}\Big(\Big(\frac{1}{2} - \frac{1}{\theta}\Big) - \frac{1}{k}\Big) < \min\{1, V_0\}\Big(\frac{1}{2} - \frac{1}{\theta}\Big).$$

The next lemma establishes an important relation between c_ε and c_0.

Lemma 3.3. *The numbers c_0 and c_ε verify the equality*

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_0. \tag{3.5}$$

Hence, there is $\varepsilon_0 > 0$ such that

$$0 < \sup_{\varepsilon\in(0,\varepsilon_0)} c_\varepsilon < \min\{1, V_0\}\Big(\Big(\frac{1}{2}-\frac{1}{\theta}\Big)-\frac{1}{k}\Big). \tag{3.6}$$

Proof. By (V1), $c_\varepsilon \geq c_0$ for all $\varepsilon \geq 0$. Then,

$$\liminf_{\varepsilon\to 0} c_\varepsilon \geq c_0. \tag{3.7}$$

Next, fix $t_\varepsilon > 0$ such that

$$t_\varepsilon w \in \mathcal{M}_\varepsilon = \{v \in X^1(\mathbb{R}^2_+)\setminus\{0\} : E'_\varepsilon(v)v = 0\}.$$

By definition of c_ε, we know that

$$c_\varepsilon \leq \max_{t\geq 0} E_\varepsilon(tw) = E_\varepsilon(t_\varepsilon w).$$

Using standard arguments as those in [20], we can prove

$$\lim_{\varepsilon\to 0} t_\varepsilon = 1 \quad\text{and}\quad \lim_{\varepsilon\to 0} E_\varepsilon(t_\varepsilon w) = J_0(w).$$

Thus,

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq J_0(w) = c_0. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\limsup_{\varepsilon\to 0} c_\varepsilon = c_0,$$

showing (3.5). Inequality (3.6) is an immediate consequence of (3.5) and Lemma 3.2. □

Lemma 3.4. *Let $\varepsilon \in (0,\varepsilon_0)$ and let $(v_n) \subset X^1(\mathbb{R}^2_+)$ be a $(\mathrm{PS})_{c_\varepsilon}$ sequence for E_ε. Then,*

$$\limsup_{n\to+\infty}\|v_n\|^2 < 1. \tag{3.9}$$

Proof. Gathering $E_\varepsilon(u_n) - \frac{1}{\theta}E'_\varepsilon(u_n)u_n = c_\varepsilon + o_n(1)$ with the definition of g, we find

$$\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int_{\mathbb{R}^2_+}|\nabla v_n|^2\,dx\,dy + \Big(\Big(\frac{1}{2}-\frac{1}{\theta}\Big)-\frac{1}{k}\Big)V_0\int_{\mathbb{R}}|v_n(x,0)|^2\,dx \leq c_\varepsilon + o_n(1),$$

from where it follows that

$$\min\{1, V_0\}\Big(\Big(\frac{1}{2}-\frac{1}{\theta}\Big)-\frac{1}{k}\Big)\limsup_{n\to+\infty}\|v_n\|^2 \leq c_\varepsilon < \min\{1, V_0\}\Big(\Big(\frac{1}{2}-\frac{1}{\theta}\Big)-\frac{1}{k}\Big),$$

and thus (3.9). □

Lemma 3.5. *For $\varepsilon \in (0,\varepsilon_0)$, the functional E_ε verifies the $(\mathrm{PS})_{c_\varepsilon}$ condition.*

Proof. Let $(v_n) \subset X^1(\mathbb{R}^2_+)$ be a $(\mathrm{PS})_{c_\varepsilon}$ sequence for E_ε, that is,

$$E_\varepsilon(v_n) \to c_\varepsilon \quad\text{and}\quad E'_\varepsilon(v_n)\to 0 \quad\text{as } n\to\infty.$$

By Lemma 3.4, (v_n) is bounded in $X^1(\mathbb{R}^2_+)$ and $\limsup_{n\to+\infty}\|v_n\|^2 < 1$. Since $X^1(\mathbb{R}^2_+)$ is reflexive, there are a subsequence of (v_n), still denoted by itself, and $v \in X^1(\mathbb{R}^2_+)$ such that

$$\begin{aligned}
&v_n \rightharpoonup v \quad \text{weakly in } X^1(\mathbb{R}^2_+) \quad \text{as } n\to\infty,\\
&v_n \to v \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}) \quad \text{for all } q\in[2,+\infty) \quad \text{as } n\to\infty,\\
&v_n(x,0)\to v(x,0) \quad \text{a.e. in } \mathbb{R} \quad \text{as } n\to\infty.
\end{aligned}$$

Moreover, by Corollary 2.3,

$$\int_{\mathbb{R}} f(v_n(x,0))\phi(x,0)\,dx \to \int_{\mathbb{R}} f(v(x,0))\phi(x,0)\,dx$$

as $n \to \infty$ for all $\phi \in C_0^\infty(\overline{\mathbb{R}^2_+})$.

Using the above limits, it is possible to prove that v is a critical point for E_ε, that is,

$$E'_\varepsilon(v)\varphi = 0 \quad \text{for all } \varphi \in X^1(\mathbb{R}^2_+).$$

Considering $\varphi = v$, we have $E'_\varepsilon(v)v = 0$, and so,

$$\int_{\mathbb{R}^2_+} |\nabla v|^2\,dx\,dy + \int_{\Lambda_\varepsilon} V(\varepsilon x)|v(x,0)|^2\,dx + \int_{(\Lambda_\varepsilon)^c} M(x, v(x,0))\,dx = \int_{\Lambda_\varepsilon} f(v(x,0))v(x,0)\,dx.$$

On the other hand, using the limit $E'_\varepsilon(v_n)v_n = o_n(1)$, we derive that

$$\int_{\mathbb{R}^2_+} |\nabla v_n|^2\,dx\,dy + \int_{\Lambda_\varepsilon} V(\varepsilon x)|v_n(x,0)|^2\,dx + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x,0))\,dx = \int_{\Lambda_\varepsilon} f(v_n(x,0))v_n(x,0)\,dx + o_n(1).$$

Since Λ_ε is bounded, by the compactness of the Sobolev embedding and Corollary 2.3, we have

$$\lim_{n\to+\infty} \int_{\Lambda_\varepsilon} f(v_n(x,0))v_n(x,0)\,dx = \int_{\Lambda_\varepsilon} f(v(x,0))v(x,0)\,dx$$

and

$$\lim_{n\to+\infty} \int_{\Lambda_\varepsilon} V(\varepsilon x)|v_n(x,0)|^2\,dx = \int_{\Lambda_\varepsilon} V(\varepsilon x)|v(x,0)|^2\,dx. \tag{3.10}$$

Therefore,

$$\limsup_{n\to+\infty}\Big(\int_{\mathbb{R}^2_+} |\nabla v_n|^2\,dx\,dy + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x,0))\,dx\Big) = \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx\,dy + \int_{(\Lambda_\varepsilon)^c} M(x, v(x,0))\,dx.$$

Now, recalling that $M(x,t) \geq 0$, Fatou's lemma leads to

$$\liminf_{n\to+\infty}\Big(\int_{\mathbb{R}^2_+} |\nabla v_n|^2\,dx\,dy + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x,0))\,dx\Big) \geq \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx\,dy + \int_{(\Lambda_\varepsilon)^c} M(x, v(x,0))\,dx.$$

Hence,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2_+} |\nabla v_n|^2\,dx\,dy = \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx\,dy \tag{3.11}$$

and

$$\lim_{n\to+\infty} \int_{(\Lambda_\varepsilon)^c} M(x, v(x,0))\,dx = \int_{(\Lambda_\varepsilon)^c} M(x, v(x,0))\,dx.$$

The last limit combined with the definition of function M gives

$$\lim_{n\to+\infty} \int_{(\Lambda_\varepsilon)^c} V(\varepsilon x)|v_n(x,0)|^2\,dx = \int_{(\Lambda_\varepsilon)^c} V(\varepsilon x)|v(x,0)|^2\,dx.$$

Gathering this limit with (3.10), we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} V(\varepsilon x)|v_n(x,0)|^2\,dx = \int_{\mathbb{R}} V(\varepsilon x)|v(x,0)|^2\,dx. \tag{3.12}$$

By (3.11) and (3.12),

$$\lim_{n\to+\infty}\|v_n\|_\varepsilon^2 = \|v\|_\varepsilon^2.$$

Since $X^1(\mathbb{R}^2_+)$ is a Hilbert space and $v_n \rightharpoonup v$ weakly in $X^1(\mathbb{R}^2_+)$ as $n \to \infty$, the above limit yields

$$v_n \to v \quad \text{in } X^1(\mathbb{R}^2_+) \quad \text{as } n \to \infty,$$

showing that E_ε verifies the $(\mathrm{PS})_{c_\varepsilon}$ condition. □

Theorem 3.6. *For $\varepsilon \in (0, \varepsilon_0)$, the functional E_ε has a non-negative critical point $v_\varepsilon \in X^1(\mathbb{R}^2_+)$ such that*

$$E_\varepsilon(v_\varepsilon) = c_\varepsilon \quad \text{and} \quad E'_\varepsilon(v_\varepsilon) = 0. \tag{3.13}$$

Proof. By Lemma 3.3, there is $\varepsilon_0 > 0$ such that E_ε verifies the $(\mathrm{PS})_{c_\varepsilon}$ condition for $\varepsilon \in (0, \varepsilon_0)$. Then, the existence of v_ε is an immediate consequence of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz (see, e.g., [30]). The function v_ε is non-negative, because $E'_\varepsilon(v_\varepsilon)(v_\varepsilon^-) = 0$ implies $v_\varepsilon^- = 0$, where $v_\varepsilon^- = \min\{v_\varepsilon, 0\}$. □

Lemma 3.7. *Decreasing ε_0, if necessary, there are $r, \beta > 0$ and $(y_\varepsilon) \subset \mathbb{R}$ such that*

$$\int_{y_\varepsilon - r}^{y_\varepsilon + r} |v_\varepsilon(x, 0)|^2\, dx \geq \beta \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{3.14}$$

Proof. First of all, we recall that since (v_ε) satisfies (3.13), there is $\alpha > 0$, which is independent of ε, such that

$$\|v_\varepsilon\|_\varepsilon^2 \geq \alpha \quad \text{for all } \varepsilon > 0. \tag{3.15}$$

To show (3.14), it is enough to prove that, for any sequence $(\varepsilon_n) \subset (0, +\infty)$ with $\varepsilon_n \to 0$, the limit

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |v_{\varepsilon_n}(x, 0)|^2\, dx = 0,$$

does not hold for any $r > 0$. Otherwise, if it holds for some $r > 0$, by Lemma 2.4,

$$\int_{\mathbb{R}} f(v_{\varepsilon_n}(\cdot, 0))v_n(x, 0)\, dx \to 0 \quad \text{as } n \to \infty,$$

implying that $\|v_{\varepsilon_n}\|_\varepsilon^2 \to 0$ as $n \to +\infty$, which contradicts (3.15). □

Lemma 3.8. *For any $\varepsilon_n \to 0$, consider the sequence $(y_{\varepsilon_n}) \subset \mathbb{R}$ given in Lemma 3.7 and $\psi_n(x, y) = v_{\varepsilon_n}(x + y_{\varepsilon_n}, y)$. Then, up to subsequence, there is $\psi \in X^1(\mathbb{R}^2_+)$ such that*

$$\psi_n \to \psi \quad \text{in } X^1(\mathbb{R}^2_+) \quad \text{as } n \to \infty. \tag{3.16}$$

Moreover, there is $x_0 \in \Lambda$ such that

$$\lim_{n\to 0} \varepsilon_n y_{\varepsilon_n} = x_0 \quad \text{and} \quad V(x_0) = V_0. \tag{3.17}$$

Proof. We begin the proof showing that $(\varepsilon_n y_{\varepsilon_n})$ is a bounded sequence. Hereafter, we denote by (y_n) and (v_n) the sequences (y_{ε_n}) and (v_{ε_n}), respectively.

Since $E'_{\varepsilon_n}(v_n)\phi = 0$ for all $\phi \in X^1(\mathbb{R}^2_+)$, we have that

$$\int_{\mathbb{R}^2_+} \nabla v_n \nabla\phi\, dx\, dy + \int_{\mathbb{R}} V(\varepsilon_n x)v_n(x, 0)\phi(x, 0)\, dx - \int_{\mathbb{R}} g_\varepsilon(x, v_n(x, 0))\phi(x, 0)\, dx = 0.$$

Then,

$$\int_{\mathbb{R}^2_+} |\nabla v_n|^2\, dx\, dy + \int_{\mathbb{R}} V(\varepsilon_n x)|v_n(x, 0)|^2\, dx - \int_{\mathbb{R}} g_\varepsilon(x, v_n(x, 0))v_n(x, 0)\, dx = 0.$$

From the definition of g, we see that

$$g_\varepsilon(x,t) \le f(t) \quad \text{for all } t \ge 0.$$

Recalling that $v_n \ge 0$, we infer that

$$\int_{\mathbb{R}^2_+} |\nabla v_n|^2\,dx\,dy + \int_{\mathbb{R}} V_0|v_n(x,0)|^2\,dx - \int_{\mathbb{R}} f(v_n(x,0))v_n(x,0)\,dx \le 0.$$

Therefore, there is $s_n \in (0,1)$ such that

$$s_n v_n \in \mathcal{M}_0 = \{v \in X^1(\mathbb{R}^2_+) \setminus \{0\} : J_0'(v)v = 0\}.$$

Using the characterization of c_0, we know that

$$c_0 \le J_0(s_n v_n) \quad \text{for all } n \in \mathbb{N}.$$

As

$$J_0(w) \le E_\varepsilon(w) \quad \text{for all } w \in X^1(\mathbb{R}^2_+),\ \varepsilon > 0,$$

it follows that

$$c_0 \le J_0(s_n v_n) \le E_{\varepsilon_n}(s_n v_n) \le \max_{s\ge 0} E_{\varepsilon_n}(s v_n) = E_{\varepsilon_n}(v_n) = c_{\varepsilon_n}.$$

Recalling that $c_{\varepsilon_n} \to c_0$ as $n \to \infty$, the last inequality gives

$$(s_n v_n) \subset \mathcal{M}_0 \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad J_0(s_n v_n) \to c_0 \quad \text{as } n \to \infty.$$

By change of variable, we also have

$$(s_n \psi_n) \subset \mathcal{M}_0 \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad J_0(s_n \psi_n) \to c_0 \quad \text{as } n \to \infty.$$

Using Ekeland's variational principle, we can assume that $(s_n v_n)$ is a $(\mathrm{PS})_{c_0}$ sequence, that is,

$$(s_n \psi_n) \subset \mathcal{M}_0 \quad \text{for all } n \in \mathbb{N}, \quad J_0(s_n \psi_n) \to c_0 \quad \text{and} \quad J_0'(s_n \psi_n) \to 0 \quad \text{as } n \to \infty.$$

A direct computation shows that (s_n) is a bounded sequence with

$$\liminf_{n\to+\infty} s_n > 0.$$

Thus, in what follows, we can assume that for some subsequence, there is $s_0 > 0$ such that $s_n \to s_0$ as $n \to \infty$. From definition of y_n and ψ_n, we know that $\psi \in X^1(\mathbb{R}^2_+) \setminus \{0\}$. Moreover, since $J_0'(s_n\psi_n) \to 0$, we also have $J_0'(s_0\psi) = 0$. Thereby, by definition of c_0, we obtain

$$c_0 \le J_0(s_0\psi).$$

On the other hand, by Fatou's lemma we obtain

$$\liminf_{n\to+\infty} J_0(s_n\psi_n) \ge J_0(s_0\psi)$$

which implies

$$J_0(s_0\psi) = c_0 \quad \text{and} \quad J_0'(s_0\psi) = 0.$$

The above equalities combined with Fatou's lemma, up to a subsequence, give

$$s_n\psi_n \to s_0\psi \quad \text{in } X^1(\mathbb{R}^2_+) \quad \text{as } n \to \infty.$$

Recalling that $s_n \to s_0 > 0$ as $n \to \infty$, we can conclude that

$$\psi_n \to \psi \quad \text{in } X^1(\mathbb{R}^2_+) \quad \text{as } n \to \infty,$$

showing (3.16).

Using the last limit, we are able to prove (3.17). First, we prove the following fact:

Claim 3.9. $\lim_{n\to+\infty}\operatorname{dist}(\varepsilon_n y_n,\overline{\Lambda})=0$.

Indeed, if the claim does not hold, there is $\delta>0$ and a subsequence of $(\varepsilon_n y_n)$, still denoted by itself, such that

$$\operatorname{dist}(\varepsilon_n y_n,\overline{\Lambda})\geq\delta\quad\text{for all } n\in\mathbb{N}.$$

Consequently, there is $r>0$ such that

$$(\varepsilon_n y_n-r,\varepsilon_n y_n+r)\subset\Lambda^c\quad\text{for all } n\in\mathbb{N}.$$

From definition of ψ_n, we have that

$$\int_{\mathbb{R}^2_+}|\nabla\psi_n|^2\,dx\,dy+\int_{\mathbb{R}}V(\varepsilon_n x+\varepsilon_n y_n)|\psi_n(x,0)|^2\,dx=\int_{\mathbb{R}}g(\varepsilon_n x+\varepsilon_n y_n,\psi_n(x,0))\psi_n(x,0)\,dx.$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}}g(\varepsilon_n x+\varepsilon_n y_n,\psi_n(x,0))\psi_n(x,0)\,dx\\&\quad\leq\int_{-r/\varepsilon_n}^{r/\varepsilon_n}g(\varepsilon_n x+\varepsilon_n y_n,\psi_n(x,0))\psi_n(x,0)\,dx+\Big(\int_{-\infty}^{-r/\varepsilon_n}+\int_{r/\varepsilon_n}^{+\infty}\Big)g(\varepsilon_n x+\varepsilon_n y_n,\psi_n(x,0))\psi_n(x,0)\,dx,\end{aligned}$$

and so,

$$\begin{aligned}&\int_{\mathbb{R}}g(\varepsilon_n x+\varepsilon_n y_n,\psi_n(x,0))\psi_n(x,0)\,dx\\&\quad\leq\frac{V_0}{k}\int_{-r/\varepsilon_n}^{r/\varepsilon_n}|\psi_n(x,0)|^2\,dx+\Big(\int_{-\infty}^{-r/\varepsilon_n}+\int_{r/\varepsilon_n}^{+\infty}\Big)f(\psi_n(x,0))\psi_n(x,0)\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^2_+}|\nabla\psi_n|^2\,dx\,dy+\int_{\mathbb{R}}V(\varepsilon_n x+\varepsilon_n y_n)|\psi_n(x,0)|^2\,dx\\&\quad\leq\frac{V_0}{k}\int_{-r/\varepsilon_n}^{r/\varepsilon_n}|\psi_n(x,0)|^2\,dx+\Big(\int_{-\infty}^{-r/\varepsilon_n}+\int_{r/\varepsilon_n}^{+\infty}\Big)f(\psi_n(x,0))\psi_n(x,0)\,dx,\end{aligned}$$

implying that

$$\int_{\mathbb{R}^2_+}|\nabla\psi_n|^2\,dx\,dy+A\int_{\mathbb{R}^N}|\psi_n(x,0)|^2\,dx\leq\Big(\int_{-\infty}^{-r/\varepsilon_n}+\int_{r/\varepsilon_n}^{+\infty}\Big)f(\psi_n(x,0))\psi_n(x,0)\,dx,\tag{3.18}$$

where $A=V_0(1-\frac{1}{k})$. By (3.16), we have

$$\Big(\int_{-\infty}^{-r/\varepsilon_n}+\int_{r/\varepsilon_n}^{+\infty}\Big)f(\psi_n(x,0))\psi_n(x,0)\,dx\to0\quad\text{as } n\to\infty,$$

and

$$\int_{\mathbb{R}^2_+}|\nabla\psi_n|^2\,dx\,dy+A\int_{\mathbb{R}}|\psi_n(x,0)|^2\,dx\to\int_{\mathbb{R}^2_+}|\nabla\psi|^2\,dx\,dy+A\int_{\mathbb{R}}|\psi(x,0)|^2\,dx>0\quad\text{as } n\to\infty,$$

which contradicts (3.18). This proves Claim 3.9.

By Claim 3.9, there are a subsequence of $(\varepsilon_n y_n)$ and $x_0\in\overline{\Lambda}$ such that

$$\lim_{n\to+\infty}\varepsilon_n y_n=x_0.$$

Claim 3.10. $x_0 \in \Lambda$.

Indeed, from definition of ψ_n, we get

$$\int_{\mathbb{R}^2_+} |\nabla \psi_n|^2 \, dx \, dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n)|\psi_n(x,0)|^2 \, dx \leq \int_{\mathbb{R}} f(\psi_n(x,0))\psi_n(x,0) \, dx.$$

Then, by (3.16),

$$\int_{\mathbb{R}^2_+} |\nabla \psi|^2 \, dx \, dy + \int_{\mathbb{R}} V(x_0)|\psi(x,0)|^2 \, dx \leq \int_{\mathbb{R}} f(\psi(x,0))\psi(x,0) \, dx.$$

Hence, there is $s_1 \in (0,1)$ such that

$$s_1 \psi \in \mathcal{M}_{V(x_0)} = \{v \in X^1(\mathbb{R}^2_+) \setminus \{0\} : \tilde{J}'_{V(x_0)} v = v\},$$

where $\tilde{J}_{V(x_0)} : X^1(\mathbb{R}^2_+) \to \mathbb{R}$ is given by

$$\tilde{J}_{V(x_0)}(v) = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla v|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V(x_0)|v(x,0)|^2 \, dx - \int_{\mathbb{R}} F(v(x,0)) \, dx.$$

If $\tilde{c}_{V(x_0)}$ denotes the mountain pass level associated with $\tilde{J}_{V(x_0)}$, we must have

$$\tilde{c}_{V(x_0)} \leq \tilde{J}_{V(x_0)}(s_1\psi) \leq \liminf_{n \to +\infty} E_{\varepsilon_n}(v_n) = \liminf_{n \to +\infty} c_{\varepsilon_n} = c_0 = \tilde{c}_{V(0)}.$$

Hence,

$$\tilde{c}_{V(x_0)} \leq \tilde{c}_{V(0)},$$

from where it follows that

$$V(x_0) \leq V(0).$$

As $V_0 = \inf_{x \in \mathbb{R}} V(x)$, the above inequality implies that

$$V(x_0) = V(0) = V_0.$$

Moreover, by (V2), we have $x_0 \notin \partial\Lambda$. Then, $x_0 \in \Lambda$, finishing the proof. □

Corollary 3.11. *Let (ψ_n) be the sequence given in Lemma 3.8. Then, $\psi_n(\cdot,0) \in L^\infty(\mathbb{R})$ and there is $K > 0$ such that*

$$|\psi_n(\cdot,0)|_\infty \leq K \quad \text{for all } n \in \mathbb{N}$$

and

$$\psi_n(\cdot,0) \to \psi(\cdot,0) \quad \text{in } L^p(\mathbb{R}) \quad \text{for all } p \in (2,+\infty) \quad \text{as } n \to \infty. \tag{3.19}$$

As an immediate consequence, the sequence $h_n(x) = g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x,0))$ must verify

$$h_n \to f(\psi(\cdot,0)) \quad \text{in } L^p(\mathbb{R}) \quad \text{for all } p \in (2,+\infty) \quad \text{as } n \to \infty. \tag{3.20}$$

Proof. In what follows, for each $L > 0$, we set

$$\psi_{n,L}(x,y) = \begin{cases} \psi_n(x,y) & \text{if } \psi_n(x,y) \leq L, \\ L & \text{if } \psi_n(x,y) \geq L, \end{cases} \quad \text{and} \quad z_{n,L} = \psi_{n,L}^{2(\beta-1)}\psi_n$$

with $\beta > 1$ to be determined later. Since

$$\int_{\mathbb{R}^2_+} \nabla \psi_n \nabla \phi \, dx \, dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n)\psi_n(x,0)\phi(x,0) \, dx$$
$$- \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x,0))\phi(x,0) \, dx = 0 \quad \text{for all } \phi \in X^1(\mathbb{R}^2_+), \ n \in \mathbb{N},$$

adapting the same approach explored in [3, Lemma 4.1] and using the fact that (ψ_n) is bounded in $X^1(\mathbb{R}^2_+)$, we conclude that there is $K > 0$ such that

$$|\psi_n(\cdot, 0)|_\infty \le K \quad \text{for all } n \in \mathbb{N}.$$

Now, the limit (3.19) is obtained by interpolation on the L^p spaces, while (3.20) follows combining the growth condition on g with (3.19). □

In what follows, we denote by $(w_n) \subset H^{1/2}(\mathbb{R})$ the sequence $(\psi_n(\cdot, 0))$, that is,

$$w_n(x) = \psi_n(x, 0) \quad \text{for all } x \in \mathbb{R}.$$

Since

$$\int_{\mathbb{R}^2_+} \nabla\psi_n \nabla\phi \, dx \, dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n)\psi_n(x, 0)\phi(x, 0) \, dx - \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0))\phi(x, 0) \, dx = 0 \quad \text{for all } \phi \in X^1(\mathbb{R}^2_+),$$

we have that w_n is a solution of the problem

$$(-\Delta)^{1/2} w_n + V(\varepsilon_n x + \varepsilon_n y_n) w_n = g(\varepsilon_n x + \varepsilon_n y_n, w_n) \quad \text{in } \mathbb{R},$$

or equivalently

$$(-\Delta)^{1/2} u + w_n = \chi_n \quad \text{in } \mathbb{R}, \tag{3.21}$$

where

$$\chi_n(x) = w_n(x) + g(\varepsilon_n x + \varepsilon_n y_n x, w_n(x)) - V(\varepsilon_n x + \varepsilon_n y_n) w_n(x), \quad x \in \mathbb{R}.$$

Denoting $\chi(x) = w(x) + f(w(x)) - V(x_0)w(x)$, we deduce from Corollary 3.11 that

$$\chi_n \to \chi \quad \text{in } L^p(\mathbb{R}) \quad \text{for all } p \in [2, +\infty) \quad \text{as } n \to \infty,$$

and that there is $k_1 > 0$ such that

$$|\chi_n|_\infty \le k_1 \quad \text{for all } n \in \mathbb{N}.$$

Motivated by some results found in [7] (see also [22]), which hold for the whole line, we deduce that

$$w_n(x) = (\mathcal{K} * \chi_n)(x) = \int_{\mathbb{R}} \mathcal{K}(x - y)\chi_n(y) \, dy,$$

where $\mathcal{K}$ is the Bessel kernel which verifies:

(K1) $\mathcal{K}$ is positive and even on $\mathbb{R} \setminus \{0\}$.
(K2) There is $C > 0$ such that $\mathcal{K}(x) \le C/|x|^2$ for all $x \in \mathbb{R} \setminus \{0\}$.
(K3) $\mathcal{K} \in L^q(\mathbb{R})$ for all $q \in [1, \infty]$.

Using the above information, we are able to prove the following result.

Lemma 3.12. *The sequence* (w_n) *verifies* $w_n(x) \to 0$ *as* $|x| \to +\infty$, *uniformly in* $n \in \mathbb{N}$.

Proof. Given $\delta > 0$, we have

$$0 \le w_n(x) \le \int_{\mathbb{R}} \mathcal{K}(x - y)|\chi_n|(y) \, dy = \Bigl(\int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty}\Bigr) \mathcal{K}(x - y)|\chi_n|(y) \, dy + \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x - y)|\chi_n|(y) \, dy.$$

By (K2), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\Bigl(\int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty}\Bigr) \mathcal{K}(x - y)|\chi_n|(y) \, dy &\le C\delta^{1/2} |\chi_n|_\infty \Bigl(\int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty}\Bigr) \frac{dy}{|x - y|^{3/2}} \\ &\le C\delta^{1/2} k_1 \Bigl(\int_{-\infty}^{x-1} + \int_{x+1}^{+\infty}\Bigr) \frac{dy}{|x - y|^{3/2}} = C_1 \delta^{1/2}.\end{aligned} \tag{3.22}$$

On the other hand,

$$\int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y)|\chi_n-\chi|(y)\,dy + \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y)|\chi|(y)\,dy.$$

Fix $q > 1$ with q sufficiently close to 1 and $q' > 2$ such that $1/q + 1/q' = 1$. By (K2) and (3.21),

$$\int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le |\mathcal{K}|_q|\chi_n-\chi|_{q'} + |\mathcal{K}|_q|\chi|_{L^{q'}(x-1/\delta,x+1/\delta)}.$$

Since

$$|\chi_n-\chi|_{q'} \to 0 \quad \text{as } n \to +\infty$$

and

$$|\chi|_{L^{q'}(x-1/\delta,x+1/\delta)} \to 0 \quad \text{as } |x| \to +\infty,$$

we deduce that there are $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le \delta \quad \text{for all } n \ge n_0,\ |x| \ge R. \tag{3.23}$$

By (3.22) and (3.23),

$$\int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le C_1\delta^d + \delta \quad \text{for all } n \ge n_0,\ |x| \ge R.$$

The same approach can be used to prove that for each $n \in \{1, \dots, n_0-1\}$, there is $R_n > 0$ such that

$$\int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le C_1\delta^d + \delta, \quad |x| \ge R_n.$$

Hence, increasing R, if necessary, we must have

$$\int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y)\,dy \le C_1\delta^d + \delta \quad \text{for } |x| \ge R, \quad \text{uniformly in } n \in \mathbb{N}.$$

Since δ is arbitrary, the proof is finished. □

Corollary 3.13. *There is $n_0 \in \mathbb{N}$ such that*

$$v_n(x,0) < a \quad \textit{for all } n \ge n_0,\ x \in \Lambda^c_{\varepsilon_n}.$$

Hence, $u_n(x) = v_n(x,0)$ is a solution of (P'_{ε_n}) for $n \ge n_0$.

Proof. By Lemma 3.8, we know that $\varepsilon_n y_n \to x_0$ for some $x_0 \in \Lambda$. Therefore, there is $r > 0$ such that up to subsequence we have

$$(r - \varepsilon_n y_n, r + \varepsilon_n y_n) \subset \Lambda \quad \text{for all } n \in \mathbb{N}.$$

Hence,

$$(y_n - r/\varepsilon_n, y_n + r/\varepsilon_n) \subset \Lambda_{\varepsilon_n} \quad \text{for all } n \in \mathbb{N},$$

or equivalently

$$\Lambda^c_{\varepsilon_n} \subset (-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty) \quad \text{for all } n \in \mathbb{N}.$$

Now, by Lemma 3.12, there is $R > 0$ such that

$$w_n(x) < a \quad \text{for all } |x| \ge R,\ n \in \mathbb{N},$$

from where it follows that

$$v_n(x,0) = \psi_n(x-y_n,0) = w_n(x-y_n) < a \quad \text{for all } x \in (-\infty, y_n - R) \cup (y_n + R, +\infty), \; n \in \mathbb{N}.$$

On the other hand, we have that

$$\Lambda^c_{\varepsilon_n} \subset (-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty) \quad \text{for all } n \in \mathbb{N}.$$

Thus, there is $n_0 \in \mathbb{N}$ such that

$$(-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty) \subset (-\infty, y_n - R) \cup (y_n + R, +\infty) \quad \text{for all } n \geq n_0,$$

implying that

$$v_n(x,0) < a \quad \text{for all } x \in \Lambda^c_{\varepsilon_n}, \; n \geq n_0,$$

finishing the proof. □

4 Proof of Theorem 1.2

By Theorem 3.6, we know that problem (AP) has a non-negative solution v_ε for all $\varepsilon > 0$. Applying Corollary 3.13, there is ε_0 such that

$$v_\varepsilon(x,0) < a \quad \text{for all } x \in \Lambda^c_\varepsilon, \; \varepsilon \in (0, \varepsilon_0),$$

that is, $v_\varepsilon(\cdot, 0)$ is a solution of (P'_ε) for $\varepsilon \in (0, \varepsilon_0)$. Thus,

$$u_\varepsilon(x) = v_\varepsilon(x/\varepsilon, 0) \quad \text{for } \varepsilon \in (0, \varepsilon_0)$$

is a solution for original problem (P_ε).

If x_ε denotes a global maximum point of u_ε, it is easy to see that there is $\tau_0 > 0$ such that

$$u_\varepsilon(x_\varepsilon) \geq \tau_0 \quad \text{for all } \varepsilon > 0.$$

In what follows, setting $z_\varepsilon = (x_\varepsilon - \varepsilon y_\varepsilon)\varepsilon^{-1}$, we have that z_ε is a global maximum point of w_ε and

$$w_\varepsilon(z_\varepsilon) \geq \tau_0 \quad \text{for all } \varepsilon > 0.$$

Now, we claim that

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0. \tag{4.1}$$

Indeed, by Lemma 3.12, we know that

$$w_{\varepsilon_n}(x) \to 0 \quad \text{as } |x| \to +\infty \quad \text{uniformly in } n \in \mathbb{N}.$$

Therefore, (z_ε) is a bounded sequence. Moreover, for some subsequence, we also know that there is $x_0 \in \Lambda$ satisfying $V(x_0) = V_0$ and

$$\varepsilon_n y_{\varepsilon_n} \to x_0 \quad \text{as } n \to \infty.$$

Hence,

$$x_{\varepsilon_n} = \varepsilon_n z_{\varepsilon_n} + \varepsilon_n y_{\varepsilon_n} \to x_0 \quad \text{as } n \to \infty,$$

implying that

$$V(x_{\varepsilon_n}) \to V_0 \quad \text{as } n \to \infty,$$

showing that (4.1) holds.

Funding: The authors were supported by INCTMAT/CNPq/Brazil. C. O. Alves was supported by CNPq/Brazil Proc. 304036/2013-7. J. M. do Ó was supported by CNPq/Brazil Proc. 407099/2013-1. O. H. Miyagaki was supported by CNPq/Brazil Proc. 304015/2014-8 and CAPES/Brazil Proc. 2531/14-3.

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